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Fixed Point and Stability Analysis of a Tripled System of Nonlinear Fractional Differential Equations with n-Nonlinear Terms

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Abstract: This research article investigates a tripled system of nonlinear fractional differential equations with n terms. The study explores this novel class of differential equations to establish existence and stability results. Utilizing Schaefer's and Banach's fixed point theorems, we derive sufficient conditions for the existence of at least one solution, as well as a unique solution. Furthermore, we apply Hyers–Ulam stability analysis to establish criteria for the stability of the system. To demonstrate the applicability of the main results, a detailed example is provided.

Keywords: tripled system; nonlinear fractional differential equations; n-term equations; fixed point; stability analysis; functional derivative

Citation: Algolam, M.S.; Osman, O.; Ali, A.; Mustafa, A.; Aldwoah, K.; Alsulami, A. Fixed Point and Stability Analysis of a Tripled System of Nonlinear Fractional Differential Equations with n-Nonlinear Terms. *Fractal Fract.* **2024**, *8*, 697. https://doi.org/10.3390/fractalfract8120697

Academic Editor: Wei-Shih Du

Received: 3 October 2024
Revised: 14 November 2024
Accepted: 25 November 2024
Published: 26 November 2024

1. Introduction

In many crucial situations, the behavior of dynamical systems is best described by fractional differential equations (FDEs), as ordinary differential equations (ODEs) may fail to capture these dynamics. Systems of differential equations are referred to as single, coupled, tripled, or m-systems of DEs, depending on the number of DEs involved. Each type of system has numerous applications and is important for mathematically modeling various phenomena. Single and coupled systems of DEs have gained importance in various applied problems, as seen in [1–8]. Numerous studies have investigated these systems, contributing significantly to the literature (see [9–19]). In [20], Taieb and Dahmani studied a coupled system of nonlinear DEs involving n-nonlinear terms, investigating the existence of solutions. On the other hand, tripled systems of DEs are rarely considered. Applications of tripled systems of DEs can be observed in gene regulatory networks, epidemiology, the dynamics of hormones in endocrine systems, food chains involving three species, three-stage life cycles, microbial community dynamics, etc. Recently, Madani et al. [21] investigated a tripled system of NFDEs. Motivated by these applications of tripled DE systems, this article investigates a tripled system of NFDEs with n-nonlinear terms. We establish existence and stability results for this system, which is described by

$$\begin{cases} {}^cD^\alpha x(t) + \sum_{i=1}^{n} f_i(t, x(t), y(t), z(t), {}^cD^\delta x(t), {}^cD^\xi y(t), {}^cD^\sigma z(t)) = 0; \quad t \in [0,1], n \in \mathbb{N}, \\ {}^cD^\beta y(t) + \sum_{i=1}^{n} g_i(t, x(t), y(t), z(t), {}^cD^\delta x(t), {}^cD^\xi y(t), {}^cD^\sigma z(t)) = 0; \quad t \in [0,1], n \in \mathbb{N}, \\ {}^cD^\gamma z(t) + \sum_{i=1}^{n} h_i(t, x(t), y(t), z(t), {}^cD^\delta x(t), {}^cD^\xi y(t), {}^cD^\sigma z(t)) = 0; \quad t \in [0,1], n \in \mathbb{N}, \\ x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \\ x'(0) = x''(0) = y'(0) = y''(0) = z'(0) = z''(0) = 0, \\ x'''(0) = J^p x(\varsigma), \quad y'''(0) = J^q y(\varrho), \quad z'''(0) = J^r z(\tau), \end{cases} \tag{1}$$

where the functions $f_i, g_i, h_i : [0,1] \times \mathbb{R}^6 \to \mathbb{R}$ are continuous for each $i = 1, 2, \dots, n$, ${}^cD^{\cdot}$ represents the Caputo derivative, with fractional orders $\alpha, \beta, \gamma \in (3,4)$, $\delta, \xi, \sigma \in (0,3)$ and $\varsigma, \varrho, \tau \in (0,1)$, J^p, J^q and J^r denote Riemann–Liouville (R-L) integrals, and $x_0, y_0, z_0 \in \mathbb{R}$.

The considered problem contains n-term DEs, which are of great interest in various scientific and engineering fields. In fact our considered problem is the generalization of the coupled system given in [20] to the tripled system of DEs. To the best of our knowledge, such problems have not yet been studied as a tripled system of DEs.

The rest of the paper is organized as follows. In Section 2, preliminary results are given. In Section 3, an auxiliary result is proved. In Section 4, main results about the solution's existence are given. In Section 5, stability results are derived. In Section 6, the derived results are applied to a general problem to validate the results. In Section 7, the conclusion is given.

2. Basic Results

The following definitions and lemmas are recalled from [1,20,22].

Definition 1. *Let $\theta : [0,T] \to \mathbb{R}$ is a continuous function. Then, the fractional-order integral of θ in the Riemann–Liouville sense is defined by*

$$J^\alpha \theta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \theta(s) ds, \quad \alpha > 0, t \geq 0, \tag{2}$$

where $\Gamma(\alpha) = \int_0^\infty e^u u^{\alpha-1} du$.

Definition 2. *Let $\theta : [0,T] \to \mathbb{R}$ is a continuous function. Then, the Caputo fractional-order derivative of θ is given by*

$${}^cD^\alpha \theta(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \theta^{(m)}(s) ds,$$

where $m - 1 < \alpha \leq m$; $m \in \mathbb{N}$.

Definition 3 ([23,24])**.** *The set G is equi-continuous if for any $\epsilon > 0$ there exists $\zeta > 0$ such that if $x \in G$, $n \in \mathbb{N}$, $t_1, t_2 \in (t_{n-1}, t_n]$ and $|t_1 - t_2| < \zeta$, we have $|x(t_1) - x(t_2)| < \epsilon$.*

Definition 4 ([25])**.** *A bounded linear operator T acting from a Banach space X into another space Y is called completely continuous if T maps weakly convergent sequences in X to norm convergent sequences in Y.*

The following lemma is adopted from [1,20,22].

Lemma 1. *If $\beta > \alpha > 0$, and $\theta \in L^1([a,b])$, then:*

$${}^cD^\alpha J^\beta \theta(t) = J^{\beta-\alpha} \theta(t),$$

In addition, ${}^{c}D^{\alpha}\theta(t) = 0$, iff the function θ is constant.

The following lemma is adopted from [1,20,22].

Lemma 2. *Assume that $\alpha > 0$, then for all $\mathbf{c}_i \in \mathbb{R}$, $(i = 1, 2, \ldots, m-1)$, we have*

$$J^{\alpha\, c}D^{\alpha}\theta(t) = \theta(t) + \mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2 t^2 + \cdots + \mathbf{c}_{m-1} t^{m-1}, \tag{3}$$

where $m = [\alpha] + 1$ is the lowest integer; $[\alpha]$ is a floor function which represents the integer part of α.

The following lemma is adopted from [1,20,22].

Lemma 3.

$$J^{\alpha} J^{\beta}\theta(t) = J^{\alpha+\beta}\theta(t), \quad {}^{c}D^{\alpha} J^{\alpha}\theta(t) = \theta(t), \quad t \in [a, b].$$

Theorem 1 ([26])**.** *(Schaefer's fixed point theorem). Let S be a norm-linear space, and let $\mathcal{W}$ be its convex subset with $0 \in \mathcal{W}$. Assume that $\mathcal{N} : \mathcal{W} \to \mathcal{W}$ is a completely continuous operator. Then, either the set*

$$\mathscr{X} = \{\theta \in \mathcal{W} : \theta = \xi\mathcal{N}\theta;\ 0 < \xi < 1\}$$

is unbounded or $\mathcal{N}$ has a fixed point in $\mathcal{W}$.

3. Auxiliary Result

In this section, an auxiliary result is proved, and it is followed by the main results. We proceed with proving a Lemma that follows.

Lemma 4. *Let $\mathcal{F}_i : [0, 1] \to \mathbb{R}$; $(i = 1, 2, \ldots, n)$ be given continuous functions. Then, the problem*

$$\begin{cases} {}^{c}D^{\alpha}w(t) + \sum_{i=1}^{n} \mathcal{F}_i(t) = 0, & t \in [0,1], 3 < \alpha < 4, n \in \mathbb{N}, \\ w(0) = w_0, \quad w'(0) = w''(0) = 0, & w'''(0) = J^{p}w(\varsigma), \quad p > 0, \varsigma \in (0,1), \end{cases} \tag{4}$$

has the solution

$$\begin{aligned} w(t) &\\ = w_0 - \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathcal{F}_i(s)ds + \frac{\Gamma(p+4)t^3}{6(\varsigma^{p+3} - \Gamma(p+4))} \left(\sum_{i=1}^{n} \int_0^{\varsigma} \frac{(\varsigma - s)^{\alpha+p-1}}{\Gamma(\alpha+p)} \mathcal{F}_i(s)ds - \frac{w_0\varsigma^{p}}{\Gamma(p+1)} \right), \end{aligned} \tag{5}$$

where $\varsigma^{p+3} - \Gamma(p+4) \neq 0$.

