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Article

Fejér-Type Fractional Integral Inequalities Involving Mittag-Leffler Function

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Abstract: Several integral inequalities of the Fejér type are derived, incorporating the generalized Mittag-Leffler function alongside the associated fractional integral operator. Consequently, generalizations of known results are achieved.

Keywords: fractional calculus; Mittag-Leffler function; Fejér inequality

1. Introduction

In the theory of differential equations, approximations and probability, inequalities involving integrals of functions and their derivatives find far-reaching applications. In recent years there has been considerable interest in the theory of fractional calculus (the theory of differential and integral operators of non-integer order), especially given that mathematical models based on fractional-order operators are considered more realistic than those based on classical calculus. Furthermore, inequalities of fractional differentiation have applications to fractional differential equations, the most important of which are determining the uniqueness of solutions to initial value problems and giving their upper bounds. The Mittag-Leffler function, with its generalizations, appears in the solutions of these equations.

The goal of this work is to obtain integral inequalities of the Fejér type, which include the generalized Mittag-Leffler function with the associated fractional integral operator, and thus achieve generalizations of known results. In doing so, these inequalities will be given for the generalized class of convex functions.

The motivation for this research is recent studies on different types of integral operators (see [1–7]) and different classes of convexity (see [8–16]). Fractional operators can describe memory and hereditary properties of various materials and processes more accurately, and, by exploring new generalizations of the Mittag-Leffler function and extending existing convexity concepts, the paper aims to broaden the understanding and application of these mathematical tools. The results obtained in this paper are expected to contribute significantly to the theory of fractional calculus and its applications, especially in solving fractional differential equations. The derived inequalities could be used in various fields such as control theory, signal processing, and other areas where fractional models are applicable. Additionally, this research might inspire further studies on other types of fractional inequalities, potentially leading to new theoretical advancements and practical applications.

2. General Background

To prove Fejer's inequality in more general settings, we need the extended generalized Mittag-Leffler function with its fractional integral operators ([1]) and the class of $(h, g; m)$-convex functions ([8]).

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Citation: Andrić, M. Fejér-Type Fractional Integral Inequalities Involving Mittag-Leffler Function. *Fractal Fract.* **2024**, *8*, 688. https://doi.org/10.3390/fractalfract8120688

Academic Editor: Ivanka Stamova

Received: 31 October 2024
Revised: 21 November 2024
Accepted: 22 November 2024
Published: 23 November 2024

2.1. On Generalizations of the Mittag-Leffler Function

In 1903, the Swedish mathematician Mittag-Leffler introduced the function $E_\rho(z)$ defined for $z \in \mathbb{C}$ and $\Re(\rho) > 0$ by the expression

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}. \tag{1}$$

Since then, it has been generalized and extended by Wiman, Prabhakar, Shukla, Prajapati, etc. Recall, $(x)_n$ is the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \tag{2}$$

B is the beta function defined by Euler's integral of the first kind

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad \Re(x), \Re(y) \tag{3}$$

and B_p is the extended beta function

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt, \qquad \Re(x), \Re(y), \Re(p) > 0. \tag{4}$$

The following function from [1] is one of those generalization of the Mittag-Leffler function:

Definition 1 ([1])**.** *Let* $\rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $r > 0$ *and* $0 < q \leq r + \Re(\rho)$*. Then, the extended generalized Mittag-Leffler function* $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ *is defined by*

$$E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta + nq, c-\delta)}{\mathrm{B}(\delta, c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{nr}}. \tag{5}$$

Remark 1. By choosing different value of the parameters, corresponding generalizations of the Mittag-Leffler function are obtained, for example:

(i) The Salim–Faraj function $E_{\rho,\sigma,r}^{\delta,\tau,q}(z)$ for $p = 0$ ([2]);
(ii) The Rahman function $E_{\rho,\sigma}^{\delta,q,c}(z;p)$ for $\tau = r = 1$ ([3]);
(iii) The Shukla–Prajapati function $E_{\rho,\sigma}^{\delta,q}(z)$ for $p = 0$ and $\tau = r = 1$ ([4]);
(iv) The Prabhakar function $E_{\rho,\sigma}^{\delta}(z)$ for $p = 0$ and $\tau = r = q = 1$ ([6]);
(v) The Wiman function $E_{\rho,\sigma}(z)$ for $p = 0$ and $\tau = r = q = \delta = 1$ ([7]);
(vi) The Mittag-Leffler function $E_\rho(z)$ for $p = 0$, $\tau = r = q = \delta = 1$ and $\sigma = 1$.

