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Advanced Differential Equations with Canonical Operators: New Criteria for the Oscillation

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Abstract: In this study, we use the integral averaging methodology, comparison with second-order differential equations, and the Riccati technique to determine the Philos-type and Hille–Nehari-type oscillation conditions of fourth-order advanced differential equations with canonical operators. In essence, these techniques supplement and generalize a wide range of established oscillation conditions. Two instance cases demonstrate the importance of our outcomes and their significant improvement.

Keywords: advanced; fourth order; p-Laplace; differential equations; oscillation

MSC: 34C10; 34K11

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1. Introduction

The oscillation terms in advanced and delay differential equations (DDEs) play a pivotal role in modeling many phenomena, especially in studying population dynamics and analyzing their environmental impacts. Laplace-type differential equations show high application value, as their uses are widely diversified in the fields of mechanical systems, electrical circuits, and the regulation of chemical processes, which highlights their importance in these practical fields [1–4]. In addition, their usefulness extends to include environmental systems and epidemiology, and they are used, in particular, in modeling population dynamics. These equations contribute to providing accurate mathematical tools for analyzing oscillatory patterns and responses in complex dynamic systems that are affected by multiple variables. Oscillation studies help in understanding how biological and chemical systems interact with external influences, which contributes to the development of accurate predictive models that are widely used in applied sciences; see [5,6].

In recent years, the study of the behavior of solutions to fractional differential equations has become an important and prominent field in science and engineering, where these criteria have gained great attention from researchers, engineers, and scientists [7–9]. This interest has led to the development of advanced models based on differential equations containing delays and fractional properties, which have proven useful in many fields. This approach is effectively used in modeling the behavior of proteins and polymers, the propagation of ultrasound, and the study of the mechanical behavior of human tissues under stress. These models provide valuable scientific insights that help improve our understanding of many biological and physical phenomena, enhancing the ability of scientists to design innovative solutions to complex applied problems; see [10,11].

The phenomenon of oscillation has attracted significant interest from researchers in a wide range of practical domains. This is mostly due to the fact that oscillation has many uses in science and engineering and that it originates from mechanical vibrations. In order to account for the impact of temporal contexts on various solutions, oscillation models can incorporate advanced terms or delays. As demonstrated by the contributions of [12–15], a significant amount of research has been conducted on the subject of oscillation in delay equations. There are not many publications that expressly address advanced oscillation, making the body of the existing literature on the subject rather small when compared to other research fields [16–24].

Oscillation theory investigates the circumstances in which differential equation solutions display recurring patterns. From basic harmonic motions to intricate, chaotic oscillations, this covers both linear and nonlinear systems [25,26]. The behavior of oscillatory systems can be predicted and controlled by researchers by looking at the characteristics of these solutions, such as amplitude, frequency, and stability. Moreover, several studies have looked into the methods and characteristics of oscillatory solutions to differential equations that involve certain generic fractional derivatives. This method's ability to provide findings for numerous fractional derivatives in a unified manner is one of its advantages. Oscillation theory has recently been extended to more complicated fields, such as fractional calculus, quantum physics, and delay differential equations with the p-Laplace-type operator [27–30]. Understanding oscillatory behavior in various circumstances is becoming more and more crucial as contemporary problems like biological rhythms, climate models, and sophisticated communication systems arise.

In this paper, we find new oscillation criteria for fourth-order advanced differential equations with the p-Laplace type operator,

$$\left(r(t)\left(x'''(t)\right)^{p-1}\right)' + \sum_{i=1}^{\ell} \eta_i(t) x^{p-1}(\sigma_i(t)) = 0, \;\; t \geq t_0, \; p > 1, \tag{1}$$

where $r \in C^1([t_0,\infty),\mathbb{R})$, $r(t) > 0$, $r'(t) \geq 0$, $\eta_i, \sigma_i \in C([t_0,\infty),\mathbb{R})$, $\eta_i(t) \geq 0$, $\sigma_i(t) \geq t$, $\lim_{t\to\infty} \sigma_i(t) = \infty$, $i = 1,2,...,\ell$.