Proof. We have

$${}^{c}D^{\alpha}w(t) = -\sum_{i=1}^{n} \mathcal{F}_i(t). \tag{6}$$

By applying Lemma 2, we have:

$$w(t) = -\sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathcal{F}_i(s)ds - \mathbf{c}_0 - \mathbf{c}_1 t - \mathbf{c}_2 t^2 - \mathbf{c}_3 t^3, \tag{7}$$

where $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \in \mathbb{R}$. Applying Lemma 3, we have

$$J^{p}w(t) = -\sum_{i=1}^{n} J^{\alpha+p}\mathcal{F}_i(\varsigma) - J^{p}\mathbf{c}_0 - \mathbf{c}_1 J^{p} t - \mathbf{c}_2 J^{p} t^2 - \mathbf{c}_3 J^{p} t^3. \tag{8}$$

Using the given conditions, we obtain $\mathbf{c}_0 = -w_0$, $\mathbf{c}_1 = \mathbf{c}_2 = 0$. Therefore, we have

$$J^p w(t) = -\sum_{i=1}^{n} J^{\alpha+p}\mathcal{F}_i(\varsigma) - J^p w_0 - \mathbf{c}_3 J^p t^3. \tag{9}$$

Using $w'''(0) = J^p w(\varsigma)$, we obtain

$$\mathbf{c}_3 = \frac{\Gamma(p+4)}{6(\varsigma^{p+3} - \Gamma(p+4))}\left(\frac{w_0\varsigma^p}{\Gamma(p+1)} - \sum_{i=1}^{n}\int_0^{\varsigma}\frac{(\varsigma-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}\mathcal{F}_i(s)ds\right). \tag{10}$$

Putting the values of $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$, we obtain (5). □

Corollary 1. *Let $\Big(x(t), y(t), z(t)\Big)$ represent the solution to the tripled system of DEs (1); then, by Lemma 4, we have*

$$\begin{cases} x(t) = x_0 - \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(t, x(t), y(t), z(t), {}^cD^{\delta}x(t), {}^cD^{\xi}y(t), {}^cD^{\sigma}z(t))ds \\ + \frac{\Gamma(p+4)t^3}{6(\varsigma^{p+3} - \Gamma(p+4))}\left(\sum_{i=1}^{n}\int_0^{\varsigma}\frac{(\varsigma-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} f_i(t, x(t), y(t), z(t), {}^cD^{\delta}x(t), {}^cD^{\xi}y(t), {}^cD^{\sigma}z(t))ds - \frac{x_0\varsigma^p}{\Gamma(p+1)}\right), \\ y(t) = y_0 - \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_i(t, x(t), y(t), z(t), {}^cD^{\delta}x(t), {}^cD^{\xi}y(t), {}^cD^{\sigma}z(t))ds \\ + \frac{\Gamma(q+4)t^3}{6(\varrho^{q+3} - \Gamma(q+4))}\left(\sum_{i=1}^{n}\int_0^{\varrho}\frac{(\varrho-s)^{\beta+q-1}}{\Gamma(\beta+q)} g_i(t, x(t), y(t), z(t), {}^cD^{\delta}x(t), {}^cD^{\xi}y(t), {}^cD^{\sigma}z(t))ds - \frac{y_0\varrho^q}{\Gamma(q+1)}\right), \\ z(t) = z_0 - \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} h_i(t, x(t), y(t), z(t), {}^cD^{\delta}x(t), {}^cD^{\xi}y(t), {}^cD^{\sigma}z(t))ds \\ + \frac{\Gamma(r+4)t^3}{6(\tau^{r+3} - \Gamma(r+4))}\left(\sum_{i=1}^{n}\int_0^{\tau}\frac{(\tau-s)^{\gamma+r-1}}{\Gamma(\gamma+r)} h_i(t, x(t), y(t), z(t), {}^cD^{\delta}x(t), {}^cD^{\xi}y(t), {}^cD^{\sigma}z(t))ds - \frac{z_0\tau^r}{\Gamma(r+1)}\right), \end{cases} \tag{11}$$

where $\varsigma^{p+3} - \Gamma(p+4) \neq 0$, $\varrho^{q+3} - \Gamma(q+4) \neq 0$, $\tau^{r+3} - \Gamma(r+4) \neq 0$.

Now, we introduce the space

$$V := \Big\{(x, y, z) : x, y, z \in C([0,1], \mathbb{R}), \quad {}^cD^{\delta}x, {}^cD^{\xi}y, {}^cD^{\sigma}z \in C([0,1], \mathbb{R})\Big\} \tag{12}$$

with the norm

$$\|(x, y, z)\|_V = \max(\|x\|, \|y\|, \|z\|, \|{}^cD^{\delta}x\|, \|{}^cD^{\xi}y\|, \|{}^cD^{\sigma}z\|), \tag{13}$$

where

$$\|x\| = \sup_{t\in[0,1]}|x(t)|, \quad \|y\| = \sup_{t\in[0,1]}|y(t)|, \quad \|z\| = \sup_{t\in[0,1]}|z(t)|, \quad \|{}^cD^{\delta}x\| = \sup_{t\in[0,1]}|{}^cD^{\delta}x(t)|,$$
$$\|{}^cD^{\xi}y\| = \sup_{t\in[0,1]}|{}^cD^{\xi}y(t)|, \quad \|{}^cD^{\sigma}z\| = \sup_{t\in[0,1]}|{}^cD^{\sigma}z(t)|.$$

Then, $(V, \|.\|_V)$ is a Banach space.

4. Main Results

In this section, we give our main results regarding the solution's existence. For the analysis of the main results, we need to impose the following assumptions:

Hypothesis 1. *Assume that the functions $f_i, g_i, h_i : [0,1] \times \mathbb{R}^6 \to \mathbb{R}$ are continuous for each $i = 1, 2, \ldots, n$.*

Hypothesis 2. *Assume that there exist non-negative and continuous functions* $\phi_j^i, \varphi_j^i, \vartheta_j^i$ *with* $j = 1,2,\ldots,6$ *and* $i = 1,2,\ldots,n$ *such that for any* $t \in [0,1]$, *and every* $(v_1, v_2, v_3, v_4, v_5, v_6)$, $(w_1, w_2, w_3, w_4, w_5, w_6) \in \mathbb{R}^6$, *the following relation holds*

$$|f_i(t, v_1, v_2, v_3, v_4, v_5, v_6) - f_i(t, w_1, w_2, w_3, w_4, w_5, w_6)| \leq \sum_{j=1}^{6} \phi_j^i(t)|v_j - w_j|,$$

$$|g_i(t, v_1, v_2, v_3, v_4, v_5, v_6) - g_i(t, w_1, w_2, w_3, w_4, w_5, w_6)| \leq \sum_{j=1}^{6} \varphi_j^i(t)|v_j - w_j|$$

$$|h_i(t, v_1, v_2, v_3, v_4, v_5, v_6) - h_i(t, w_1, w_2, w_3, w_4, w_5, w_6)| \leq \sum_{j=1}^{6} \vartheta_j^i(t)|v_j - w_j|,$$

with $\mu_j^i = \sup_{t\in[0,1]}\phi_j^i(t)$, $\rho_j^i = \sup_{t\in[0,1]}\varphi_j^i(t)$, $\eta_j^i = \sup_{t\in[0,1]}\vartheta_j^i(t)$ *for* $j = 1,2,\ldots,6$, *and* $i = 1,2,\ldots,n$.

Hypothesis 3. *Assume that there exist non-negative functions* $\zeta_i(t), \xi_i(t), \gamma_i(t) \in C([0,1],\mathbb{R})$ *such that*

$$|f_i(t, v_1, v_2, v_3, v_4, v_5, v_6)| \leq \zeta_i(t), \quad |g_i(t, v_1, v_2, v_3, v_4, v_5, v_6)| \leq \xi_i(t), \quad |h_i(t, v_1, v_2, v_3, v_4, v_5, v_6)| \leq \gamma_i(t),$$

with $L_i = \sup_{t\in[0,1]}\zeta_i(t)$, $M_i = \sup_{t\in[0,1]}\xi_i(t)$, $N_i = \sup_{t\in[0,1]}\gamma_i(t)$, $\quad i = 1,2,\ldots,n$.

We introduce the notions assuming that the denominators are different from zero:

$$\mathbf{C}_1 := \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3} - \Gamma(p+4)|\Gamma(\alpha+p+1)}$$
$$\mathbf{C}_2 := \frac{1}{\Gamma(\beta+1)} + \frac{\Gamma(q+4)\varrho^{\beta+q}}{6|\varrho^{q+3} - \Gamma(q+4)|\Gamma(\beta+q+1)}$$
$$\mathbf{C}_3 := \frac{1}{\Gamma(\gamma+1)} + \frac{\Gamma(r+4)\tau^{\gamma+r}}{6|\tau^{r+3} - \Gamma(r+4)|\Gamma(\gamma+r+1)},$$
$$\mathbf{C}_4 := \frac{1}{\Gamma(\alpha-\delta+1)} + \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3} - \Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)},$$
$$\mathbf{C}_5 := \frac{1}{\Gamma(\beta-\xi+1)} + \frac{\Gamma(q+4)\varrho^{\beta+q}}{2|\varrho^{q+3} - \Gamma(q+4)|\Gamma(4-\xi)\Gamma(\beta+q+1)},$$
$$\mathbf{C}_6 := \frac{1}{\Gamma(\gamma-\sigma+1)} + \frac{\Gamma(r+4)\tau^{\gamma+r}}{2|\tau^{r+3} - \Gamma(r+4)|\Gamma(4-\sigma)\Gamma(\gamma+r+1)},$$
$$S_1 := \sum_{i=1}^{n}(\mu_1^i + \mu_2^i + \mu_3^i + \mu_4^i + \mu_5^i + \mu_6^i), \quad S_2 := \sum_{i=1}^{n}(\rho_1^i + \rho_2^i + \rho_3^i + \rho_4^i + \rho_5^i + \rho_6^i),$$
$$S_3 := \sum_{i=1}^{n}(\eta_1^i + \eta_2^i + \eta_3^i + \eta_4^i + \eta_5^i + \eta_6^i),$$
$$Q_1 := |x_0| + \frac{(p+3)(p+2)(p+1)|x_0|\varsigma^p}{6|\varsigma^{p+3} - \Gamma(p+4)|}, \quad Q_2 := |y_0| + \frac{(q+3)(q+2)(q+1)|y_0|\varrho^q}{6|\varrho^{q+3} - \Gamma(q+4)|},$$
$$Q_3 := |z_0| + \frac{(r+3)(r+2)(r+1)|z_0|\tau^r}{6|\tau^{r+3} - \Gamma(r+4)|}, \quad Q_4 := |x_0| + \frac{(p+3)(p+2)(p+1)|x_0|\varsigma^p}{2|\varsigma^{p+3} - \Gamma(p+4)|\Gamma(4-\delta)},$$
$$Q_5 := |y_0| + \frac{(q+3)(q+2)(q+1)|y_0|\varrho^q}{2|\varrho^{q+3} - \Gamma(q+4)|\Gamma(4-\xi)}, \quad Q_6 := |z_0| + \frac{(r+3)(r+2)(r+1)|z_0|\tau^r}{2|\tau^{r+3} - \Gamma(r+4)|\Gamma(4-\eta)}.$$