Fractional integral operators are generalizations of classical integral operators, extending the concept of an integral to non-integer orders. There are several well-known forms of the fractional operators that have been studied extensively for their applications: Riemann–Liouville, Caputo, Weyl, Erdély–Kober, Hadamard, and Katugampola are just a few. Here, we consider fractional integral operators for which the function $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ is their kernel, namely the left-sided $\varepsilon_{a^+,\rho,\sigma,\tau}^{\omega,\delta,c,q,r}f$ and the right-sided $\varepsilon_{b^-,\rho,\sigma,\tau}^{\omega,\delta,c,q,r}f$ operator.

By $L_1[a,b]$ we denote the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|dt < \infty$, where $[a,b]$ is a finite interval in $\mathbb{R}$:

Definition 2 ([1]). *Let $\omega,\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then, the left-sided and the right-sided generalized fractional integral operators $\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ and $\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ are defined by*

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) = \int_a^x (x-t)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(\omega(x-t)^{\rho};p)f(t)dt, \tag{6}$$

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p) = \int_x^b (t-x)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(\omega(t-x)^{\rho};p)f(t)dt. \tag{7}$$

Remark 2. With different choices of parameters, these operators generalize known results. We give examples for the left-sided fractional integral operator (the same applies to the right-sided operator):

(i) The Salim–Faraj fractional integral operator $\varepsilon^{\omega,\delta,q,r}_{a^+,\rho,\sigma,\tau}f(x)$ for $p = 0$ ([2]);
(ii) The Rahman fractional integral operator $\varepsilon^{\omega,\delta,q,c}_{a^+,\rho,\sigma}f(x;p)$ for $\tau = r = 1$ ([3]);
(iii) The Srivastava–Tomovski fractional integral operator $\varepsilon^{\omega,\delta,q}_{a^+,\rho,\sigma}f(x)$ for $p = 0$ and $\tau = r = 1$ ([5]);
(iv) The Prabhakar fractional integral operator $\varepsilon(\rho,\sigma;\delta;\omega)f(x)$ for $p = 0$ and $\tau = r = q = 1$ ([6]);
(v) The Wiman fractional integral operator $\varepsilon^{\omega}_{a^+,\rho,\sigma}f(x)$ for $p = 0$ and $\tau = r = q = \delta = 1$;
(vi) The Mittag-Leffler fractional integral operator $\varepsilon^{\omega}_{a^+,\rho}f(x)$ for $p = 0$, $\tau = r = q = \delta = 1$ and $\sigma = 1$;
(vii) The Riemann–Liouville fractional integral of order $\sigma > 0$ for $p = \omega = 0$ ([17])

$$J^{\sigma}_{a^+}f(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t)\,dt, \quad x \in (a,b]. \tag{8}$$

In [1,18], one can see more details about the properties of this generalized form of the Mittag-Leffler function and its fractional integral operators, whereas here we list only the results needed in this paper:

Proposition 1 ([18]). *Let $\alpha,\omega,\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\alpha),\Re(\rho),\Re(\sigma),\Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$.*

(i) *If the function $f \in L_1[a,b]$ is symmetric about $\frac{a+b}{2}$, then*

$$(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f)(b;p) = (\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f)(a;p). \tag{9}$$

In particular,

$$(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1)(b;p) = (\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1)(a;p). \tag{10}$$

(ii) *Furthermore,*

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(t-a)^{\alpha-1}\right)(b;p) = (\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(b-t)^{\alpha-1})(a;p), \tag{11}$$

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(b-t)^{\alpha-1}\right)(b;p) = (\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(t-a)^{\alpha-1})(a;p). \tag{12}$$

In particular,

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{0^+,\rho,\sigma,\tau}t^{\alpha-1}\right)(1;p) = (\varepsilon^{\omega,\delta,c,q,r}_{1^-,\rho,\sigma,\tau}(1-t)^{\alpha-1})(0;p), \tag{13}$$

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{0^+,\rho,\sigma,\tau}(1-t)^{\alpha-1}\right)(1;p) = (\varepsilon^{\omega,\delta,c,q,r}_{1^-,\rho,\sigma,\tau}t^{\alpha-1})(0;p). \tag{14}$$

2.2. Convexity Classes

Convexity is a simple and natural notion which can be traced back to Archimedes, who noticed that the perimeter of a convex figure is smaller than the perimeter of any other convex figure surrounding it. Classes of convexities refer to various generalized notions of convex functions. Each class extends the idea of convexity in different ways, allowing for a wider range of functions with specific properties and applications. One of them is the class of $(h, g; m)$-convex functions. It unifies a certain range of convexity, thus enabling generalizations of known results:

Definition 3 ([8])**.** *Let h be a non-negative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$; let g be a positive function on $I \subseteq \mathbb{R}$; and let $m \in (0,1]$. A function $f : I \to \mathbb{R}$ is said to be an $(h, g; m)$-convex function if it is non-negative and if*

$$f(tx + m(1-t)y) \leq h(t)f(x)g(x) + m\,h(1-t)f(y)g(y) \tag{15}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

If (15) holds in the reversed sense, then f is said to be an $(h, g; m)$-concave function.

Remark 3. Depending on the specific choices of the functions h, g and the parameter m, a class of $(h, g; m)$-convex functions can be specialized into various specific classes of functions. In addition to ordinary convexity, there are, among others, the following classes:

(i) P-functions for $h, g \equiv 1$ and $m = 1$ ([9]);
(i) s-convex functions in the second sense for $h(\lambda) = \lambda^s$, $g \equiv 1$ and $m = 1$ ([10]);
(ii) Godunova–Levin functions for $h(\lambda) = \lambda^{-1}$, $g \equiv 1$ and $m = 1$ ([11]),
(iii) h-convex functions for $g \equiv 1$ and $m = 1$ ([12]);
(iv) m-convex functions for $h(\lambda) = \lambda$ and $g \equiv 1$ ([13]);
(v) $(h - m)$-convex functions for $g \equiv 1$ ([14]);
(vi) (s, m)-Godunova–Levin functions in the second sense for $h(\lambda) = \lambda^{-s}$ and $g \equiv 1$ ([15]);
(vii) Exponentially s-convex functions in the second sense for $h(\lambda) = \lambda^s$, $g(x) = e^{-\alpha x}$ and $m = 1$ ([9]);
(vii) Exponentially (s, m)-convex functions in the second sense for $h(\lambda) = \lambda^s$ and $g(x) = e^{-\alpha x}$ ([16]).

For more information on the properties of this convexity class, refer to [8,19]. Here, we present one result that is applied in this paper:

Lemma 1 ([19])**.** *Let f be a non-negative $(h, g; m)$-convex function on $[0, \infty)$ where h is a non-negative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0,1]$. Then, for all $t \in (a, b)$, there exists $\lambda \in (0,1)$ such that*

$$f(a + b - t) \leq [h(\lambda) + h(1-\lambda)]\left[f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right] - f(t). \tag{16}$$

If f is an $(h, g; m)$-concave function, then the reversed inequality holds.

2.3. Fejér's Inequalities

The Fejér inequalities are a generalization of Hermite–Hadamard inequalities obtained with a weight function φ, giving greater importance or influence to different elements within a dataset or model. In some references, they are called the left and the right Féjer inequalities.

Theorem 1 (Fejér's inequalities)**.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function and let $\varphi : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then,*

$$f\left(\frac{a+b}{2}\right) \int_a^b \varphi(x)dx \le \int_a^b f(x)\varphi(x)\,dx \le \frac{f(a)+f(b)}{2} \int_a^b \varphi(x)dx. \tag{17}$$

One of the objectives of this paper is to offer fractional generalizations of Fejér's inequalities presented in [19]:

Theorem 2 (The left Fejér inequality for an $(h, g; m)$-convex function, [19])**.** *Let f be a non-negative $(h, g; m)$-convex function on $[0, \infty)$ where h is a non-negative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Furthermore, let $\varphi : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a+b}{2}$, $0 \le a < b < \infty$, and let $f, g, h \in L_1[a, b]$. Then, the following inequality holds:*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) \int_a^b \varphi(x)dx \\ &\le h\left(\frac{1}{2}\right) \int_a^b f(x)g(x)\varphi(x)dx + mh\left(\frac{1}{2}\right) \int_a^b f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\varphi(x)dx. \end{aligned} \tag{18}$$

Theorem 3 (The right Fejér inequality for an $(h, g; m)$-convex function citem0)**.** *Suppose that the assumptions of Theorem 2 hold. Then,*

$$\begin{aligned} &\int_a^b f(x)\varphi(x)dx \\ &\le \frac{1}{2}\left[f(a)g(a) + f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right] \\ &\quad \times \int_a^b h\left(\frac{x-a}{b-a}\right)\varphi(x)dx. \end{aligned} \tag{19}$$

2.4. Notation

To avoid complicated manuscript form, throughout this paper we will use the following simplified notation for the generalized extended Mittag-Leffler function with its fractional integral operators:

$$\boldsymbol{E}(z; p) := E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z; p)$$

and

$$\begin{aligned} (\boldsymbol{\varepsilon}_{a^+} f)(x; p) &:= \left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x; p), \\ (\boldsymbol{\varepsilon}_{b^-} f)(x; p) &:= \left(\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f\right)(x; p). \end{aligned}$$

The conditions for the parameters $\rho, \sigma, \tau, \omega, \delta, c, q, r$ will be included in all theorems.

