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Significant Study of Fuzzy Fractional Inequalities with Generalized Operators and Applications

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Abstract: There are many techniques for the extension and generalization of fractional theories, one of which improves fractional operators by means of their kernels. This paper is devoted to the most general concept of interval-valued functions, studying fractional integral operators for interval-valued functions, along with the multi-variate extension of the Bessel–Maitland function, which acts as kernel. We discuss the behavior of Hermite–Hadamard Fejér (HHF)-type inequalities by using the convex fuzzy interval-valued function (C-FIVF) with generalized fuzzy fractional operators. Also, we obtain some refinements of Hermite–Hadamard(H-H)-type inequalities via convex fuzzy interval-valued functions (C-FIVFs). Our results extend and generalize existing findings from the literature.

Keywords: convexity; fractional integral operators; Hermite–Hadamard inequalities

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1. Introduction

Due to the gradual development of fractional calculus, many problems related to non-integer models have been resolved. Special functions make great contributions to overcoming such issues, which are continuing to trouble scientific communities. The most important special functions are gamma functions, beta functions, and hypergeometric functions. For the advanced innovations of multi-index series, special functions have extended the theory of fractional operators by means of their kernels. Special functions have the potential to be helpful in many domains of mathematics, physics, and engineering. One of the most essential functions is the Bessel function [1–3].

Fractional calculus is the generalization of natural calculus, which plays an important role in applied, pure, and computational mathematics. The advancement of fractional analysis in several areas of mathematics has raised the requirement for fractional operators. To address this issue, numerous academics have sought to implement series-type special functions as kernels to build generalized fractional operators and achieve the modified generation of inequalities. Recent scientific work on fractional analysis has made significant contributions in a multitude of disciplines, including those related to control operator theory, biology, computer structure optimization, physics, signal–image processing, and fluid dynamics [4,5].

The convex function plays a key role in improving fractional calculus in various areas of mathematics. Convexity has been extensively researched since it is helpful in many fields of mathematics, including optimization. Control optimization and inequalities theory have a strong connection as a result of the convex function that was analyzed and discovered to have well-known inequalities by the researchers [6–8]. The extensions and generalizations of Hermite–Hadamard-type inequalities and its refinements have great promise for developing the theory of analysis [9–12]. Many scholars have performed extensive work on their significance and uses in the area of analysis [13–15].

Let $\backepsilon : K \to R$ be a convex function; then, a H-H inequality [16–18] is defined for $[\jmath, \leftthreetimes] \in K \subseteq \mathbb{R}, \jmath < \leftthreetimes, \jmath, \leftthreetimes \in \mathbb{R}$ as follows:

$$\backepsilon\left(\frac{\jmath + \leftthreetimes}{2}\right) \leq \frac{1}{\leftthreetimes - \jmath} \int_{\jmath}^{\leftthreetimes} \backepsilon(z)dz \leq \frac{\backepsilon(\jmath) + \backepsilon(\leftthreetimes)}{2}. \tag{1}$$

Let K be the subset of reals, i.e., $K \subset \mathbb{R}$ be the convex set, and the convex function [19] $\backepsilon : K \to \mathbb{R}$ for $\sigma \in [0, 1], \forall\ u, v \in K$ is defined as follows:

$$\backepsilon\Big[\sigma u + (1 - \sigma)v\Big] \leq \sigma\backepsilon(u) + (1 - \sigma)\backepsilon(v). \tag{2}$$

The function is said to be concave for $\backepsilon$ if we have the reversed inequality (2).

Since Hanson's original discovery, a considerable amount of work has been performed, and this work has broadened the role and uses of invexity in nonlinear optimization and other areas of the pure and practical sciences. The fundamental characteristics of pre-invex functions and their application to optimization, variational inequalities, and equilibrium issues have been researched by Weir, Mond, and Noor. The possibility that pre-invex functions and invex sets are not convex functions and convex sets is widely recognized.

Research on set-valued analysis is widely recognised for its theoretical and practical significance. Many advancements in set-valued analysis have been driven by control theory and dynamical games. Since the early 1960s, advancements have been made in both areas of mathematical programming, as well as optimal control theory. A specific type of analysis known as interval analysis was created to address interval uncertainty, which can be present in many computer or mathematical models of deterministic real-world systems.

Moore was the first to introduce the idea of interval analysis in 1966 [20].The critical analysis of interval-valued functions and its applications has been peformed by many researchers in various fields, including those related to mathematical economy and control theory. Recently, Zhang et al. [21] discussed Jensen's inequalities and their refinements on a set-valued function (convex), as well as a fuzzy-valued function, and new versions of well-known inequalities have been discussed for convex fuzzy number mapping [22].

In [23], Ghosh et al. introduced the concept of applying fixed-order models and variable order models on the intervals and showed that the relation of the convex cone is actually the partially ordered relation on the intervals, which has an immense role in obtaining the optimal solutions of physical problems with interval-valued functions. Many researchers have worked to discuss the optimal solutions on behalf of semi-locally pseudo-convex mappings and also established the effect of their properties on the interval analysis [24]. The fuzzy concept is used for different purposes. A fuzzy decision-making approach is suggested for challenges with many objectives. Its primary characteristics are the expression of imprecise and uncertain outcomes through fuzzy probability and the representation of the decision-maker's preference structure through fuzzy connectives. The decision-maker and computer work together to derive the preference structure. New methods based on vague set theory for dealing with multi-criteria fuzzy decision-making situations have also been studied. With regard to a set of criteria, the suggested methodologies enable the degrees of sustainability and non-sustainability of each alternative to be represented by vague values.

Due to the significance in a variety of domains, integral inequalities have drawn a lot of attention over the last twenty years. According to modern research, many inequalities are used in terms of fuzzy interval-valued functions, which indicates that this method is fascinating from a theoretical and practical stand point, since it makes it possible to convert fuzzy integral inequalities into actual integral inequalities. In all of those, Hermite–Hadamard (H-H) inequalities generate significant connections between various classes of convex functions and are crucial in numerous mathematical domains. Thus, tools from the classical real analysis can be used to deal with classes of non-deterministic situations that differ from the ones previously examined. Convexity and its generalizations have a novel approach for optimization in fuzzy domains because, when the optimal condition of convexity is characterised, we obtain fuzzy variation inequalities. As a result, fuzzy complementary problem theory and variation inequality have formed a strong structural relationship with mathematical problems, and this is a cordial relationship. Several authors have made contributions to this unique and captivating area.

Let $\eth : [\eth, v] \subset \mathbb{R} \to \Omega_c^+$ be the positive and bounded intervals in $\mathbb{R}$ for all $\alpha \in [\eth, v]$, given by $\eth(\alpha) = [\eth(\alpha)_*, \eth(\alpha)^*]$, where $\eth(\alpha)^*$ is a concave function, and $\eth(\alpha)_*$ is said to be convex function if it is integrable and satisfies the following relation:

$$\eth\left(\frac{\alpha+\Lambda}{2}\right) \supseteq (IR)\frac{1}{\Lambda-\alpha}\int_{\alpha}^{\Lambda}\eth(z)dz \supseteq \frac{\eth(\alpha)+\eth(\Lambda)}{2}. \quad (3)$$

Most of the research pertaining to developing the relations of inequalities and their refinements with fuzzy convexities involve successfully implementing fuzzy fractional operators, which have great importance, despite being novel, in the field of fuzzy inequalities [25–29].

The inclusion relation for the Hermite–Hadamard inequality [30] is defined as follows

$$\eth\left(\frac{\alpha+\Lambda}{2}\right) \supseteq \frac{1}{2\varsigma(v,\eth)^{\nu'+1}}\left[\mathcal{I}_{v,v'}^{\eth+}(\Omega,\eth) + \mathcal{I}_{\eth,v'}^{v-}(\Omega,\eth)\right] \supseteq \frac{\eth(\alpha)+\eth(\Lambda)}{2}. \quad (4)$$

There are several notable uses of fuzzy set theory; for example, it can be used to deal with problems involving vague, subjective, and uncertain evaluations; in qualifying the linguistic characteristics of given data; and in making decisions for individual or group collaboration. The current work is more focused on achieving more advancements in fuzzy fractional inequalities through using the generalized Bessel–Maitland function perform as a kernel. The use of fuzzy number analysis in the main findings also introduces a new path in the investigation of inequalities. We develop novel Hermite–Hadamard- and Fejér-type inequalities based on the recently presented idea of fuzzy fractional operators [31] via the extension of generalized fractional integral operators.