We intend to find a solution of (1) a function $x(t) : [t_x,\infty) \to \mathbb{R}, t_x \geq t_0$, such that $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. We assume that (1) possesses such a solution. If the solution to (1) is neither positive in the end nor negative in the end, it is called oscillatory; otherwise, this solution is called non-oscillatory. Equation (1) itself is called oscillatory if all of its solutions are oscillatory.

Definition 1. *Let*

$$M = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \text{ and } M_0 = \{(t,s) \in \mathbb{R}^2 : t > s \geq t_0\}.$$

$G_i \in C(D,\mathbb{R})$ is a kernel function. It is written by $G \in y$, which indicates that it belongs to the function class y if, for $i = 1,2$,

(i) $G_i(t,s) = 0$ *for* $t \geq t_0$, $G_i(t,s) > 0$, $(t,s) \in M_0$.

$G_i(t,s)$ is a non-positive function that has a partial derivative and is also continuous $\partial G_i/\partial s$ on r_0 and $\delta, \xi \in C^1([t_0,\infty),(0,\infty))$ and $g_i \in C(M_0,\mathbb{R})$, such that

$$\frac{\partial}{\partial s} G_1(t,s) + \frac{\delta'(s)}{\delta(s)} G_1(t,s) = g_1(t,s) G_1^{(p-1)/((p-1)+1)}(t,s) \tag{2}$$

and

$$\frac{\partial}{\partial s} G_2(t,s) + \frac{\xi'(s)}{\xi(s)} G_2(t,s) = g_2(t,s)\sqrt{G_2(t,s)}. \tag{3}$$

The topic of creating non-oscillation and/or oscillation conditions for differential equations with diverging arguments has been a very active study subject since it was first examined in Fite's foundational paper [31]. The vast majority of the research to date has focused on studying delay differential equations; in contrast, relatively few studies have examined equations with advanced terms.

$$\left(r(t)\left(x'(t)\right)^{\gamma}\right)' + \eta(t)x^{\gamma}(\sigma(t)) = 0, \tag{4}$$

and its specific cases or generalizations, primarily, are concerned with canonical equations; see [32–34]. As a result, current research has tried to enhance the established oscillation criteria. In particular, Chatzarakis et al. [35] explored the the oscillation and asymptotic behavior of (4) in the non-canonical case

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(s)} \mathrm{d}s < \infty, \tag{5}$$

and they also used several iterative methods to create new oscillation criteria.

Some recent works have attempted to complement asymptotic behavior. In particular, Agarwal et al. [36] created a novel comparison principle for equations with the canonical type and advanced terms,

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\gamma}\right)' + \eta(t)x^{\gamma}(\sigma(t)) = 0, \tag{6}$$

under

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(s)} \mathrm{d}s = \infty. \tag{7}$$

Many authors have studied the oscillatory behavior of (6). In particular, Agarwal and Grace [37] showed that if

$$\int_{t_0}^{\infty} \eta(s) \mathrm{d}s < \infty$$

then (6) is oscillatory, while Agarwal et al. [38] proved that if

$$\limsup_{t\to\infty} t^{\gamma(n-1)} \int_{t}^{\infty} \eta(s) \mathrm{d}s > ((n-1)!)^{\gamma}$$

then (6) is oscillatory.

We shall be studying canonical equations with p-Laplace type operators in this work, so let us concentrate on them.

The main motivation for this work was to contribute to the development of the oscillation theory for fourth-order advanced equations. The objective of this paper was to extend the results in [36–38] by obtaining new conditions for (1) under

$$\int_{t_0}^{\infty} \frac{1}{r^{1/(p-1)}(s)} \mathrm{d}s = \infty. \tag{8}$$

by using the integral averaging and Riccati techniques and the comparison method. Furthermore, we include illustrated instances that show the theoretical significance and practical implementation of our criteria.

2. Some Auxiliary Lemmas

We begin this section with two preliminary lemmas.