For $n, m \in (0, 1]$, $0 \le a < b < \infty$, and two functions f and g defined on $[0, \infty]$, we will use the following notation:

$$M^{a,b}_{n,m}(f, g) := nf\left(\frac{a}{n}\right)g\left(\frac{a}{n}\right) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

3. Fractional Integral Inequalities of the Fejér Type for $(h, g; m)$-Convex Functions

In this section, we prove fractional generalizations of Fejér's inequalities for $(h, g; m)$-convex functions, using the extended generalized Mittag-Leffler function with its fractional integral operators, in the real domain. Furthermore, we present some similar results.

We start with the generalization of the first Fejér inequality:

Theorem 4. *Let $\omega \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let f be a non-negative $(h, g; m)$-convex function on $[0, \infty)$, where h is a non-negative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Furthermore, let $\varphi : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a+b}{2}$, $0 \leq a < b < \infty$, and let $f, g, h \in L_1[a, b]$. Then, the following inequality holds*

$$\begin{aligned}(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)\, f\left(\frac{a+b}{2}\right) \quad &\leq \quad h\left(\frac{1}{2}\right)(\boldsymbol{\varepsilon}_{b^-} f g \varphi)(a;p) \\ &+ m\, h\left(\frac{1}{2}\right) \int_a^b f\left(\frac{t}{m}\right) g\left(\frac{t}{m}\right)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt. \end{aligned} \tag{20}$$

Proof. Let f be an $(h, g; m)$-convex function on $[0, \infty)$, $m \in (0, 1]$ and $t \in [a, b]$. Applying the $(h, g; m)$-convexity of f and

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}(a+b-t) + m\frac{1}{2}\frac{t}{m}\right),$$

we obtain

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(a+b-t)g(a+b-t) + mf\left(\frac{t}{m}\right)g\left(\frac{t}{m}\right)\right].$$

In the following step, we will need to multiply both sides of the above inequality by $(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)$ and integrate on $[a, b]$ with respect to the variable t, which gives us

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\int_a^b (b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt \\ &\leq \quad h\left(\frac{1}{2}\right)\int_a^b f(a+b-t)\, g(a+b-t)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt \\ &\qquad + m\, h\left(\frac{1}{2}\right)\int_a^b f\left(\frac{t}{m}\right)g\left(\frac{t}{m}\right)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt. \end{aligned}$$

The left side of the inequality is equal to $f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)$, and from the symmetry of the weight φ, that is $\varphi(t) = \varphi(a+b-t)$, with a substitution $u = a+b-t$, we obtain

$$\begin{aligned} &\int_a^b f(a+b-t)\, g(a+b-t)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(a+b-t)dt \\ &= \quad \int_a^b f(u)g(u)\varphi(u)(u-a)^{\sigma-1}\mathbf{E}(\omega(u-a)^{\rho};p)du \;=\; (\boldsymbol{\varepsilon}_{b^-} f g \varphi)(a;p). \end{aligned}$$

This completes the proof. □

In the following result, we use Lemma 1.

Theorem 5. *Suppose that the assumptions of Theorem 4 hold. Then, for every $\lambda \in (0, 1)$ there is the representation*

$$(\boldsymbol{\varepsilon}_{b^-} f\varphi)(a;p) + (\boldsymbol{\varepsilon}_{a^+} f\varphi)(b;p) \leq [h(\lambda) + h(1-\lambda)]M^{a,b}_{1,m}(f,g)(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p). \tag{21}$$

Proof. Let f be an $(h, g; m)$-convex function on $[0, \infty)$, $m \in (0, 1]$ and $\lambda \in (0, 1)$. If we multiply the inequality (16) by $(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)$ and integrate on $[a, b]$ with respect to the variable t, we obtain

$$
\begin{aligned}
&\int_a^b f(a+b-t)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^\rho;p)\varphi(t)dt \\
&\leq [h(\lambda)+h(1-\lambda)]\left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b (b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^\rho;p)\varphi(t)dt \\
&\quad -\int_a^b f(t)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^\rho;p)\varphi(t)dt.
\end{aligned}
$$

If we use the symmetric property of φ and the substitution $u = a+b-t$ on the left side of the above inequality, then we obtain $(\boldsymbol{\varepsilon}_{b^-} f\varphi)(a;p)$. The rest follows easily. □

Let us prove a result similar to Lemma 1 from which we will obtain Theorem 6.