Before going to the fixed point results, we define the integral operator $Z : V \to V$ by

$$Z(x,y,z)(t) := (Z_1(x,y,z)(t), Z_2(x,y,z)(t), Z_3(x,y,z)(t)), \quad t \in [0,1], \tag{14}$$

such that

$$Z_1(x,y,z)(t) = x_0 - \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(t,x(t),y(t),z(t),^c D^{\delta}x(t),^c D^{\xi}y(t),^c D^{\sigma}z(t))ds$$
$$+ \frac{\Gamma(p+4)t^3}{6(\varsigma^{p+3}-\Gamma(p+4))}\left(\sum_{i=1}^{n}\int_0^{\varsigma} \frac{(\varsigma-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} f_i(t,x(t),y(t),z(t),^c D^{\delta}x(t),^c D^{\xi}y(t),^c D^{\sigma}z(t))ds - \frac{x_0\varsigma^p}{\Gamma(p+1)}\right),$$

$$Z_2(x,y,z)(t) = y_0 - \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_i(t,x(t),y(t),z(t),^c D^{\delta}x(t),^c D^{\xi}y(t),^c D^{\sigma}z(t))ds$$
$$+ \frac{\Gamma(q+4)t^3}{6(\varrho^{q+3}-\Gamma(q+4))}\left(\sum_{i=1}^{n}\int_0^{\varrho} \frac{(\varrho-s)^{\beta+q-1}}{\Gamma(\beta+q)} g_i(t,x(t),y(t),z(t),^c D^{\delta}x(t),^c D^{\xi}y(t),^c D^{\sigma}z(t))ds - \frac{y_0\varrho^q}{\Gamma(q+1)}\right),$$

and

$$Z_3(x,y,z)(t) = z_0 - \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} h_i(t,x(t),y(t),z(t),^c D^{\delta}x(t),^c D^{\xi}y(t),^c D^{\sigma}z(t))ds$$
$$+ \frac{\Gamma(r+4)t^3}{6(\tau^{r+3}-\Gamma(r+4))}\left(\sum_{i=1}^{n}\int_0^{\tau} \frac{(\tau-s)^{\gamma+r-1}}{\Gamma(\gamma+r)} h_i(t,x(t),y(t),z(t),^c D^{\delta}x(t),^c D^{\xi}y(t),^c D^{\sigma}z(t))ds - \frac{z_0\tau^r}{\Gamma(r+1)}\right),$$

where $\varsigma^{p+3} - \Gamma(p+4) \neq 0$, $\varrho^{q+3} - \Gamma(q+4) \neq 0$, $\tau^{r+3} - \Gamma(r+4) \neq 0$.

Next, we investigate uniqueness of the solution for the proposed problem which is based on the Banach fixed point theorem.

Theorem 2. *Let (Hypothesis 1) and (Hypothesis 2) hold. If the inequality*

$$\max\left(\mathbf{C}_1 S_1, \mathbf{C}_2 S_2, \mathbf{C}_3 S_3, \mathbf{C}_4 S_1, \mathbf{C}_5 S_2, \mathbf{C}_6 S_3\right) < 1 \tag{15}$$

is satisfied for the notions defined, then the integral operator $Z : V \to V$ has a unique fixed point in Banach space V, defined by (12).

Proof. For the required result, it is necessary to show that Z is contractive.

Let $(x_1,y_1,z_1),(x_2,y_2,z_2) \in V$; then, for each $t \in [0,1]$, we have, after the triangle inequality,

$$\begin{aligned}
&|Z_1(x_1,y_1,z_1)(t) - Z_1(x_2,y_2,z_2)(t)| \\
&\leq \sup_{t\in[0,1]} \sum_{i=1}^{n}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big| f_i(s,x_1(s),y_1(s),z_1(s),^c D^{\delta}x_1(s),^c D^{\xi}y_1(s),^c D^{\sigma}z_1(s)) \\
&- f_i(s,x_2(s),y_2(s),z_2(s),^c D^{\delta}x_2(s),^c D^{\xi}y_2(s),^c D^{\sigma}z_2(s))\Big| ds \\
&+ \frac{\Gamma(p+4)}{6|\varsigma^{p+3}-\Gamma(p+4)|}\sum_{i=1}^{n}\int_0^{\varsigma} \frac{(\varsigma-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}\Big| f_i(s,x_1(s),y_1(s),z_1(s),^c D^{\delta}x_1(s),^c D^{\xi}y_1(s),^c D^{\sigma}z_1(s)) \\
&- f_i(s,x_2(s),y_2(s),z_2(s),^c D^{\delta}x_2(s),^c D^{\xi}y_2(s),^c D^{\sigma}z_2(s))\Big| ds \\
&\leq \frac{1}{\Gamma(\alpha+1)}\sup_{t\in[0,1]} \sum_{i=1}^{n}\Big| f_i(t,x_1(t),y_1(t),z_1(t),^c D^{\delta}x_1(t),^c D^{\xi}y_1(t),^c D^{\sigma}z_1(t)) \\
&- f_i(t,x_2(t),y_2(t),z_2(t),^c D^{\delta}x_2(t),^c D^{\xi}y_2(t),^c D^{\sigma}z_2(t))\Big| \\
&+ \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\sup_{t\in[0,1]} \sum_{i=1}^{n}\Big| f_i(t,x_1(t),y_1(t),z_1(t),^c D^{\delta}x_1(t),^c D^{\xi}y_1(t),^c D^{\sigma}z_1(t)) \\
&- f_i(t,x_2(t),y_2(t),z_2(t),^c D^{\delta}x_2(t),^c D^{\xi}y_2(t),^c D^{\sigma}z_2(t))\Big|,
\end{aligned} \tag{16}$$

By using (Hypothesis 2), (16) implies that, according to definition (13), we have

$$
\begin{aligned}
&\|Z_1(x_1,y_1,z_1)-Z_1(x_2,y_2,z_2)\| \\
&\le \sum_{i=1}^{n}(\mu_1^i+\mu_2^i+\mu_3^i+\mu_4^i+\mu_5^i+\mu_6^i)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\right) \\
&\times \max\Big(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|+\|^cD^\delta x_1-^cD^\delta x_2\|+\|^cD^\delta y_1-^cD^\delta y_2\|+\|^cD^\delta z_1-^cD^\delta z_2\|\Big),
\end{aligned}
$$

Hence,

$$
\|Z_1(x_1,y_1,z_1)-Z_1(x_2,y_2,z_2)\| \le \mathbf{C}_1S_1\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V. \tag{17}
$$

Similarly,

$$
\|Z_2(x_1,y_1,z_1)-Z_2(x_2,y_2,z_2)\| \le \mathbf{C}_2S_2\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V, \tag{18}
$$

$$
\|Z_3(x_1,y_1,z_1)-Z_3(x_2,y_2,z_2)\| \le \mathbf{C}_3S_3\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V. \tag{19}
$$

On the other side,

$$
\begin{aligned}
&\left|^cD^\delta Z_1(x_1,y_1,z_1)(t)-^cD^\delta Z_1(x_2,y_2,z_2)(t)\right| \\
&\le \frac{1}{\Gamma(\alpha-\delta+1)}\sup_{t\in[0,1]}\sum_{i=1}^{n}\Big|f_i(t,x_1(t),y_1(t),z_1(t),^cD^\delta x_1(t),^cD^\xi y_1(t),^cD^\sigma z_1(t)) \\
&-f_i(t,x_2(t),y_2(t),z_2(t),^cD^\delta x_2(t),^cD^\xi y_2(t),^cD^\sigma z_2(t))\Big| \\
&+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)} \\
&\times \sup_{t\in[0,1]}\sum_{i=1}^{n}\Big|f_i(t,x_1(t),y_1(t),z_1(t),^cD^\delta x_1(t),^cD^\xi y_1(t),^cD^\sigma z_1(t)) \\
&-f_i(t,x_2(t),y_2(t),z_2(t),^cD^\delta x_2(t),^cD^\xi y_2(t),^cD^\sigma z_2(t))\Big|.
\end{aligned} \tag{20}
$$

Consequently,

$$
\begin{aligned}
&\|^cD^\delta Z_1(x_1,y_1,z_1)-^cD^\delta Z_1(x_2,y_2,z_2)\| \\
&\le \sum_{i=1}^{n}(\mu_1^i+\mu_2^i+\mu_3^i+\mu_4^i+\mu_5^i+\mu_6^i)\left[\frac{1}{\Gamma(\alpha-\delta+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)}\right] \\
&\times \max\Big(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|+\|^cD^\delta x_1-^cD^\delta x_2\|+\|^cD^\delta y_1-^cD^\delta y_2\|+\|^cD^\delta z_1-^cD^\delta z_2\|\Big).
\end{aligned} \tag{21}
$$

Therefore,

$$
\|^cD^\delta Z_1(x_1,y_1,z_1)-^cD^\delta Z_1(x_2,y_2,z_2)\| \le \mathbf{C}_4S_1\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V. \tag{22}
$$