Lemma 1 ([39]). *Let E and $D > 0$ be constants. Then,*

$$\frac{p-1^{p-1}}{p^{p}} \frac{E^{p}}{D^{(p-1)}} \geq Ex - Dx^{p/(p-1)}.$$

Lemma 2 ([36]). *Let $x \in C^{a}([t_0,\infty),(0,\infty))$, $x^{(a-1)}(t)x^{(a)}(t) \leq 0$, and $x^{(a)}$ be of a fixed sign and not identically zero on $[t_0,\infty)$. If $\lim_{t\to\infty} x(t) \neq 0$ then*

$$x(t) \geq \frac{\varepsilon}{(a-1)!} t^{a-1} \left| x^{(a-1)}(t) \right|$$

for every $t \geq t_{\varepsilon}$ and $\varepsilon \in (0,1)$.

Lemma 3 ([40]). *The function x is identified as the ultimate positive solution to (1). Thus, we find two cases:*

$$\begin{array}{ll} (\mathbf{S}_1) & x'(t) > 0, x''(t) > 0,\ x'''(t) > 0 \text{ and } x^{(4)}(t) < 0, \\ (\mathbf{S}_2) & x'(t) > 0, x'''(t) > 0, x''(t) < 0 \text{ and } x^{(4)}(t) < 0, \end{array}$$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Lemma 4 ([41]). *Let the equation*

$$\left[r(t)\left(x'(t)\right)^{\theta} \right]' + \eta(t) x^{\theta}(\omega(t)) = 0, \quad t \geq t_0, \tag{9}$$

where $\theta > 0$ is the odd-to-positive integer ratio, r , $\eta \in C([t_0,\infty),\mathbb{R}^+)$ is only non-oscillatory if and when a number $t \geq t_0$, and a function $\zeta \in C^1([t,\infty),\mathbb{R})$, fulfilling the inequality

$$\zeta'(t) + \gamma r^{-1/\theta}(t)(\zeta(t))^{(1+\theta)/\theta} + \eta(t) \leq 0, \quad \text{on } [t,\infty).$$

3. Main Results

We will find certain oscillation conditions of the Philos tyoe and the Hille–Nehari type for (1) in this section.

In this theorem, we obtain a Philos-type oscillation criterion for (1) by the integral averaging technique:

Theorem 1 (Let (8) hold). *If $\delta, \xi \in C^1([t_0,\infty),\mathbb{R})$, such that*

$$\limsup_{t\to\infty} \frac{1}{G_1(t,t_1)} \int_{t_1}^{t} \left(G_1(t,s)\delta(s) \sum_{i=1}^{\ell} \eta_i(s) - \varnothing(s) \right) \mathrm{d}s = \infty \tag{10}$$

and

$$\limsup_{t\to\infty} \frac{1}{G_2(t,t_1)} \int_{t_1}^{t} \left(G_2(t,s)\xi(s) \int_{t}^{\infty} \left(\frac{1}{r(s)} \varphi(s) \right)^{1/(p-1)} \mathrm{d}s - \frac{\xi(s) g_2^2(t,s)}{4} \right) \mathrm{d}s = \infty, \tag{11}$$

where

$$\varnothing(s) = \frac{g_1^p(t,s) G_1^{p-1}(t,s)}{p^p} \frac{2^{(p-1)}\delta(s) r(s)}{(\varepsilon s^2)^{p-1}},$$

for all $\varepsilon \in (0,1)$, and

$$\varphi(s) = \int_{\zeta}^{\infty} \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s$$

then (1) is oscillatory.

Proof. The function x is identified as the ultimate positive solution to (1). Using Lemma 3, we see cases$(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

We assume that $(\mathbf{S}_1)$ holds. Using Lemma 2, we find

$$x'(t) \geq \frac{\varepsilon}{2} t^2 x'''(t). \tag{12}$$

Let us know the Riccati formula as follows:

$$\psi(t) := \delta(t) r(t) x^{1-p}(t) \big(x'''(t)\big)^{p-1}; \tag{13}$$

we see that $\psi(t) > 0$ for $t \geq t_1$, where $\delta \in C^1([t_0, \infty), (0, \infty))$, and

$$\begin{aligned} \psi'(t) \quad = \quad & \delta'(t) r(t) x^{1-p}(t) \big(x'''(t)\big)^{p-1} + \delta(t) x^{1-p}(t) \Big(r \big(x'''\big)^{p-1}\Big)'(t) \\ & -(p-1)\delta(t) x^{2(1-p)}(t) x^{p-2}(t) x'(t) r(t) \big(x'''(t)\big)^{p-1}. \end{aligned}$$

