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Observer Design for Fractional-Order Polynomial Fuzzy Systems Depending on a Parameter

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Abstract: For fractional-order systems, observer design is remarkable for the estimation of unavailable states from measurable outputs. In addition, the nonlinear dynamics and the presence of parameters that can vary over different operating conditions or time, such as load or temperature, increase the complexity of the observer design. In view of the aforementioned factors, this paper investigates the observer design problem for a class of Fractional-Order Polynomial Fuzzy Systems (FORPSs) depending on a parameter. The Caputo–Hadamard derivative is considered in this study. First, we prove the practical Mittag-Leffler stability, using the Lyapunov methods, for the general case of Caputo–Hadamard Fractional-Order Systems (CHFOSs) depending on a parameter. Secondly, based on this stability theory, we design an observer for the considered class of FORPSs. The state estimation error is ensured to be practically generalized Mittag-Leffler stable by solving Sum Of Squares (SOSs) conditions using the developed SOSTOOLS.

Keywords: Mittag-Leffler function; Caputo–Hadamard derivative; observer design

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Citation: Gassara, H.; Rhaima, M.; Mchiri, L.; Ben Makhlouf, A. Observer Design for Fractional-Order Polynomial Fuzzy Systems Depending on a Parameter. *Fractal Fract.* **2024**, *8*, 693. https://doi.org/10.3390/fractalfract8120693

Academic Editor: Aldo Jonathan Muñoz-Vazquez, Sakthivel Ramalingam and Parivallal Arumugam

Received: 10 September 2024
Revised: 19 November 2024
Accepted: 23 November 2024
Published: 25 November 2024

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1. Introduction

The last decade has seen a significant increase in research into fractional-order systems. Dynamical systems that have been differentiated or integrated can be better described by fractional-order models [1–5]. Furthermore, in the physical world, fractional-order state equations are usually used to describe a number of physical systems [6,7], such as the fractional Langevin equation [8] and the fractional model of nonlinear Duffing oscillator [9]. For control systems, stability analysis is one of the most important issues [10–13]. In the literature, stability and stabilization problems for fractional-order systems have been extensively studied [14–17]. Practical stability is one kind of stability that has been studied; this notion was discussed in [18,19].

Due to its effectiveness at approximating nonlinear dynamics, the Takagi-Sugeno Method (TSM) [20] is extensively employed in the literature to represent various classes of Nonlinear Systems (NSs). By adopting the Linear Matrix Inequality (LMI) approach, many results have used TSM for different categories of integer-order NSs such as regular NS, singular NS, delayed NS, and stochastic NSs. Recent developments have extended the TSM to tackle stability problems of fractional-order NSs. Lin et al., in [21], developed a stabilizing static output feedback controller. In [22], an adaptive observer, based on sliding mode technique, is designed for a class of descriptor systems. The problem of the stabilization of singular NSs is treated in [23]. Taking into account input saturations and uncertainties, the authors in [24] developed an adaptive control for delayed fractional-order NSs described by the Takagi-Sugeno fuzzy model.

Although the LMI approach has proved its ability for studying stability, an alternative to this approach, called Sum Of Squares (SOSs) [25], has been appearing since 2002. This method generalizes the LMI approach by resolving polynomial matrix inequalities, which can be viewed as a more general form of linear inequalities. Accordingly, in 2007, polynomial fuzzy models extend Takagi-Sugeno fuzzy models by incorporating local models with polynomial matrices in place of constant matrices [26]. To date, extensive research has been conducted to address the issues of stability and control in integer-order polynomial fuzzy models using an SOS approach. For instance, by approximating membership functions adopting the sector nonlinearity of the control input, Lam et al. [27] proposed new relaxed stability results for Polynomial Fuzzy Systems (PFSs). Based on the line integral polynomial fuzzy Lyapunov function, Saenz et al. [28] investigated the stabilization problem associated with the disturbance attenuation for PFSs. In addition, by adopting the Positivstellensatz, the domain of the polynomial variables that represent the membership functions is constrained in order to provide less conservative SOS conditions. However, it is important to note that very little attention has been paid to fractional-order polynomial fuzzy models. For instance, in a recent work cited in [29], an observer was designed for FORPSs with a Caputo derivative. However, there are no works proposed for Caputo–Hadamard FORPSs depending on a parameter. Thus, motivated by the above interpretation, our work offers the following contributions:

- Based on the advantages of polynomial fuzzy models and by adopting the Caputo–Hadamard fractional-order derivative, a new class of Caputo–Hadamard Fractional-Order Polynomial Fuzzy Systems (CHFORPSs) depending on a parameter is considered in this study.
- The Practical Generalized Mittag-Leffler stability problem has not yet been explored for the general case of Caputo–Hadamard Fractional-Order systems depending on a parameter in the literature. Therefore, this paper tackles and resolves this gap.
- Compared to recent work [18], this paper addresses the Caputo–Hadamard derivative rather than the Caputo derivative, which presents a greater challenge due to its increased complexity. Additionally, our design accounts for the presence of s nonlinear function depending on a parameter.

Notations: $\mathcal{M}_{x(\zeta)}, \mathcal{F}_{x(\zeta)}$, and $\mathcal{SOS}\int$ are the sets of polynomial matrices in $x(\zeta)$ and the polynomial function in $x(\zeta)$ and SOSs.

2. Preliminaries

In this section, we provide specific definitions and lemmas, as outlined in [14].