Similarly, we get

$$
\|^cD^\xi Z_2(x_1,y_1,z_1)-^cD^\xi Z_2(x_2,y_2,z_2)\| \le \mathbf{C}_5S_2\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V. \tag{23}
$$

and

$$
\|^cD^\sigma Z_3(x_1,y_1,z_1)-^cD^\sigma Z_3(x_2,y_2,z_2)\| \le \mathbf{C}_6S_3\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V. \tag{24}
$$

According to the norm $\|\,\|_V$ defined by (13), we have:

$$\begin{aligned}&\|Z(x_1,y_1,z_1)-Z(x_2,y_2,z_2)\|_V\\&\leq \max\Big(\|Z_1(x_1,y_1,z_1)-Z_1(x_2,y_2,z_2)\|,\|^cD^{\delta}Z_1(x_1,y_1,z_1)-^c D^{\delta}Z_1(x_2,y_2,z_2)\|,\\&\|Z_2(x_1,y_1,z_1)-Z_2(x_2,y_2,z_2)\|,\|^cD^{\xi}Z_2(x_1,y_1,z_1)-^c D^{\xi}Z_2(x_2,y_2,z_2)\|,\\&\|Z_3(x_1,y_1,z_1)-Z_3(x_2,y_2,z_2)\|,\|^cD^{\sigma}Z_3(x_1,y_1,z_1)-^c D^{\sigma}Z_3(x_2,y_2,z_2)\|\Big).\end{aligned} \tag{25}$$

Using (17)–(19) and (22)–(24), we have

$$\begin{aligned}&\|Z(x_1,y_1,z_1)-Z(x_2,y_2,z_2)\|_V\\&\leq \max\left(\mathbf{C}_1S_1,\mathbf{C}_2S_2,\mathbf{C}_3S_3,\mathbf{C}_4S_1,\mathbf{C}_5S_2,\mathbf{C}_6S_3\right)\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_V.\end{aligned} \tag{26}$$

Thus, Z is contractive by using (15) and therefore it has a unique fixed point. Consequently, the proposed problem solution is a unique one. □

In the next theorem, we prove that the proposed problem solution is at least one. This is a very important result because demonstrating the existence of a solution guarantees that the problem is solvable, which is essential for mathematical modeling. This result provides a basis for further mathematical analysis, such as that for uniqueness, and stability.

Theorem 3. *Let Hypothesis 1 and 3 be satisfied and let operator Z be well-defined. Then, there is at least one confirmed solution for problem (1).*

Proof. The proof of the theorem is based on Schaefer's fixed point theorem [26]. By (H_1), the functions f_i, g_i, h_i are continuous and hence the operator Z is continuous. To show that Z is completely continuous, it is necessary that the following is true:

(I) It maps bounded sets of V into bounded sets of V;
(II) It is equi-continuous.

To prove (I), we take the finite set $\mathcal{D}=\{(x,y,z)\in V:\|(x,y,z)\|_V\leq\lambda\}$. For $(x,y,z)\in\mathcal{D}$, and for each $t\in[0,1]$, we have

$$\begin{aligned}&|Z_1(x,y,z)(t)|\\&\leq |x_0|+\sup_{t\in[0,1]}\sum_{i=1}^{n}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|f_i(s,x(s),y(s),z(s),^cD^{\delta}x(s),^cD^{\xi}y(s),^cD^{\sigma}z(s))\Big|ds\\&+\frac{\Gamma(p+4)}{6|\varsigma^{p+3}-\Gamma(p+4)|}\sum_{i=1}^{n}\int_0^{\varsigma}\frac{(\varsigma-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}\Big|f_i(s,x(s),y(s),z(s),^cD^{\delta}x(s),^cD^{\xi}y(s),^cD^{\sigma}z(s))\Big|ds\\&+\sup_{t\in[0,1]}\frac{(p+3)(p+2)(p+1)t^3|x_0|\varsigma^p}{6|\varsigma^{p+3}-\Gamma(p+4)|}\leq|x_0|+\frac{1}{\Gamma(\alpha+1)}\sup_{t\in[0,1]}\sum_{i=1}^{n}\Big|f_i(t,x(t),y(t),z(t),^cD^{\delta}x(t),^cD^{\xi}y(t),^cD^{\sigma}z(t))\Big|\\&+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\sup_{t\in[0,1]}\sum_{i=1}^{n}\Big|f_i(t,x(t),y(t),z(t),^cD^{\delta}x(t),^cD^{\xi}y(t),^cD^{\sigma}z(t))\Big|\\&+\frac{(p+3)(p+2)(p+1)|x_0|\varsigma^p}{6|\varsigma^{p+3}-\Gamma(p+4)|}.\end{aligned} \tag{27}$$

By using (H_3), we have

$$\begin{aligned}&\|Z_1(x,y,z)\|\\&\leq\sum_{i=1}^{n}L_i\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\right)+|x_0|+\frac{(p+3)(p+2)(p+1)|x_0|\varsigma^p}{6|\varsigma^{p+3}-\Gamma(p+4)|}.\end{aligned} \tag{28}$$

Therefore,

$$\|Z_1(x,y,z)\| \leq \mathbf{C}_1 \sum_{i=1}^{n} L_i + Q_1, \tag{29}$$

Similarly,

$$\|Z_2(x,y,z)\| \leq \mathbf{C}_2 \sum_{i=1}^{n} M_i + Q_2, \tag{30}$$

$$\|Z_3(x,y,z)\| \leq \mathbf{C}_3 \sum_{i=1}^{n} N_i + Q_3. \tag{31}$$

On the other side,

$$\begin{aligned}
&\left|{}^cD^{\delta} Z_1(x,y,z)(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha-\delta+1)} \sup\nolimits_{t\in[0,1]} \sum_{i=1}^{n} \left| f_i(t,x(t),y(t),z(t),{}^c D^{\delta}x(t),{}^c D^{\xi}y(t),{}^c D^{\sigma}z(t)) \right| \\
&+ \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)} \sup\nolimits_{t\in[0,1]} \sum_{i=1}^{n} \left| f_i(t,x(t),y(t),z(t),{}^c D^{\delta}x(t),{}^c D^{\xi}y(t),{}^c D^{\sigma}z(t)) \right| \\
&+ \frac{(p+3)(p+2)(p+1)|x_0|\varsigma^{p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)}.
\end{aligned} \tag{32}$$

Consequently,

$$\begin{aligned}
\|{}^cD^{\delta} Z_1(x,y,z)\| &\leq \sum_{i=1}^{n} L_i \left[\frac{1}{\Gamma(\alpha-\delta+1)} + \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)} \right] \\
&+ \frac{(p+3)(p+2)(p+1)|x_0|\varsigma^{p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)}.
\end{aligned} \tag{33}$$

Therefore,

$$\|{}^cD^{\delta} Z_1(x,y,z)\| \leq \mathbf{C}_4 \sum_{i=1}^{n} L_i + Q_4. \tag{34}$$

Similarly,

$$\|{}^cD^{\xi} Z_2(x,y,z)\| \leq \mathbf{C}_5 \sum_{i=1}^{n} M_i + Q_5, \tag{35}$$

$$\|{}^cD^{\sigma} Z_3(x,y,z)\| \leq \mathbf{C}_6 \sum_{i=1}^{n} N_i + Q_6, \tag{36}$$

By using (29)–(31), (34)–(36), we write

$$\begin{aligned}
\|Z(x,y,z)\|_V \leq \max \Bigg(&\mathbf{C}_1 \sum_{i=1}^{n} L_i + Q_1, \mathbf{C}_2 \sum_{i=1}^{n} M_i + Q_2, \mathbf{C}_3 \sum_{i=1}^{n} N_i + Q_3, \\
&\mathbf{C}_4 \sum_{i=1}^{n} L_i + Q_4, \mathbf{C}_5 \sum_{i=1}^{n} M_i + Q_5, \mathbf{C}_6 \sum_{i=1}^{n} N_i + Q_6 \Bigg),
\end{aligned} \tag{37}$$

Consequently,

$$\|Z(x,y,z)\|_V < \infty. \tag{38}$$

Hence, we prove that Z(V) is bounded. Thus, it is proof of (I). In the next section, we prove (II): that Z_1 is equi-continuous.