When (12) and (13) are combined, we obtain

$$\begin{aligned} \psi'(t) \quad \leq \quad & \frac{\delta'(t)}{\delta(t)} \psi(t) + \delta(t) \frac{\Big(r(t)(x'''(t))^{p-1}\Big)'}{x^{p-1}(t)} \\ & -(p-1)\delta(t) \frac{\varepsilon}{2} t^2 \frac{r(t)(x'''(t))^p}{x^p(t)} \\ \leq \quad & \frac{\delta'(t)}{\delta(t)} \psi(t) + \delta(t) \frac{\Big(r(t)(x'''(t))^{(p-1)}\Big)'}{x^{(p-1)}(t)} \\ & - \frac{(p-1)\varepsilon t^2}{2(\delta(t) r(t))^{\frac{1}{(p-1)}}} \psi^{\frac{p}{p-1}}(t). \end{aligned} \tag{14}$$

From (1) and (14), we obtain

$$\psi'(t) \leq \frac{\delta'(t)}{\delta(t)} \psi(t) - \delta(t) \frac{\sum_{i=1}^{\ell} \eta_i(t) x^{p-1}(\sigma_i(t))}{x^{p-1}(t)} - \frac{(p-1)\varepsilon t^2}{2(\delta(t) r(t))^{\frac{1}{p-1}}} \psi^{\frac{p}{p-1}}(t).$$

Note that $x'(t) > 0$ and $\sigma_i(t) \geq t$. Thus,

$$\psi'(t) \leq \frac{\delta'(t)}{\delta(t)} \psi(t) - \delta(t) \sum_{i=1}^{\ell} \eta_i(t) - \frac{(p-1)\varepsilon t^2}{2(\delta(t) r(t))^{\frac{1}{(p-1)}}} \psi(t)^{\frac{p}{(p-1)}}. \tag{15}$$

When (15) is multiplied by $G_1(t,s)$ and the resulting inequality from t_1 to t is integrated, we discover that

$$\begin{aligned} \int_{t_1}^{t} G_1(t,s)\delta(s) \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \quad \leq \quad & \psi(t_1) G_1(t,t_1) + \int_{t_1}^{t} \left(\frac{\partial}{\partial s} G_1(t,s) + \frac{\delta'(s)}{\delta(s)} G_1(t,s) \right) \psi(s) \mathrm{d}s \\ & - \int_{t_1}^{t} \frac{(p-1)\varepsilon s^2}{2(\delta(s) r(s))^{\frac{1}{(p-1)}}} G_1(t,s) \psi^{\frac{p}{(p-1)}}(s) \mathrm{d}s. \end{aligned}$$

From (2), we see

$$\begin{aligned} \int_{t_1}^{t} G_1(t,s)\delta(s) \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \quad \leq \quad & \psi(t_1) G_1(t,t_1) + \int_{t_1}^{t} g_1(t,s) G_1^{(p-1)/P}(t,s) \psi(s) \mathrm{d}s \\ & - \int_{t_1}^{t} \frac{(p-1)\varepsilon s^2}{2(\delta(s) r(s))^{\frac{1}{(p-1)}}} G_1(t,s) \psi^{\frac{p}{p-1}}(s) \mathrm{d}s, \end{aligned} \tag{16}$$

with $D = (p-1)\varepsilon s^2 / \left(2(\delta(s)r(s))^{\frac{1}{p-1}} \right) G_1(t,s)$, $E = g_1(t,s) G_1^{(p-1)/P}(t,s)$, and $x = \psi(s)$; by Lemma 1 we find

$$g_1(t,s)G_1^{(p-1)/P}(t,s)\psi(s) - \frac{(p-1)\varepsilon s^2}{2(\delta(s)r(s))^{\frac{1}{(p-1)}}} G_1(t,s)\psi^{\frac{p}{(p-1)}}(s)$$
$$\leq \frac{g_1^P(t,s)G_1^{(p-1)}(t,s)}{p^P} \frac{2^{(p-1)}\delta(s)r(s)}{(\varepsilon s^2)^{p-1}},$$

which, with (16), gives

$$\frac{1}{G_1(t,t_1)} \int_{t_1}^{t} \left(G_1(t,s)\delta(s) \sum_{i=1}^{\ell} \eta_i(s) - \varnothing(s) \right) \mathrm{d}s \leq \psi(t_1).$$

This contradicts (10).