Definition 1 ([14])**.** *The Hadamard fractional integral of a locally integrable function Ψ of order $\delta > 0$ is given by:*

$$I^{\delta}\Psi(\tau) = \frac{1}{\Gamma(\delta)} \int_1^{\tau} \big(\log \frac{\tau}{\nu}\big)^{\delta-1} \frac{\Psi(\nu)}{\nu} d\nu, \ \tau \geq 1. \tag{1}$$

Definition 2 ([14])**.** *The Caputo fractional derivative with order $0 < \delta < 1$ for an absolutely continuous function $h : [t_0, \infty) \to \mathbb{R}$ is as follows:*

$$^{C_H}D_1^{\delta} h(\vartheta) = \frac{1}{\Gamma(1-\delta)} \int_1^{\vartheta} \left(\log \frac{\vartheta}{\nu}\right)^{-\delta} h'(\nu) d\nu, \ \vartheta \geq 1. \tag{2}$$

Lemma 1 ([15])**.** *Let $\delta \in (0,1)$ and $S \in \mathbb{R}^{n \times n}$ be a Symmetric Positive Definite (SPD) matrix. Then:*

$$\frac{1}{2}\, ^{C_H}D_1^{\delta}(x^T(\sigma) S x(\sigma)) \leq x^T(\sigma) S\, ^{C_H}D_1^{\delta} x(\sigma), \ \sigma \geq 1. \tag{3}$$

Definition 3 ([14]). *The Mittag-Leffler function is given by:*

$$E_{c_1,c_2}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(kc_1 + c_2)}, \tag{4}$$

where $c_1, c_2 > 0$, $z \in \mathbb{C}$.
When $c_2 = 1$, we write $E_{c_1,1}(z) := E_{c_1}(z)$.

Lemma 2 ([18]). *For $c \in (0,1)$ and $r \in \mathbb{R}$, we have*

$$\int_0^\sigma (\sigma - s)^{c-1} E_{c,c}(r(\sigma - s)^c) ds = \sigma^c E_{c,c+1}(r\sigma^c).$$

Remark 1. *If $r < 0$, then $\sigma \longmapsto \sigma^c E_{c,c+1}(r\sigma^c)$ is a bounded function.*

The solution of the FOS:

$$\begin{aligned} {}^{CH}D_1^\delta y(\zeta) &= \theta y + m(\zeta),\ \rho \geq 1, \\ y(1) &= y_0, \end{aligned} \tag{5}$$

is given by [15]:

$$y(\zeta) = E_\delta\left(\theta\left(\log\zeta\right)^\delta\right) y_0 + \int_1^\zeta \left(\log\frac{\zeta}{l}\right)^{\delta-1} E_{\delta,\delta}\left(\theta\left(\log\frac{\zeta}{l}\right)^\delta\right) m(l)\frac{dl}{l}.$$

Definition 4 ([25]). *Let $b(x(\zeta)) \in \mathcal{F}_{x(\zeta)}$. If $\exists\ \{b_1(x(\zeta)), b_2(x(\zeta)), \ldots, b_m(x(\zeta))\} \in \mathcal{F}_{x(\zeta)}$ such that*

$$b(x(\zeta)) = \sum_{j=1}^m b_j^2(x(\zeta)). \tag{6}$$

then $b(x(\zeta)) \in \mathcal{SOS}\int$, which implies that $b(x(\zeta)) \geq 0$.

Lemma 3 ([26]). *Let $X(x(\zeta)) \in \mathcal{M}_{x(\zeta)}$. If $e^T X(x(\zeta))e \in \mathcal{SOS}\int$, where e is a vector independent of $x(\zeta)$; then, $M(x(\zeta)) \geq 0$, $\forall x(\zeta)$.*

3. Main Results

3.1. CHFORPSs Depending on a Parameter Description

We consider the following CHFORPSs depending on a parameter:
Model rule $s (s \in \mathbb{S} = \{1, 2, \cdots, f\})$: If $N_1(\zeta)$ is Q_{s1} and $\cdots$ and $N_p(\zeta)$ is Q_{sp}, then

$$\begin{cases} {}^{CH}D_1^\delta x(\zeta) &= A_s(x(\zeta))x(\zeta) + B_s(x(\zeta))u(\zeta) + f_s(\zeta, x(\zeta), \epsilon), \quad \zeta \geq 1, \\ y(\zeta) &= Cx(\zeta), \end{cases} \tag{7}$$

where $x(\zeta)$ is the state vector, $u(\zeta)$ is the input vector, $y(\zeta)$ is the measured output vector, $\{A_s(x(\zeta)), B_s(x(\zeta))\} \in \mathcal{M}_{x(\zeta)}$, C is a constant matrix, and $f_s(\zeta, x(\zeta), \epsilon)$ are functions depending on a parameter ϵ.

Remark 2. *The model (7) employs a polynomial fuzzy structure; it is distinguished from Takagi-Sugeno models by its use of nonlinear polynomial functions in the local models. This approach introduces polynomial nonlinearity directly into each rule, allowing for a compact yet accurate representation of complex dynamics with fewer rules than a Takagi-Sugeno model would require. Furthermore, the term $f_s(\zeta, x(\zeta), \epsilon)$ depends not only on ζ and $x(\zeta)$ but also on a parameter ϵ, which allows one to represent a broader range of uncertainties or disturbances.*

Remark 3. *It is noted that this paper represents the first comprehensive investigation into the observer-based control of CHFORPSs that includes nonlinear functions $f_s(\zeta, x(\zeta), \epsilon)$ depending on a parameter ϵ.*