2. Preliminaries

In this section, we will discuss the basic definitions and concepts.

Definition 1. *Let $\jmath \in \mathbb{N}$ and $\leftthreetimes \in \mathbb{C}$, the Pochammer's symbol [32], be defined as noted below*

$$(\leftthreetimes)_\jmath = \left\{ \begin{array}{ll} 1, & \text{if } \jmath = 0, \leftthreetimes \neq 0 \\ \leftthreetimes(\leftthreetimes+1)\cdots(\leftthreetimes+\jmath-1), & \text{if } \jmath \geq 1, \end{array} \right\}$$

Definition 2. *The integral representation of the gamma function [32] is defined for $\Re(\sigma) > 0$ as given below*

$$\Gamma(\sigma) = \int_0^\infty z^{\sigma-1}e^{-z}dz.$$

Definition 3. *The beta function [33] is defined for $\Re(g) > 0$, as well as $\Re(\imath) > 0$, as follows*

$$\begin{aligned} \beta(g,\imath) &= \int_0^1 \rtimes^{g-1}(1-\rtimes)^{\imath-1}d\rtimes \\ &= \frac{\Gamma(\imath)\Gamma(g)}{\Gamma(\imath+g)}. \end{aligned}$$

Definition 4. *Let $\Re(\bowtie) > 0$, $\Re(h) > 0$, and $\Re(c) > 0$; then, the extended form of the beta function [34] is defined as follows*

$$\mathbf{B}_c(\bowtie, h) = \int_0^1 t^{\bowtie-1}(1-t)^{h-1} exp(\frac{-c}{t(1-t)})dt.$$

We obtain the classical beta function if we let $c = 0$.

Definition 5. *The multi-variate Bessel–Maitland function with eight parameters [35] is defined as follows*

$$\mathbf{J}^{\beth,\imath,\sigma,\vartheta}_{\xi,\hbar,m,\sigma}(\curlywedge) = \sum_{q=0}^{\infty} \frac{(\imath)_{\hbar q}(\vartheta)_{\sigma q}(-\curlywedge)^q}{\Gamma(\xi q + \beth + 1)(\sigma)_{mq}}, \tag{5}$$

where $\xi, \beth, \hbar, m, \sigma$ are complex numbers, for which $\Re(\imath) > 0$, $\Re(\vartheta) > 0$, $\Re(\beth) \geq -1$, $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$; $m, \hbar, \sigma \geq 0$, and $\hbar, m > \Re(\xi) + \sigma$.

Definition 6. *The generalized form of the mutli-variate Bessel–Maitland function [35] is defined as follows*

$$\mathbf{J}^{\Xi,\hbar,m,\sigma,c}_{v,\jmath,\rho,\wp}(\Omega; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\jmath + \hbar n, c - \jmath)(c)_{\hbar n}(\wp)_{\sigma n}}{\beta(\jmath, c - \jmath)\Gamma(\Xi n + v + 1)(\rho)_{mn}}(-\Omega)^n. \tag{6}$$

where $\Re(\nu) \geq -1$, $\Re(\Xi) > 0$, $\jmath, \Xi, \wp, \nu, \rho, c \in \mathbb{C}$, $\Re(\jmath) > 0$, $\Re(\wp) > 0$, $\Re(\rho) > 0$; $\sigma, \hbar, m \geq 0$, and $m, \hbar > \Re(\Xi) + \sigma$.

Proposition 1 ([26])**.** *The partial order relation $\preccurlyeq$ on set $\mathbb{F}_0$ is defined as follows*

$$\Xi \preccurlyeq \alpha \iff [\Xi]^\imath \leq_I [\alpha]^\imath \ \forall \ \imath \in [0, 1], \tag{7}$$

where $\Xi, \alpha \in \mathbb{F}_0$.

The Hukuhara difference of Ξ and $\rtimes$ for $\ell \in \mathbb{F}_0$ and $\Xi = \rtimes \tilde{+} \ell$; then, the difference of ℓ is defined as given below

$$(\ell)^*(\imath) = (\Xi \tilde{-} \rtimes)^*(\imath) = \Xi^*(\imath) - \rtimes^*(\imath),$$
$$(\ell)_*(\imath) = (\Xi \tilde{-} \rtimes)_*(\imath) = \Xi_*(\imath) - \rtimes_*(\imath).$$

Let the set Q represent the partition on $[\eth, v]$; it can be written as

$$Q = \eth = \jmath_1 < \jmath_2 < \jmath_3 < \jmath_4 < \jmath_5 < \ldots .\jmath_k = v.$$

The maximum length of the sub-interval containing Q is defined as follows

$$mesh(Q) = max(\jmath_j - \jmath_{j-1} : j = 1, 2, 3 \ldots k). \tag{8}$$

The Riemann sum of α over the partition Q is written as

$$S(\alpha, p, \sigma) = \sum_{j=1}^{k} \alpha(\zeta_j)(\jmath_j - \jmath_{j-1}).$$

Definition 7. *The Riemann-integrable function [36], $\alpha : [\jmath, \imath] \to \mathbb{R}_I$, on the interval $[\jmath, \imath]$ is defined for $B \in \mathbb{R}_I$, $\epsilon, \sigma > 0$ as follows*

$$d(S(\alpha, Q, \sigma), I) < \epsilon.$$

The Riemann sum over the partition Q is denoted by $I = (IR) \int_\jmath^\imath \alpha(t)dt$.

Theorem 1 ([20]). *Let $\supset : [\eth, v] \subseteq \mathbb{R} \to \mathbb{R}_I$ be a real-valued function. It is said to be an integrable if $\supset(\Xi) = [\supset_*, \supset^*]$ on $[\jmath, \imath]$, $\supset_*$ and $\supset^*$ are both integrable functions over $[\jmath, \imath]$, such that*

$$(IR)\int_{\jmath}^{\imath}\supset(\jmath)dx = \left[(R)\int_{\jmath}^{\imath}\supset_*(\jmath),(R)\int_{\jmath}^{\imath}\supset^*(\Xi)dx\right], \tag{9}$$

where $R_{[c,d]}, IR_{[\jmath,\imath]}$ represent the real integrable and generalized integrable functions.

Definition 8 ([37]). *Let $\supset :\subsetneq \mathbb{R} \to \mathbb{F}_{\not\vdash}$ be a fuzzy IVF, and each $\triangleright \in [0,1]$ be the $\triangleright$-levels on $\supset : k \subseteq \mathbb{R} \to \Omega_c$ as $\supset_{\triangleright}(\triangleleft) = \left[\supset_*(\triangleleft,\triangleright), \supset^*(\triangleleft,\triangleright)\right]$, if for all $\triangleleft \in \Omega$. These real-valued functions $\supset_*(\triangleleft,\triangleright), \supset^*(\triangleleft,\triangleright) : \Omega \to \mathbb{R}$ are also called upper and lower functions of $\supset$.*

Remark 1. *For each $\triangleright \in [0,1]$, consider the continuous function $\zeta : k \subseteq R \to \mathbb{F}_{\not\vdash}$ at $\triangleleft \in \Omega$ if both functions $\zeta_*(\triangleleft,\triangleright), \zeta^*(\triangleleft,\triangleright)$ (left and right real-valued functions) are continuous at $\triangleleft \in K$.*

Now, we discuss some definitions and well-known properties of fuzzy interval-valued functions.

Let η_C of $\mathbb{R}$ and $\triangleright \in \eta_C$; then, the bounded and closed intervals can be described as given below

$$\triangleright = [\triangleright_*, \triangleright^*] = \{\alpha \in \mathbb{R} | \triangleright_* \preccurlyeq \alpha \preccurlyeq \triangleright^*\}, (\triangleright_*, \triangleright^* \in \mathbb{R}).$$

If we have $\triangleright_* = \triangleright^*$, then $\triangleright$ is degenerate.