Lemma 2. *Suppose that the assumptions of Lemma 1 hold. Then, for all $t \in (a,b)$ there exists $\lambda \in (0,1)$ such that*

$$
f(a+b-t) \leq h(\lambda)M_{1,1}^{a,b}(f,g) + h(1-\lambda)M_{m,m}^{a,b}(f,g) - f(t). \tag{22}
$$

If f is an $(h,g;m)$-concave function, then the reversed inequality holds.

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $t \in (a,b)$. Then there exists $\lambda \in (0,1)$ such that $t = \lambda a + (1-\lambda)b$. Therefore,

$$
\begin{aligned}
f(a+b-t)+f(t) &= f((1-\lambda)a+\lambda b) + f(\lambda a + (1-\lambda)b) \\
&\leq h(\lambda)f(b)g(b) + mh(1-\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) \\
&\quad + h(\lambda)f(a)g(a) + mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \\
&= h(\lambda)[f(a)g(a)+f(b)g(b)] \\
&\quad + h(1-\lambda)\left[mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right].
\end{aligned}
$$

This completes the proof. □

Theorem 6. *Suppose that the assumptions of Theorem 4 hold. Then, for every $\lambda \in (0,1)$ there is the representation*

$$
\begin{aligned}
&(\boldsymbol{\varepsilon}_{b^-} f\varphi)(a;p) + (\boldsymbol{\varepsilon}_{a^+} f\varphi)(b;p) \\
&\leq \left[h(\lambda)M_{1,1}^{a,b}(f,g) + h(1-\lambda)M_{m,m}^{a,b}(f,g)\right](\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p).
\end{aligned} \tag{23}
$$

Proof. The result is obtained by multiplying the inequality (22) by $(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^\rho;p)\varphi(t)$, which we integrate on $[a,b]$ with respect to the variable t, and by using the symmetric property of φ. □

Remark 4. If we compare Lemmas 1 and 2, we see that the same results are obtain for $m = 1$. The same applies to Theorems 5 and 6. In all other cases, we obtain different results that depend on the functions h and g.

If additionally the function f is symmetric about $\frac{a+b}{2}$, then from Proposition 1, we obtain the following results.

Corollary 1. *Suppose that the assumptions of Theorem 4 hold with the symmetric property of the function f. Then, for every $\lambda \in (0,1)$, there are representations*

$$(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p) \leq \frac{1}{2}[h(\lambda)+h(1-\lambda)]M^{a,b}_{1,m}(f,g)(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) \tag{24}$$

and

$$(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p) \quad \leq \quad \frac{1}{2}\Big[h(\lambda)M^{a,b}_{1,1}(f,g)+h(1-\lambda)M^{a,b}_{m,m}(f,g)\Big](\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p). \tag{25}$$

We continue with the generalization of the second Fejér inequality:

Theorem 7. *Suppose that the assumptions of Theorem 4 hold. Then, the following inequality holds:*

$$\begin{aligned}
&(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)\\
&\leq \quad M^{a,b}_{1,1}(f,g)\int_a^b h\left(\frac{b-t}{b-a}\right)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt\\
&\qquad +M^{a,b}_{m,m}(f,g)\int_a^b h\left(\frac{t-a}{b-a}\right)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt.
\end{aligned} \tag{26}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m\in(0,1]$ and $t\in[a,b]$. First, we use

$$t=\frac{b-t}{b-a}a+m\frac{t-a}{b-a}\frac{b}{m},$$

$$a+b-t=\frac{b-t}{b-a}b+m\frac{t-a}{b-a}\frac{a}{m}.$$

from which, with the $(h,g;m)$-convexity, we obtain

$$\begin{aligned}
f(t)+f(a+b-t) \quad &\leq \quad h\left(\frac{b-t}{b-a}\right)f(a)g(a)+mh\left(\frac{t-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\\
&\qquad +h\left(\frac{b-t}{b-a}\right)f(b)g(b)+mh\left(\frac{t-a}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\\
&= \quad [f(a)g(a)+f(b)g(b)]h\left(\frac{b-t}{b-a}\right)\\
&\qquad +m\left[f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]h\left(\frac{t-a}{b-a}\right).
\end{aligned}$$