Assume that $(\mathbf{S}_2)$ holds. Defining

$$\vartheta(t) := \xi(t) x^{-1}(t) x'(t),$$

we see that $\vartheta(t) > 0$ for $t \geq t_1$, where $\xi \in C^1([t_0,\infty),(0,\infty))$. By differentiating $\vartheta(t)$, we find

$$\vartheta'(t) = \xi'(t)\xi^{-1}(t)\vartheta(t) + \xi(t)x^{-1}(t)x''(t) - \xi^{-1}(t)\vartheta^2(t). \tag{17}$$

Using $x'(t) > 0$ and integrating (1) from t to h, we now obtain

$$r(h)\left(x'''(h)\right)^{(p-1)} - r(t)\left(x'''(t)\right)^{(p-1)} = -\int_t^h \sum_{i=1}^{\ell} \eta_i(s) x^{p-1}(\sigma_i(s)) ds;$$

$x'(t) > 0$ and $\sigma_i(t) \geq t$ indicate that we have

$$r(h)\left(x'''(h)\right)^{p-1} - r(t)\left(x'''(t)\right)^{p-1} \leq -x^{p-1}(t) \int_t^u \sum_{i=1}^{\ell} \eta_i(s) ds.$$

Letting $h \to \infty$, we see that

$$r(t)\left(x'''(t)\right)^{p-1} \geq x^{p-1}(t) \int_t^\infty \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s$$

and so

$$x'''(t) \geq x(t) \left(\frac{1}{r(t)} \int_t^\infty \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \right)^{1/(p-1)}.$$

Integrating again from t to ∞, we obtain

$$x''(t) + x(t) \int_t^\infty \left(\frac{1}{r(\zeta)} \int_\zeta^\infty \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}\zeta \leq 0. \tag{18}$$

When (17) and (18) are combined, we obtain

$$\vartheta'(t) \leq \frac{\xi'(t)}{\xi(t)} \vartheta(t) - \xi(t) \int_t^\infty \left(\frac{1}{r(\zeta)} \int_\zeta^\infty \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}\zeta - \frac{1}{\xi(t)} \vartheta^2(t). \tag{19}$$

With (19) multiplied by $G_2(t,s)$ and the resulting inequality from t_1 integrated to t, we obtain

$$\begin{aligned}\int_{t_1}^{t} G_2(t,s)\xi(s)\int_t^\infty\left(\frac{1}{r(\zeta)}\varphi(s)\right)^{1/(p-1)}\mathrm{d}\zeta\mathrm{d}s &\le \vartheta(t_1)G_2(t,t_1)\\ &\quad+\int_{t_1}^{t}\left(\frac{\partial}{\partial s}G_2(t,s)+\frac{\xi'(s)}{\xi(s)}G_2(t,s)\right)\vartheta(s)\mathrm{d}s\\ &\quad-\int_{t_1}^{t}\frac{1}{\xi(s)}G_2(t,s)\vartheta^2(s)\mathrm{d}s.\end{aligned}$$

Thus, from (3), we obtain

$$\begin{aligned}\int_{t_1}^{t} G_2(t,s)\xi(s)\int_t^\infty\left(\frac{1}{r(\zeta)}\varphi(s)\right)^{1/(p-1)}\mathrm{d}\zeta\mathrm{d}s &\le \vartheta(t_1)G_2(t,t_1)+\int_{t_1}^{t} g_2(t,s)\sqrt{G_2(t,s)}\vartheta(s)\mathrm{d}s\\ &\quad-\int_{t_1}^{t}\frac{1}{\xi(s)}G_2(t,s)\vartheta^2(s)\mathrm{d}s\\ &\le \vartheta(t_1)G_2(t,t_1)+\int_{t_1}^{t}\frac{\xi(s)g_2^2(t,s)}{4}\mathrm{d}s,\end{aligned}$$

and so

$$\frac{1}{G_2(t,t_1)}\int_{t_1}^{t}\left(G_2(t,s)\xi(s)\int_t^\infty\left(\frac{1}{r(s)}\varphi(s)\right)^{1/(p-1)}\mathrm{d}s-\frac{\xi(s)g_2^2(t,s)}{4}\right)\mathrm{d}s\le\vartheta(t_1).$$

This runs counter to (11). The theorem's proof is finished. □

Now, we discuss an application of Theorem 1.