In this work, all the intervals will be non-degenerate. If $\triangleright_* \geq 0$, then we say that the interval $[\triangleright_*, \triangleright^*]$ is positive and is denoted and defined as

$$\Omega_C^+ = \{[\triangleright_*, \triangleright^*] : [\triangleright_*, \triangleright^*] \in \Omega_C \ \textit{and} \ \triangleright_* \geq 0\}.$$

Remark 2 ([38]). *The property with respect to the relation " $\preccurlyeq_1$ " is defined on Ω_C as follows*

$$[\sigma_*, \sigma^*] \preccurlyeq_1 [\Xi_*, \Xi^*] \ \textit{if and only if} \ \triangleleft_* \preccurlyeq \Xi_*, \triangleleft^* \preccurlyeq \tau^*$$

for all $[\triangleleft_*, \triangleleft^*], [\Xi_*, \Im^*] \in \Omega_C$ is an order relation, and $[\triangleleft_*, \triangleleft^*], [\Xi_*, \Xi^*] \in \Omega_C$; then, $[\triangleleft_*, \triangleleft^*] \preccurlyeq_1 [\Xi_*, \Xi^*]$ if and only if $\triangleleft_* \preccurlyeq \Xi_*$ or $\triangleleft^* \preccurlyeq \Xi^*$, $\triangleleft < \Xi$.

Definition 9 ([14]). *Let $\hbar : M \times M \to \mathbb{R}$ be a bi-function, and $\eth, v \in M$, $\lambda \in [0,1]$; then, the invex set $M \subseteq \mathbb{R}$ is defined as given below*

$$\jmath + \lambda\hbar(\eth, \jmath) \in M.$$

Definition 10 ([14]). *Let M be an invex set with aspects of ς; then, the real-valued function $\wp : M \to \mathbb{R}$ is said to be pre-invex function for $\jmath, \curlywedge \in M$ if the following inequality is satisfied*

$$\wp(\curlywedge + \Lambda\varsigma(\jmath, \curlywedge)) \leq \Lambda\wp(\jmath) + (1 - \Lambda)\wp(\curlywedge), \tag{10}$$

where $\Lambda \in [0,1]$.

Definition 11 ([39]). *Let $\supset : K \to \mathbb{R}$ be a real-valued function that is said to be a convex fuzzy interval-valued function for $\sigma \in [0,1]$, $\forall \eth, v \in K$ if we have*

$$\supset\left[\sigma\eth + (1-\sigma)v\right] \preccurlyeq \sigma\supset(\eth)\tilde{+}(1-\sigma)\supset(v). \tag{11}$$

The function $\supset$ is concave if the inequality (11) is reversed.

Remark 3. *We have inequality 2 if $\imath = 1$ and $\supset_*(\jmath, \imath) = \supset^*(\jmath, \imath)$.*

Definition 12 ([40]). *Let $\eth : M \to \mathbb{R}$ be a function; then, a pre-invex fuzzy interval-valued (FIV) function is defined for $\jmath, \curlywedge \in M$ and $\Xi \in [0,1]$ if the following relation is satisfied*

$$\eth(\curlywedge + \Xi\varsigma(\jmath, \curlywedge)) \preccurlyeq \Xi\eth(\jmath)\widetilde{+}(1-\Xi)\eth(\curlywedge), \tag{12}$$

where M is an open invex set with respect to ς. We have a pre-incave fuzzy interval-valued (FIV) function if we reverse the inequality (12).

Definition 13 ([35]). *The multi-variate version of fractional integral operators is defined as given below*

$$\left(\mathfrak{T}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp;p^+}f\right)(\jmath,r) = \int_p^\jmath (\jmath-t)^v \mathbf{J}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp}(\Psi(\jmath-t)^{\rtimes};r)f(t)dt, (\jmath > p),$$

and

$$\left(\mathfrak{T}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp;q^-}f\right)(\jmath,r) = \int_\jmath^q (t-\jmath)^v \mathbf{J}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp}(\Psi(t-\jmath)^{\rtimes};r)f(t)dt, (\jmath < q),$$

where $\nu, \rtimes, \rho, \zeta, \wp, c \in \mathbb{C}$, $\Re(\rtimes) > 0$, $\Re(\nu) \geq -1$, $\Re(\rho) > 0$, $\Re(\zeta) > 0$, $\Re(\wp) > 0$; $\hbar, \bowtie, m \geq 0$ and $\hbar, m > \Re(\rtimes) + \bowtie$.

Remark 4. *If we replace $r = 0$, $\Psi = 0$, and $\rtimes = \rtimes - 1$ in definition (13), then we obtain the left- and right-sided Riemann–Liouville fractional integral operators.*

Definition 14 ([30]). *The multi-variate versions of the fuzzy fractional integral operators based on the $\imath$ level are defined as given below*

$$\left[\left(\mathfrak{T}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp;p^+}f\right)(\jmath,r)\right]^\imath = \int_p^\jmath (\jmath-t)^v \mathbf{J}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp}(\Psi(\jmath-t)^{\rtimes};r)f_\imath(t)dt$$

$$= \int_p^\jmath (\jmath-t)^v \mathbf{J}^{\rtimes,\hbar,m,\bowtie,c}_{v,\zeta,\rho,\wp}(\Psi(\jmath-t)^{\rtimes};r)[f_*(t,\imath), f^*(t,\imath)]dt, (\jmath > p),$$

where $\rtimes, \nu, \zeta, \rho, \wp, c \in \mathbb{C}$, $\Re(\rtimes) > 0$, $\Re(\nu) \geq -1$, $\Re(\zeta) > 0$, $\Re(\rho) > 0$, $\Re(\wp) > 0$; $\hbar, m, \bowtie \geq 0$ and $m, \hbar > \Re(\rtimes) + \bowtie$,

Similarly, Cortez [30] defined the generalized fractional operators based on left–right end point functions.

Remark 5. *If we replace $r = 0$, $\Psi = 0$, and $\rtimes = \rtimes - 1$ in definition (14), then we obtain the left end point Riemann–Liouville fuzzy fractional integral operator.*

Remark 6.

$$\begin{aligned}
&\int_v^u (v-x)^{v'} \mathcal{J}^{\Xi,\hbar,m,\bowtie,c}_{v',\zeta,\rho,\imath}(\Psi(\frac{v-x}{\varsigma(v,\eth)})^{\Xi};p)1dx = (\Im^{v-}_{\eth,v'})(\Psi',1)\\
&\int_u^v (x-u)^{v'} \mathcal{J}^{\Xi,\hbar,m,\bowtie,c}_{v',\zeta,\rho,\imath}(\Psi(\frac{x-u}{\varsigma(v,\eth)})^{\Xi};p)\eth_*(x,\imath)dx +\\
&\int_v^u (v-x)^{v'} \mathcal{J}^{\Xi,\hbar,m,\bowtie,c}_{v',\zeta,\rho,\imath}(\Psi(\frac{v-x}{\varsigma(v,\eth)})^{\Xi};p)\eth_*(x,\imath)dx\\
&= (\Im^{\eth+}_{v,v'})(\Psi';\eth_*) + (\Im^{v-}_{\eth,v'})(\Psi';\eth_*)
\end{aligned} \tag{13}$$

The following notations are frequently used in our paper

$$(\mathcal{T}^{\eth+}_{v,v'})(\Psi,\eth) = (\Im^{\Xi,\hbar,m,\bowtie,c}_{v',\zeta,\rho,\wp;\eth^+}\eth)(v,p)$$

$$\left(\mathcal{T}^{v^-}_{\eth,v'}\right)(\Psi,\eth) = \left(\Im^{\Xi,\hbar,m,\bowtie,c}_{v',\zeta,\rho,\wp;v^-}\eth\right)(\eth,p).$$

3. Analyzing Behavior of H-H Fejér Inequalities Through Convex FIV Functions

Here, we analyze the behavior of generalized Hermite–Hadamard-Fejér inequalities by using convex fuzzy interval-valued functions (C-FIVFs).

