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Article

Solution for Time-Fractional Coupled Burgers Equations by Generalized-Laplace Transform Methods

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Abstract: In this work, nonlinear time-fractional coupled Burgers equations are solved utilizing a computational method, which is called the double and triple generalized-Laplace transform and decomposition method. We discuss the proof of triple generalized-Laplace transform for a Caputo fractional derivative. We have given four examples to show the precision and adequacy of the suggested approach. The results show that this method is easy and accurate when compared to the A domain decomposition method (ADM), homotopy perturbation method (HPM), and generalized differential transform method (GDTM). Finally, we have sketched the graphics for all these examples.

Keywords: double and triple generalized-Laplace transform; inverse double and triple generalized-Laplace transform; singular Burgers equation; coupled Burgers equation; generalized-Laplace transform; decomposition methods

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1. Introduction

Generalized classical calculus is a fractional calculus. Several research areas such as science and engineering have used fractional differential equations to solve these problems. For instance, Burgers equation is widely utilized to explain numerous physical problems, namely, unidirectional propagation of weakly nonlinear acoustic waves, shock waves in the viscous medium, flow systems, and electromagnetic waves, etc. The time-fractional Burgers equation is a type of diffusion convection equation. The Burgers equation was presented by the author of [1] in 1915 and subsequently discussed by the researcher in [2]. Many studies have been conducted by mathematicians to find out the numerical or analytical solution of time-space fractional Burgers equations such as, e.g., [3,4]; the authors in [4], implemented the A domian decomposition method and Pade approximation technique, and the homotopy perturbation method was utilized to obtain exact solutions for nonlinear Burgers equation [5]. The space–time-fractional Burgers equation for various initial conditions were solved by the variational iteration method (VIM) in [6,7]. The time-fractional coupled Burgers equations were discussed using the homotopy perturbation Sumudu transform method (HPSTM) in [8]. The authors in [9] studied a series solution for two-dimensional Burgers equations, employing the homotopy perturbation method. The author in [10] investigated the solutions of the two-dimensional nonlinear Burgers differential equations employing the Laplace decomposition method (LDM). The authors in [11] used the G-Laplace transform method to find the exact solution of the Burgers equations. In [12], the approximate solutions of the coupled Burgers equations were obtained by applying the homotopy perturbation method and Laplace transform. The authors in [13,14] obtained the approximate solution of the viscous coupled Burgers equation using cubic and cubic B-spline collocation method.

This study aims to introduce a new technique named the double and triple generalized-Laplace transform decomposition method. This technique is used to obtain the solution of fractional coupled Burgers equation and to the assist mathematicians in solving numerous equations related to physics and engineering in the future.

Remarks: Through this study, we use the following abbreviations:

(1) (GLT) in place of "Generalized-Laplace transform".
(2) (DGLT) in place of "double Generalized-Laplace transform".
(3) (IDGLT) in place of "inverse double Generalized-Laplace transform".
(4) (TGLT) in place of "triple Generalized-Laplace transform".
(5) (ITGLT) in place of "inverse triple Generalized-Laplace transform".
(6) (DM) "decomposition method".
(7) (DGLTDM) in place of "double Generalized-Laplace transform decomposition method".
(8) (TGLTDM) in place of "triple Generalized-Laplace transform decomposition method".

Now, we recall the following definitions which are useful in this work.

Definition 1. *The fractional derivative of $f(\chi,t)$ in the Caputo sense is denoted by*

$$D_t^{\sigma} f(\chi,t) = \begin{cases} \frac{1}{\Gamma(k-\sigma)} \int_0^t (t-\tau)^{k-\sigma-1} \frac{\partial^k f(\chi,\tau)}{\partial \tau^k} d\tau, & k-1<\sigma<k, \\ \frac{\partial^k f(\chi,t)}{\partial t^k}, & \text{for } k=\sigma\in\mathbb{N} \end{cases}$$

For more details, see [15].

Definition 2 ([16])**.** *The partial fractional integrals and caputo derivatives of a function $f(\chi,t)$, where $(\chi,t)\in\mathbb{R}^+\times\mathbb{R}^+$ are granted by*

$$D_t^{\beta} f(\chi,t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\lambda)^{n-\beta-1} \frac{\partial^n}{\partial \lambda^n} f(\chi,\lambda) d\lambda, \tag{1}$$

where $n-1<\beta\leq n, 0<\beta$.

Definition 3 ([17])**.** *Let $f(t)$ be integrable, for all $t\geq 0$. The (GLT) G_α of the function $f(t)$ is given by*

$$F(s) = G_\alpha(f) = s^\alpha \int_0^\infty f(t) e^{-\frac{t}{s}} dt, \tag{2}$$

for $s\in\mathbb{C}$ and $\alpha\in Z$.

The following symbols are using in this work:

$$G_2 = G_\chi G_t,\ G_3 = G_\chi G_\gamma G_t,\ G_2^{-1} = G_p^{-1} G_s^{-1},\ G_3^{-1} = G_p^{-1} G_q^{-1} G_s^{-1}.$$

Definition 4 ([18])**.** *The (DGLT) of the function $f(\chi,t)$ is defined as*

$$G_2[f(\chi,t)] = F(p,s) = p^\alpha s^\alpha \int_0^\infty \int_0^\infty e^{-\frac{\chi}{p}-\frac{t}{s}} f(\chi,t) dt\, d\chi, \tag{3}$$

where $\alpha\in\mathbb{Z}$, $p,s\in\mathbb{C}$ and the symbol G_2 indicate transform of χ and t, respectively, and the function $F(p,s)$ is denoted by the (DGLT) of the $f(\chi,t)$.

Definition 5 ([18])**.** *The (IDGLT) is given by*

$$G_2^{-1}(F(p,s)) = f(\chi,t) = \frac{1}{(2\pi i)^2} \int_{\tau-i\infty}^{\tau+i\infty} \int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{\frac{1}{p}\chi+\frac{1}{s}t} F(p,s) ds\, dp,$$

where G_2^{-1} indicates (IDGLT).

Definition 6 ([19]). *The (TGLT) of the function $f(\chi,\gamma,t)$ is defined as*

$$G_3[f(\chi,\gamma,t)] = F(p,q,s) = p^\alpha q^\alpha s^\alpha \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{\chi}{p}-\frac{\gamma}{q}-\frac{t}{s}} f(\chi,\gamma,t)dt\,d\gamma d\chi, \tag{4}$$

where $\alpha \in \mathbb{Z}$, $p,q,s \in \mathbb{C}$ and the symbol G_3 indicate the transform of χ,γ and t, respectively, and the function $F(p,q,s)$ is denoted as the (TGLT) of the $f(\chi,\gamma,t)$.

Definition 7 ([19]). *The inverse (TGLT)*

$$G_3^{-1}(F(p,q,s)) = \frac{1}{(2\pi i)^3} \int_{\tau-i\infty}^{\tau+i\infty} \int_{\varsigma-i\infty}^{\varsigma+i\infty} \int_{\eta-i\infty}^{\delta+i\infty} e^{\frac{1}{p}\chi+\frac{1}{q}\gamma+\frac{1}{s}t} F(p,q,s)ds\,dqdp,$$

where G_3^{-1} denotes (ITGLT).

The advantage of the (TGLT) is that it is useful to produce some transformations from Definition 6 as follows:

1. If we set $\alpha = 0, s = \frac{1}{s}, q = \frac{1}{q}$ and $p = \frac{1}{p}$, we obtain triple Laplace transform:

$$L_\chi L_\gamma L_v(f(\chi,\gamma,t)) = F(p,q,s) = \int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\gamma,t)e^{-(p\chi+q\gamma+st)}dtd\gamma d\chi, \tag{5}$$

2. If we set $\alpha = 0, \quad \sigma = \frac{1}{p}$, $\rho = \frac{1}{q}$ and substituting s by ϖ, we obtain double Laplace–Yang Transform:

$$L_\chi L_\gamma \gamma_t(f(\chi,\gamma,t)) = F(\sigma,\rho,\varpi) = \int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\gamma,t)e^{-\left(\sigma\chi+\rho\gamma+\frac{t}{\varpi}\right)}dtd\gamma d\chi, \tag{6}$$

3. At $\alpha = -1$ and substituting p,q,s by u,v,μ, respectively, we obtain a triple Sumudu Transform:

$$S_\chi S_\gamma S_t(f(\chi,\gamma,t)) = F(u,v,\mu) = \frac{1}{uv\mu}\int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\gamma,t)e^{-\left(\frac{\chi}{u}+\frac{\gamma}{v}+\frac{t}{\mu}\right)}dtd\gamma d\chi. \tag{7}$$

Theorem 1. *If (DGLT) of the function $f(\chi,t)$ is given by $G_2[f(\chi,t)] = F(p,s)$, then $(DGLT)$ of $\frac{\partial f(\chi,t)}{\partial t}$ and $\frac{\partial^2 f(\chi,t)}{\partial t^2}$ are given by*

$$G_2\left[\frac{\partial f(\chi,t)}{\partial t}\right] = \frac{F(p,s)}{s} - s^\alpha F(p,0), \tag{8}$$

and

$$G_2\left[\frac{\partial^2 f(\chi,t)}{\partial^2 t}\right] = \frac{F(p,s)}{s^2} - s^{\alpha-1}F(p,0) - s^\alpha \frac{\partial F(p,0)}{\partial t}. \tag{9}$$

Proof. By using definition of (DGLT) for $\frac{\partial f(\chi,t)}{\partial t}$, we have

$$G_\chi G_t\left[\frac{\partial f(\chi,t)}{\partial t}\right] = p^\alpha \int_0^\infty e^{-\frac{\chi}{p}} \left[s^\alpha \int_0^\infty e^{-\frac{t}{s}} \frac{\partial f(\chi,t)}{\partial t}dt\right] d\chi, \tag{10}$$

and the integral inside the bracket is calculated as follows:

$$\begin{aligned} s^{\alpha}\int_{0}^{\infty} e^{-\frac{t}{s}}\frac{\partial f(\chi,t)}{\partial t}dt &= s^{\alpha}\left[e^{-\frac{t}{s}}f(\chi,t)\right]_{0}^{\infty}+\frac{s^{\alpha}}{s}\int_{0}^{\infty} e^{-\frac{t}{s}}f(\chi,t)dt \\ &= \frac{1}{s}F(p,t)-s^{\alpha}f(\chi,0) \end{aligned} \tag{11}$$

By substituting Equation (11) into Equation (10), we have

$$G_2\left[\frac{\partial f(\chi,t)}{\partial t}\right]=p^{\alpha}\int_{0}^{\infty} e^{-\frac{\chi}{p}}\left[\frac{1}{s}F(p,t)-s^{\alpha}f(\chi,0)\right]dt,$$

and thus,

$$G_2\left[\frac{\partial f(\chi,t)}{\partial t}\right]=\frac{1}{s}F(p,s)-s^{\alpha}F(p,0). \tag{12}$$

In a similar way, one can easily see that

$$G_2\left[\frac{\partial^2 f(\chi,t)}{\partial^2 t}\right]=\frac{F(p,s)}{s^2}-s^{\alpha-1}F(p,0)-s^{\alpha}\frac{\partial F(p,0)}{\partial t}.$$

□

Theorem 2. *The (DGLT) of the Caputo fractional derivative* $D_t^{\beta}f(\chi,t)$ *is denoted by*

$$G_{\chi}G_t\left[D_t^{\beta}f(\chi,t)\right]=\frac{F(p,s)}{s^{\beta}}-s^{\alpha}\sum_{k=1}^{n}\frac{1}{s^{\beta-k}}G_{\chi}\left[\frac{\partial^{k-1}f(\chi,0)}{\partial t^{k-1}}\right] \tag{13}$$

where $n-1<\beta\leq n, 0<\beta$.