Example 1. *Examine the equation*

$$\left(t\left(x'''(t)\right)\right)'+\frac{3t\eta_0}{t^3}x(t+2)+\frac{\eta_0\left(3t-t^2\right)}{t^3}x(t+2)=0, \tag{20}$$

where $t\ge 1,\eta_0>0$. *Let* $p=2,\ r(t)=t,\ \eta(t)=3t\eta_0/t^3+\eta_0\left(3t-t^2\right)/t^3$, *and* $\sigma(t)=t+2$. *If we set* $g_1(t,s)=\delta(s)=1,\ G_1(t,s)=t$ *then*

$$\begin{aligned}&\int_{t_0}^{\infty}\frac{1}{r^{1/p-1}(s)}\mathrm{d}s\\ =&\int_{t_0}^{\infty}\frac{1}{s}\mathrm{d}s=\infty,\end{aligned}$$

and

$$\begin{aligned}\varnothing(s) &= \frac{g_1^p(t,s)G_1^{p-1}(t,s)}{p^p}\frac{2^{p-1}\delta(s)r(s)}{(\varepsilon s^2)^{p-1}},\\ &= \frac{s}{4}\frac{2s}{\varepsilon s^2}=1/2\varepsilon,\end{aligned}$$

where $\varepsilon\in(0,1)$. *Also, we see that*

$$\begin{aligned}\varphi(s) &= \int_t^\infty\sum_{i=1}^{\ell}\eta_i(s)\mathrm{d}s\\ &= \int_t^\infty\frac{\mathrm{d}s}{s}=\infty.\end{aligned}$$

From Theorem 1, we ascertain that (20) is oscillatory.

Theorem 2. *We assume (8) is true. If the formulas of equations*

$$\left(\frac{2r^{\frac{1}{p-1}}(t)}{(\varepsilon t^2)^{(p-1)}}\left(x'(t)\right)^{p-1}\right)' + \sum_{i=1}^{\ell} \eta_i(t) x^{p-1}(t) = 0 \tag{21}$$

and

$$x''(t) + x(t)\int_t^\infty \left(\frac{1}{r(\zeta)}\int_\zeta^\infty \sum_{i=1}^{\ell} \eta_i(s)\mathrm{d}s\right)^{1/(p-1)} \mathrm{d}\zeta = 0 \tag{22}$$

are oscillatory then (1) is oscillatory.

Proof. The function x is identified as the ultimate positive solution to (1). Using Lemma 3, we see cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$. Let case $(\mathbf{S}_1)$ hold.

Theorem 1 indicates that (15) is true. Setting $\delta(t) = 1$ in (15) yields

$$\psi'(t) + \frac{(p-1)\varepsilon t^2}{2r^{\frac{1}{p-1}}(t)}\psi^{\frac{p}{p-1}}(t) + \sum_{i=1}^{\ell} \eta_i(t) \le 0.$$

Equation (21) is non-oscillatory, which is a contradiction, as demonstrated by Lemma 4. We assume that $(\mathbf{S}_2)$ is true. Theorem 1 leads us to the conclusion that (19) is true.

If $\xi(t) = \varsigma = 1$ in (19), we obtain

$$\vartheta'(t) + \vartheta^2(t) + \int_t^\infty \left(\frac{1}{r(\zeta)}\int_\zeta^\infty \sum_{i=1}^{\ell} \eta_i(s)\mathrm{d}s\right)^{1/(p-1)} \mathrm{d}\zeta \le 0.$$

Thus, we observe that Equation (22) is contradictory, as it is non-oscillatory. The theorem's proof is finished. □

Now, using $p = 2$, on Theorem 2, we derive the Hille–Nehari-type oscillation condition for (1).