Theorem 2. *Let $\Lambda : [m,n] \to F.$ be a convex fuzzy interval-valued function on $[m,n]$, whose $\dagger$-levels define the family of interval-valued functions $\Lambda_\dagger : [m,n] \subset \Re \to \vartheta_{c^+}$ given by $\Lambda_\dagger(\S) = [\Lambda_*(\S,\dagger), \Lambda^*(\S,\dagger)]$ for all $\dagger \in [m,n]$ and for all $\dagger \in [0,1]$.If $\Lambda \in ([m,n], F.)$ and $\rho : [m,n] \to \Re$,$\rho(\S) \geq 0$, symmetry with respect to $\frac{m+n}{2}$ is achieved for the multi-variate fractional integral defined in (13), with the multi-variate Bessel–Maitland function as its kernel; then, we have*

$$\left[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda\rho)\widetilde{+}\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda\rho)\right] \leq \frac{\Lambda(m,\dagger)\widetilde{+}\Lambda(n,\dagger)}{2}\left[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho) + \mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\right].$$

Proof. Let a convex fuzzy interval-valued function $'\Lambda'$ and $\Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\rho\left(\Upsilon m + (1-\Upsilon)n\right) \geq 0$ for each $\dagger \in [0,1]$; then, we obtain

$$\Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\Lambda_*\left(\Upsilon m + (1-\Upsilon)n,\dagger\right)\rho\left(\Upsilon m + (1-\Upsilon)n\right) \leq \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})$$
$$\left[\Upsilon\Lambda_*(m,\dagger) + (1-\Upsilon)\Lambda_*(n,\dagger)\rho\left(\Upsilon m + (1-\Upsilon)n\right)\right] \tag{14}$$

Also, we have

$$\Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\Lambda_*\left(\Upsilon m + (1-\Upsilon)n,\dagger\right)\rho\left(\Upsilon m + (1-\Upsilon)n\right) \leq \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})$$
$$\left[(1-\Upsilon)\Lambda_*(m,\dagger) + \Upsilon\Lambda_*(n,\dagger)\rho\left(\Upsilon m + (1-\Upsilon)n\right)\right]. \tag{15}$$

By adding Equations (14) and (15) and then integrating with respect to Υ over interval [0,1], we obtain

$$\int_0^1 \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\Lambda_*\left(\Upsilon m + (1-\Upsilon)n,\dagger\right)\rho\left(\Upsilon m + (1-\Upsilon)n\right)d\,\Upsilon +$$
$$\int_0^1 \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\Lambda_*\left((1-\Upsilon)m + \Upsilon n,\dagger\right)\rho\left((1-\Upsilon)m + \Upsilon n\right)d\Upsilon \leq$$
$$\int_0^1 \Big[\Upsilon^{v'}\,\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\Lambda_*(m,\dagger)\{\Upsilon\rho\left(\Upsilon m + (1-\Upsilon)nd\,\Upsilon + (1-\Upsilon)\rho\left((1-\Upsilon)m + \Upsilon n\right\}$$
$$+\,\Upsilon^{v'}\,\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\Lambda_*(n,\dagger)\{(1-\Upsilon)\rho\left(\Upsilon m + (1-\Upsilon)n + \Upsilon\rho\left((1-\Upsilon)m + \Upsilon n\right\}\Big]d\,\Upsilon$$
$$= \Lambda_*(m,\dagger)\int_0^1 \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\rho\left(\Upsilon m + (1-\Upsilon)n\right)d\,\Upsilon + \Lambda_*(n,\dagger)\int_0^1 \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma} \times$$
$$(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\rho\left((1-\Upsilon)m + \Upsilon n\right)d\Upsilon,$$

ρ is symmetric, so we have

$$= \left[\Lambda_*(m,\dagger) + \Lambda_*(n,\dagger)\right]\int_0^1 \Upsilon^{v'}\mathbf{J}^{\leftthreetimes,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\leftthreetimes};\mathbf{p})\rho\left((1-\Upsilon)m + \Upsilon n\right)d\,\Upsilon$$
$$= \left[\Lambda_*(m,\dagger) + \Lambda_*(n,\dagger)\right]\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\leftthreetimes n + v' + 1)(\rho)_{mn}}(-\omega)^n\int_0^1 \Upsilon^{v'+\leftthreetimes^n}$$

$$\rho\big((1-\Upsilon)m+\Upsilon n\big)d\Upsilon \leq \frac{\Lambda_*(m,\dagger)+\Lambda_*(n,\dagger)}{2(n-m)^{v'+\curlywedge^n+1}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big]. \tag{16}$$

Now, for the left side,

$$= \int_0^1 \Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\rho\big(\Upsilon m+(1-\Upsilon)n\big)d\,\Upsilon + \int_0^1 \Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\Lambda_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)\rho\big((1-\Upsilon)m+\Upsilon n\big)d\,\Upsilon\,.$$

If we continue the same process for solving the right side, we have

$$= \frac{1}{(n-m)^{v'+\curlywedge^n+1}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda_*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda_*\rho)\Big] \tag{17}$$

By combining (16) and (17), we have

$$\frac{\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda_*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda_*\rho)\Big]}{(n-m)^{v'+\curlywedge^n+1}} \leq \frac{\Lambda_*(m,\dagger)+\Lambda_*(n,\dagger)}{2(n-m)^{v'+\curlywedge^n+1}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big]. \tag{18}$$

Similarly, solving for $\Lambda^*(\S,\dagger)$ gives

$$\frac{\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda^*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda^*\rho)\Big]}{(n-m)^{v'+\curlywedge^n+1}} \leq \frac{\Lambda^*(m,\dagger)+\Lambda^*(n,\dagger)}{2(n-m)^{v'+\curlywedge^n+1}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big]. \tag{19}$$

By combining (18) and (19), we obtain the required result

$$\frac{1}{(n-m)^{v'+\curlywedge^n+1}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda^*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda^*\rho),\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda_*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda_*\rho)\Big] \leq_I \frac{1}{(n-m)^{v'+\curlywedge^n+1}}\Big[\frac{\Lambda^*(m,\dagger)+\Lambda^*(n,\dagger)}{2},\frac{\Lambda_*(m,\dagger)+\Lambda_*(n,\dagger)}{2}\Big]\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big]$$

$$\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda\rho)\widetilde{+}\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda\rho)\Big] \leq \frac{\Lambda(m,\dagger)\widetilde{+}\Lambda(n,\dagger)}{2}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big].$$

□

Corollary 1. *If we replace* $\mathbf{p}=0$, $\omega=0$, *and* $\curlywedge=\curlywedge-1$ *in theorem (2), then we have the well-known inequality [41].*

Theorem 3. *Let* $\Lambda:[m,n]\to F.$ *be a convex fuzzy interval-valued function on* $[m,n]$ *, whose* $\dagger$*-levels define the family of interval valued functions* $\Lambda_\dagger:[m,n]\subset\Re\to\vartheta_{c^+}$ *given by* $\Lambda_\dagger(\S)=[\Lambda_*(\S,\dagger),\Lambda^*(\S,\dagger)]$ *for all* $\dagger\in[m,n]$ *and for all* $\dagger\in[0,1]$*.If* $\Lambda\in([m,n],F.)$*, and* $\rho:[m,n]\to\Re,\rho(\S)\geq 0$*, symmetric with respect to* $\frac{m+n}{2}$*; then, for the multi-variate fractional integral defined in (13) with the multi-index Bessel–Maitland function as its kernel, we have*

$$\Lambda\Big(\frac{m+n}{2}\Big)\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big] \preccurlyeq \Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda\rho)\Big]. \tag{20}$$

If the inequality is reversed, then Λ *is a concave fuzzy interval-valued function.*

Proof. Let Λ be a convex fuzzy interval-valued function; then, for each $\dagger \in [0,1]$,

$$\Lambda_*\left(\frac{m+n}{2},\dagger\right) \leq \frac{1}{2}\left(\Lambda_*(\Upsilon m+(1-\Upsilon)n,\dagger)+\Lambda_*((1-\Upsilon)m+\Upsilon n,\dagger)\right) \tag{21}$$