Next, we multiply the above by $(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)$ and integrate on $[a,b]$ with respect to the variable t

$$\begin{aligned}
&\int_a^b [f(t)+f(a+b-t)](b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt\\
&\leq \quad [f(a)g(a)+f(b)g(b)]\int_a^b h\left(\frac{b-t}{b-a}\right)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt\\
&\qquad +\left[mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b h\left(\frac{t-a}{b-a}\right)(b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho};p)\varphi(t)dt.
\end{aligned}$$

From the symmetry of the weight φ, with a substitution $u=a+b-t$, the left side of the inequality follows. The right side is identical to the one above, written in simplified notation. □

From Theorem 4 and Theorem 7, using specific choices for the functions h, g, as well as the parameter m, the fractional integral inequalities of the Fejér type can be deduced for several forms of convexity, such as those mentioned in Remark 3. In [16,20,21], fractional integral inequalities of the Fejér type were made for m, $(h-m)$ and exponentially (s,m)-

convex functions in the second sense, but only with an additional symmetry condition of the function f. Thus, the inequalities presented here are their extensions and generalizations.

As an example, we observe the class of h-convex functions from [12]. This class is obtained by setting $g \equiv 1$ and $m = 1$ in the $(h, g; m)$-convexity:

Corollary 2. *Let $\omega \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let f be a non-negative h-convex functions on $[0, \infty)$, where h is a non-negative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$. Furthermore, let $\varphi : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a+b}{2}$, $0 \leq a < b < \infty$, and let $f, h \in L_1[a, b]$. Then, the following inequality holds:*

$$\begin{aligned}
&(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)\, f\left(\frac{a+b}{2}\right) \\
&\leq h\left(\frac{1}{2}\right)[(\boldsymbol{\varepsilon}_{b^-} f\varphi)(a;p) + (\boldsymbol{\varepsilon}_{a^+} f\varphi)(b;p)] \\
&\leq h\left(\frac{1}{2}\right)[f(a) + f(b)] \int_a^b \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right](b-t)^{\sigma-1}\mathbf{E}(\omega(b-t)^{\rho}; p)\varphi(t)dt.
\end{aligned} \tag{27}$$

Corollary 3. *Suppose that the assumptions of Corollary 2 hold with*

$$h(\lambda) + h(1-\lambda) \leq 1, \quad \lambda \in (0, 1). \tag{28}$$

Then

$$\begin{aligned}
(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)\, f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)[(\boldsymbol{\varepsilon}_{b^-} f\varphi)(a;p) + (\boldsymbol{\varepsilon}_{a^+} f\varphi)(b;p)] \\
&\leq h\left(\frac{1}{2}\right)[f(a) + f(b)](\boldsymbol{\varepsilon}_{a^+}\varphi).
\end{aligned} \tag{29}$$

Furthermore, from Theorems 4 and 7, we can also derive Fejér inequalities for different fractional operators (observe Remark 2).

4. Fejér-Type Estimations

In the subsequent discussion, we provide certain fractional integral identities and produce Fejér-type estimations for the generalized fractional integral operators. They generalize the results for Riemann–Liouville fractional integrals applied to convex functions, which were obtained by İscan in [22]. Furthermore, they generalize the results given for all operators and all classes of convex functions that can be deduced from Definition 2 and Definition 3, respectively.

Lemma 3. *Let $\omega \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with $0 \leq a < b < \infty$ and $f' \in L_1[a, b]$. Let $\varphi : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then,*

$$\begin{aligned}
&[f(a) + f(b)](\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) - [(\boldsymbol{\varepsilon}_{a^+} f\varphi)(b;p) + (\boldsymbol{\varepsilon}_{b^-} f\varphi)(a;p)] \\
&= \int_a^b \left[\int_a^t (b-s)^{\sigma-1}\mathbf{E}(\omega(b-s)^{\rho}; p)\varphi(s)ds - \int_t^b (s-a)^{\sigma-1}\mathbf{E}(\omega(s-a)^{\rho}; p)\varphi(s)ds\right] f'(t)dt.
\end{aligned} \tag{30}$$