Proof. Utilizing Equation (3), we obtain

$$\begin{aligned} G_{\chi}G_t\left[D_t^{\beta}f(\chi,t)\right] &= p^{\alpha}s^{\alpha}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{t}{s}}D_t^{\beta}f(\chi,t)dt\,d\chi \\ &= p^{\alpha}s^{\alpha}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{t}{s}}\frac{1}{\Gamma(n-\beta)}\int_0^t(t-\lambda)^{n-\beta-1}\frac{\partial^n f(\chi,\lambda)}{\partial\lambda^n}d\lambda dt\,d\chi \\ &= \frac{p^{\alpha}s^{\alpha}}{\Gamma(n-\beta)}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{t}{s}}\int_{\lambda}^{\infty}(t-\lambda)^{n-\beta-1}\frac{\partial^n f(\chi,\lambda)}{\partial\lambda^n}dt d\chi d\lambda. \end{aligned} \tag{14}$$

Assuming $r=t-\lambda$ in Equation (22), we gain

$$G_{\chi}G_t\left[D_t^{\beta}f(\chi,t)\right]=\frac{p^{\alpha}s^{\alpha}}{\Gamma(n-\beta)}\int_0^{\infty}e^{-\frac{\chi}{p}}\int_0^{\infty}\frac{\partial^n f(\chi,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}\left[\int_0^{\infty}r^{n-\beta-1}e^{-\frac{r}{s}}dr\right]d\lambda d\chi. \tag{15}$$

The gamma function is defined by the integral inside the bracket as

$$\int_0^{\infty}r^{n-\beta-1}e^{-\frac{r}{s}}dr=\frac{\Gamma(n-\beta)}{\left(\frac{1}{s}\right)^{n-\beta}}$$

Hence, Equation (23) becomes

$$G_{\chi}G_t\left[D_t^{\beta}f(\chi,t)\right]=\frac{p^{\alpha}s^{\alpha}}{\Gamma(n-\beta)}\int_0^{\infty}e^{-\frac{\chi}{p}}\int_0^{\infty}\frac{\partial^n f(\chi,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}\left[\frac{\Gamma(n-\beta)}{\left(\frac{1}{s}\right)^{n-\beta}}\right]d\lambda d\chi. \tag{16}$$

By rewriting Equation (24), we obtain

$$G_\chi G_t\left[D_t^\beta f(\chi,t)\right] = s^{n-\beta}\left[p^{\alpha+1}s^{\alpha+1}\int_0^\infty e^{-\frac{\chi}{p}}\int_0^\infty \frac{\partial^n f(\chi,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}d\lambda d\chi\right]. \tag{17}$$

We define the integral inside the bracket in the following form:

$$\begin{aligned} G_\chi G_t\left[\frac{\partial^n f(\chi,t)}{\partial t^n}\right] &= p^{\alpha+1}s^{\alpha+1}\int_0^\infty e^{-\frac{\chi}{p}}\int_0^\infty \frac{\partial^n f(\chi,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}d\lambda d\chi \\ &= \frac{f(p,s)}{s^n} - s^\alpha\sum_{k=1}^{n}\frac{1}{s^{n-k}}G_\chi\left[\frac{\partial^{k-1}f(\chi,0)}{\partial t^{k-1}}\right]. \end{aligned} \tag{18}$$

By substituting Equation (26) into Equation (25), we obtain

$$G_\chi G_t\left[D_t^\beta f(\chi,t)\right] = \frac{F(p,s)}{s^\beta} - s^\alpha\sum_{k=1}^{n}\frac{1}{s^{\beta-k}}G_\chi\left[\frac{\partial^{k-1}f(\chi,0)}{\partial t^{k-1}}\right].$$

The proof is complete. □

One can easily obtain the (GDLT) of the functions $\frac{\partial^\beta f(\chi,t)}{\partial t^\beta}$ and $\frac{\partial^{2\beta} f(\chi,t)}{\partial t^{2\beta}}$ from Theorem 2 as follows:

$$G_2\left[\frac{\partial^\beta f(\chi,t)}{\partial t^\beta}\right] = \frac{F(p,s)}{s^\beta} - s^{\alpha-\beta+1}F(p,0), \quad 0<\beta\le 1 \tag{19}$$

$$\begin{aligned} G_2\left[\frac{\partial^{2\beta} f(\chi,t)}{\partial t^{2\beta}}\right] &= \frac{F(p,s)}{s^{2\beta}} - s^{\alpha-2\beta+1}F(p,0) - s^{\alpha-2\beta+2}F_t(p,0), \\ 0 &< \beta\le 1. \end{aligned} \tag{20}$$

Definition 8. *The (TGLT) of the partial derivative* $\frac{\partial^n\omega(\chi,\gamma,t)}{\partial t^n}$ *is denoted by*

$$G_3\left[\frac{\partial^n\omega(\chi,\gamma,t)}{\partial t^n}\right] = \frac{F(p,q,s)}{s^n} - s^\alpha\sum_{k=1}^{n}\frac{1}{s^{n-k}}G_2\left[\frac{\partial^{k-1}\omega(\chi,\gamma,0)}{\partial t^{k-1}}\right].$$

Theorem 3. *The (TGLT) of th Caputo fractional derivative* $D_t^\beta\omega(\chi,\gamma,t)$ *is given by*

$$G_3\left[D_t^\beta\omega(\chi,\gamma,t)\right] = \frac{F(p,q,s)}{s^\beta} - s^\alpha\sum_{k=1}^{n}\frac{1}{s^{\beta-k}}G_2\left[\frac{\partial^{k-1}\omega(\chi,\gamma,0)}{\partial t^{k-1}}\right], \tag{21}$$

where $n-1<\beta\le n, 0<\beta$.

Proof. Using the definition of (TGLT) for $D_t^\beta\omega(\chi,\gamma,t)$, we have

$$\begin{aligned} & G_3\left[D_t^\beta\omega(\chi,\gamma,t)\right] \\ =\ & p^\alpha q^\alpha s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{\chi}{p}-\frac{\gamma}{q}-\frac{t}{s}}D_t^\beta\omega(\chi,\gamma,t)dt\,d\gamma d\chi \\ =\ & p^\alpha q^\alpha s^\alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{\chi}{p}-\frac{\gamma}{q}-\frac{t}{s}}\frac{1}{\Gamma(n-\beta)}\int_0^t(t-\lambda)^{n-\beta-1}\frac{\partial^n\omega(\chi,\gamma,\lambda)}{\partial\lambda^n}d\lambda dt\,d\gamma d\chi \\ =\ & \frac{p^\alpha q^\alpha s^\alpha}{\Gamma(n-\beta)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{\chi}{p}-\frac{\gamma}{q}-\frac{t}{s}}\int_\lambda^\infty(t-\lambda)^{n-\beta-1}\frac{\partial^n\omega(\chi,\gamma,\lambda)}{\partial\lambda^n}dtd\gamma d\chi d\lambda. \end{aligned} \tag{22}$$

By putting $r = t - \lambda$ in Equation (22), we obtain

$$
\begin{aligned}
&G_3\left[D_t^{\beta}\omega(\chi,\gamma,t)\right] \\
=\;& \frac{p^{\alpha}q^{\alpha}s^{\alpha}}{\Gamma(n-\beta)}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{\gamma}{q}}\int_0^{\infty}\frac{\partial^n\omega(\chi,\gamma,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}\left[\int_0^{\infty}r^{n-\beta-1}e^{-\frac{r}{s}}dr\right]d\lambda d\gamma d\chi. \quad (23)
\end{aligned}
$$

The gamma function is defined by the integral inside the bracket as follows:

$$
\int_0^{\infty}r^{n-\beta-1}e^{-\frac{r}{s}}dr = \frac{\Gamma(n-\beta)}{\left(\frac{1}{s}\right)^{n-\beta}}.
$$

Hence, Equation (23) becomes

$$
\begin{aligned}
&G_3\left[D_t^{\beta}\omega(\chi,\gamma,t)\right] \\
=\;& \frac{p^{\alpha}q^{\alpha}s^{\alpha}}{\Gamma(n-\beta)}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{\gamma}{q}}\int_0^{\infty}\frac{\partial^n\omega(\chi,\gamma,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}\left[\frac{\Gamma(n-\beta)}{\left(\frac{1}{s}\right)^{n-\beta}}\right]d\lambda d\gamma d\chi. \quad (24)
\end{aligned}
$$

By rewriting Equation (24), one can obtain

$$
\begin{aligned}
&G_3\left[D_t^{\beta}\omega(\chi,\gamma,t)\right] \\
=\;& s^{n-\beta}\left[p^{\alpha}q^{\alpha}s^{\alpha}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{\gamma}{q}}\int_0^{\infty}\frac{\partial^n\omega(\chi,\gamma,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}d\lambda d\gamma d\chi\right]. \quad (25)
\end{aligned}
$$

The integral inside the bracket in the above equation is defined by

$$
\begin{aligned}
G_3\left[\frac{\partial^n\omega(\chi,\gamma,t)}{\partial t^n}\right] &= p^{\alpha}q^{\alpha}\int_0^{\infty}\int_0^{\infty}e^{-\frac{\chi}{p}-\frac{\gamma}{q}}\left[s^{\alpha}\int_0^{\infty}\frac{\partial^n\omega(\chi,\gamma,\lambda)}{\partial\lambda^n}e^{-\frac{\lambda}{s}}d\lambda\right]d\gamma d\chi \\
&= \frac{F(p,q,s)}{s^n} - s^{\alpha}\sum_{k=1}^{n}\frac{1}{s^{n-k}}G_{\chi}G_{\gamma}\left[\frac{\partial^{k-1}\omega(\chi,\gamma,0)}{\partial t^{k-1}}\right]. \quad (26)
\end{aligned}
$$

By replacing Equation (26) into Equation (25), we have

$$
G_3\left[D_t^{\beta}\omega\right] = \frac{F(p,q,s)}{s^{\beta}} - s^{\alpha}\sum_{k=1}^{n}\frac{1}{s^{\beta-k}}G_2\left[\frac{\partial^{k-1}\omega(\chi,\gamma,0)}{\partial t^{k-1}}\right].
$$

The proof is complete. □

2. Double Generalized-Laplace Transform Decomposition Method and Time-Fractional Burgers Equation (DGLTDM)

Here, we explain the solutions to two problems by utilizing the (DGLTDM).