Multiplying the Equation (21) with $\Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\rho\big(\Upsilon m+(1-\Upsilon)n\big)$ and integrating over $\dagger \in [0,1]$ such that $\rho\big(\Upsilon m+(1-\Upsilon)n\big)=\rho\big((1-\Upsilon)m+\Upsilon n\big)$, we have

$$\begin{aligned}&\Lambda_*(\frac{m+n}{2},\dagger)\int_0^1 \Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\rho\big(\Upsilon m+(1-\Upsilon)n\big)d\Upsilon \leq \int_0^1 \Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}\\&(\omega\Upsilon^{\curlywedge};\mathbf{p})\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\rho\big(\Upsilon m+(1-\Upsilon)n\big)d\,\Upsilon+\int_0^1 \Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\\&\Lambda_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)\rho\big((1-\Upsilon)m+\Upsilon n\big)d\Upsilon\end{aligned} \tag{22}$$

$$\begin{aligned}&\Lambda_*(\frac{m+n}{2},\dagger)\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\curlywedge n+v'+1)(\rho)_{mn}}(-\omega)^n\int_0^1 \Upsilon^{v'+\curlywedge^n}\rho\big(\Upsilon m+(1-\Upsilon)n\big)d\Upsilon\\&\leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\curlywedge n+v'+1)(\rho)_{mn}}(-\omega)^n\bigg[\int_0^1 \Upsilon^{v'+\curlywedge^n}\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\\&\rho\big((1-\Upsilon)m+\Upsilon n\big)d\,\Upsilon+\int_0^1 \Upsilon^{v'+\curlywedge^n}\Lambda_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)\rho\big((1-\Upsilon)m+\Upsilon n\big)d\,\Upsilon\bigg]\end{aligned} \tag{23}$$

Let $\S=(1-\Upsilon)m+\Upsilon n$, and

$$\begin{aligned}&\int_0^1 \Upsilon^{v'+\curlywedge^n}\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\rho\big((1-\Upsilon)m+\Upsilon n\big)d\,\Upsilon+\int_0^1 \Upsilon^{v'+\curlywedge^n}\\&\Lambda_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)\rho\big((1-\Upsilon)m+\Upsilon n\big)d\Upsilon\end{aligned}$$

$$\Lambda_*(\frac{m+n}{2},\dagger)\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big] \leq \Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda_*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda_*\rho)\Big] \tag{24}$$

Similarly, for Λ^*

$$\Lambda^*(\frac{m+n}{2},\dagger)\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big] \leq \Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda^*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda^*\rho)\Big] \tag{25}$$

from Equations (24) and (25), we have the required result

$$\begin{aligned}&\Big[\Lambda_*(\frac{m+n}{2},\dagger),\Lambda^*(\frac{m+n}{2},\dagger)\Big]\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big] \leq_I \Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda_*\rho)+\\&\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda_*\rho),\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda^*\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda^*\rho)\Big]\end{aligned} \tag{26}$$

$$\Lambda(\frac{m+n}{2})\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\rho)\Big] \preccurlyeq \Big[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda\rho)+\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda\rho)\Big].$$

□

Corollary 2. *If we replace* $\mathbf{p}=0$, $\omega=0$, *and* $\curlywedge=\curlywedge-1$ *in theorem (3), then we have the well-known inequality [41].*

Theorem 4. *Let $\Lambda, \mathbb{U} : [m,n] \to F$. Both are convex fuzzy interval-valued functions on $[m,n]$, whose $\dagger$-levels define the family of interval-valued functions $\mathbb{U}_{\dagger}, \Lambda_{\dagger} : [m,n] \subset \Re \to \vartheta_{c^{+}}$, which are given by $\Lambda_{\dagger}(\S) = [\Lambda_{*}(\S,\dagger), \Lambda^{*}(\S,\dagger)], \mathbb{U}_{\dagger}(\S) = [\mathbb{U}_{*}(\S,\dagger), \mathbb{U}^{*}(\S,\dagger)]$ for all $\dagger \in [m,n]$ and for all $\dagger \in [0,1]$, as $\Lambda, \mathbb{U}$ and $\Lambda \widetilde{\times} \mathbb{U} \in L([m,n], F.)$; then, for the multi-variate fractional integral operators defined in (13), with the multi-index Bessel–Maitland function as its kernel, we have*

$$\frac{\Theta}{2(n-m)^{\Theta}}\left[\mathbf{J}^{m^{+}}_{n,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U}))+\mathbf{J}^{n^{-}}_{m,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U}))\right] \leq \Bigg[\Upsilon(m,n)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p}) \\ \{\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)}\}\widetilde{+}2\Psi(m,n)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{\Theta}{(\Theta+1)(\Theta+2)}\}\Bigg] \tag{27}$$

where $\Theta = v' + \curlywedge n + 1$

$\Upsilon(m,n) = \Lambda(m)\widetilde{\times}\,\mathbb{U}\,(m)\widetilde{+}\Lambda(n)\widetilde{\times}\,\mathbb{U}\,(n), \Psi(m,n) = \Lambda(m)\widetilde{\times}\,\mathbb{U}\,(n)\widetilde{+}\Lambda(n)\widetilde{\times}\,\mathbb{U}\,(m)$

and

$\Upsilon_{\dagger}(m,n) = [\Upsilon_{*}((m,n),\dagger), \Upsilon^{*}((m,n),\dagger)], \Psi_{\dagger}(m,n) = [\Psi_{*}((m,n),\dagger), \Psi\Psi^{*}((m,n),\dagger)$

Proof. Let $\Lambda, \mathbb{U}$ be convex fuzzy Interval-valued functions for each $\dagger \in [0,1]$

$$\Lambda_{*}\bigg(\Upsilon m+(1-\Upsilon)n,\dagger\bigg) \leq \Upsilon\Lambda_{*}(m,\dagger)+(1-\Upsilon)\Lambda_{*}(n,\dagger)$$

$$\mathbb{U}_{*}\bigg(\Upsilon m+(1-\Upsilon)n,\dagger\bigg) \leq \Upsilon\,\mathbb{U}_{*}\,(m,\dagger)+(1-\Upsilon)\,\mathbb{U}_{*}\,(n,\dagger)$$

since $\widetilde{0} \preccurlyeq \Lambda(\S), \widetilde{0} \preccurlyeq \mathbb{U}(\S)$; then, via the definition of the convex fuzzy Interval-valued function,

$$\Lambda_{*}\bigg(\Upsilon m+(1-\Upsilon)n,\dagger\bigg)\times\mathbb{U}_{*}\bigg(\Upsilon m+(1-\Upsilon)n,\dagger\bigg) \leq (\Upsilon\Lambda_{*}(m,\dagger)+(1-\Upsilon)\Lambda_{*}(n,\dagger)) \\ \times(\Upsilon\,\mathbb{U}_{*}\,(m,\dagger)+(1-\Upsilon)\,\mathbb{U}_{*}\,(n,\dagger)) \\ =\Upsilon^{2}\Lambda_{*}(m,\dagger)\times\mathbb{U}_{*}(m,\dagger)+(1-\Upsilon^{2})\Lambda_{*}(n,\dagger)\times\mathbb{U}_{*}(n,\dagger)+\Upsilon(1-\Upsilon)\Lambda_{*}(m,\dagger) \\ \times\mathbb{U}_{*}\,(n,\dagger)+\Upsilon(1-\Upsilon)\Lambda_{*}(n,\dagger)\times\mathbb{U}_{*}(m,\dagger) \tag{28}$$

Now,

$$\Lambda_{*}\bigg((1-\Upsilon)m+\Upsilon n,\dagger\bigg)\times\mathbb{U}_{*}\bigg((1-\Upsilon)m+\Upsilon n,\dagger\bigg) \leq (1-\Upsilon^{2})\Lambda_{*}(m,\dagger)\,\mathbb{U}_{*}\,(m,\dagger) \\ +\Upsilon^{2}\,\Lambda_{*}(n,\dagger)\times\mathbb{U}_{*}(n,\dagger)+\Upsilon(1-\Upsilon)\Lambda_{*}(m,\dagger)\times\mathbb{U}_{*}(n,\dagger)+ \\ \Upsilon(1-\Upsilon)\Lambda_{*}(n,\dagger)\times\mathbb{U}_{*}(m,\dagger) \tag{29}$$