Proof. We use integration by parts to obtain

$$\int_a^b\left(\int_a^t (b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)ds\right)f'(t)dt$$
$$= \left(\int_a^t (b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)ds\right)f(t)\Big|_a^b - \int_a^b (b-t)^{\sigma-1}\boldsymbol{E}(\omega(b-t)^\rho;p)\varphi(t)f(t)dt$$
$$= f(b)(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) - (\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p) \tag{31}$$

and

$$\int_a^b\left(-\int_t^b (s-a)^{\sigma-1}\boldsymbol{E}(\omega(s-a)^\rho;p)\varphi(s)ds\right)f'(t)dt$$
$$= \left(-\int_t^b (s-a)^{\sigma-1}\boldsymbol{E}(\omega(s-a)^\rho;p)\varphi(s)ds\right)f(t)\Big|_a^b - \int_a^b (t-a)^{\sigma-1}\boldsymbol{E}(\omega(t-a)^\rho;p)\varphi(t)f(t)dt$$
$$= f(a)(\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p) - (\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p). \tag{32}$$

From Proposition 1 follows $(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) = (\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)$. Finally, (30) is obtained by adding (31) and (32). □

Remark 5. The equality $(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) = (\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)$ gives us

$$(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) = \frac{(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) + (\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)}{2} = (\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)$$

and from the symmetry of the weight φ we obtain

$$\int_t^b (s-a)^{\sigma-1}\boldsymbol{E}(\omega(s-a)^\rho;p)\varphi(s)ds = \int_a^{a+b-t}(b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)ds.$$

Hence, (30) can be written as

$$\frac{f(a)+f(b)}{2}[(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p) + (\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)] - [(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p) + (\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)]$$
$$= \int_a^b\left[-\int_t^{a+b-t}(b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)ds\right]f'(t)dt. \tag{33}$$

Furthermore,

$$\left|\int_t^{a+b-t}(b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)ds\right|$$
$$\le \begin{cases}\int_t^{a+b-t}\left|(b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)\right|ds, & t\in\left[a,\frac{a+b}{2}\right]\\ \int_{a+b-t}^{t}\left|(b-s)^{\sigma-1}\boldsymbol{E}(\omega(b-s)^\rho;p)\varphi(s)\right|ds, & t\in\left[\frac{a+b}{2},b\right]\end{cases}. \tag{34}$$

In the following, let

$$||\varphi||_\infty = \sup_{t\in[a,b]}|\varphi(t)|. \tag{35}$$

The absolute convergence of the Mittag-Leffler series in (5), for all values of t provided that $q < r + \Re(\rho)$, was proven in [1]. Hence, if we set

$$M = \sum_{n=0}^{\infty}\left|\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{t^n}{(\tau)_{nr}}\right|, \tag{36}$$

then

$$|E_{\rho,\sigma,\tau}^{\delta,c,q,r}(t;p)| = |\boldsymbol{E}(t;p)| \le M.$$

Theorem 8. *Let $\omega \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function with $0 \leq a < b < \infty$. Let $|f'|$ be a non-negative $(h,g;m)$-convex function on $[0,\infty)$, where h is a non-negative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. Furthermore, let $\varphi : [a,b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $\frac{a+b}{2}$ and let $f', g, h \in L_1[a,b]$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}[(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)]-[(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)]\right| \\
&\leq \frac{M||\varphi||_\infty}{\sigma}\left[|f'(a)g(a)|+m\left|f'\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right|\right] \\
&\quad \times \int_a^{\frac{a+b}{2}}[(b-t)^\sigma-(t-a)^\sigma]\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]dt, \qquad (37)
\end{aligned}$$

where M is defined by (36).

Proof. From (33) and (34), we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}[(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)]-[(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)]\right| \\
&\leq \int_a^b\left|\int_t^{a+b-t}(b-s)^{\sigma-1}\mathbf{E}(\omega(b-s)^\rho;p)\varphi(s)ds\right||f'(t)|dt \\
&\leq \int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\sigma-1}\mathbf{E}(\omega(b-s)^\rho;p)\varphi(s)\right|ds\right)|f'(t)|dt \\
&\quad + \int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\sigma-1}\mathbf{E}(\omega(b-s)^\rho;p)\varphi(s)\right|ds\right)|f'(t)|dt, \qquad (38)
\end{aligned}$$

whereas from (35) and (36) we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}[(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)]-[(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)]\right| \\
&\leq M||\varphi||_\infty\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(b-s)^{\sigma-1}ds\right)|f'(t)|dt+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t(b-s)^{\sigma-1}ds\right)|f'(t)|dt\right] \\
&= \frac{M||\varphi||_\infty}{\sigma}\left[\int_a^{\frac{a+b}{2}}[(b-t)^\sigma-(t-a)^\sigma]|f'(t)|dt+\int_{\frac{a+b}{2}}^b[(t-a)^\sigma-(b-t)^\sigma]|f'(t)|dt\right]. \qquad (39)
\end{aligned}$$