Assumption 1. *We suppose that:*

- *$A_s(x(\zeta))$ and $B_s(x(\zeta))$ are solely dependent on measurable variables, i.e., $A_s(x(\zeta)) = A_s(y(\zeta))$ and $B_s(x(\zeta)) = B_s(y(\zeta))$.*
- *Each $f_s(\zeta, x(\zeta), \epsilon)(s \in \mathbb{S})$ verifies the following condition:*

$$\|f_s(\zeta, x_1(\zeta), \epsilon) - f_s(\zeta, x_2(\zeta), \epsilon)\| \leq \delta_1(\epsilon)\|x_1(\zeta) - x_2(\zeta)\| + \delta_2(\epsilon)\mu(\zeta), \tag{8}$$

  where $\mu(\zeta)$ is a continuous function and $\delta_1(\epsilon) > 0,\ \delta_2(\epsilon) > 0$ such that $\lim_{\epsilon \to 0} \delta_q(\epsilon) = 0$, $(q = 1, 2)$.
- *$N_j(\zeta)$, $j = 1, \ldots, p$ are measurable.*

Remark 4. *In Assumption 1, we consider polynomial matrices that depend only on measurable states. This assumption is commonly used in numerous engineering systems, such as the inverted pendulum and tunnel diode electronic circuit, and so on [30].*

Remark 5. *If $\mu(\zeta) = 0$ and the scalar $\delta_1(\epsilon) = \delta_1$ does not depend on ϵ, then (8) simplifies to the following Lipschitz condition:*

$$\|f_s(\zeta, x_1(\zeta), \epsilon) - f_s(\zeta, x_2(\zeta), \epsilon)\| \leq \delta_1 \|x_1(\zeta) - x_2(\zeta)\|. \tag{9}$$

The complete polynomial fuzzy CHFOSs can be expressed as follows:

$$^{CH}D_1^{\delta} x(\zeta) = \sum_{s=1}^{f} \beta_s(N(\zeta))\Big(A_s(y(\zeta))x(\zeta) + B_s(y(\zeta))u(\zeta) + f_s(\zeta, x(\zeta), \epsilon)\Big) \tag{10}$$

where

$$\beta_s(N(\zeta)) = \frac{\prod_{\varsigma=1}^{p} Q_{s\varsigma}(N_\varsigma(\zeta))}{\sum_{\iota=1}^{r} \prod_{\varsigma=1}^{p} Q_{s\varsigma}(N_\varsigma(\zeta))} \quad \text{in which} \quad N(\zeta) = [N_1(\zeta), \ldots, N_p(\zeta)].$$

It is clear that

$$\beta_s(N(\zeta)) \geq 0, \quad \sum_{s=1}^{f} \beta_s(N(\zeta)) = 1. \tag{11}$$

In the particular case, when $A_s(y(\zeta)$ and $B_s(y(\zeta))$ are constant, FORPSs reduce to the following Fractional-Order Takagi-Sugeno Fuzzy System (FOTSS):

$$^{CH}D_1^{\delta} x(\zeta) = \sum_{s=1}^{f} \beta_s(N(\zeta))\Big(A_s x(\zeta) + B_s u(\zeta) + f_s(\zeta, x(\zeta), \epsilon)\Big) \tag{12}$$

3.2. Practical Generalized Mittag-Leffler Stability of the General Case of CHFOSs

Consider the $\epsilon-$CHFOSs

$$\begin{cases} ^{CH}D_1^{\delta} x(\zeta) & = F(\zeta, x(\zeta), \epsilon), \quad \zeta \geq 1 \\ x(1) & = x_0. \end{cases} \tag{13}$$

Definition 5. *The ϵ−CHFOSs (13) is said to be ϵ^*−Practically Generalized Mittag-Leffler stable if for all $0 < \epsilon < \epsilon^*$ there are positive scalars $c_1(\epsilon), c_2(\epsilon)$ and $r(\epsilon)$ such that:*

$$\|x_\epsilon(\zeta)\| \leq c_1(\epsilon)\|x_0\| \Big[E_\delta\big(-c_2(\epsilon)(\log(\zeta))^\delta\big)\Big]^{\frac{1}{2}} + r(\epsilon), \quad \forall\zeta \geq 1 \tag{14}$$

where $\lim_{\epsilon\to 0} r(\epsilon) = 0$ and $0 < c_1(\epsilon) \leq \rho_1,\ \rho_2 \leq c_2(\epsilon) \leq \rho_3$ in which ρ_1, ρ_2, and ρ_3 are positive scalars.

Theorem 1. *For given $\epsilon^* > 0$, suppose that for all $0 < \epsilon < \epsilon^*$ there is $V_\epsilon \in C^1\Big([1,\infty) \times \mathbb{R}^n, \mathbb{R}\Big)$ such that*

$$\begin{cases} a_1(\epsilon)\|x\|^2 \leq V_\epsilon(\zeta, x) \leq a_2(\epsilon)\|x\|^2 + r_1(\epsilon) \\ {}^{C_H}D_1^\delta V_\epsilon(\zeta, x_\epsilon(\zeta)) \leq -a_3(\epsilon)\|x_\epsilon(\zeta)\|^2 + \mu(\zeta)r_2(\epsilon) \end{cases} \tag{15}$$

where $\mu \in \mathcal{C}\Big([1,+\infty), \mathbb{R}_+\Big)$ and the positive scalars $a_\iota(\epsilon), r_N(\epsilon)$ $(\iota = 1,2,3; N = 1,2)$, satisfying the following conditions:

- $\forall\epsilon \in]0,\epsilon^*]$, $\frac{a_3(\epsilon)}{a_2(\epsilon)} \geq \lambda,\ \ 0 < \frac{a_2(\epsilon)}{a_1(\epsilon)} \leq K$ *where* $\lambda, K > 0$.
- $\zeta \to \int_1^\zeta (\log(\zeta) - \log(s))^{\delta-1} E_{\delta,\delta}(-\lambda(\log(\zeta) - \log(s))^\delta) \frac{\mu(s)}{s} ds$ *is a bounded function.*
- $\lim_{\epsilon\to 0} c(\epsilon) = 0$ *where* $c(\epsilon) = r_1(\epsilon)\frac{a_2(\epsilon)+(\mathcal{M}_1+\mathcal{M}_2)a_3(\epsilon)}{a_1(\epsilon)a_2(\epsilon)} + r_2(\epsilon)\frac{(\mathcal{M}_1+\mathcal{M}_2)}{a_1(\epsilon)}$ *in which*

$$\mathcal{M}_1 = \sup_{s\geq 1}\Big((\log(s))^\delta E_{\delta,\delta+1}\big(-\lambda(\log(s))^\delta\big)\Big)$$

and

$$\mathcal{M}_2 = \int_1^\zeta (\log(\zeta) - \log(s))^{\delta-1} E_{\delta,\delta}(-\lambda(\log(\zeta) - \log(s))^\delta) \frac{\mu(s)}{s} ds.$$

then, the system (13) is ϵ^*−Practically Generalized Mittag-Leffler stable.

Proof. We obtain from (15)

$$\begin{aligned} {}^{C_H}D_1^\delta V_\epsilon(\zeta, x_\epsilon(\zeta)) &\leq -\frac{a_3(\epsilon)}{a_2(\epsilon)} V_\epsilon(\zeta, x_\epsilon(\zeta)) + \rho(\zeta)l(\epsilon) \\ &\leq -\lambda V_\epsilon(t, x_\epsilon(t)) + \rho(\zeta)l(\epsilon),\ \forall\zeta \geq 1, \end{aligned} \tag{16}$$

where $\rho(\zeta) = \big(\mu(\zeta) + 1\big)$ and $l(\epsilon) = r_2(\epsilon) + \frac{r_1(\epsilon)a_3(\epsilon)}{a_2(\epsilon)}$.

Let consider the function $h(\zeta)$ given by

$$h(\zeta) = {}^{C_H}D_1^\delta V_\epsilon(\zeta, x_\epsilon(\zeta)) + \lambda V_\epsilon(\zeta, x_\epsilon(\zeta)). \tag{17}$$

Therefore,

$$\begin{aligned} V_\epsilon(\zeta, x_\epsilon(\zeta)) \ &= \ E_\delta\big(-\lambda(\log(\zeta))^\delta\big) V_\epsilon(1, x_\epsilon(1)) \\ &\quad + \int_1^\zeta (\log(\zeta) - \log(s))^{\delta-1} E_{\delta,\delta}(-\lambda(\log(\zeta) - \log(s))^\delta) \frac{h(s)}{s} ds, \end{aligned} \tag{18}$$

then,

$$\begin{aligned} V_\epsilon(\zeta, x_\epsilon(\zeta)) \ &\leq \ E_\delta\big(-\lambda(\log(\zeta))^\delta\big) V_\epsilon(1, x_\epsilon(1)) \\ &\quad + l(\epsilon) \int_1^\zeta (\log(\zeta) - \log(s))^{\delta-1} E_{\delta,\delta}(-\lambda(\log(\zeta) - \log(s))^\delta) \frac{\rho(s)}{s} ds. \end{aligned} \tag{19}$$

Hence,

$$V_\epsilon(\zeta, x_\epsilon(\zeta)) \leq E_\delta\big(-\lambda(\log(\zeta))^\delta\big) V_\epsilon(1, x_\epsilon(1)) + Ml(\epsilon),\ \forall\ \zeta \geq 1, \tag{20}$$

where $M = \mathcal{M}_1 + \mathcal{M}_2$.

By (15), we have:

$$\|x_\epsilon(\zeta)\|^2 \leq \frac{1}{a_1(\epsilon)} E_\delta\big(-\lambda(\log(\zeta))^\delta\big)\big(a_2(\epsilon)\|x_\epsilon(1)\|^2 + r_1(\epsilon)\big) + \frac{Ml(\epsilon)}{a_1(\epsilon)}\ , \forall\, \zeta \geq 1. \quad (21)$$

Since $E_\delta\big(-\lambda m^\delta\big) \leq 1$, $\forall m \geq 0$, so

$$\|x_\epsilon(\zeta)\|^2 \leq \frac{a_2(\epsilon)}{a_1(\epsilon)} E_\delta\big(-\lambda(\log(\zeta))^\delta\big)\|x_\epsilon(1)\|^2 + c(\epsilon)\ , \forall\, \zeta \geq 1 \quad (22)$$

Therefore,

$$\|x_\epsilon(\zeta)\| \leq \Big[\frac{a_2(\epsilon)}{a_1(\epsilon)} E_\delta\big(-\lambda(\log(\zeta))^\delta\big)\|x_\epsilon(1)\|^2\Big]^{\frac{1}{2}} + r(\epsilon)\ , \forall\, \zeta \geq 1, \quad (23)$$

with $r(\epsilon) = \sqrt{c(\epsilon)}$. Hence, the system (13) is ϵ^*-Practically Generalized Mittag-Leffler stable. $\square$

4. Observer Design for FORPSs

Developing the following observer in the form of a polynomial fuzzy model for CHFOSs: Model rule $s (s \in \mathbb{S} = \{1, 2, \cdots, f\})$: If $N_1(\zeta)$ is Q_{s1} and $\cdots$ and $N_p(\zeta)$ is Q_{sp} then

$$\begin{cases} {}^{CH}D_1^\delta \hat{x}(\zeta) & = A_s(y(\zeta))\hat{x}(\zeta) + B_s(y(\zeta))u(\zeta) + f_s(\zeta, \hat{x}(\zeta), \epsilon) \\ & +L_s(y(\zeta))(y(\zeta) - \hat{y}(\zeta)), \quad \zeta \geq 1, \\ \hat{y}(\zeta) & = C\hat{x}(\zeta), \end{cases} \quad (24)$$

where $\hat{x}(\zeta)$ and $\hat{y}(\zeta)$ are the estimates of $x(\zeta)$ and $y(\zeta)$, respectively. $L_s(y(\zeta)$ are the polynomial observer gains.