By adding Equations (28) and (29), we have

$$\Lambda_{*}\bigg(\Upsilon m+(1-\Upsilon)n,\dagger\bigg)\times\mathbb{U}_{*}\bigg(\Upsilon m+(1-\Upsilon)n,\dagger\bigg)+\Lambda_{*}\bigg((1-\Upsilon)m+\Upsilon n,\dagger\bigg)\times\mathbb{U}_{*} \\ \bigg((1-\Upsilon)m+\Upsilon n,\dagger\bigg) \leq [\,\Upsilon^{2}+(1-\Upsilon^{2})]\bigg[\Lambda_{*}(m,\dagger)\times\mathbb{U}_{*}(m,\dagger)+\Lambda_{*}(n,\dagger)\times\mathbb{U}_{*}(n,\dagger)\bigg] \\ +2\,\Upsilon\,(1-\Upsilon)\bigg[\Lambda_{*}(m,\dagger)\times\mathbb{U}_{*}(n,\dagger)+\Lambda_{*}(n,\dagger)\times\mathbb{U}_{*}(m,\dagger)\bigg] \tag{30}$$

Multiplying both sides by $\Upsilon^{v'}\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})$ of Equation (30) and integrating the resulting inequality on $(0,1)$ with respect to Υ, we have

$$\int_0^1 \Upsilon^{v'} \mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\times\mathbb{U}_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)d\,\Upsilon$$
$$+\int_0^1 \Upsilon^{v'} \mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\Lambda_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)\times\mathbb{U}_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)d\,\Upsilon$$
$$\leq \Upsilon_*\big((m,n),\dagger\big)\int_0^1 \Upsilon^{v'} \mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\big[\,\Upsilon^2+(1-\Upsilon^2)\big]d\,\Upsilon+2\Psi_*\big((m,n),\dagger\big) \quad (31)$$
$$\times\int_0^1 \Upsilon^{v'} \mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})\,\Upsilon\,(1-\Upsilon)d\Upsilon$$

After solving (38), we have

$$\frac{\Big[\mathbf{J}^{m+}_{n,v'}(\omega,(\Lambda_*\times\mathbb{U}_*),\dagger)+\mathbf{J}^{n-}_{m,v'}(\omega,(\Lambda_*\times\mathbb{U}_*),\dagger)\Big]}{(n-m)^{\Theta}}\leq\Big[\Upsilon_*\big((m,n),\dagger\big)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})$$
$$\{\tfrac{2}{\Theta}(\tfrac{1}{2}-\tfrac{\Theta}{(\Theta+1)(\Theta+2)})\}+2\Psi_*\big((m,n),\dagger\big)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\tfrac{2}{\Theta}(\tfrac{\Theta}{(\Theta+1)(\Theta+2)}\}\Big] \quad (32)$$

Similarly, for $(\Lambda^*\times\mathbb{U}^*,\dagger)$, we have

$$\frac{\Big[\mathbf{J}^{m+}_{n,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)+\mathbf{J}^{n-}_{m,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)\Big]}{(n-m)^{\Theta}}\leq\Big[\Upsilon^*\big((m,n),\dagger\big)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})$$
$$\{\tfrac{2}{\Theta}(\tfrac{1}{2}-\tfrac{\Theta}{(\Theta+1)(\Theta+2)})\}+2\Psi^*\big((m,n),\dagger\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\tfrac{2}{\Theta}(\tfrac{\Theta}{(\Theta+1)(\Theta+2)}\}\Big] \quad (33)$$

By combining (32) and (33), we obtain

$$\left[\frac{\mathbf{J}^{m+}_{n,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)+\mathbf{J}^{n-}_{m,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger),\mathbf{J}^{m+}_{n,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)+\mathbf{J}^{n-}_{m,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)}{(n-m)^{\Theta}}\right]$$
$$\leq_I\{\tfrac{2}{\Theta}(\tfrac{1}{2}-\tfrac{\Theta}{(\Theta+1)(\Theta+2)})\}\big[\Upsilon_*\big((m,n),\dagger\big),\Upsilon^*\big((m,n),\dagger\big)\big]+\tfrac{2}{\Theta}(\tfrac{\Theta}{(\Theta+1)(\Theta+2)})$$
$$\big[\Psi_*\big((m,n),\Psi^*(m,n),\dagger\big)\big] \quad (34)$$

So, we have the required result

$$\frac{\Theta}{2(n-m)^{\Theta}}\Big[\mathbf{J}^{m+}_{n,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U}))+\mathbf{J}^{n-}_{m,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U}))\Big]\leq\Big[\Upsilon(m,n)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})$$
$$\{\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)}\}\widetilde{+}2\Psi(m,n)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{\Theta}{(\Theta+1)(\Theta+2)}\}\Big] \quad (35)$$

□

Corollary 3. *If we replace* $\mathbf{p}=0$, $\omega=0$, *and* $\curlywedge=\curlywedge-1$ *in theorem* (4), *then we have the well-known inequality [41].*

Theorem 5. *Let* $\Lambda,\mathbb{U}:[m,n]\to F$. *Both are convex fuzzy interval-valued functions on* $[m,n]$ *, whose* $\dagger$*-levels define the family of interval-valued functions* $\mathbb{U}_\dagger,\Lambda_\dagger:[m,n]\subset\Re\to\vartheta_{c^+}$ *given by* $\Lambda_\dagger(\S)=[\Lambda_*(\S,\dagger),\Lambda^*(\S,\dagger)],\mathbb{U}_\dagger(\S)=[\mathbb{U}_*(\S,\dagger),\mathbb{U}^*(\S,\dagger)]$ *for all* $\dagger\in[m,n]$ *and for all* $\dagger\in[0,1]$, *as* $\Lambda,\mathbb{U}$ *and* $\Lambda\widetilde{\times}\mathbb{U}\in L([m,n],F.)$; *then, for the multi-variate fractional integral operators* (13), *we have*

$$\frac{1}{\Theta}\Lambda(\frac{m+n}{2})\widetilde{\times}\,\mathbb{U}\,(\frac{m+n}{2})\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p}) \preccurlyeq \frac{1}{4(n-m)^{\Theta}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U})+\mathbf{J}^{n^-}_{m,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U}))\Big]+\Upsilon(m,n)\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{1}{2\Theta}(\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)})\}+\Psi(m,n)\,\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{1}{2\Theta}(\frac{\Theta}{(\Theta+1)(\Theta+2)}\} \quad (36)$$

where $\Theta = v' + \measuredangle n + 1$
$\Upsilon(m,n) = \Lambda(m)\widetilde{\times}\,\mathbb{U}\,(m)\widetilde{+}\Lambda(n)\widetilde{\times}\,\mathbb{U}\,(n), \Psi(m,n) = \Lambda(m)\widetilde{\times}\,\mathbb{U}\,(n)\widetilde{+}\Lambda(n)\widetilde{\times}\,\mathbb{U}\,(m)$
and
$\Upsilon_{\dagger}(m,n) = [\Upsilon_*((m,n),\dagger),\Upsilon^*((m,n),\dagger)], \Psi_{\dagger}(m,n) = [\Psi_*((m,n),\dagger),\Psi^*((m,n),\dagger)$

Proof. Let $\Lambda, \mathbb{U}$ be convex fuzzy Interval-valued functions; then, for each $\dagger \in [0,1]$

$$\Lambda_*\left(\tfrac{m+n}{2},\dagger\right)\times\mathbb{U}_*\left(\tfrac{m+n}{2},\dagger\right) \le \tfrac{1}{4}\Big[\Lambda_*(\Upsilon m+(1-\Upsilon)n,\dagger)\times\mathbb{U}_*(\Upsilon m+(1-\Upsilon)n,\dagger) +\Lambda_*(\Upsilon m+(1-\Upsilon)n,\dagger)\times\mathbb{U}_*((1-\Upsilon)m+\Upsilon n,\dagger)\Big]+\tfrac{1}{4}\Big[\Lambda_*((1-\Upsilon)m+\Upsilon n,\dagger)\times \mathbb{U}_*(\Upsilon m+(1-\Upsilon)n,\dagger)+\Lambda_*((1-\Upsilon)m+\Upsilon n,\dagger)\times\mathbb{U}_*((1-\Upsilon)m+\Upsilon n,\dagger)\Big] \quad (37)$$

$$\Lambda_*\left(\tfrac{m+n}{2},\dagger\right)\times\mathbb{U}_*\left(\tfrac{m+n}{2},\dagger\right) \le \tfrac{1}{4}\Big[\Lambda_*(\Upsilon m+(1-\Upsilon)n,\dagger)\times\mathbb{U}_*(\Upsilon m+(1-\Upsilon)n,\dagger) +\Lambda_*(\Upsilon m+(1-\Upsilon)n,\dagger)\times\mathbb{U}_*((1-\Upsilon)m+\Upsilon n,\dagger)\Big]+\tfrac{1}{4}\Big[\big(\Upsilon\Lambda_*(m,\dagger)+(1-\Upsilon)\Lambda_*(n,\dagger)\big)\times \big((1-\Upsilon)\,\mathbb{U}_*\,(m,\dagger)+\Upsilon\,\mathbb{U}_*\,(n,\dagger)\big)+\big((1-\Upsilon)\Lambda_*(m,\dagger)+ \Upsilon\Lambda_*(n,\dagger)\big)\times\big(\Upsilon\,\mathbb{U}_*\,(m,\dagger)+(1-\Upsilon)\,\mathbb{U}_*\,(n,\dagger)\big)\Big]$$