Since $|f'|$ is $(h,g;m)$-convex on $[a,b]$ where h is a non-negative function, then for $t \in [a,b]$ we have

$$\begin{aligned}
|f'(t)| &= \left|f'\left(\frac{b-t}{b-a}a+m\frac{t-a}{b-a}\frac{b}{m}\right)\right| \\
&\leq h\left(\frac{b-t}{b-a}\right)|f'(a)g(a)|+mh\left(\frac{t-a}{b-a}\right)\left|f'\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right|. \qquad (40)
\end{aligned}$$

Finally, from (38) and (40) follows

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}[(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)]-[(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)]\right| \\
&\leq \frac{M||\varphi||_\infty}{\sigma}\left\{\int_a^{\frac{a+b}{2}}[(b-t)^\sigma-(t-a)^\sigma]\right. \\
&\quad\times\left[|f'(a)g(a)|h\left(\frac{b-t}{b-a}\right)+m\left|f'\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right|h\left(\frac{t-a}{b-a}\right)\right]dt \\
&\quad+\int_{\frac{a+b}{2}}^b[(t-a)^\sigma-(b-t)^\sigma] \\
&\quad\left.\times\left[|f'(a)g(a)|h\left(\frac{b-t}{b-a}\right)+m\left|f'\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right|h\left(\frac{t-a}{b-a}\right)\right]dt\right\}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_a^{\frac{a+b}{2}}[(b-t)^\sigma-(t-a)^\sigma]h\left(\frac{b-t}{b-a}\right)dt &= \int_{\frac{a+b}{2}}^b[(t-a)^\sigma-(b-t)^\sigma]h\left(\frac{t-a}{b-a}\right)dt, \\
\int_a^{\frac{a+b}{2}}[(b-t)^\sigma-(t-a)^\sigma]h\left(\frac{t-a}{b-a}\right)dt &= \int_{\frac{a+b}{2}}^b[(t-a)^\sigma-(b-t)^\sigma]h\left(\frac{b-t}{b-a}\right)dt.
\end{aligned}$$

This completes the proof. □

Theorem 9. *Suppose that the assumptions of Theorem 8 hold with the condition* (28). *Then,*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}[(\boldsymbol{\varepsilon}_{a^+}\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}\varphi)(a;p)]-[(\boldsymbol{\varepsilon}_{a^+}f\varphi)(b;p)+(\boldsymbol{\varepsilon}_{b^-}f\varphi)(a;p)]\right| \\
&\leq \frac{M||\varphi||_\infty(b-a)^{\sigma+1}}{\sigma(\sigma+1)}\left[1-\frac{1}{2^\sigma}\right]\left[|f'(a)g(a)|+m\left|f'\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right|\right]. \qquad (41)
\end{aligned}$$

Proof. From the hypothesis (28) we have

$$h\left(\frac{t-a}{b-a}\right) \leq 1-h\left(\frac{b-t}{b-a}\right).$$

Observe that

$$\int_a^{\frac{a+b}{2}}[(b-t)^\sigma-(t-a)^\sigma]dt = \int_{\frac{a+b}{2}}^b[(t-a)^\sigma-(b-t)^\sigma]dt = \frac{(b-a)^{\sigma+1}}{\sigma+1}\left[1-\frac{1}{2^\sigma}\right].$$

This provides the required inequality. □

5. Conclusions

This research was on Fejér-type inequalities for the class of $(h,g;m)$-convex functions that include the generalized Mittag-Leffler function with the corresponding fractional integral operator. We gave an overview of the published results of the studied problems, and pointed out the contribution of the obtained results in this work.

Potential future research could involve fractional generalizations of other inequalities, such as Jensen's inequality or the Lah-Ribarič inequality. From these, we could derive inequalities by Hermite–Hadamard, Fejér, Giaccardi, Popoviciu, and Petrović. Such advancements might inspire new theoretical investigations in this area, especially considering the established applications of these problems in initial boundary value problems that describe irregular processes.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Acknowledgments: This research was partially supported through project KK.01.1.1.02.0027 (a project co-financed by the Croatian Government and the European Union through the European Regional Development Fund—the Competitiveness and Cohesion Operational Programme).

Conflicts of Interest: The author declares no conflicts of interest.

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