For any $0 \leq t_1 < t_2 \leq 1$, we consider

$$\begin{aligned}
&|Z_1(x,y,z)(t_2)-Z_1(x,y,z)(t_1)| \\
&\le \frac{1}{\Gamma(\alpha+1)}\left((t_2-t_1)^{\alpha}+(t_2^{\alpha}-t_1^{\alpha})\right)\sup_{t\in[0,1]}\sum_{i=1}^{n}\left|f_i(t,x(t),y(t),z(t),{}^cD^{\delta}x(t),{}^cD^{\xi}y(t),{}^cD^{\sigma}z(t))\right| \\
&+\frac{1}{\Gamma(\alpha+1)}\left((t_2-t_1)^{\alpha}\right)\sup_{t\in[0,1]}\sum_{i=1}^{n}\left|f_i(t,x(t),y(t),z(t),{}^cD^{\delta}x(t),{}^cD^{\xi}y(t),{}^cD^{\sigma}z(t))\right| \\
&+\frac{\Gamma(p+4)(t_2^3-t_1^3)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\sup_{t\in[0,1]}\sum_{i=1}^{n}\left|f_i(t,x(t),y(t),z(t),{}^cD^{\delta}x(t),{}^cD^{\xi}y(t),{}^cD^{\sigma}z(t))\right| \\
&+\frac{(p+3)(p+2)(p+1)(t_2^3-t_1^3)|x_0|\varsigma^{p}}{6|\varsigma^{p+3}-\Gamma(p+4)|}.
\end{aligned} \tag{39}$$

By using Hypothesis 3 and taking supremum, we have

$$\begin{aligned}
&\|Z_1(x,y,z)(t_2)-Z_1(x,y,z)(t_1)\| \le \sum_{i=1}^{n}L_i\left(\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}+\frac{2(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}(t_2^3-t_1^3)}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\right) \\
&+\frac{(p+3)(p+2)(p+1)\varsigma^{p}|x_0|(t_2^3-t_1^3)}{6|\varsigma^{p+3}-\Gamma(p+4)|}.
\end{aligned} \tag{40}$$

Similarly,

$$\begin{aligned}
&\|Z_2(x,y,z)(t_2)-Z_2(x,y,z)(t_1)\| \le \sum_{i=1}^{n}M_i\left(\frac{t_2^{\beta}-t_1^{\beta}}{\Gamma(\beta+1)}+\frac{2(t_2-t_1)^{\beta}}{\Gamma(\beta+1)}+\frac{\Gamma(q+4)\varrho^{\beta+q}(t_2^3-t_1^3)}{6|\varrho^{q+3}-\Gamma(q+4)|\Gamma(\beta+q+1)}\right) \\
&+\frac{(q+3)(q+2)(q+1)\varrho^{q}|y_0|(t_2^3-t_1^3)}{6|\varrho^{q+3}-\Gamma(q+4)|},
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
&\|Z_3(x,y,z)(t_2)-Z_3(x,y,z)(t_1)\| \le \sum_{i=1}^{n}N_i\left(\frac{t_2^{\gamma}-t_1^{\gamma}}{\Gamma(\gamma+1)}+\frac{2(t_2-t_1)^{\gamma}}{\Gamma(\gamma+1)}+\frac{\Gamma(r+4)\tau^{\gamma+r}(t_2^3-t_1^3)}{6|\tau^{r+3}-\Gamma(r+4)|\Gamma(\gamma+r+1)}\right) \\
&+\frac{(r+3)(r+2)(r+1)\tau^{r}|z_0|(t_2^3-t_1^3)}{6|\tau^{r+3}-\Gamma(r+4)|}.
\end{aligned} \tag{42}$$

On the other side, we have

$$\begin{aligned}
&\left|{}^cD^{\delta}Z_1(x,y,z)(t_2)-{}^cD^{\delta}Z_1(x,y,z)(t_1)\right| \\
&\le \frac{1}{\Gamma(\alpha-\delta+1)}\left((t_2-t_1)^{\alpha-\delta}+(t_2^{\alpha-\delta}-t_1^{\alpha-\delta})\right)\sup_{t\in[0,1]}\sum_{i=1}^{n}\left|f_i(t,x(t),y(t),z(t),{}^cD^{\delta}x(t),{}^cD^{\xi}y(t),{}^cD^{\sigma}z(t))\right| \\
&+\frac{(t_2-t_1)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\sup_{t\in[0,1]}\sum_{i=1}^{n}\left|f_i(t,x(t),y(t),z(t),{}^cD^{\delta}x(t),{}^cD^{\xi}y(t),{}^cD^{\sigma}z(t))\right| \\
&+\frac{\Gamma(p+4)(t_2^{3-\delta}-t_1^{3-\delta})\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)}\sup_{t\in[0,1]}\sum_{i=1}^{n}\left|f_i(t,x(t),y(t),z(t),{}^cD^{\delta}x(t),{}^cD^{\xi}y(t),{}^cD^{\sigma}z(t))\right| \\
&+\frac{(p+3)(p+2)(p+1)(t_2^{3-\delta}-t_1^{3-\delta})|x_0|\varsigma^{p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)}.
\end{aligned} \tag{43}$$

Consequently,

$$\begin{aligned}&\|{}^cD^{\delta}Z_1(x,y,z)(t_2)-{}^cD^{\delta}Z_1(x,y,z)(t_1)\|\\&\le\sum_{i=1}^{n}L_i\bigg(\frac{(t_2^{\alpha-\delta}-t_1^{\alpha-\delta})}{\Gamma(\alpha-\delta+1)}+\frac{2(t_2-t_1)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}+\frac{\Gamma(p+4)(t_2^{3-\delta}-t_1^{3-\delta})\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)}\bigg)\\&+\frac{(p+3)(p+2)(p+1)(t_2^{3-\delta}-t_1^{3-\delta})|x_0|\varsigma^p}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)}.\end{aligned}\tag{44}$$

Similarly,

$$\begin{aligned}&\|{}^cD^{\xi}Z_2(x,y,z)(t_2)-{}^cD^{\xi}Z_2(x,y,z)(t_1)\|\\&\le\sum_{i=1}^{n}M_i\bigg(\frac{(t_2^{\beta-\xi}-t_1^{\beta-\xi})}{\Gamma(\beta-\xi+1)}+\frac{2(t_2-t_1)^{\beta-\xi}}{\Gamma(\beta-\xi+1)}+\frac{\Gamma(q+4)(t_2^{3-\xi}-t_1^{3-\xi})\varrho^{\beta+q}}{2|\varrho^{q+3}-\Gamma(q+4)|\Gamma(4-\xi)\Gamma(\beta+q+1)}\bigg)\\&+\frac{(q+3)(q+2)(q+1)(t_2^{3-\xi}-t_1^{3-\xi})|y_0|\varrho^q}{2|\varrho^{q+3}-\Gamma(q+4)|\Gamma(4-\xi)},\end{aligned}\tag{45}$$

and

$$\begin{aligned}&\|{}^cD^{\sigma}Z_3(x,y,z)(t_2)-{}^cD^{\sigma}Z_3(x,y,z)(t_1)\|\\&\le\sum_{i=1}^{n}N_i\bigg(\frac{(t_2^{\gamma-\sigma}-t_1^{\gamma-\sigma})}{\Gamma(\gamma-\sigma+1)}+\frac{2(t_2-t_1)^{\gamma-\sigma}}{\Gamma(\gamma-\sigma+1)}+\frac{\Gamma(r+4)(t_2^{3-\sigma}-t_1^{3-\sigma})\tau^{\gamma+r}}{2|\tau^{r+3}-\Gamma(r+4)|\Gamma(4-\sigma)\Gamma(\gamma+r+1)}\bigg)\\&+\frac{(r+3)(r+2)(r+1)(t_2^{3-\sigma}-t_1^{3-\sigma})|z_0|\tau^r}{2|\tau^{r+3}-\Gamma(r+4)|\Gamma(4-\sigma)}.\end{aligned}\tag{46}$$

Looking at the inequalities, (40)–(42), and (44)–(46), we observe that

$$\|Z(x,y,z)(t_2)-Z(x,y,z)(t_1)\|_V\to 0,\quad as\ t_1\to t_2.\tag{47}$$

Combining (I), (II) and applying the Arzel*à*–Ascoli theorem [23,24], we have that Z is a completely continuous operator.

Now, it remains necessary to show that the set defined by

$$\mathscr{B}=\{(x,y,z)\in V:(x,y,z)=\kappa Z(x,y,z),0<\kappa<1\},$$

is bounded. Let $(x,y,z)\in\mathscr{B}$. Then, by definition, $(x,y,z)=\kappa Z(x,y,z)$. Explicitly, we write $x(t)=\kappa Z_1(x,y,z)(t)$, $y(t)=\kappa Z_2(x,y,z)(t)$ and $z(t)=\kappa Z_3(x,y,z)(t)$. Thus, we have

$$\begin{aligned}&\frac{1}{\kappa}x(t)=|Z_1(x,y,z)(t)|\\&\le|x_0|+\sup_{t\in[0,1]}\sum_{i=1}^{n}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|f_i(s,x(s),y(s),z(s),{}^cD^{\delta}x(s),{}^cD^{\xi}y(s),{}^cD^{\sigma}z(s))\Big|ds\\&+\frac{\Gamma(p+4)}{6|\varsigma^{p+3}-\Gamma(p+4)|}\sum_{i=1}^{n}\int_0^{\varsigma}\frac{(\varsigma-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}\Big|f_i(s,x(s),y(s),z(s),{}^cD^{\delta}x(s),{}^cD^{\xi}y(s),{}^cD^{\sigma}z(s))\Big|ds.\end{aligned}\tag{48}$$

From (28), we write

$$\begin{aligned}&\frac{1}{\kappa}x(t)=\|Z_1(x,y,z)\|\\&\le\sum_{i=1}^{n}L_i\bigg(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\bigg)+|x_0|+\frac{(p+3)(p+2)(p+1)|x_0|\varsigma^p}{6|\varsigma^{p+3}-\Gamma(p+4)|}.\end{aligned}\tag{49}$$

Therefore,

$$\frac{1}{\kappa}x(t) \leq \mathbf{C}_1 \sum_{i=1}^{n} L_i + Q_1 \Rightarrow \|x\| \leq \kappa\left(\mathbf{C}_1 \sum_{i=1}^{n} L_i + Q_1\right), \tag{50}$$

Similarly,

$$\|y\| \leq \kappa\left(\mathbf{C}_2 \sum_{i=1}^{n} M_i + Q_2\right), \tag{51}$$

$$\|z\| \leq \kappa\left(\mathbf{C}_3 \sum_{i=1}^{n} N_i + Q_3\right). \tag{52}$$

On the other side, we obtain

$$\|{}^cD^{\delta}x\| \leq \kappa\left(\mathbf{C}_4 \sum_{i=1}^{n} L_i + Q_4\right), \tag{53}$$

$$\|{}^cD^{\xi}y\| \leq \kappa\left(\mathbf{C}_5 \sum_{i=1}^{n} M_i + Q_5\right), \tag{54}$$

$$\|{}^cD^{\sigma}z\| \leq \kappa\left(\mathbf{C}_6 \sum_{i=1}^{n} N_i + Q_6\right). \tag{55}$$