In this work, we deal with the time-fractional Burgers equation containing the initial condition granted by

$$
\begin{aligned}
D_t^{\beta}\omega - \omega_{\chi\chi} + \eta\omega\omega_{\chi} + \zeta\left(\omega^2\right)_{\chi} &= h(\chi,t), \\
\omega(\chi,0) &= h_1(\chi). \quad (27)
\end{aligned}
$$

The first problem: To show the basic idea of (DGLTDM), we consider Equation (27) can be written in the following form:

$$
\begin{aligned}
D_t^{\beta}\omega - L\omega(\chi,t) + N\omega(\chi,t) &= f(\chi,t), \\
\omega(\chi,0) &= f_1(\chi). \quad (28)
\end{aligned}
$$

Here, L and N are linear and nonlinear differential operators, respectively. The following steps are needed to solve Equation (28).

Step 1: Using the (DGLT) on both sides of Equation (28), it becomes :

$$G_2\left[D_t^{\beta}\omega\right] - G_2[L\omega(\chi,t) - N\omega(\chi,t)] = G_2[f(\chi,t)]. \tag{29}$$

Employing Theorem 1 and Equation (19), in the above equation, we obtain

$$\frac{\omega(p,s)}{s^{\beta}} = s^{\alpha-\beta+1}\omega(p,0) + G_2[L\omega(\chi,t) - N\omega(\chi,t)] + G_2[f(\chi,t)]. \tag{30}$$

Step 2: Multiplying Equation (30) by s^{β}, and using the inverse of (IDGLT), we acquire

$$\begin{aligned}\omega(\chi,t) &= f_1(\chi) + G_2^{-1}\left[s^{\beta}[G_2[L\omega(\chi,t) - N\omega(\chi,t)]]\right] \\ &\quad + G_2^{-1}\left[s^{\beta}F(p,s)\right].\end{aligned} \tag{31}$$

Step 3: The solution of Equation (28) is given by infinite series as follows:

$$\omega(\chi,t) = \sum_{n=0}^{\infty}\omega_n(\chi,t). \tag{32}$$

By replacing Equation (32) into Equation (31), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\omega_n(\chi,t) = f_1(\chi) &\quad + G_2^{-1}\left[s^{\beta}F(p,s)\right] \\ &\quad + G_2^{-1}\left[s^{\beta}\left[G_2\left[L\left(\sum_{n=0}^{\infty}\omega_n(\chi,t)\right) - \sum_{n=0}^{\infty}N\omega_n(\chi,t)\right]\right]\right].\end{aligned} \tag{33}$$

Step 4: Now, by using Equation (33), we derive the following:

$$\omega_0 = f_1(\chi) + G_2^{-1}\left[s^{\beta}F(p,s)\right], \tag{34}$$

and

$$\begin{aligned}\omega_1 &= G_2^{-1}\left[s^{\beta}[G_2[L\omega_0 - N\omega_0(\chi,t)]]\right] \\ \omega_2 &= G_2^{-1}\left[s^{\beta}[G_2[L\omega_1 - N\omega_1(\chi,t)]]\right] \\ \omega_3 &= G_2^{-1}\left[s^{\beta}[G_2[L\omega_2 - N\omega_2(\chi,t)]]\right] \\ &\;\;\vdots \\ \omega_n &= G_2^{-1}\left[s^{\beta}[G_2[L\omega_{n-1} - N\omega_{n-1}(\chi,t)]]\right].\end{aligned} \tag{35}$$

The solution of Equation (28) is obtained by substituting Equation (35) into Equation (32) as follows:

$$\omega(\chi,t) = \lim_{n\to 1}\omega_n(\chi,t) = \omega_0(\chi,t) + \omega_1(\chi,t) + \omega_2(\chi,t) + \ldots + \omega_n(\chi,t).$$

We assume the inverse of Equations (34) and (35) exists.

Convergence:

Theorem 4. *Let B be a Banach space. The series solution of Equation (35) is convergent if there exists k,* $0 \leq k < 1$*, such that* $\|\omega_n(\chi,t)\| \leq k\|\omega_{n-1}(\chi,t)\|$ *for all* $n \in \mathbb{N}$*.*

Proof. By defining sequence S_n of partial sums of the series of Equation (35) as follows:

$$\begin{aligned}
S_0 &= \omega_0(\chi,t) \\
S_1 &= \omega_0(\chi,t)+\omega_1(\chi,t) \\
S_2 &= \omega_0(\chi,t)+\omega_1(\chi,t)+\omega_2(\chi,t) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
S_n &= \omega_0(\chi,t)+\omega_1(\chi,t)+\omega_2(\chi,t)+\ldots+\omega_n(\chi,t),
\end{aligned}$$

we show that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Banach space B. Therefore, we consider

$$\begin{aligned}
\|S_{n+1}-S_n\| &= \|\omega_{n+1}(\chi,t)\| \leq k\|\omega_n(\chi,t)\| \leq k^2\|\omega_{n-1}(\chi,t)\| \leq \ldots \\
&\leq k^{n+1}\|\omega_0(\chi,t)\|.
\end{aligned}$$

By using the above triangle inequality for $n \geq m$, we have

$$\begin{aligned}
\|S_n-S_m\| &= \|(S_n-S_{n-1})+(S_{n-1}-S_{n-2})+\ldots+(S_{m+1}-S_m)\|, \\
&\leq \|S_n-S_{n-1}\|+\|S_{n-1}-S_{n-2}\|+\ldots+\|S_{m+1}-S_m\|, \\
&\leq k^n\|\omega_0(\chi,t)\|+k^{n-1}\|\omega_0(\chi,t)\|+\ldots+k^{m+1}\|\omega_0(\chi,t)\|, \\
&= k^{m+1}\left(\frac{1-k^{n-m}}{1-k}\right)\|\omega_0(\chi,t)\|.
\end{aligned}$$

From $0 \leq k < 1$, we see that $1-k^{n-m} \leq 1$; therefore,

$$\|S_n-S_m\| \leq \frac{k^{m+1}}{1-k}\|\omega_0(\chi,t)\|.$$

Since $\|\omega_0(\chi,t)\|$ is bounded, $\|S_n-S_m\| \to 0$ at $n,m \to \infty$. Therefore, the sequence $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space B, and the series solution of Equation (35) converges. □

To illustrate this technique for time-fractional Burgers equations, we offer the following example:

Example 1. *Consider the following time-fractional Burgers equation contains the initial condition*

$$\begin{aligned}
D_t^{\beta}\omega &= \omega_{\chi\chi}+2\omega\omega_{\chi}-\left(\omega^2\right)_{\chi}, \\
\omega(\chi,0) &= \sin(\chi),\ 0<\ \beta \leq 1.
\end{aligned} \tag{36}$$

By using the previous steps, we obtain

$$\omega_0 = \sin(\chi), \tag{37}$$

and

$$\omega_{n+1} = G_2^{-1}\left[s^{\beta}G_2[\omega_{n\chi\chi}+2A_n-B_n]\right]. \tag{38}$$

The Adomain polynomials of the nonlinear term F(u) can be estimated by the following expression:

$$Q_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0},\quad n=0,1,2,\cdots,$$

where Adomian polynomials Q_n of the nonliear term $F(u)$ are defined by

$$\begin{aligned}
Q_0 &= F(u_0), \\
Q_1 &= u_1 F'(u_0), \\
Q_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1 F''(u_0), \\
Q_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\
Q_4 &= u_4 F'(u_0) + \left(\frac{1}{2!} u_2^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) \\
&\quad + \frac{1}{4!} u_1^4 F^{(4)}(u_0).
\end{aligned}$$

Now, it is easy to generate A_n and B_n from the above equation, as follows:

$$\begin{aligned}
A_0 &= \omega_0 \omega_{0\chi}, \\
A_1 &= \omega_0 \omega_{1\chi} + \omega_1 \omega_{0\chi}, \\
A_2 &= \omega_0 \omega_{2\chi} + \omega_1 \omega_{1\chi} + \omega_2 \omega_{0\chi},
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= (\omega_0)^2_{\chi}, \\
B_1 &= 2\omega_0 \omega_{1\chi} + 2\omega_1 \omega_{0\chi}, \\
B_2 &= 2\omega_0 \omega_{2\chi} + 2\omega_1 \omega_{1\chi} + 2\omega_2 \omega_{0\chi}.
\end{aligned}$$

According to the GDLTDM, we obtain the following:

$$\begin{aligned}
\omega_1 &= G_2^{-1}\left[s^{\beta} G_2 \left[\omega_{0\chi\chi} + 2A_0 - B_0 \right] \right] \\
&= G_2^{-1}\left[s^{\beta} G_2 [-\sin(\chi)] \right] \\
&= -G_2^{-1}\left[\frac{p^{\alpha+2}}{1+p^2} s^{\alpha+\beta+1} \right], \\
\omega_1 &= -\frac{t^{\beta}}{\Gamma(\beta+1)} \sin(\chi).
\end{aligned}$$

At $n = 1$, we have

$$\begin{aligned}
\omega_2 &= G_2^{-1}\left[s^{\beta} G_2 \left[\frac{t^{\beta}}{\Gamma(\beta+1)} \sin(\chi) \right] \right] \\
&= G_2^{-1}\left[\frac{p^{\alpha+2}}{1+p^2} s^{\alpha+2\beta+1} \right]. \\
&= \frac{t^{2\beta}}{\Gamma(2\beta+1)} \sin(\chi),
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
\omega_3 &= G_2^{-1}\left[s^{\beta} G_2 \left[\frac{t^{2\beta}}{\Gamma(2\beta+1)} \sin(\chi) \right] \right], \\
&= -G_2^{-1}\left[\frac{p^{\alpha+2}}{1+p^2} s^{\alpha+3\beta+1} \right], \\
&= -\frac{t^{3\beta}}{\Gamma(3\beta+1)} \sin(\chi).
\end{aligned}$$

The series solution of Equation (36) can be found in the following form:

$$\omega(\chi,t) = \sin(\chi) - \frac{t^{\beta}}{\Gamma(\beta+1)}\sin(\chi) + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin(\chi) - \frac{t^{3\beta}}{\Gamma(3\beta+1)}\sin(\chi) + \cdots.$$

By putting $\beta = 1$, the exact solution of Equation (36) is given by the following:

$$\omega(\chi,t) = \sin(\chi)e^{-t}.$$

Figure 1: We demonstrate the comparison between the exact solution and the obtained numerical solution for Equation (36). At $t = 1$ and $\beta = 1$, we obtained the accurate solution. By taking various values of β, for instance, ($\beta = 0.75$, $\beta = 0.85$ and $\beta = 0.95$), we obtained the estimate solutions.