Remark 6. *The polynomial observer (36) includes the function $f_s(\zeta, x(\zeta), \epsilon)$, which depends on the parameter ϵ. This dependency introduces complexity to the design conditions. Furthermore, the use of the Caputo–Hadamard derivative, which has distinct characteristics compared to the Caputo derivative, further complicates the design by introducing different memory effects.*

The overall observer can then be expressed through fuzzy blending as follows:

$$\begin{aligned} {}^{CH}D_1^\delta \hat{x}(\zeta) &= \sum_{s=1}^{f} \beta_s(N(\zeta))\Big(A_s(y(\zeta))\hat{x}(\zeta) + B_s(y(\zeta))u(\zeta) \\ &+ f_s(\zeta, \hat{x}(\zeta), \epsilon) + L_s(y(\zeta))(y(\zeta) - \hat{y}(\zeta))\Big). \end{aligned} \quad (25)$$

Let $\tilde{x}(\zeta) = x(\zeta) - \hat{x}(\zeta)$ and $\tilde{f}_s(\zeta, \hat{x}(\zeta), \epsilon) = f_s(\zeta, \hat{x}(\zeta), \epsilon) - \hat{f}_s(\zeta, \hat{x}(\zeta), \epsilon)$, we obtain

$${}^{C_H}D_1^\delta \tilde{x}(\zeta) = \sum_{s=1}^{f} \beta_s(N(\zeta))\Big(\big(A_s(y(\zeta)) - L_s(y(\zeta))C\big)\tilde{x}(\zeta) + \tilde{f}_s(\zeta, \hat{x}(\zeta), \epsilon)\Big). \quad (26)$$

Theorem 2. *For given scalar $\eta > 0$, the error $\tilde{x}(\zeta)$ in (26) is ϵ^*-Practically Generalized Mittag-Leffler stable if there are $Q = Q^T > 0$ and $\mathcal{Y}_s(y(\zeta)) \in \mathcal{F}_{y(\zeta)}$ such that the following conditions are satisfied:*

$$-e^T\big(\mathcal{Q}A_s(y(\zeta)) - \mathcal{Y}_s(y(\zeta))C + A_s(y(\zeta))^T\mathcal{Q} - C^T\mathcal{Y}_s^T(y(\zeta)) + \eta I\big)e \in \mathcal{SOS}\int, \quad (27)$$

where e is a vector independent of $y(\zeta)$.

$$\zeta \to \int_1^\zeta (\log(\zeta) - \log(s))^{\delta-1} E_{\delta,\delta}\big(-\frac{\eta}{2\|\mathcal{Q}\|}(\log(\zeta) - \log(s))^\delta\big)\frac{\mu^2(s)}{s}ds \quad (28)$$

is bounded.

In this case, the polynomial gains $L_s(y(\zeta))$ are given as follows: $L_s(y(\zeta)) = \mathcal{Q}^{-1}\mathcal{Y}_s(y(\zeta))$.

Proof. We select the following Lyapunov functional candidate:

$$\mathcal{V}(\tilde{x}(\zeta)) = \tilde{x}(\zeta)^T \mathcal{Q}\tilde{x}(\zeta), \tag{29}$$

Based on Lemma 1 and Equation (26), we obtain

$$\begin{aligned} {}^{C_H}D_1^{\delta}\mathcal{V}(\xi) &\leq 2\,{}^{C_H}D_1^{\delta}\tilde{x}(\zeta)^T \mathcal{Q}\tilde{x}(\zeta) \\ &= \sum_{s=1}^{f} \beta_s(N(\zeta))\{\tilde{x}(\zeta)^T \Omega_s(y(\zeta))\tilde{x}(\zeta) + 2\tilde{x}(\zeta)^T \mathcal{Q}\tilde{f}_s(\zeta,\hat{x}(\zeta),\epsilon)\} \end{aligned} \tag{30}$$

where: $\Omega_s(y(\zeta)) = \mathcal{Q}A_s(y(\zeta)) - \mathcal{Y}_s(y(\zeta))C + A_s(y(\zeta))^T\mathcal{Q} - C^T\mathcal{Y}_s(y(\zeta))^T$.

According to (8), we obtain

$$2\tilde{x}(\zeta)^T \mathcal{Q}\tilde{f}_s(\zeta,\hat{x}(\zeta),\epsilon) \quad \leq \quad 2\|\tilde{x}(\zeta)\| \times \|\mathcal{Q}\| \left(\delta_1(\epsilon)\|\tilde{x}(\zeta)\| + \delta_2(\epsilon)\mu(\zeta)\right). \tag{31}$$

By letting η_1 such that $0 < \eta_1 < \frac{\eta}{4}$, we obtain

$$2\|\tilde{x}(\zeta)\| \times \|\mathcal{Q}\|\delta_2(\epsilon)\mu(\zeta) \leq \eta_1\|\tilde{x}(\zeta)\|^2 + \frac{\left(\|\mathcal{Q}\|\delta_2(\epsilon)\mu(\zeta)\right)^2}{\eta_1}. \tag{32}$$