The results (50)–(55), imply that

$$\begin{aligned}\|(x,y,z)\|_V \leq \kappa \max\Bigg(&\mathbf{C}_1 \sum_{i=1}^{n} L_i + Q_1, \mathbf{C}_2 \sum_{i=1}^{n} M_i + Q_2, \mathbf{C}_3 \sum_{i=1}^{n} N_i + Q_3,\\ &\mathbf{C}_4 \sum_{i=1}^{n} L_i + Q_4, \mathbf{C}_5 \sum_{i=1}^{n} M_i + Q_5, \mathbf{C}_6 \sum_{i=1}^{n} N_i + Q_6\Bigg) < \infty.\end{aligned} \tag{56}$$

Hence, the set $\mathscr{B}$ is bounded. Therefore, by Theorem 1, there is one or more solutions to problem (1). □

5. Hyers–Ulam (H-U) Stability

In this section, we perform Hyers–Ulam stability analysis for tripled systems of DEs (1). Let $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. Then, for some $t \in [0,1]$, we construct the set of inequalities in unknowns $\overline{x}(t), \overline{y}(t)$, and $\overline{z}(t)$, as:

$$\begin{cases} \left|{}^cD^{\alpha}\overline{x}(t) + \sum_{i=1}^{n} f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t))\right| \leq \epsilon_1, \\ \left|{}^cD^{\alpha}\overline{y}(t) + \sum_{i=1}^{n} g_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t))\right| \leq \epsilon_2, \\ \left|{}^cD^{\alpha}\overline{z}(t) + \sum_{i=1}^{n} h_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t))\right| \leq \epsilon_3, \end{cases} \tag{57}$$

where

$$(\overline{x}, \overline{y}, \overline{z}) \in V := \left\{(\overline{x}, \overline{y}, \overline{z}) : \overline{x}, \overline{y}, \overline{z} \in C([0,1], \mathbb{R}), \quad {}^cD^{\delta}\overline{x}, {}^cD^{\xi}\overline{y}, {}^cD^{\sigma}\overline{z} \in C([0,1], \mathbb{R})\right\}. \tag{58}$$

From [27], we adopt the following definitions of H-U stability.

Definition 5. *The tripled system of DEs (1) is said to be H-U stable if there exists a positive real number κ such that for any solution $(\overline{x}, \overline{y}, \overline{z})$ of the inequality (57), there exists a unique solution (x,y,z) of (1) satisfying*

$$|(\overline{x}, \overline{y}, \overline{z}) - (x,y,z)| \leq \kappa\epsilon, \ t \in [0,1],$$

where $\epsilon = \max(\epsilon_1, \epsilon_2, \epsilon_3)$.

Definition 6. *The tripled system of DEs (1) is said to be generalized H-U stable if there exists a real function* $\mathcal{F} \in C(\mathbb{R}_+, \mathbb{R}_+)$, *with* $\mathcal{F}(0) = 0$, *such that for any solution* $(\overline{x}, \overline{y}, \overline{z})$ *of the inequality (57), and a unique solution* (x, y, z) *of (1), the following condition satisfies*

$$|(\overline{x}, \overline{y}, \overline{z}) - (x, y, z)| \leq \mathcal{F}(\epsilon),\ t \in [0, 1].$$

We make the following remark to obtain the corresponding perturbed problem with small perturbation functions. It is used to establish bounds on the perturbation's effect on the system, and to quantify the relationship between the perturbation and the resulting change in the system's behavior.

Remark 1. $(\overline{x}, \overline{y}, \overline{z})$ *is a solution of the inequality (57), if there exist functions* $v_1, v_2, v_3 \in C([0, 1], \mathbb{R})$ *which are dependent of* $\overline{x}, \overline{y}, \overline{z}$, *respectively, such that for* $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, *we have*

(i) $|v_1(t)| \leq \epsilon_1$, $|v_2(t)| \leq \epsilon_2$, $|v_3(t)| \leq \epsilon_3$, $\quad t \in [0, 1]$,

(ii)

$$\begin{cases} {}^cD^{\alpha}\overline{x}(t) = -\sum_{i=1}^{n} f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t)) + v_1(t), \\ {}^cD^{\alpha}\overline{y}(t) = -\sum_{i=1}^{n} g_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t)) + v_2(t), \\ {}^cD^{\alpha}\overline{z}(t) = -\sum_{i=1}^{n} h_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t)) + v_3(t). \end{cases} \tag{59}$$

By Remark 1, we have the following problem with small perturbation functions

$$\begin{cases} {}^cD^{\alpha}\overline{x}(t) = -\sum_{i=1}^{n} f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t)) + v_1(t), \\ {}^cD^{\alpha}\overline{y}(t) = -\sum_{i=1}^{n} g_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t)) + v_2(t), \\ {}^cD^{\alpha}\overline{z}(t) = -\sum_{i=1}^{n} h_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^cD^{\delta}\overline{x}(t), {}^cD^{\xi}\overline{y}(t), {}^cD^{\sigma}\overline{z}(t)) + v_3(t) \\ \overline{x}(0) = \overline{x}_0, \quad \overline{y}(0) = \overline{y}_0, \quad \overline{z}(0) = \overline{z}_0, \\ \overline{x}'(0) = \overline{x}''(0) = \overline{y}'(0) = \overline{y}''(0) = \overline{z}'(0) = \overline{z}''(0) = 0, \\ \overline{x}'''(0) = J^p\overline{x}(\varsigma), \quad \overline{y}'''(0) = J^q\overline{y}(\varrho), \quad \overline{z}'''(0) = J^r\overline{z}(\tau). \end{cases} \tag{60}$$

Corollary 2. *Let* $\Big(\overline{x}(t), \overline{y}(t), \overline{z}(t)\Big)$ *represent the solution to the tripled system of perturbed DEs (60). Then, by Lemma 4, we have*

$$\begin{cases} \overline{x}(t) = \overline{x}_0 - \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) ds \\ + \frac{\Gamma(p+4)t^3}{6(\varsigma^{p+3} - \Gamma(p+4))} \left(\sum_{i=1}^{n} \int_0^{\varsigma} \frac{(\varsigma - s)^{\alpha+p-1}}{\Gamma(\alpha+p)} f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) ds - \frac{\overline{x}_0 \varsigma^p}{\Gamma(p+1)} \right) \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s) ds + \frac{\Gamma(p+4)t^3}{6(\varsigma^{p+3} - \Gamma(p+4))} \int_0^{\varsigma} \frac{(\varsigma - s)^{\alpha+p-1}}{\Gamma(\alpha+p)} v_1(s) ds, \\ \overline{y}(t) = \overline{y}_0 - \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) ds \\ + \frac{\Gamma(q+4)t^3}{6(\varrho^{q+3} - \Gamma(q+4))} \left(\sum_{i=1}^{n} \int_0^{\varrho} \frac{(\varrho - s)^{\beta+q-1}}{\Gamma(\beta+q)} g_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) ds - \frac{\overline{y}_0 \varrho^q}{\Gamma(q+1)} \right), \\ + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} v_2(t) ds + \frac{\Gamma(q+4)t^3}{6(\varrho^{q+3} - \Gamma(q+4))} \int_0^{\varrho} \frac{(\varrho - s)^{\beta+q-1}}{\Gamma(\beta+q)} v_2(t) ds, \\ \overline{z}(t) = \overline{z}_0 - \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} h_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) ds \\ + \frac{\Gamma(r+4)t^3}{6(\tau^{r+3} - \Gamma(r+4))} \left(\sum_{i=1}^{n} \int_0^{\tau} \frac{(\tau - s)^{\gamma+r-1}}{\Gamma(\gamma+r)} h_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) ds - \frac{\overline{z}_0 \tau^r}{\Gamma(r+1)} \right) \\ + \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} v_3(t) ds + \frac{\Gamma(r+4)t^3}{6(\tau^{r+3} - \Gamma(r+4))} \int_0^{\tau} \frac{(\tau - s)^{\gamma+r-1}}{\Gamma(\gamma+r)} v_3(t) ds \end{cases} \tag{61}$$

where $\varsigma^{p+3} - \Gamma(p+4) \neq 0$, $\varrho^{q+3} - \Gamma(q+4) \neq 0$, $\tau^{r+3} - \Gamma(r+4) \neq 0$.

Theorem 4. *Let* (H_1) *and* (H_2) *hold and let* $\varsigma^{p+3} - \Gamma(p+4) \neq 0$, $\varrho^{q+3} - \Gamma(q+4) \neq 0$, *and* $\tau^{r+3} - \Gamma(r+4) \neq 0$. *If the inequality*

$$\max \left(\mathbf{C}_1 S_1, \mathbf{C}_2 S_2, \mathbf{C}_3 S_3, \mathbf{C}_4 S_1, \mathbf{C}_5 S_2, \mathbf{C}_6 S_3 \right) < 1, \tag{62}$$

is satisfied, then the proposed problem is H-U stable and consequently it is generalized H-U stable.