Figure 2: We demonstrate the result of the functions $\omega(\chi,t)$ in three-dimensional space.

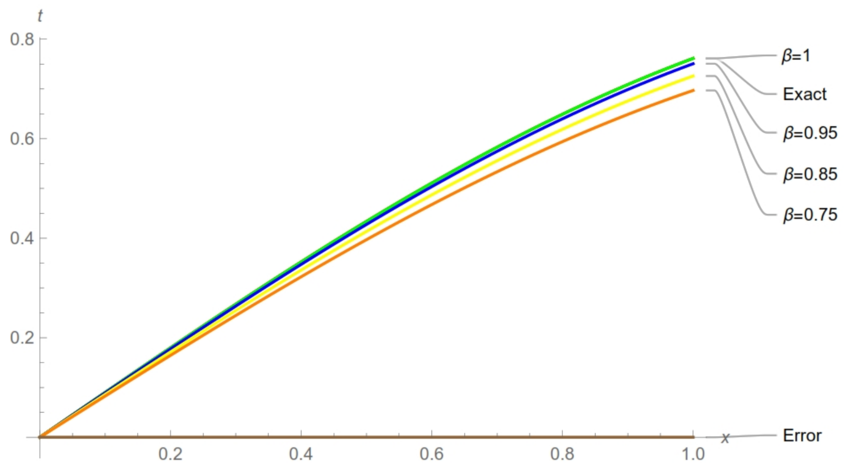

Figure 1. Comparison between exact and numerical solutions.

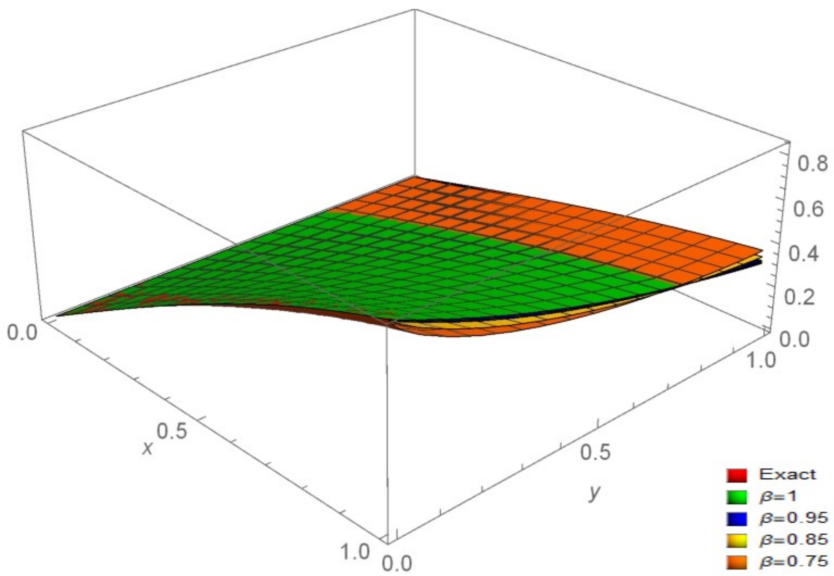

Figure 2. The surface of the function $\omega(x,t)$.

Table 1 Shows the numerical solution for different values of β for the function $\omega(x,t)$.

Table 1. Comparison between exact and approximation solutions.

t	x	$\beta = 0.75$	$\beta = 0.85$	$\beta = 0.95$	$\beta = 1$	Exact
	0.00	1.2832	1.25528	1.23157	1.2214	1.2214
	0.25	1.64767	1.61181	1.58137	1.56831	1.56831
0.1	0.50	2.11564	2.06961	2.03052	2.01375	2.01375
	0.75	2.71654	2.65743	2.60724	2.58571	2.58571
	1.00	3.48811	3.41221	3.34776	3.32011	3.320112

The second problem:

To illustrate the essential idea of this method, we consider the following system of time-fractional coupled Burgers equation with the initial conditions of the form:

$$\begin{aligned} D_t^{\beta} u - u_{\chi\chi} + \eta u u_{\chi} + \zeta (u\omega)_{\chi} &= f(\chi, t), \\ D_t^{\beta} \omega - \omega_{\chi\chi} + \eta \omega \omega_{\chi} + \mu (u\omega)_{\chi} &= h(\chi, t), \end{aligned} \tag{39}$$

and

$$u(\chi, 0) = f_1(\chi), \quad \omega(\chi, 0) = h_1(\chi) \tag{40}$$

for $t > 0$. Here, $f(\chi, t)$, $h(\chi, t)$, $f_1(\chi)$ and $h_1(\chi)$ are given functions, η is a real constant, ζ and μ are arbitrary constants depending on the system parameters such as Peclet number, Stokes velocity of particles due to gravity and Brownian diffusivity [20]. The next steps are needed to solve Equation (39).

Step 1: Taking the (DGLT) on both sides of Equation (39) and single generalized-Laplace transform for Equation (40) and multiplying the outcome by s^{β}, we obtain

$$U(p, s) = s^{\alpha+1} F_1(p) + s^{\beta} F(p, s) + s^{\beta} G_2 \left[u_{\chi\chi} - \eta u u_{\chi} - \zeta (u\omega)_{\chi} \right], \tag{41}$$

and

$$W(p, s) = s^{\alpha+1} H_1(p) + s^{\beta} H(p, s) + s^{\beta} G_2 \left[\omega_{\chi\chi} - \eta \omega \omega_{\chi} - \mu (u\omega)_{\chi} \right]. \tag{42}$$

where $U(p, s)$ and $W(p, s)$ are double generalized-Laplace transform of $u(\chi, t)$ and $\omega(\chi, t)$, respectively.

Step 2: The (DGLTDM) defined the solution of Equation (39) according to the following forms:

$$u(\chi, t) = \sum_{n=0}^{\infty} u_n(\chi, t), \quad \omega(\chi, t) = \sum_{n=0}^{\infty} \omega_n(\chi, t). \tag{43}$$

We can obtain Adomian's polynomials A_n, C_n and D_n, respectively, as follows:

$$C_n = \sum_{n=0}^{\infty} u_n u_{n\chi}, \tag{44}$$

and

$$D_n = \sum_{n=0}^{\infty} u_n \omega_n, \tag{45}$$

where A_n is mentioned in Example 1. The Adomian polynomials for the nonlinear term $\omega\omega_{\chi}$, uu_{χ}, and $u\omega$ are given by

$$\begin{aligned} A_0 &= \omega_0 \omega_{0\chi}, \\ A_1 &= \omega_0 \omega_{1\chi} + \omega_1 \omega_{0\chi}, \\ A_2 &= \omega_0 \omega_{2\chi} + \omega_1 \omega_{1\chi} + \omega_2 \omega_{0\chi}, \\ A_3 &= \omega_0 \omega_{3\chi} + \omega_1 \omega_{2\chi} + \omega_2 \omega_{1\chi} + \omega_3 \omega_{0\chi}, \\ A_4 &= \omega_0 \omega_{4\chi} + \omega_1 \omega_{3\chi} + \omega_2 \omega_{2\chi} + \omega_3 \omega_{1\chi} + \omega_4 \omega_{0\chi}. \end{aligned} \tag{46}$$

$$\begin{aligned} C_0 &= u_0 u_{0\chi}, \\ C_1 &= u_0 u_{1\chi} + u_1 u_{0\chi}, \\ C_2 &= u_0 u_{2\chi} + u_1 u_{1\chi} + u_2 u_{0\chi}, \\ C_3 &= u_0 u_{3\chi} + u_1 u_{2\chi} + u_2 u_{1\chi} + u_3 u_{0\chi}, \\ C_4 &= u_0 u_{4\chi} + u_1 u_{3\chi} + u_2 u_{2\chi} + u_3 u_{1\chi} + u_4 u_{0\chi}. \end{aligned} \tag{47}$$

and

$$\begin{aligned} D_0 &= u_0 \omega_0, \\ D_1 &= u_0 \omega_1 + u_1 \omega_0, \\ D_2 &= u_0 \omega_2 + u_1 \omega_1 + u_2 \omega_0. \\ D_3 &= u_0 \omega_3 + u_1 \omega_2 + u_2 \omega_1 + u_3 \omega_0, \\ D_3 &= u_0 \omega_4 + u_1 \omega_3 + u_2 \omega_2 + u_3 \omega_1 + u_4 \omega_0. \end{aligned} \tag{48}$$

Operating with the (IDGLT) on both sides of Equations (41) and (42) and using Equations (43)–(45), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(\chi, t) &= f_1(\chi) + G_2^{-1}\left[s^{\beta} F(p, s)\right] \\ &\quad + G_2^{-1}\left[\left[s^{\beta} G_2\left(\sum_{n=0}^{\infty} u_{n\chi\chi}\right)\right]\right] \\ &\quad - G_2^{-1}\left[s^{\beta} G_2\left[\eta \sum_{n=0}^{\infty} C_n\right]\right] \\ &\quad - G_2^{-1}\left[s^{\beta} G_2\left[\zeta\left(\sum_{n=0}^{\infty} D_n\right)_{\chi}\right]\right], \end{aligned} \tag{49}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \omega_n(\chi, t) &= h_1(\chi) + G_2^{-1}\left[s^{\beta} H(p, s)\right] \\ &\quad + G_2^{-1}\left[s^{\beta} G_2\left[\sum_{n=0}^{\infty} \omega_{n\chi\chi}\right]\right] \\ &\quad - G_2^{-1}\left[s^{\beta} G_2\left[\eta \sum_{n=0}^{\infty} A_n\right]\right] \\ &\quad - G_2^{-1}\left[s^{\beta} G_2\left[\mu\left(\sum_{n=0}^{\infty} D_n\right)_{\chi}\right]\right]. \end{aligned} \tag{50}$$