Therefore,

$$2\tilde{x}(\zeta)^T \mathcal{Q}\tilde{f}_s(\zeta,\hat{x}(\zeta),\epsilon) \quad \leq \quad \left(\eta_1 + 2\delta_1(\epsilon)\|\mathcal{Q}\|\right)\|\tilde{x}(\zeta)\|^2 + \frac{\left(\|\mathcal{Q}\|\delta_2(\epsilon)\mu(\zeta)\right)^2}{\eta_1}. \tag{33}$$

Since $\lim_{\epsilon\to 0}\delta_1(\epsilon) = 0$, there is $\epsilon^* > 0$ such that $\forall\epsilon \in (0,\epsilon^*]$, $2\delta_1(\epsilon)\|\mathcal{Q}\| < \frac{\eta}{4}$; then, we obtain

$$2\tilde{x}(\zeta)^T \mathcal{Q}\tilde{f}_s(\zeta,\hat{x}(\zeta),\epsilon) \quad \leq \quad \frac{\eta}{2}\|\tilde{x}(\zeta)\|^2 + \frac{\left(\|Q\|\delta_2(\epsilon)\mu(\zeta)\right)^2}{\eta_1}. \tag{34}$$

Taking into account the condition (27), we obtain

$${}^{C_H}D_1^{\delta}\mathcal{V}(\xi) \quad \leq -\frac{\eta}{2}\|\tilde{x}(\zeta)\|^2 + \frac{\left(\|Q\|\delta_2(\epsilon)\mu(\zeta)\right)^2}{\eta_1}. \tag{35}$$

It follows from Theorem 1 that the error system is ϵ^*-Practically Generalized Mittag-Leffler stable. □

We propose the Algorithm 1 for Theorem 1.

Algorithm 1 Steps for Solving Theorem 1

1: Solve the SOS conditions (27) using SOSTOOLS for the known system matrices $A_s(y(\zeta))$ and C.
2: Ensure condition (28) for $f_s(\zeta,x(\zeta),\epsilon)$.
3: Compute $x(\zeta)$ and $\hat{x}(\zeta)$ using the polynomial gains $L_s(y(\zeta))$ from Step 1 and the system's initial conditions.

In the particular case of CHFOTSS, the polynomial observer (36) reduces to the following form: Model rule $s(s \in \mathbb{S} = \{1,2,\cdots,f\})$: If $N_1(\zeta)$ is Q_{s1} and $\cdots$ and $N_p(\zeta)$ is Q_{sp} then

$$\begin{cases} {}^{CH}D_1^{\delta}\hat{x}(\zeta) & = A_s\hat{x}(\zeta) + B_s u(\zeta) + f_s(\zeta, \hat{x}(\zeta), \epsilon) \\ & +L_s(y(\zeta) - \hat{y}(\zeta)), \quad \zeta \geq 1, \\ \hat{y}(\zeta) & = C\hat{x}(\zeta), \end{cases} \tag{36}$$

where L_s are constant instead of polynomial matrices. Therefore, we obtain:

$${}^{CH}D_1^{\delta}\tilde{x}(\zeta) = \sum_{s=1}^{f} \beta_s(N(\zeta))\Big((A_s - L_s C)\tilde{x}(\zeta) + \tilde{f}_s(\zeta, \hat{x}(\zeta), \epsilon)\Big). \tag{37}$$

Consequently, we obtain the following Corollary:

Corollary 1. *For given scalar $\eta > 0$, the error $\tilde{x}(\zeta)$ in (37) is ϵ^*-Practically Generalized Mittag-Leffler stable if there are matrices $\mathcal{Q} = \mathcal{Q}^T > 0, \mathcal{Y}_s$ such that conditions (28) and the following LMIs are satisfied:*

$$\mathcal{Q}A_s - \mathcal{Y}_s C + A_s^T \mathcal{Q} - C^T \mathcal{Y}_s^T + \eta I < 0 \tag{38}$$

In this case, the gains L_s are given as follows: $L_s = \mathcal{Q}^{-1}\mathcal{Y}_s$.

5. Illustrative Example

Consider the following Caputo–Hadamard Fractional-Order NS:

$$\begin{cases} {}^{CH}D_1^{\delta}x_1(\zeta) = \sin(x_1(\zeta)) - 0.5x_1(\zeta)x_2(\zeta) + u(\zeta) + \epsilon^2 \frac{\zeta^2}{\zeta^4+1}\sin(x_2(\zeta)), \\ {}^{CH}D_1^{\delta}x_2(\zeta) = -1.5x_1^2(\zeta) - 2x_2(\zeta) - x_1^2(\zeta)x_2(\zeta) + u(\zeta) + \epsilon^2 \frac{\zeta^2}{\zeta^4+1}\sin(x_2(\zeta)), \ \zeta \geq 1, \end{cases} \tag{39}$$

where $\delta = 0.9$ and $u(\zeta)$ is defined as

$$u(\zeta) = \begin{cases} 7\sin(20\pi\zeta), & 1 \leq \zeta < 3, \\ 9(\zeta - 3), & 3 \leq \zeta \leq 4. \end{cases}$$

First, we develop a Takagi-Sugeno fuzzy model that can precisely characterize the behavior of the NSs (39). In order to achieve this, we suppose that the premise variables are:

$$N_1(\zeta) = \frac{\sin(x_1(\zeta))}{x_1(\zeta)}, \quad N_2(\zeta) = -0.5x_1(\zeta), \quad N_3(\zeta) = -2 - x_1^2(\zeta),$$

We have $max(N_1(\zeta)) = 1$ and $min(N_1(\zeta)) = -0.2172, \forall x_1(\zeta)$. To obtain the maximum and the minimum of $N_2(\zeta)$ and $N_3(\zeta)$, the state $x_1(\zeta)$ is restricted to be bounded. Therefore, we assume that $|x_1(\zeta)| < \overline{m}_1$. Then, we can obtain

$$\begin{aligned} \overline{N}_2 &= max(N_2(\zeta)) = 0.5\overline{m}_1, \quad \overline{N}_3 = max(N_3(\zeta)) = -2, \\ \underline{N}_2 &= min(N_2(\zeta)) = -0.5\overline{m}_1, \quad \underline{N}_3 = min(N_3(\zeta)) = -2 - \overline{m}_1^2. \end{aligned}$$