Proof.

$$\begin{aligned} & |\overline{x}(t) - x(t)| \\ & \leq \frac{1}{\Gamma(\alpha+1)} \sup_{t \in [0,1]} \sum_{i=1}^{n} \Big| f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) \\ & - f_i(t, x(t), y(t), z(t), {}^c D^{\delta} x(t), {}^c D^{\xi} y(t), {}^c D^{\sigma} z(t)) \Big| + \frac{1}{\Gamma(\alpha+1)} |v_1| \\ & + \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3} - \Gamma(p+4)|\Gamma(\alpha+p+1)} \sup_{t \in [0,1]} \sum_{i=1}^{n} \Big| f_i(t, \overline{x}(t), \overline{y}(t), \overline{z}(t), {}^c D^{\delta} \overline{x}(t), {}^c D^{\xi} \overline{y}(t), {}^c D^{\sigma} \overline{z}(t)) \\ & - f_i(t, x(t), y(t), z(t), {}^c D^{\delta} x(t), {}^c D^{\xi} y(t), {}^c D^{\sigma} z(t)) \Big| + \frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3} - \Gamma(p+4)|\Gamma(\alpha+p+1)} |v_1|, \end{aligned} \tag{63}$$

On using (H_2), (63) implies that

$$\begin{aligned}&\|\overline{x}-x\|\\&\leq\sum_{i=1}^{n}(\mu_1^i+\mu_2^i+\mu_3^i+\mu_4^i+\mu_5^i+\mu_6^i)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\right)\\&\times\max\Big(\|\overline{x}-x\|+\|\overline{y}-y\|+\|\overline{z}-z\|+\|^cD^\delta\overline{x}-^cD^\delta\overline{x}\|+\|^cD^\delta\overline{y}-^cD^\delta\overline{y}\|+\|^cD^\delta\overline{z}-^cD^\delta\overline{z}\|\Big)\\&+\frac{1}{\Gamma(\alpha+1)}\epsilon_1+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{6|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(\alpha+p+1)}\epsilon_1,\end{aligned}$$

Hence,

$$\|\overline{x}-x\|\leq\mathbf{C}_1S_1\|(\overline{x}-x,\overline{y}-y,\overline{z}-z)\|_V+\mathbf{C}_1\epsilon_1.\qquad(64)$$

Similarly,

$$\|\overline{y}-y\|\leq\mathbf{C}_2S_2\|(\overline{x}-x,\overline{y}-y,\overline{z}-z)\|_V+\mathbf{C}_2\epsilon_2,\qquad(65)$$

$$\|\overline{z}-z\|\leq\mathbf{C}_3S_3\|(\overline{x}-x,\overline{y}-y,\overline{z}-z)\|_V+\mathbf{C}_3\epsilon_3.\qquad(66)$$

On the other hand, we obtain

$$\begin{aligned}&\|^cD^\delta\overline{x}-^cD^\delta x\|\\&\leq\sum_{i=1}^{n}(\mu_1^i+\mu_2^i+\mu_3^i+\mu_4^i+\mu_5^i+\mu_6^i)\left[\frac{1}{\Gamma(\alpha-\delta+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)}\right]\\&\times\max\Big(\|\overline{x}-x\|+\|\overline{y}-y\|+\|\overline{z}-z\|+\|^cD^\delta\overline{x}-^cD^\delta\overline{x}\|+\|^cD^\delta\overline{y}-^cD^\delta\overline{y}\|+\|^cD^\delta\overline{z}-^cD^\delta\overline{z}\|\Big)\\&+\frac{\epsilon_1}{\Gamma(\alpha-\delta+1)}+\frac{\Gamma(p+4)\varsigma^{\alpha+p}}{2|\varsigma^{p+3}-\Gamma(p+4)|\Gamma(4-\delta)\Gamma(\alpha+p+1)}\epsilon_1.\end{aligned}\qquad(67)$$

Therefore,

$$\|^cD^\delta\overline{x}-^cD^\delta x\|\leq\mathbf{C}_4S_1\|(\overline{x}-x,\overline{y}-y,\overline{z}-z)\|_V+\mathbf{C}_4\epsilon_1.\qquad(68)$$

Similarly,

$$\|^cD^\delta\overline{y}-^cD^\delta y\|\leq\mathbf{C}_5S_2\|(\overline{x}-x,\overline{y}-y,\overline{z}-z)\|_V+\mathbf{C}_5\epsilon_2,\qquad(69)$$

$$\|^cD^\delta\overline{z}-^cD^\delta z\|\leq\mathbf{C}_6S_3\|(\overline{x}-x,\overline{y}-y,\overline{z}-z)\|_V+\mathbf{C}_6\epsilon_3.\qquad(70)$$

By the norm $\|\|_V$ defined in (13), we have

$$\|\overline{x}-x,\overline{y}-y,\overline{z}-z\|_V=\max\Big(\|\overline{x}-x\|,\|\overline{y}-y\|,\|\overline{z}-z\|,\|^cD^\delta\overline{x}-^cD^\delta\overline{x}\|,\|^cD^\delta\overline{y}-^cD^\delta\overline{y}\|,\|^cD^\delta\overline{z}-^cD^\delta\overline{z}\|\Big).\qquad(71)$$

Using (64)–(66), (68)–(70), we have

$$\begin{aligned}&\|\overline{x}-x,\overline{y}-y,\overline{z}-z\|_V\\&\leq\max\Big(\mathbf{C}_1S_1,\mathbf{C}_2S_2,\mathbf{C}_3S_3,\mathbf{C}_4S_1,\mathbf{C}_5S_2,\mathbf{C}_6S_3\Big)\|\overline{x}-x,\overline{y}-y,\overline{z}-z\|_V+\max\Big(\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3,\mathbf{C}_4,\mathbf{C}_5,\mathbf{C}_6\Big)\epsilon,\end{aligned}$$

where $\epsilon = \max\left(\epsilon_1, \epsilon_2, \epsilon_3\right)$. This implies that

$$\|\overline{x}-x, \overline{y}-y, \overline{z}-z\|_V \leq \frac{\max\left(\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\right)}{1-\max\left(\mathbf{C}_1 S_1, \mathbf{C}_2 S_2, \mathbf{C}_3 S_3, \mathbf{C}_4 S_1, \mathbf{C}_5 S_2, \mathbf{C}_6 S_3\right)}\epsilon := \Omega\epsilon.$$

where

$$\Omega = \frac{\max\left(\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\right)}{1-\max\left(\mathbf{C}_1 S_1, \mathbf{C}_2 S_2, \mathbf{C}_3 S_3, \mathbf{C}_4 S_1, \mathbf{C}_5 S_2, \mathbf{C}_6 S_3\right)} > 0.$$

By Definition 5, the H-U stability criteria are satisfied. Therefore, the proposed problem is H-U stable. □

Corollary 3. *If we set*

$$\|\overline{x}(t) - x(t)\|_F \leq \mathcal{F}(\epsilon), \tag{72}$$

where $\mathcal{F} \in C(\mathbb{R}_+, \mathbb{R}_+)$, with $\mathcal{F}(0) = 0$, then in this case the tripled system (1) is generalized H-U stable.

6. Application

In this section, we apply the main results to the following general problem to illustrate their applicability.

Example 1.

$$\begin{cases} {}^cD^{\frac{7}{2}}x(t) = \dfrac{1}{7\pi^3+t^2}\left(\cos x(t) + \cos^c D^{\frac{5}{2}}x(t) + \sin y(t) + \sin^c D^{\frac{3}{2}}y(t) + \sin z(t) + \sin^c D^{\frac{1}{2}}z(t) + \ln(1+t)\right) \\ + \dfrac{|x(t)|+|y(t)|+|z(t)| +^c D^{\frac{1}{2}}x(t) +^c D^{\frac{3}{2}}y(t) +^c D^{\frac{5}{2}}z(t)}{(6t^3+40)(e^{-5t}+|x(t)|+|y(t)|+|z(t)| +^c D^{\frac{1}{2}}x(t) +^c D^{\frac{3}{2}}y(t) +^c D^{\frac{5}{2}}z(t))}, \\ {}^cD^{\frac{16}{5}}y(t) = \dfrac{1}{28(t^3+1)}\left(\dfrac{|x(t)|}{(1+|x(t)|)} + \dfrac{|y(t)|}{(1+|y(t)|)} + \dfrac{|z(t)|}{(1+|z(t)|)} + \dfrac{t^4|{}^cD^{\frac{1}{2}}x(t)|}{\pi^4(1+|{}^cD^{\frac{1}{2}}x(t)|)}\right. \\ \left. + \dfrac{t|{}^cD^{\frac{3}{2}}y(t)|}{\pi^2(1+|{}^cD^{\frac{3}{2}}y(t)|)} + \dfrac{t^3|{}^cD^{\frac{5}{2}}z(t)|}{\pi^2(1+|{}^cD^{\frac{5}{2}}z(t)|)}\right) \\ + \dfrac{1}{18e^{-t^3}}\left(\sin x(t) + \dfrac{t^3}{9\pi^3}\sin 2\pi^c D^{\frac{5}{2}}x(t) + \sin y(t) + \dfrac{t^2}{\pi}\sin 2\pi^c D^{\frac{1}{2}}y(t) + \sin z(t) + \dfrac{t^5}{5\pi^4}\sin 2\pi^c D^{\frac{3}{2}}z(t)\right) \\ {}^cD^{\frac{10}{3}}z(t) = \dfrac{e^{-t}}{10} + \dfrac{x(t)|\cos(t)|}{t^2+20} + e^{-t}\dfrac{y(t)|\cos(t)|}{3t^3+30} + e^{-t^5}\dfrac{z(t)|\cos(t)|}{5t^4+40} + \dfrac{|{}^cD^{\frac{1}{2}}\cos(x)|}{2t^2+28} \\ + \dfrac{|{}^cD^{\frac{3}{2}}\cos(y)|}{6t^3+30} + \dfrac{|{}^cD^{\frac{1}{2}}\cos(z)|}{5t^4+45} \\ + \dfrac{e^{-t}}{\sqrt{60+t^3}}\left(\dfrac{x(t)}{3} + \dfrac{y(t)}{7} + \dfrac{z(t)}{5} + {}^cD^{\frac{5}{2}}x(t) + {}^cD^{\frac{3}{2}}y(t) + {}^cD^{\frac{1}{2}}z(t)\right), \\ x(0) = \sqrt{5}, y(0) = \sqrt{3}, z(0) = \sqrt{4}, x'(0) = x''(0) = y'(0) = y''(0) = z'(0) = z''(0) = 0, \\ x'''(0) = J^{\frac{3}{2}}(\dfrac{1}{3}), \quad y'''(0) = J^{\frac{4}{3}}(\dfrac{1}{2}), \quad z'''(0) = J^{\frac{1}{5}}(\dfrac{3}{4}). \end{cases} \tag{73}$$

Here $\alpha = \frac{7}{2}, \beta = \frac{16}{5}, \gamma = \frac{10}{3}, \delta = \frac{5}{2}, \xi = \frac{3}{2}, \sigma = \frac{1}{2}, p = \frac{3}{2}, q = \frac{4}{3}, r = \frac{1}{5}$. *Clearly, we have* $(\frac{1}{3})^{\frac{3}{2}+3} - \Gamma(\frac{3}{2}+4) \neq 0$, $(\frac{1}{2})^{\frac{4}{3}+3} - \Gamma(\frac{4}{3}+4) \neq 0$, *and* $(\frac{3}{4})^{\frac{1}{5}+3} - \Gamma(\frac{1}{5}+4) \neq 0$.