On matching both sides of Equations (49) and (50), we can obtain

$$\begin{aligned} u_0 &= f_1(\chi) + G_2^{-1}\left[s^{\beta} F(p, s)\right], \\ \omega_0 &= h_1(\chi) + G_2^{-1}\left[s^{\beta} H(p, s)\right]. \end{aligned} \tag{51}$$

In general, the recursive relations are given by

$$\begin{aligned} u_{n+1} &= G_2^{-1}\left[s^{\beta} G_2[u_{n\chi\chi}]\right] - G_2^{-1}\left[s^{\beta} G_2[\eta C_n]\right] \\ &\quad - G_2^{-1}\left[s^{\beta} G_2\left[\zeta (D_n)_{\chi}\right]\right], \end{aligned} \tag{52}$$

and

$$\begin{aligned}\omega_{n+1} &= G_2^{-1}\left[s^{\beta}G_2[\omega_{n\chi\chi}]\right] - G_2^{-1}\left[s^{\beta}G_2[\eta A_n]\right] \\ &\quad -G_2^{-1}\left[s^{\beta}G_2\left[\mu(D_n)_{\chi}\right]\right].\end{aligned} \tag{53}$$

Here , we assume that the (IDGLT) concerning p and s exists for each term on the right-hand side of the above equations. To demonstrate this approach for time-fractional coupled Burgers equations, we offer the following examples:

Example 2. *Consider the following homogeneous form of a coupled Burgers equation [12]*

$$\begin{aligned}D_t^{\beta}u - u_{\chi\chi} - 2uu_{\chi} + (u\omega)_{\chi} &= 0, \\ D_t^{\beta}\omega - \omega_{\chi\chi} - 2\omega\omega_{\chi} + (u\omega)_{\chi} &= 0,\end{aligned} \tag{54}$$

with initial condition

$$u(\chi,0) = \sin\chi, \quad \omega(\chi,0) = \sin\chi. \tag{55}$$

By using Equations (51)–(53), we have

$$\begin{aligned}u_0 &= \sin\chi, \;\; \omega_0 = \sin\chi, \\ u_1 &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{\partial^2 u_0}{\partial\chi^2} + 2u_0u_{0\chi} - (u_0\omega_0)_{\chi}\right]\right], \\ u_1 &= G_2^{-1}\left[s^{\beta}G_2[-\sin\chi]\right] = G_2^{-1}\left[-\frac{s^{\alpha+\beta+1}p^{\alpha+2}}{1+p^2}\right] = -\frac{t^{\beta}}{\Gamma(\beta+1)}\sin\chi, \\ \omega_1 &= G_2^{-1}\left[s^{\beta}G_2[-\sin\chi]\right] = G_2^{-1}\left[-\frac{s^{\alpha+\beta+1}p^{\alpha+2}}{1+p^2}\right] = -\frac{t^{\beta}}{\Gamma(\beta+1)}\sin\chi, \\ u_2 &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{\partial^2 u_1}{\partial\chi^2} + 2(u_0u_{1\chi} + u_1u_{0\chi}) - (u_0\omega_1 + u_1\omega_0)_{\chi}\right]\right], \\ &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{t^{\beta}}{\Gamma(\beta+1)}\sin\chi - \frac{2t^{\beta}}{\Gamma(\beta+1)}\sin 2\chi + \frac{2t^{\beta}}{\Gamma(\beta+1)}\sin 2\chi\right]\right], \\ &= G_2^{-1}\left[\frac{s^{\alpha+2\beta+1}p^{\alpha+2}}{1+p^2}\right], \\ u_2 &= \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin\chi.\end{aligned}$$

In a similar manner,

$$\omega_2 = \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin\chi.$$

We continue in the same way to obtain the following solutions:

$$\begin{aligned}u(\chi,t) &= \sin\chi - \frac{t^{\beta}}{\Gamma(\beta+1)}\sin\chi + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin\chi - \frac{2t^{3\beta}}{\Gamma(3\beta+1)}\sin\chi + \cdots, \\ \omega(\chi,t) &= \sin\chi - \frac{t^{\beta}}{\Gamma(\beta+1)}\sin\chi + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\sin\chi - \frac{2t^{3\beta}}{\Gamma(3\beta+1)}\sin\chi + \cdots.\end{aligned}$$

By putting $\beta = 1$, we obtain

$$\begin{aligned}u(\chi,t) &= u_0 + u_2 + u_3 + \cdots = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\sin\chi, \\ \omega(\chi,t) &= \omega_0 + \omega_2 + \omega_3 + \cdots = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\sin\chi,\end{aligned}$$

and thus, the exact solutions become

$$u(\chi,t) = e^{-t}\sin\chi, \quad \omega(\chi,t) = e^{-t}\sin\chi.$$

Figure 3: This shows the comparison between the exact solution and the gained numerical solution for Equation (54). At $t = 1$ and $\beta = 1$, we obtained the accurate solution; by taking diverse values of β such as ($\beta = 0.75$, $\beta = 0.85$ and $\beta = 0.95$), we obtained the estimated solutions.

Figure 4: We demonstrate the result of the functions $\omega(\chi,t) = u(\chi,t)$ in three-dimensional space.

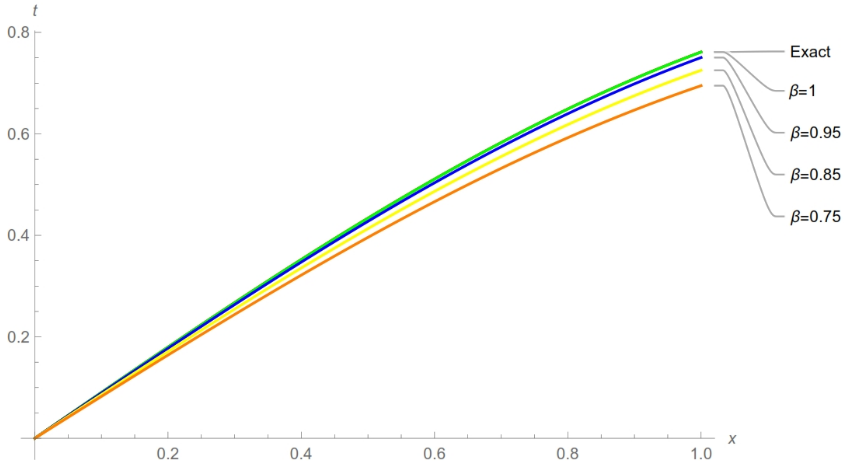

Figure 3. Comparison between exact and numerical solutions.

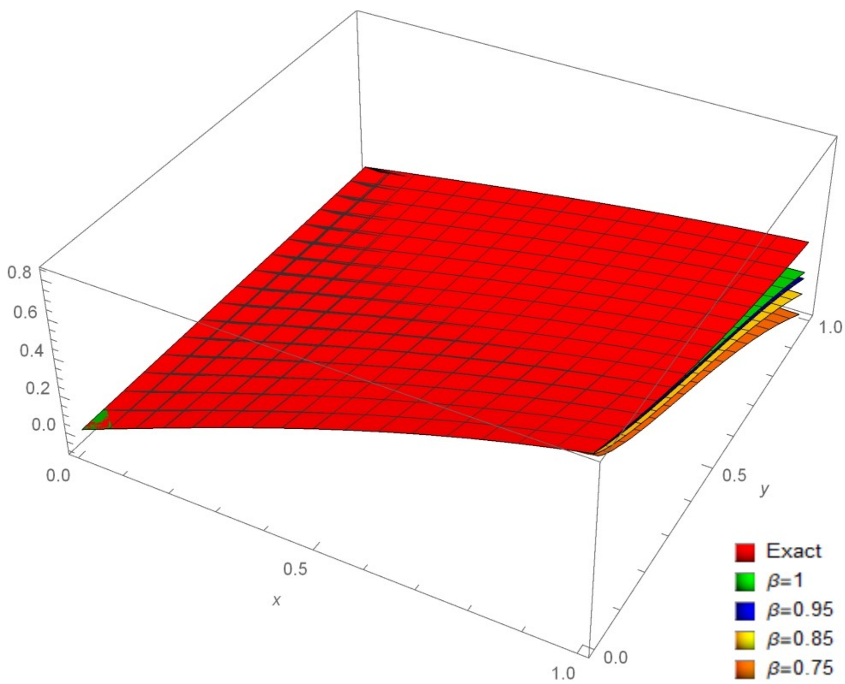

Figure 4. The surface of the function $\omega(x,t) = u(x,t)$.

Table 2 Shows the numerical solution for different values of β for the function $\omega(\chi,t) = u(\chi,t)$.

Table 2. Comparison between exact and approximation solutions.

t	x	$\beta = 0.75$	$\beta = 0.85$	$\beta = 0.95$	$\beta = 1$	Exact
	0.00	0.00	0.00	0.00	0.000	0.00
	0.25	0.204328	0.213246	0.220639	0.223818	0.22386
0.1	0.50	0.395952	0.413233	0.42756	0.43372	0.433802
	0.75	0.562957	0.587528	0.607897	0.616656	0.616772
	1.00	0.694961	0.725293	0.750438	0.761251	0.761394

Example 3 ([21]). *Consider the following system of singular fractional coupled Burgers equation with the initial conditions of the form:*

$$\begin{aligned} D_t^{\beta} u - \frac{1}{\chi}(\chi u_{\chi})_{\chi} - 2uu_{\chi} + (u\omega)_{\chi} &= -\chi^2 e^{-t} - 4e^{-t}, \\ D_t^{\beta} \omega - \frac{1}{\chi}(\chi \omega_{\chi})_{\chi} - 2\omega\omega_{\chi} + (u\omega)_{\chi} &= -\chi^2 e^{-t} - 4e^{-t}, \end{aligned} \tag{56}$$

and

$$u(\chi, 0) = \chi^2, \quad \omega(\chi, 0) = \chi^2. \tag{57}$$

By applying the above-mentioned steps, we have

$$\begin{aligned} U(p,s) &= 2s^{\alpha+1}p^{\alpha+3} + s^{\beta} G_2\left[\frac{1}{\chi}(\chi u_{\chi})_{\chi} + 2uu_{\chi} - (u\omega)_{\chi}\right] \\ &\quad - s^{\beta} G_2\left[\left(\chi^2+4\right)\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\right], \end{aligned} \tag{58}$$

and

$$\begin{aligned} \omega(p,s) &= 2s^{\alpha+1}p^{\alpha+3} + s^{\beta} G_2\left[\omega_{\chi\chi} + 2\omega\omega_{\chi} - (u\omega)_{\chi}\right] \\ &\quad - s^{\beta} G_2\left[\left(\chi^2+4\right)\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\right]. \end{aligned} \tag{59}$$