Based on the concept of sector nonlinearity, we establish the following CHFOTSS:

$${}^{CH}D_1^{\delta}x(\zeta) = \sum_{s=1}^{8} \beta_s(N(\zeta))\{A_s x(\zeta) + Bu(\zeta) + f(\zeta, x(\zeta), \epsilon)\}, \tag{40}$$

where

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 1 & \overline{N}_2 \\ -1.5 & \overline{N}_3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & \overline{N}_2 \\ -1.5 & \underline{N}_3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & \underline{N}_2 \\ -1.5 & \underline{N}_3 \end{bmatrix}, \\
A_4 &= \begin{bmatrix} -0.2172 & \overline{N}_2 \\ -1.5 & \overline{N}_3 \end{bmatrix}, A_5 = \begin{bmatrix} -0.2172 & \overline{N}_2 \\ -1.5 & \underline{N}_3 \end{bmatrix}, A_6 = \begin{bmatrix} -0.2172 & \underline{N}_2 \\ -1.5 & \underline{N}_3 \end{bmatrix}, \\
A_7 &= \begin{bmatrix} 1 & \underline{N}_2 \\ -1.5 & \overline{N}_3 \end{bmatrix}, A_8 = \begin{bmatrix} -0.2172 & \underline{N}_2 \\ -1.5 & \overline{N}_3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\beta_1(N(\zeta)) &= \overline{\kappa}_1 \times \overline{\kappa}_2 \times \overline{\kappa}_3, \quad \beta_2(N(\zeta)) = \overline{\kappa}_1 \times \overline{\kappa}_2 \times \underline{\kappa}_3, \quad \beta_3(N(\zeta)) = \overline{\kappa}_1 \times \underline{\kappa}_2 \times \underline{\kappa}_3, \\
\beta_4(N(\zeta)) &= \underline{\kappa}_1 \times \overline{\kappa}_2 \times \overline{\kappa}_3, \quad \beta_5(N(\zeta)) = \underline{\kappa}_1 \times \overline{\kappa}_2 \times \underline{\kappa}_3, \quad \beta_6(N(\zeta)) = \underline{\kappa}_1 \times \underline{\kappa}_2 \times \underline{\kappa}_3, \\
\beta_7(N(\zeta)) &= \overline{\kappa}_1 \times \underline{\kappa}_2 \times \overline{\kappa}_3, \quad \beta_8(N(\zeta)) = \underline{\kappa}_1 \times \underline{\kappa}_2 \times \overline{\kappa}_3, f(\zeta, x(\zeta), \epsilon) = \begin{bmatrix} \epsilon^2 \frac{\zeta^2}{\zeta^4+1} \sin(x_2) \\ \epsilon^4 \cos(x_1) \end{bmatrix},
\end{aligned}
$$

in which

$$
\begin{aligned}
&\overline{\kappa}_1 = \frac{x_1(\zeta) - \sin(x_1(\zeta))}{1.2172 x_1(\zeta)}, \quad \underline{\kappa}_1 = \frac{\sin(x_1(\zeta)) + 0.2172 x_1(\zeta)}{1.2172 x_1(\zeta)}, \\
&\overline{\kappa}_2 = \frac{\overline{N}_2 - N_2(\zeta)}{\overline{N}_2 - \underline{N}_2}, \quad \underline{\kappa}_2 = \frac{N_2(\zeta) - \underline{N}_2}{\overline{N}_2 - \underline{N}_2}, \quad , \quad \overline{\kappa}_3 = \frac{\overline{N}_3 - N_3(\zeta)}{\overline{N}_3 - \underline{N}_3}, \quad \underline{\kappa}_3 = \frac{N_3(\zeta) - \underline{N}_3}{\overline{N}_3 - \underline{N}_3}.
\end{aligned}
$$

By taking into account only the premise variable $N_1(\zeta)$, the NSs (39) could be described by the following CHFORPSs:

$$
{}^{CH}D_1^{\delta} x(\zeta) = \sum_{s=1}^{2} \beta_s(N(\zeta)) \{ A_s(x(\zeta)) x(\zeta) + Bu(\zeta) + f(\zeta, x(\zeta), \epsilon) \}, \tag{41}
$$

$$
\begin{aligned}
A_1(x(\zeta)) &= \begin{bmatrix} 1 & -0.5 x_1(\zeta) \\ -1.5 & -2 - x_1^2(\zeta) \end{bmatrix}, A_2(x_1(\zeta)) = \begin{bmatrix} -0.2172 & -0.5 x_1(\zeta) \\ -1.5 & -2 - x_1^2(\zeta) \end{bmatrix} \\
\beta_1(N(\zeta)) &= \frac{\sin(x_1(\zeta)) + 0.2172 x_1(\zeta)}{1.2172 x_1(\zeta)}, \quad \beta_2(N(\zeta)) = \frac{x_1(\zeta) - \sin(x_1(\zeta))}{1.2172 x_1(\zeta)}
\end{aligned}
$$

Table 1 highlights two main advantages of the CHFORPS model over the CHFOTSS model. The first one is the reduction of rules and consequently the system description probably with fewer computational costs. The second one is the system validity. In fact, CHFOTSS is only valid in a limited domain where $x_1(\zeta) \in [-\overline{m}_1, \overline{m}_1]$. However, CHFORPSs apply for $x_1(\zeta) \in (-\infty, +\infty)$, signifying they can handle a larger range of values for $x_1(\zeta)$.