Multiplying both sides by $\Upsilon^{v'}\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\measuredangle};\mathbf{p})$ of equation and integrating the resulting inequality on $(0,1)$ with respect to Υ, we have

$$\Lambda_*\left(\frac{m+n}{2},\dagger\right)\times\mathbb{U}_*\left(\frac{m+n}{2},\dagger\right)\int_0^1\Upsilon^{v'}\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\measuredangle};\mathbf{p})d\,\Upsilon \le \frac{1}{4}\Big[\int_0^1\Upsilon^{v'}\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\measuredangle};\mathbf{p})\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\times\mathbb{U}_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)d\,\Upsilon +\int_0^1\Upsilon^{v'}\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\measuredangle};\mathbf{p})\Lambda_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)\times\mathbb{U}_*\Big((1-\Upsilon)m+\Upsilon n,\dagger\Big)d\,\Upsilon\Big] +\frac{1}{4}\Big[\Upsilon_*((m,n),\dagger)\int_0^1\Upsilon^{v'}\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\measuredangle};\mathbf{p})\big[\,\Upsilon^2+(1-\Upsilon^2)\big]d\,\Upsilon+\Psi_*((m,n),\dagger)\int_0^1\Upsilon^{v'} \times\mathbf{J}^{\measuredangle,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega\Upsilon^{\measuredangle};\mathbf{p})\{\Upsilon(1-\Upsilon)+(1-\Upsilon)\Upsilon\}d\,\Upsilon\Big] \quad (38)$$

$$\Lambda_*\left(\tfrac{m+n}{2},\dagger\right)\times\mathbb{U}_*\left(\tfrac{m+n}{2},\dagger\right)\textstyle\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\measuredangle n+v'+1)(\rho)_{mn}}(-\omega)^n\int_0^1\Upsilon^{v'+\measuredangle n}d\Upsilon \le \tfrac{1}{4}\Big[\textstyle\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\measuredangle n+v'+1)(\rho)_{mn}}(-\omega)^n\int_0^1\Upsilon^{v'+\measuredangle n}\Lambda_*\Big(\Upsilon m+(1-\Upsilon)n,\dagger\Big)\times\mathbb{U}_*$$

$$\left(\Upsilon m+(1-\Upsilon)n,\dagger\right)d\,\Upsilon+\int_0^1\Upsilon^{v'+\curlywedge n}\Lambda_*\left((1-\Upsilon)m+\Upsilon n,\dagger\right)\times\mathbb{U}_*\left((1-\Upsilon)m+\Upsilon n,\dagger\right)d\,\Upsilon\Big]$$
$$+\frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\curlywedge n+v'+1)(\rho)_{mn}}(-\omega)^n\{\Upsilon_*((m,n),\dagger)\int_0^1\Upsilon^{v'+\curlywedge n}[\Upsilon^2+(1-\Upsilon^2)]d\Upsilon\}\right.$$
$$\left.+\{\Psi_*((m,n),\dagger)\int_0^1\Upsilon^{v'+\curlywedge n}(\Upsilon(1-\Upsilon)+(1-\Upsilon)\Upsilon)d\Upsilon\}\right] \tag{39}$$

$$\frac{1}{\Theta}\Lambda_*\left(\frac{m+n}{2},\dagger\right)\times\mathbb{U}_*\left(\frac{m+n}{2},\dagger\right)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\leq\frac{1}{4(n-m)^{\Theta}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,(\Lambda_*\times\mathbb{U}_*),\dagger)$$
$$+\mathbf{J}^{n^-}_{m,v'}(\omega,(\Lambda_*\times\mathbb{U}_*),\dagger)\Big]+\frac{1}{4}\Big[\Upsilon_*((m,n),\dagger)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{2}{\Theta}(\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)})\}$$
$$+2\Psi_*((m,n),\dagger)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{2}{\Theta}(\frac{\Theta}{(\Theta+1)(\Theta+2)}\}\Big] \tag{40}$$

where

$$\Theta=v'+\curlywedge n+1$$

Similarly, for $\Lambda^*\times\mathbb{U}^*$,

$$\frac{1}{\Theta}\Lambda^*\left(\frac{m+n}{2},\dagger\right)\times\mathbb{U}^*\left(\frac{m+n}{2},\dagger\right)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\leq\frac{1}{4(n-m)^{\Theta}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)$$
$$+\mathbf{J}^{n^-}_{m,v'}(\omega,(\Lambda^*\times\mathbb{U}^*),\dagger)\Big]+\frac{1}{4}\Big[\Upsilon^*((m,n),\dagger)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{2}{\Theta}(\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)})\}$$
$$+2\Psi^*((m,n),\dagger)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{2}{\Theta}(\frac{\Theta}{(\Theta+1)(\Theta+2)}\}\Big] \tag{41}$$

By combining (40) and (41), we have

$$\frac{1}{\Theta}\left[\Lambda_*\left(\frac{m+n}{2},\dagger\right)\times\mathbb{U}_*\left(\frac{m+n}{2},\dagger\right),\Lambda^*\left(\frac{m+n}{2},\dagger\right)\times\mathbb{U}^*\left(\frac{m+n}{2},\dagger\right)\right]\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})$$
$$\leq_I\frac{1}{4(n-m)^{\Theta}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,[(\Lambda_*\times\mathbb{U}_*),\dagger),(\Lambda^*\times\mathbb{U}^*),\dagger)+\mathbf{J}^{n^-}_{m,v'}(\omega,[(\Lambda_*\times\mathbb{U}_*),\dagger),(\Lambda^*\times\mathbb{U}^*),\dagger)]\Big]$$
$$+[\Upsilon_*((m,n),\dagger),\Upsilon^*((m,n),\dagger)]\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{1}{2\Theta}(\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)})\}$$
$$+[\Psi_*((m,n),\dagger),\Psi^*((m,n),\dagger)]\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{1}{2\Theta}(\frac{\Theta}{(\Theta+1)(\Theta+2)}\}$$

Thus, we obtain the required result

$$\frac{1}{\Theta}\Lambda(\frac{m+n}{2})\widetilde{\times}\mathbb{U}(\frac{m+n}{2})\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\preccurlyeq\frac{1}{4(n-m)^{\Theta}}\Big[\mathbf{J}^{m^+}_{n,v'}(\omega,(\Lambda\widetilde{\times}\mathbb{U})+\mathbf{J}^{n^-}_{m,v'}$$
$$(\omega,(\Lambda\widetilde{\times}\mathbb{U}))\Big]+\Upsilon(m,n)\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{1}{2\Theta}(\frac{1}{2}-\frac{\Theta}{(\Theta+1)(\Theta+2)})\}+\Psi(m,n)$$
$$\mathbf{J}^{\curlywedge,\xi,w,\sigma,c}_{v',\eta,\rho,\gamma}(\omega;\mathbf{p})\{\frac{1}{2\Theta}(\frac{\Theta}{(\Theta+1)(\Theta+2)}\} \tag{42}$$

□

Corollary 4. *If we replace* $\mathbf{p}=0$, $\omega=0$, *and* $\curlywedge=\curlywedge-1$ *in theorem* (5), *then we have the well-known inequality [41].*

4. Significant Behavior of Hermite–Hadamard Fractional Inequalities with Convex FIV Functions

In this section, we develop generalized versions of Hermite–Hadamard fractional integral inequalities for convex fuzzy interval-valued functions (C-FIVFs).