Hence,

$$\begin{aligned} U(p,s) &= 2s^{\alpha+1}p^{\alpha+3} + s^{\beta} G_2\left[\frac{1}{\chi}(\chi u_{\chi})_{\chi} + 2uu_{\chi} - (u\omega)_{\chi}\right] \\ &\quad - \left[\left(2p^{\alpha+3} + 4p^{\alpha+1}\right)\left(s^{\beta+\alpha+1} - s^{\beta+\alpha+3} + s^{\beta+\alpha+5} - \cdots\right)\right], \end{aligned} \tag{60}$$

and

$$\begin{aligned} \omega(p,s) &= 2s^{\alpha+1}p^{\alpha+3} + s^{\beta} G_2\left[\omega_{\chi\chi} + 2\omega\omega_{\chi} - (u\omega)_{\chi}\right] \\ &\quad - \left[\left(2p^{\alpha+3} + 4p^{\alpha+1}\right)\left(s^{\beta+\alpha+1} - s^{\beta+\alpha+3} + s^{\beta+\alpha+5} - \cdots\right)\right]. \end{aligned} \tag{61}$$

Operating the (IDGLT) for Equations (60) and (61), and applying Equations (43) and (33), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(\chi, t) &= \chi^2 + G_2^{-1}\left[s^{\beta} G_2\left[\frac{1}{\chi}\left(\chi \sum_{n=0}^{\infty} u_{n\chi}\right)_{\chi} + 2\sum_{n=0}^{\infty} C_n - \sum_{n=0}^{\infty} D_n\right]\right] \\ &\quad - \left[\left(\chi^2+4\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} - \cdots\right)\right], \end{aligned} \tag{62}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\omega_n(\chi,t) &= \chi^2+G_2^{-1}\left[s^{\beta}G_2\left[\frac{1}{\chi}\left(\chi\sum_{n=0}^{\infty}\omega_{n\chi}\right)_{\chi}+2\sum_{n=0}^{\infty}A_n-\sum_{n=0}^{\infty}D_n\right]\right]\\ &\quad -\left[\left(\chi^2+4\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right)\right]. \end{aligned}\tag{63}$$

Then, we determine the iteration components as

$$\begin{aligned}u_0 &= \chi^2-\left(\chi^2+4\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right)\\ \omega_0 &= \chi^2-\left[\left(\chi^2+4\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right)\right],\end{aligned}$$

and

$$\begin{aligned}u_{n+1} &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{1}{\chi}(\chi u_{n\chi})_{\chi}+2C_n-D_n\right]\right],\\ \omega_{n+1} &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{1}{\chi}(\chi\omega_{n\chi})_{\chi}+2A_n-D_n\right]\right],\end{aligned}$$

where $n\geq 0$*. The rest terms are given by, at* $n=0$,

$$\begin{aligned}u_1 &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{1}{\chi}(\chi u_{0\chi})_{\chi}+2u_0u_{0\chi}-(u_0\omega_0)_{\chi}\right]\right],\\ &= G_2^{-1}\left[s^{\beta}G_2\left[4-4\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right)\right]\right],\\ &= G_2^{-1}\left[\left[4p^{\alpha+1}s^{\alpha+\beta+1}-4p^{\alpha+1}\left(s^{2\beta+\alpha+1}-s^{2\beta+\alpha+1}+s^{2\beta+\alpha+1}-\cdots\right)\right]\right],\\ u_1 &= 4\frac{t^{\beta}}{\Gamma(\beta+1)}-4\left(\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}+\frac{t^{2\beta+2}}{\Gamma(2\beta+3)}-\cdots\right).\end{aligned}$$

In a similar manner,

$$\begin{aligned}\omega_1 &= G_2^{-1}\left[s^{\beta}G_2\left[\frac{1}{\chi}(\chi\omega_{0\chi})_{\chi}+2\omega_0\omega_{0\chi}-(u_0\omega_0)_{\chi}\right]\right],\\ &= G_2^{-1}\left[s^{\beta}G_2\left[4-4\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right)\right]\right],\\ &= G_2^{-1}\left[\left[4p^{\alpha+1}s^{\alpha+\beta+1}-4p^{\alpha+1}\left(s^{2\beta+\alpha+1}-s^{2\beta+\alpha+1}+s^{2\beta+\alpha+1}-\cdots\right)\right]\right],\\ \omega_1 &= 4\frac{t^{\beta}}{\Gamma(\beta+1)}-4\left(\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}+\frac{t^{2\beta+2}}{\Gamma(2\beta+3)}-\cdots\right).\end{aligned}$$

Similarly, at $n=1$

$$u_2=G_2^{-1}\left[s^{\beta}G_2[0]\right]=0,\ \omega_2=0.$$

In a similar manner,

$$u_3=0,\ \omega_3=0\cdot$$

As a collection of all terms, we obtain

$$\begin{aligned}u(\chi,t) &= u_0+u_1+u_2+u_3+\cdots,\\ \omega(\chi,t) &= \omega_0+\omega_1+\omega_2+\omega_3+\cdots.\end{aligned}$$

Thus, the approximation solution of Equation (56) is defined by

$$\begin{aligned} u(\chi,t) &= \chi^2-\left(\chi^2+4\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right) \\ &\quad +4\frac{t^{\beta}}{\Gamma(\beta+1)}-4\left(\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}+\frac{t^{2\beta+2}}{\Gamma(2\beta+3)}-\cdots\right), \\ \omega(\chi,t) &= \chi^2-\left(\chi^2+4\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}-\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}-\cdots\right) \\ &\quad +4\frac{t^{\beta}}{\Gamma(\beta+1)}-4\left(\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}+\frac{t^{2\beta+2}}{\Gamma(2\beta+3)}-\cdots\right). \end{aligned}$$

Our exact solution can be obtained by putting $\beta = 1$, as follows:

$$u(\chi,t) = \chi^2\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right),$$

and

$$\omega(\chi,t) = \chi^2\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right).$$

Hence,

$$\begin{aligned} u(\chi,t) &= \chi^2 e^{-t}, \\ \omega(\chi,t) &= \chi^2 e^{-t}. \end{aligned}$$

Figure 5: This illustrates the contrast between the exact solution and the obtained numerical solution for Equation (54). At $t = 1$ and $\beta = 1$, we achieved the accurate solution. By taking different values of β, for instance, ($\beta = 0.75$, $\beta = 0.85$ and $\beta = 0.95$), we obtained the estimated solutions.

Figure 6: We demonstrate the result of the functions $\omega(\chi,t) = u(\chi,t)$ in three-dimensional space.

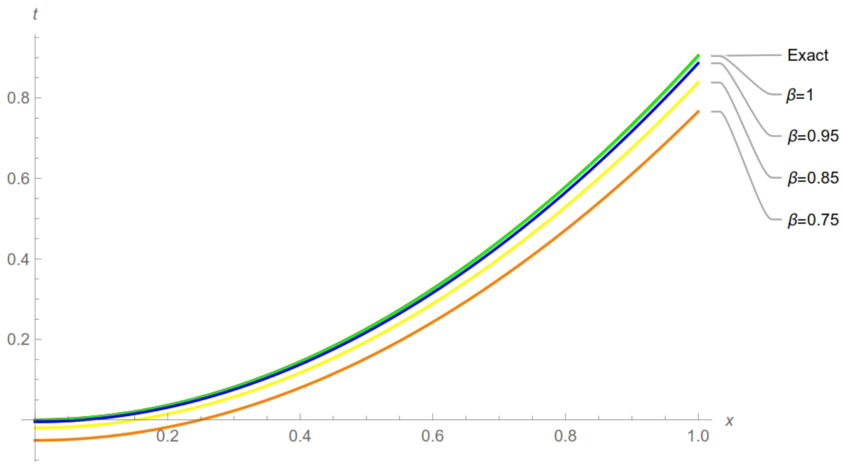

Figure 5. Comparison between exact and numerical solutions.

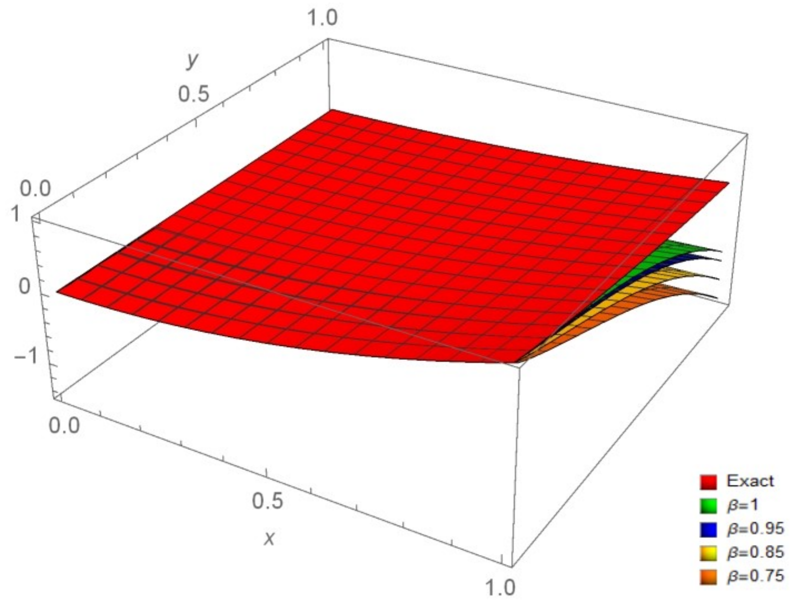

Figure 6. The surface of the function $\psi(x,t)$.

Table 3 Explain the numerical solution for different values of β for the function $\psi(x,t)$.