Theorem 3. *Suppose $p = 2$ and that*

$$\int_{t_0}^\infty \frac{\varepsilon t^2}{2r(t)}\mathrm{d}t = \infty$$

and

$$\liminf_{t\to\infty}\left(\int_{t_0}^t \frac{\varepsilon s^2}{2r(s)}\mathrm{d}s\right)\int_t^\infty \sum_{i=1}^{\ell} \eta_i(s)\mathrm{d}s > \frac{1}{4}, \tag{23}$$

for some constant $\varepsilon \in (0,1)$,

$$\liminf_{t\to\infty} t\int_{t_0}^t \int_v^\infty \left(\frac{1}{r(\zeta)}\int_\zeta^\infty \sum_{i=1}^{\ell} \eta_i(s)\mathrm{d}s\right)\mathrm{d}\zeta\mathrm{d}v > \frac{1}{4}; \tag{24}$$

consequently, every solution to (1) is oscillatory.

Example 2. *Let equation*

$$x^{(4)}(t) + \left(3\eta_0^3 - 7\right)/t^2 x(\nu t) + 3\eta_0^3/t^2 x(\nu t) - \left(7/t^2 + \eta_0/t^4\right)x(\nu t) = 0. \tag{25}$$

Let $p = 2,\ t, \nu \geq 1, \eta_0 > 0,\ r(t) = 1,\ \eta(t) = (3\eta_0^3 - 7)/t^2 + 3\eta_0^3/t^2 - (7/t^2 + \eta_0/t^4)$, *and* $\sigma(t) = \nu t$. *If we set* $\nu = 2$ *then*

$$\begin{aligned} & \int_{t_0}^{\infty} \frac{1}{r^{1/p-1}(s)} \mathrm{d}s \\ = & \int_{t_0}^{\infty} \mathrm{d}s = \infty, \end{aligned}$$

and condition (23) becomes

$$\begin{aligned} & \liminf_{t\to\infty} \left(\int_{t_0}^{t} \frac{\varepsilon s^2}{2r(s)} \mathrm{d}s \right) \int_{t}^{\infty} \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \\ = & \liminf_{t\to\infty} \left(\frac{\varepsilon}{2} \int_{t_0}^{t} s^2 \mathrm{d}s \right) \int_{t}^{\infty} \frac{\eta_0}{s^4} ds \\ = & \infty; \end{aligned}$$

also, condition (24) becomes

$$\begin{aligned} & \liminf_{t\to\infty} t \int_{t_0}^{t} \int_{v}^{\infty} \left(\frac{1}{r(\zeta)} \int_{\zeta}^{\infty} \sum_{i=1}^{\ell} \eta_i(s) \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}\zeta \mathrm{d}v \\ = & \liminf_{t\to\infty} t \left(\frac{\eta_0}{6t} \right), \\ = & \frac{\eta_0}{6} > \frac{1}{4} > 1.5. \end{aligned}$$

From Theorem 3, we ascertain that (25) is oscillatory.

4. Recommendations

Since there are not many publications that expressly address advanced oscillation, making the body of the existing literature on the subject rather small when compared to other research fields, the authors recommend using a number of different methods that give researchers strong tools for examining the dynamics and stability of systems while offering insightful information on the oscillatory behavior of differential equations with advanced terms. These approaches' comparative advantages and uses demonstrate the variety of approaches available for studying differential equations, which will eventually improve our comprehension of their complex behaviors.

We will intensify our efforts in future work on studying the oscillatory properties of advanced differential equations of different orders with deviating arguments in canonical and non-canonical cases.

5. Conclusions

The objective of this paper was to obtain new Philos-type and Hille–Nehari-type oscillation criteria for (1) by using the Riccati technique, integral averaging, and comparison with second-order differential equations. Several previous criteria were greatly simplified and enhanced by ours. A few examples were provided to demonstrate the outcomes.

In future work, we will study fourth-order differential equations in their non-canonical form, to find oscillatory properties that will contribute to enriching oscillation theory.

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