Table 1. Comparison between CHFOTSS and CHFORPSs.

	Number of Rules	System's Validity Domain
CHFOTSS	8	$x_1(\zeta) \in [-\overline{m}_1, \overline{m}_1]$
CHFORPSs	2	$x_1(\zeta) \in (-\infty, +\infty)$

We assume that $x_1(\zeta)$ is measurable. Then, the output equation is

$$
y(\zeta) = Cx(\zeta) \tag{42}
$$

where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Since $y(\zeta) = x_1(\zeta)$, then $A_s(x(\zeta))$ and $N(\zeta)$ are measurable. Furthermore, $f(\zeta, x(\zeta), \epsilon)$ satisfies condition (8) for $\delta_1(\epsilon) = \epsilon^2$, $\delta_2(\epsilon) = \epsilon^4$, and $\mu(\zeta) = 2$.

By solving the SOS conditions in Theorem 1, we obtain

$$
\begin{aligned}
Q &= \begin{bmatrix} 0.5173 & 0.02601 \\ 0.02601 & 0.3608 \end{bmatrix}, \\
L_1(y(\zeta)) &= \begin{bmatrix} 0.4325y^2(\zeta) + 0.868 - 0.02217y^2(\zeta) - 0.4066 \end{bmatrix}, \\
L_2(y(\zeta)) &= \begin{bmatrix} 0.432y^2(\zeta) + 0.3098 - 0.02225y^2(\zeta) - 0.4095 \end{bmatrix}.
\end{aligned}
$$

Table 2 demonstrates the advantage of the SOS approach compared to the LMI approach. Actually, the first one gives an unbounded domain of feasibility in which $m_1 = +\infty$, while the second one provides a limited domain in which $m_1 = +\infty$.

Table 2. Comparison between LMI approach and SOS approach.

	Domain of Feasibility
LMI approach	$\overline{m}_1 = 4$
SOS approach	$\overline{m}_1 = +\infty$

For the simulations, we consider $\epsilon = 10^{-4}$; then, $2\delta_1(\epsilon)\|Q\| = 7.4283 \times 10^{-7} < \frac{\eta}{4} = \frac{10^{-3}}{4}$. Figure 1 shows $x(\zeta)$ and $\hat{x}(\zeta)$ for the initial conditions $x(\zeta) = \begin{bmatrix} -1 & 4 \end{bmatrix}^T$ and $\hat{x}(\zeta) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, while Figure 2 shows them for the initial conditions $x(\zeta) = \begin{bmatrix} -9 & -2 \end{bmatrix}^T$ and $\hat{x}(\zeta) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

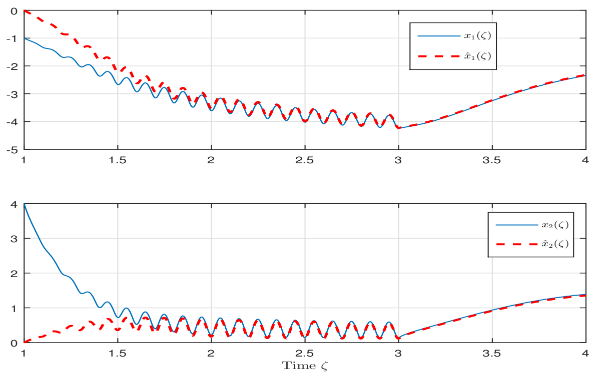

Figure 1. Time evolution of $x(\zeta)$ and $\hat{x}(\zeta)$ for the initial conditions $x(\zeta) = \begin{bmatrix} -1 & 4 \end{bmatrix}^T$ and $\hat{x}(\zeta) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

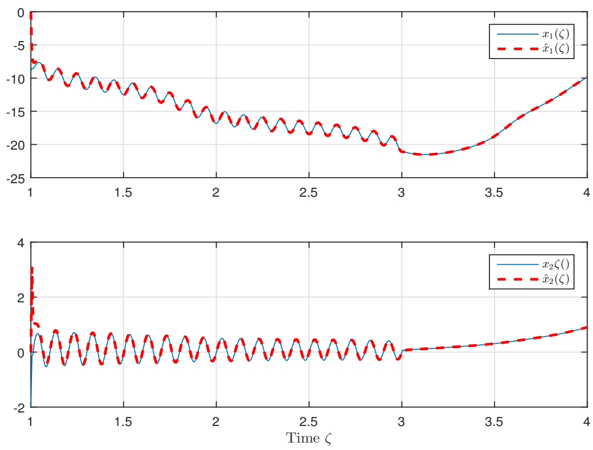

Figure 2. Time evolution of $x(\zeta)$ and $\hat{x}(\zeta)$ for the initial conditions $x(\zeta) = \begin{bmatrix} -9 & -2 \end{bmatrix}^T$ and $\hat{x}(\zeta) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

6. Conclusions

The problem of observer design for FORPSs depending on a parameter is investigated in this work. First, we propose a practical generalized Mittag-Leffler stability analysis of the general case of Caputo–Hadamard Fractional-Order systems depending on a parameter. Building upon this essential analysis, an observer is designed for the considered class of FORPSs. The practical generalized Mittag-Leffler stability of the state estimation error is achieved by solving a set of SOSs using SOSTOOLS.

Author Contributions: H.G.: Methodology, Investigation. M.R.: Conceptualization, Software, Investigation. L.M.: Writing—review and editing, Visualization. A.B.M.: Validation, Writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Saud University through Researchers Supporting Project number (RSPD2024R683), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: No underlying data were collected or produced in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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