Table 3. Comparison between exact and approximation solutions.

t	x	$\beta = 0.75$	$\beta = 0.85$	$\beta = 0.95$	$\beta = 1$	Exact
	0.00	−0.0509492	−0.0200888	−0.00492833	−0.0008333	0.00
	0.25	0.0000985564	0.0335443	0.0507572	0.0557083	0.0565523
0.1	0.50	0.153242	0.1944443	0.217814	0.225333	0.226209
	0.75	0.408481	0.462609	0.496241	0.508042	0.508971
	1.00	0.765815	0.838041	0.88604	0.903833	0.904837

3. Triple Generalized-Laplace Transform Decomposition Method and Fractional Coupled Burgers Equation (TGLTDM)

Here, we explain the solutions of the singular time-fractional coupled system of Burgers equation by applying the triple Generalized- Laplace transform decomposition method (TGLTDM):

To demonstrate the essential idea of this method, we consider the following singular time-fractional coupled system of Burgers equation with the initial conditions of the form:

$$\begin{aligned} D_t^{\beta} u - \frac{1}{\chi}(\chi u_{\chi})_{\chi} - \frac{1}{\gamma}(\gamma u_{\gamma})_{\gamma} + \frac{\eta}{\chi} u u_{\chi} + \frac{\zeta}{\gamma} u_{\gamma}\omega &= f(\chi,\gamma,t), \\ D_t^{\beta} \omega - \frac{1}{\chi}(\chi \omega_{\chi})_{\chi} - \frac{1}{\gamma}(\gamma \omega_{\gamma})_{\gamma} + \frac{\eta}{\chi} u \omega_{\chi} + \frac{\zeta}{\gamma} \omega_{\gamma}\omega &= h(\chi,\gamma,t), \end{aligned} \quad (64)$$

and

$$u(\chi,0) = f_1(\chi,\gamma), \qquad \omega(\chi,0) = h_1(\chi,\gamma) \quad (65)$$

for $t > 0$. Here, $f(\chi,\gamma,t)$, $h(\chi,\gamma,t)$, $f_1(\chi,\gamma)$ and $h_1(\chi,\gamma)$ are known functions, η is a real constant, ζ and μ are arbitrary constants. To obtain the solution of Equation (64), the following steps are applied.

Step 1: Applying the (TGLT) on both sides of Equation (64) and (DGLT) for Equation (65) and multiplying the outcome by s^{β}, one can obtain

$$\begin{aligned} U(p,q,s) &= s^{\alpha+1}F_1(p,q) + s^{\beta}F(p,q,s) \\ &+ s^{\beta}G_3\left[\frac{1}{\chi}(\chi u_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma u_{\gamma})_{\gamma} - \frac{\eta}{\chi}uu_{\chi} - \frac{\zeta}{\gamma}u_{\gamma}\omega\right], \end{aligned} \tag{66}$$

and

$$\begin{aligned} \omega(p,q,s) &= s^{\alpha+1}H_1(p,q) + s^{\beta}H(p.q,s) \\ &+ s^{\beta}G_3\left[\frac{1}{\chi}(\chi \omega_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma \omega_{\gamma})_{\gamma} - \frac{\eta}{\chi}u\omega_{\chi} - \frac{\zeta}{\gamma}\omega_{\gamma}\omega\right]. \end{aligned} \tag{67}$$

Step 2: The (TGLTDM) defined the solution of the time-space fractional coupled Burgers equation in the following forms:

$$u(\chi,\gamma,t) = \sum_{n=0}^{\infty} u_n(\chi,\gamma,t), \quad \omega(\chi,\gamma,t) = \sum_{n=0}^{\infty} \omega_n(\chi,\gamma,t). \tag{68}$$

We can obtain Adomian's polynomials F_n, E_n and R_n, respectively, as follows:

$$E_n = \sum_{n=0}^{\infty} u_{n\gamma}\omega_n, \quad F_n = \sum_{n=0}^{\infty} u_n\omega_{n\chi}, \quad R_n = \sum_{n=0}^{\infty} \omega_n\omega_{n\gamma} \tag{69}$$

where C_n is mentioned in Equation (47). The Adomian polynomials for the nonlinear term $u_{\gamma}\omega$, $u\omega_{\chi}$, and $\omega\omega_{\gamma}$ are defined by

$$\begin{aligned} E_0 &= u_0\omega_{0\gamma} \\ E_1 &= u_0\omega_{1\gamma} + u_1\omega_{0\gamma}, \\ E_2 &= u_0\omega_{2\gamma} + u_1\omega_{1\gamma} + u_2\omega_{0\gamma}, \\ E_3 &= u_0\omega_{3\gamma} + u_1\omega_{2\gamma} + u_2\omega_{1\gamma} + u_3\omega_{0\gamma}, \end{aligned} \tag{70}$$

$$\begin{aligned} F_0 &= u_0\omega_{0\chi} \\ F_1 &= u_0\omega_{1\chi} + u_1\omega_{0\chi}, \\ F_2 &= u_0\omega_{2\chi} + u_1\omega_{1\chi} + u_2\omega_{0\chi}, \\ F_3 &= u_0\omega_{3\chi} + u_1\omega_{2\chi} + u_2\omega_{1\chi} + u_3\omega_{0\chi}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} R_0 &= \omega_0\omega_{\gamma} \\ R_1 &= \omega_0\omega_{1\gamma} + \omega_1\omega_{0\gamma} \\ R_2 &= \omega_0\omega_{2\gamma} + \omega_1\omega_{1\gamma} + \omega_2\omega_{0\gamma}. \\ R_3 &= \omega_0\omega_{3\gamma} + \omega_1\omega_{2\gamma} + \omega_2\omega_{1\gamma} + \omega_3\omega_{0\gamma}. \end{aligned} \tag{72}$$

Step 3: Applying the (ITGLT) on both sides of Equations (66) and (67) and using Equations (44) and (69), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(\chi,\gamma,t) &= f_1(\chi,\gamma) + G_3^{-1}\Big[s^{\beta}F(p,q,s)\Big] \\ &+G_3^{-1}\left[\left[s^{\beta}G_3\left(\frac{1}{\chi}\left(\chi\sum_{n=0}^{\infty}u_{n\chi}\right)\right)_{\chi}\right]\right] \\ &+G_3^{-1}\left[\left[s^{\beta}G_3\left(\frac{1}{\gamma}\left(\gamma\sum_{n=0}^{\infty}u_{n\gamma}\right)\right)_{\gamma}\right]\right] \\ &-G_3^{-1}\left[s^{\beta}G_3\left[\frac{\eta}{\chi}\sum_{n=0}^{\infty}C_n\right]\right] \\ &-G_3^{-1}\left[s^{\beta}G_3\left[\zeta\left(\sum_{n=0}^{\infty}E_n\right)_{\chi}\right]\right], \end{aligned} \tag{73}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \omega_n(\chi,\gamma,t) &= h_1(\chi,\gamma) + G_3^{-1}\Big[s^{\beta}H(p,q,s)\Big] \\ &+G_3^{-1}\left[\left[s^{\beta}G_3\left(\frac{1}{\chi}\left(\chi\sum_{n=0}^{\infty}\omega_{n\chi}\right)\right)_{\chi}\right]\right] \\ &+G_3^{-1}\left[\left[s^{\beta}G_3\left(\frac{1}{\gamma}\left(\gamma\sum_{n=0}^{\infty}\omega_{n\gamma}\right)\right)_{\gamma}\right]\right] \\ &-G_3^{-1}\left[s^{\beta}G_3\left[\frac{\eta}{\chi}\sum_{n=0}^{\infty}F_n\right]\right] \\ &-G_3^{-1}\left[s^{\beta}G_3\left[\zeta\left(\sum_{n=0}^{\infty}R_n\right)_{\chi}\right]\right]. \end{aligned} \tag{74}$$

Step 4: On comparing both sides of Equations (73) and (74), we can obtain

$$\begin{aligned} u_0(\chi,\gamma,t) &= f_1(\chi,\gamma) + G_p^{-1}G_s^{-1}\Big[s^{\beta}F(p,q,s)\Big], \\ \omega_0(\chi,\gamma,t) &= h_1(\chi,\gamma) + G_p^{-1}G_s^{-1}\Big[s^{\beta}H(p,q,s)\Big]. \end{aligned} \tag{75}$$

In general, the rest of the recursive relations are given by

$$\begin{aligned} u_{n+1} &= G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}(\chi u_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma u_{\gamma})_{\gamma} - \frac{\eta}{\chi}C_n - \frac{\zeta}{\gamma}E_n\right]\right], \\ \omega_{n+1} &= G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}(\chi \omega_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma \omega_{\gamma})_{\gamma} - \frac{\eta}{\chi}F_n - \frac{\zeta}{\gamma}R_n\right]\right], \end{aligned} \tag{76}$$

We assume that the (ITGLT) concerning p, q, and s exists for each term on the right-hand side of Equations (75) and (76). In the next example, we apply (TGLTDM) to solve the singular time-fractional coupled Burgers equation.

Example 4 ([22])**.** *Consider the following system of singular fractional coupled Burgers equation with the initial conditions of the form:*

$$\begin{aligned} D_t^{\beta} u - \frac{1}{\chi}(\chi u_{\chi})_{\chi} - \frac{1}{\gamma}(\gamma u_{\gamma})_{\gamma} + \frac{1}{\chi} u u_{\chi} + \frac{1}{\gamma} u_{\gamma}\omega &= \left(\chi^2 - \gamma^2\right) e^{-t} \\ D_t^{\beta} \omega - \frac{1}{\chi}(\chi \omega_{\chi})_{\chi} - \frac{1}{\gamma}(\gamma \omega_{\gamma})_{\gamma} + \frac{1}{\chi} u \omega_{\chi} + \frac{1}{\gamma} \omega_{\gamma}\omega &= \left(\chi^2 - \gamma^2\right) e^{-t}, \end{aligned} \tag{77}$$

and

$$u(\chi, \gamma, 0) = \chi^2 - \gamma^2, \quad \omega(\chi, \gamma, 0) = \chi^2 - \gamma^2. \tag{78}$$

By applying the above-mentioned method, we have

$$\begin{aligned} U(p,q,s) &= s^{\alpha+1}\left(2! p^{\alpha+3} q^{\alpha+1} - 2! q^{\alpha+3} p^{\alpha+1}\right) \\ &+ s^{\beta} G_3\left[\frac{1}{\chi}(\chi u_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma u_{\gamma})_{\gamma} - \frac{1}{\chi} u u_{\chi} - \frac{1}{\gamma} u_{\gamma}\omega\right] \\ &+ s_{\chi}^{\beta} G_3\left[\left(\chi^2 - \gamma^2\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right], \end{aligned} \tag{79}$$

and

$$\begin{aligned} \omega(p,q,s) &= s^{\alpha+1}\left(2! p^{\alpha+3} q^{\alpha+1} - 2! q^{\alpha+3} p^{\alpha+1}\right) \\ &+ s^{\beta} G_3\left[\frac{1}{\chi}(\chi \omega_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma \omega_{\gamma})_{\gamma} - \frac{1}{\chi} u \omega_{\chi} - \frac{1}{\gamma} \omega_{\gamma}\omega\right] \\ &+ s_{\chi}^{\beta} G_3\left[\left(\chi^2 - \gamma^2\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right]. \end{aligned} \tag{80}$$