Theorem 6. *Let $\Lambda : [m,n] \to F.$ be a convex fuzzy interval-valued function on $[m,n]$, whose $\dagger$-levels define the family of interval valued functions $\Lambda_\dagger : [m,n] \subset \Re \to \vartheta_{c^+}$, given by $\Lambda_\dagger(\S) = [\Lambda_*(\S,\dagger), \Lambda^*(\S,\dagger)]$ for all $\dagger \in [m,n]$ and for all $\dagger \in [0,1]$.If $\Lambda \in ([m,n], F.)$; then, for the generalized fractional Integral, we have*

$$\Lambda\left(\frac{m+n}{2}\right)\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega;\mathbf{p}) \preccurlyeq \frac{(v'+\curlywedge n+1)}{2(n-m)^{v'+\curlywedge n+1}}[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda)\widetilde{+}\mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda)]$$
$$\preccurlyeq \frac{\Lambda(m)\widetilde{+}\Lambda(n)}{2}\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega;\mathbf{p})$$

Proof. Let the convex fuzzy interval-valued function be $\Lambda : [m,n] \to \mathbf{F}$ so that we have

$$2\Lambda\left(\frac{m+n}{2}\right) \preccurlyeq \Lambda\big(\Upsilon m + (1-\Upsilon)n\big)\widetilde{+}\Lambda\big((1-\Upsilon)m+\Upsilon n\big) \tag{43}$$

for each $\dagger \in [0,1]$

$$2\Lambda_*\left(\frac{m+n}{2},\dagger\right) \le \Lambda_*\big(\Upsilon m + (1-\Upsilon)n,\dagger\big) + \Lambda_*\big((1-\Upsilon)m+\Upsilon n,\dagger\big) \tag{44}$$

By multiplying both sides by $\Upsilon^{v'}\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega\Upsilon^{\curlywedge};\mathbf{p})$ and integrating the resulting inequality on $(0,1)$ with respect to Υ, we have

$$2\int_0^1 \lambda^{v'}\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega\lambda^{\curlywedge};\mathbf{p})\Lambda_*\left(\frac{m+n}{2},\dagger\right)d\lambda \le \int_0^1 \lambda^{v'}\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega\lambda^{\curlywedge};\mathbf{p})\Lambda_*\big(\Upsilon m +$$
$$(1-\Upsilon)n,\dagger\big)d\lambda + \int_0^1 \lambda^{v'}\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega\lambda^{\curlywedge};\mathbf{p})\Lambda_*\big((1-\Upsilon)m+\Upsilon n,\dagger\big)d\lambda \tag{45}$$

$$2\Lambda_*\left(\frac{m+n}{2},\dagger\right)\sum_{n=0}^{\infty}\frac{\beta_p(\daleth+\hbar n,c-\daleth)(c)_{\hbar n}(\gamma)_{\sigma n}}{\beta(\daleth,c-\daleth)\Gamma(\curlywedge n+v'+1)(\rho)_{mn}}(-\omega)^n\int_0^1 \lambda^{v'+\curlywedge n}d\lambda \le$$
$$\sum_{n=0}^{\infty}\frac{\beta_p(\daleth+\hbar n,c-\daleth)(c)_{\hbar n}(\gamma)_{\sigma n}}{\beta(\daleth,c-\daleth)\Gamma(\curlywedge n+v'+1)(\rho)_{mn}}(-\omega)^n\left[\int_0^1 \lambda^{v'+\curlywedge n}\Lambda_*\big(\Upsilon m+(1-\Upsilon)n,\dagger\big)d\lambda\right. \tag{46}$$
$$\left.+\int_0^1 \lambda^{v'+\curlywedge n}\Lambda_*\big((1-\Upsilon)m+\Upsilon n,\dagger\big)d\lambda\right]$$

Let the interval $[m,n]$ and $\ddagger = \Upsilon m + (1-\Upsilon)n$, $\S = (1-\Upsilon)m + \Upsilon n$, where

$$\ddagger, \S \in [m,n]$$

$$\frac{2\Lambda_*\left(\frac{m+n}{2},\dagger\right)\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega;\mathbf{p})}{(v'+\curlywedge n+1)} \le \frac{[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda_*(n,\dagger)) + \mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda_*(m,\dagger))]}{(n-m)^{v'+\curlywedge n+1}} \tag{47}$$

Similarly, solving for $\Lambda^*(\ddagger,\dagger)$

$$\frac{2\Lambda^*\left(\frac{m+n}{2},\dagger\right)\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\daleth,\rho,\gamma}(\omega;\mathbf{p})}{(v'+\curlywedge n+1)} \le \frac{[\mathbf{J}^{m^+}_{n,v'}(\omega,\Lambda^*(n,\dagger)) + \mathbf{J}^{n^-}_{m,v'}(\omega,\Lambda^*(m,\dagger))]}{(n-m)^{v'+\curlywedge n+1}} \tag{48}$$

from the Equations (47) and (48), we have

$$\frac{2}{(v' + \curlywedge n + 1)} \left[\Lambda^* (\frac{m+n}{2}, \dagger), \Lambda^* (\frac{m+n}{2}, \dagger) \right] \mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\dashv,\rho,\gamma}(\omega; \mathbf{p}) \leq_I \frac{1}{(n-m)^{v'+\curlywedge n+1}}$$
$$\left[\mathbf{J}^{m+}_{n,v'}(\omega, [\Lambda^*(n,\dagger), \Lambda_*(n,\dagger)]) + \mathbf{J}^{n-}_{m,v'}(\omega, [\Lambda^*(m,\dagger), \Lambda_*(n,\dagger)]) \right] \tag{48}$$

Now, we have

$$\frac{2\Lambda\left(\frac{m+n}{2}\right)\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\dashv,\rho,\gamma}(\omega; \mathbf{p})}{(v' + \curlywedge n + 1)} \preccurlyeq \frac{[\mathbf{J}^{m+}_{n,v'}(\omega, \Lambda) \widetilde{+} \mathbf{J}^{n-}_{m,v'}(\omega, \Lambda)]}{(n-m)^{v'+\curlywedge n+1}} \tag{49}$$

Through using a way that is similar to the one we used to obtain the second part of the inequality, we have

$$\frac{[\mathbf{J}^{m+}_{n,v'}(\omega, \Lambda) \widetilde{+} \mathbf{J}^{n-}_{m,v'}(\omega, \Lambda)]}{(n-m)^{v'+\curlywedge n+1}} \preccurlyeq \frac{\Lambda(m) \widetilde{+} \Lambda(n)}{(v' + \curlywedge n + 1)} \mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\dashv,\rho,\gamma}(\omega; \mathbf{p}) \tag{50}$$

By combining (49) and (50), we obtain

$$\Lambda\left(\frac{m+n}{2}\right)\mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\dashv,\rho,\gamma}(\omega; \mathbf{p}) \preccurlyeq \frac{(v' + \curlywedge n + 1)}{2(n-m)^{v'+\curlywedge n+1}} [\mathbf{J}^{m+}_{n,v'}(\omega, \Lambda) \widetilde{+} \mathbf{J}^{n-}_{m,v'}(\omega, \Lambda)]$$
$$\preccurlyeq \frac{\Lambda(m) \widetilde{+} \Lambda(n)}{2} \mathbf{J}^{\curlywedge,\hbar,w,\sigma,c}_{v',\dashv,\rho,\gamma}(\omega; \mathbf{p})$$

This is the required result. □

Corollary 5. *If we replace* $\mathbf{p} = 0$, $\omega = 0$, *and* $\curlywedge = \curlywedge - 1$ *in theorem* (6), *then we have the well known-inequality [41].*

5. Conclusions

In this article, we discussed the multi-variate versions of fuzzy fractional operators and their implementation in the well-known inequalities to derive more refinements. The behavior of renowned inequalities, such as the Hermite–Hadamard-type inequalities and the Hermite–Hadamard Fejér (HHF)-type inequalities, and their refinements for convex fuzzy interval-valued functions, considering the implementation of multi-variate versions of fuzzy fractional integral operators, were also discussed. Numerous tasks in the analysis sector could be accomplished by enhancing the convex functions and generalized fuzzy fractional operators. We hope that many researchers in many academic fields will find this method helpful in completing their projects.

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