Therefore,

$$\begin{aligned} U(p,q,s) &= s^{\alpha+1}\left(2! p^{\alpha+3} q^{\alpha+1} - 2! q^{\alpha+3} p^{\alpha+1}\right) \\ &+ s^{\beta} G_3\left[\frac{1}{\chi}(\chi u_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma u_{\gamma})_{\gamma} - \frac{1}{\chi} u u_{\chi} - \frac{1}{\gamma} u_{\gamma}\omega\right] \\ &+ \left[\left(2! p^{\alpha+3} q^{\alpha+1} - 2! q^{\alpha+3} p^{\alpha+1}\right)\left(s^{\beta+\alpha+1} + s^{\beta+\alpha+2} + s^{\beta+\alpha+3} + \cdots\right)\right], \end{aligned} \tag{81}$$

and

$$\begin{aligned} \omega(p,q,s) &= s^{\alpha+1}\left(2! p^{\alpha+3} q^{\alpha+1} - 2! q^{\alpha+3} p^{\alpha+1}\right) \\ &+ s^{\beta} G_3\left[\frac{1}{\chi}(\chi \omega_{\chi})_{\chi} + \frac{1}{\gamma}(\gamma \omega_{\gamma})_{\gamma} - \frac{1}{\chi} u \omega_{\chi} - \frac{1}{\gamma} \omega_{\gamma}\omega\right] \\ &+ \left[\left(2! p^{\alpha+3} q^{\alpha+1} - 2! q^{\alpha+3} p^{\alpha+1}\right)\left(s^{\beta+\alpha+1} + s^{\beta+\alpha+2} + \cdots\right)\right]. \end{aligned} \tag{82}$$

Operating an (ITGLT) for Equations (81) and (82), and using the polynomial series solution, we obtain the following:

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(\chi, \gamma, t) &= \chi^2 - \gamma^2 \\ &+ G_3^{-1}\left[s^{\beta} G_3\left[\frac{1}{\chi}\left(\chi \sum_{n=0}^{\infty} u_{n\chi}\right)_{\chi} + \frac{1}{\gamma}\left(\gamma \sum_{n=0}^{\infty} u_{n\gamma}\right)_{\chi}\right]\right] \\ &+ G_3^{-1}\left[s^{\beta} G_3\left[-\frac{1}{\chi}\sum_{n=0}^{\infty} C_n - \frac{1}{\gamma}\sum_{n=0}^{\infty} E_n\right]\right] \\ &+ \left[\left(\chi^2 - \gamma^2\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \ldots\right)\right], \end{aligned} \tag{83}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}\omega_n(\chi,\gamma,t) &= \chi^2-\gamma^2 \\
&+G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}\left(\chi\sum_{n=0}^{\infty}\omega_{n\chi}\right)_{\chi}+\frac{1}{\gamma}\left(\gamma\sum_{n=0}^{\infty}\omega_{n\gamma}\right)_{\chi}\right]\right] \\
&+G_3^{-1}\left[s^{\beta}G_3\left[-\frac{1}{\chi}\sum_{n=0}^{\infty}F_n-\frac{1}{\gamma}\sum_{n=0}^{\infty}R_n\right]\right] \\
&+\left[\left(\chi^2-\gamma^2\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\cdots\right)\right]. \qquad (84)
\end{aligned}$$

Then, we determine the iteration components as

$$\begin{aligned}
u_0 &= \left(\chi^2-\gamma^2\right)+\left(\chi^2-\gamma^2\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\cdots\right), \\
\omega_0 &= \left(\chi^2-\gamma^2\right)+\left(\chi^2-\gamma^2\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\cdots\right),
\end{aligned}$$

and

$$\begin{aligned}
u_{n+1} &= G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}(\chi u_{n\chi})_{\chi}+\frac{1}{\gamma}(\gamma u_{n\gamma})_{\chi}-C_n-E_n\right]\right], \\
\omega_{n+1} &= G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}(\chi\omega_{n\chi})_{\chi}+\frac{1}{\gamma}(\gamma\omega_{n\gamma})_{\chi}-F_n-R_n\right]\right],
\end{aligned}$$

where $n\geq 0$*. The remaining terms are given by the following, at* $n=0$ *:*

$$\begin{aligned}
u_1 &= G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}(\chi u_{0\chi})_{\chi}+\frac{1}{\gamma}(\gamma u_{0\gamma})_{\gamma}-\frac{1}{\chi}u_0u_{0\chi}-\frac{1}{\gamma}u_{0\gamma}\omega_0\right]\right], \\
&= G_3^{-1}\left[s^{\beta}G_3[0]\right]. \\
u_1 &= 0,
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
\omega_1 &= G_3^{-1}\left[s^{\beta}G_3\left[\frac{1}{\chi}(\chi\omega_{0\chi})_{\chi}+\frac{1}{\gamma}(\chi\gamma\omega_{0\gamma})_{\gamma}-\frac{1}{\chi}u\omega_{\chi}-\frac{1}{\gamma}\omega_0\omega_{0\gamma}\right]\right], \\
&= G_3^{-1}\left[s^{\beta}G_3[0]\right], \\
\omega_1 &= 0.
\end{aligned}$$

Similarly, at $n=1$*,*

$$u_2=G_3^{-1}\left[s^{\beta}G_3[0]\right]=0,\ \omega_2=0.$$

In a similar manner,

$$u_3=0,\ \omega_3=0.$$

As a collection of all terms, we obtain

$$\begin{aligned}
u(\chi,\gamma,t) &= u_0+u_1+u_2+u_3+\cdots, \\
\omega(\chi,\gamma,t) &= \omega_0+\omega_1+\omega_2+\omega_3+\cdots.
\end{aligned}$$

Thus, the approximate solution of Equation (77) is defined by

$$u(\chi,\gamma,t) = \left(\chi^2-\gamma^2\right)+\left(\chi^2-\gamma^2\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\cdots\right),$$
$$\omega(\chi,\gamma,t) = \left(\chi^2-\gamma^2\right)+\left(\chi^2-\gamma^2\right)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\cdots\right).$$

The exact solution can be obtained by substituting with $\beta = 1$, as follows:

$$u(\chi,\gamma,t) = \left(\chi^2-\gamma^2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right),$$
$$\omega(\chi,\gamma,t) = \left(\chi^2-\gamma^2\right)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right).$$

Hence,

$$\omega(\chi,\gamma,t) = \left(\chi^2-\gamma^2\right)e^t,$$
$$\omega(\chi,\gamma,t) = \left(\chi^2-\gamma^2\right)e^t.$$

Figure 7: This illustrates the contrast between the exact solution and the obtained numerical solution for Equation (54). At $t = 1$ and $\beta = 1$, we obtained the accurate solution. By taking different values of β, for instance, ($\beta = 0.75$, $\beta = 0.85$ and $\beta = 0.95$), we obtained the estimated solutions.

Figure 8: We demonstrate the result of the functions $\omega(\chi,t) = u(\chi,t)$ in three-dimensional space.

Table 4 Indicates the numerical solution for different values of β for the function $\psi(x,t)$.

Table 4. Comparison between exact and approximation solutions.

y	t	x	$\beta = 0.75$	$\beta = 0.85$	$\beta = 0.95$	$\beta = 1$	Exact
		0.00	−0.0118283	−0.0114159	−0.0110883	−0.0109517	−0.0110517
		0.25	0.0620988	0.0599333	0.0582137	0.0574963	0.0580215
0.10	0.10	0.50	0.28388	0.273981	0.26612	0.26284	0.265241
		0.75	0.653516	0.630727	0.61263	0.60508	0.610607
		1.00	1.17101	1.13017	1.09774	1.08422	1.09412

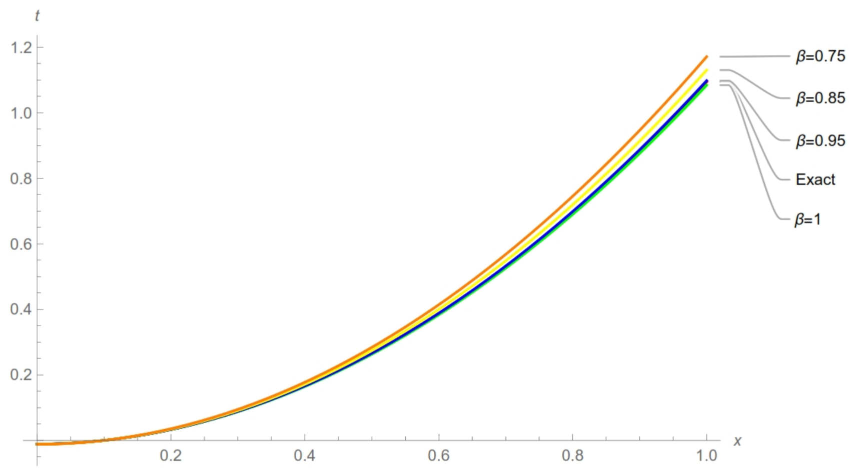

Figure 7. Comparison between exact and numerical solutions.

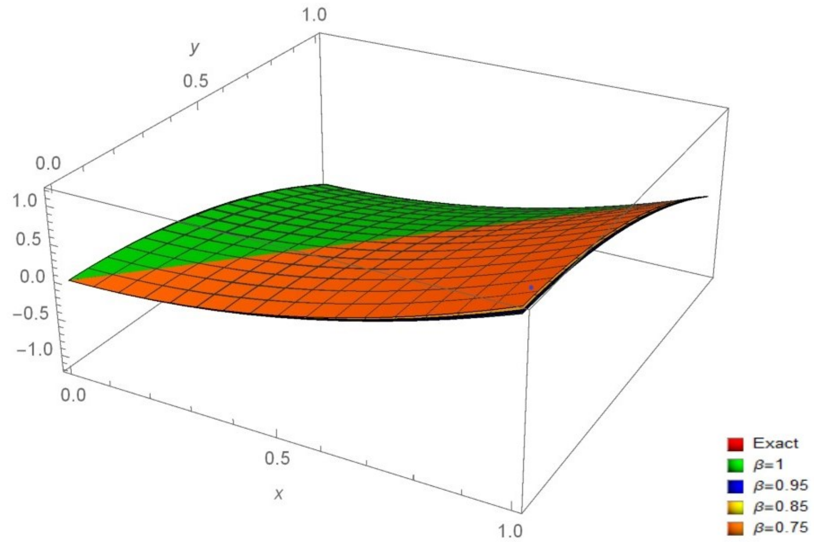

Figure 8. The surface of the function $\psi(x,t)$.

4. Conclusions

This study proposes a new double and triple generalized-Laplace transform decomposition method (DGLTDM and TGLTDM), a novel method combining DGLT, TGLT, and DM to obtain the solutions of regular and singular fractional Burgers equations. This combination produces a strong method. The ability and precision of the proposed plan are confirmed through examples. This technique can be used for many difficult linear and nonlinear FPDEs and systems of FPDE that do not require linearity. Furthermore, in the future, we plan to employ this technique to solve various scientific problems related to our research field.

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