We set the functions

$$f_1(t, v_1, v_2, v_3, v_4, v_5, v_6) = \frac{1}{7\pi^3 + t^2}\bigg(\cos v_1 + \cos v_2 + \sin v_3 + \sin v_4 + \sin v_5 + \sin v_6 + \ln(1+t) \bigg),$$
$$f_2(t, v_1, v_2, v_3, v_4, v_5, v_6) = \frac{|v_1| + |v_2| + |v_3| + |v_4| + |v_5| + |v_6|}{(6t^3 + 40)(e^{-5t} + |v_1| + |v_2| + |v_3| + |v_4| + |v_5| + |v_6|)}.$$

So, for $(v_1, v_2, v_3, v_4, v_5, v_6), (w_1, w_2, w_3, w_4, w_5, w_6) \in \mathbb{R}^6$ *and* $t \in [0,1]$, *we have*

$$\begin{aligned} &|f_1(t, v_1, v_2, v_3, v_4, v_5, v_6) - f_1(t, w_1, w_2, w_3, w_4, w_5, w_6)| \\ &\leq \frac{1}{7\pi^3 + t^2}\bigg(|v_1 - w_1| + |v_2 - w_2| + |v_3 - w_3| + |v_4 - w_4| + |v_5 - w_5| + |v_6 - w_6| \bigg), \\ &|f_2(t, v_1, v_2, v_3, v_4, v_5, v_6) - f_2(t, w_1, w_2, w_3, w_4, w_5, w_6)| \\ &\leq \frac{1}{(6t^3 + 40)}\bigg(|v_1 - w_1| + |v_2 - w_2| + |v_3 - w_3| + |v_4 - w_4| + |v_5 - w_5| + |v_6 - w_6| \bigg). \end{aligned}$$

We take

$$\begin{aligned} &\phi_1^1(t) = \phi_2^1(t) = \phi_3^1(t) = \phi_4^1(t) = \phi_5^1(t) = \phi_6^1(t) = \frac{1}{7\pi^3 + t^2}, \\ &\phi_1^2(t) = \phi_2^2(t) = \phi_3^2(t) = \phi_4^2(t) = \phi_5^2(t) = \phi_6^2(t) = \frac{1}{(6t^3 + 40)}, \\ &\mu_1^1 = \mu_2^1 = \mu_3^1 = \mu_4^1 = \mu_5^1 = \mu_6^1 = \frac{1}{7\pi^3}, \\ &\mu_1^2 = \mu_2^2 = \mu_3^2 = \mu_4^2 = \mu_5^2 = \mu_6^2 = \frac{1}{40}. \end{aligned}$$

Similarly, we set

$$\begin{aligned} &g_1(t, v_1, v_2, v_3, v_4, v_5, v_6) \\ &= \frac{1}{28(t^3+1)}\bigg(\frac{|v_1|}{(1+|v_1|)} + \frac{|v_2|}{(1+|v_2|)} + \frac{|v_3|}{(1+|v_3|)} + \frac{t^4|v_4|}{\pi^4(1+|v_4|)} + \frac{t|v_5|}{\pi^2(1+|v_5|)} + \frac{t^3|v_6|}{\pi^2(1+|v_6|)} \bigg), \\ &g_2(t, v_1, v_2, v_3, v_4, v_5, v_6) = \frac{1}{18e^{-t^3}}\bigg(\sin v_1 + \frac{t^3}{9\pi^3}\sin(2\pi v_2) + \sin v_3 + \frac{t^2}{\pi}\sin(2\pi v_4) + \sin v_5 + \frac{t^5}{5\pi^4}\sin(2\pi v_6) \bigg). \end{aligned}$$

So, for $(v_1, v_2, v_3, v_4, v_5, v_6), (w_1, w_2, w_3, w_4, w_5, w_6) \in \mathbb{R}^6$ *and* $t \in [0,1]$, *we have*

$$\begin{aligned} &|g_1(t, v_1, v_2, v_3, v_4, v_5, v_6) - g_1(t, w_1, w_2, w_3, w_4, w_5, w_6)| \\ &\leq \frac{1}{28(t^3+1)}|v_1 - w_1| + \frac{1}{28(t^3+1)}|v_2 - w_2| + \frac{1}{28(t^3+1)}|v_3 - w_3| + \frac{t^4}{28\pi^4(t^3+1)}|v_4 - w_4| \\ &+ \frac{t}{28\pi^2(t^3+1)}|v_5 - w_5| + \frac{t^3}{28\pi^2(t^3+1)}|v_6 - w_6|. \end{aligned}$$

Here, we take

$$\varphi_1^1(t) = \varphi_2^1(t) = \varphi_3^1(t) = \frac{1}{28(t^3+1)}, \varphi_4^1(t) = \frac{t^4}{28\pi^4(t^3+1)}, \varphi_5^1(t) = \frac{t}{28\pi^2(t^3+1)}, \varphi_6^1(t) = \frac{t^3}{28\pi^2(t^3+1)}.$$

The corresponding supremum values are given by

$$\rho_1^1 = \rho_2^1 = \rho_3^1 = \frac{1}{28}, \quad \rho_4^1 = \frac{1}{28\pi^4}, \quad \rho_5^1 = \rho_6^1 = \frac{1}{28\pi^2}.$$

Similarly, from

$$\begin{aligned}
&|g_2(t, v_1, v_2, v_3, v_4, v_5, v_6) - g_2(t, w_1, w_2, w_3, w_4, w_5, w_6)| \\
&\leq \frac{1}{18e^{-t^3}}|v_1 - w_1| + \frac{t^3}{162\pi^3 e^{-t^3}}|v_2 - w_2| + \frac{1}{18e^{-t^3}}|v_3 - w_3| + \frac{t^2}{18\pi e^{-t^3}}|v_4 - w_4| \\
&+ \frac{1}{18e^{-t^3}}|v_5 - w_5| + \frac{t^5}{90\pi^4 e^{-t^3}}|v_6 - w_6|\Big),
\end{aligned}$$

we have

$$\varphi_1^2(t) = \varphi_3^2(t) = \varphi_5^2(t) = \frac{1}{18e^{-t^3}}, \quad \varphi_2^2(t) = \frac{t^3}{162\pi^3}e^{-t^3}, \varphi_4^2(t) = \frac{t^2}{18\pi e^{-t^3}}, \varphi_6^2(t) = \frac{t^5}{90\pi^4}e^{-t^3}.$$

The corresponding supremum values are:

$$\rho_1^2 = \rho_3^2 = \rho_5^2 = \frac{e}{18}, \quad \rho_2^2 = \frac{e}{162\pi^3}, \quad \rho_4^2 = \frac{e}{18\pi}, \quad \rho_6^2 = \frac{e}{90\pi^4}.$$

We set the functions

$$\begin{aligned}
h_1(t, v_1, v_2, v_3, v_4, v_5, v_6) &= \frac{e^{-t}}{10} + \frac{v_1|\cos(t)|}{t^2 + 20} + e^{-t}\frac{v_2|\cos(t)|}{3t^3 + 30} + e^{-t^5}\frac{v_3|\cos(t)|}{5t^4 + 40} + \frac{|{}^cD^{\frac{1}{2}}\cos v_4|}{2t^2 + 28} \\
&+ \frac{|{}^cD^{\frac{3}{2}}\cos v_5|}{6t^3 + 30} + \frac{|{}^cD^{\frac{1}{2}}\cos v_6|}{5t^4 + 45}, \\
h_2(t, v_1, v_2, v_3, v_4, v_5, v_6) &= \frac{e^{-t}}{\sqrt{60 + t^3}}\left(\frac{v_1}{3} + \frac{v_2}{7} + \frac{v_3}{5} + v_4 + v_5 + v_6\right).
\end{aligned}$$

So, for $(v_1, v_2, v_3, v_4, v_5, v_6), (w_1, w_2, w_3, w_4, w_5, w_6) \in \mathbb{R}^6$ *and* $t \in [0, 1]$, *we have*

$$\begin{aligned}
&|h_1(t, v_1, v_2, v_3, v_4, v_5, v_6) - h_1(t, w_1, w_2, w_3, w_4, w_5, w_6)| \\
&\leq \frac{1}{t^2 + 20}|v_1 - w_1| + \frac{1}{e^t(3t^3 + 30)}|v_2 - w_2| + \frac{1}{e^{t^5}(5t^4 + 40)}|v_3 - w_3| + \frac{1}{2t^2 + 28}|v_4 - w_4| \\
&+ \frac{1}{6t^3 + 30}|v_5 - w_5| + \frac{1}{5t^4 + 45}|v_6 - w_6|.
\end{aligned}$$

We have

$$\begin{aligned}
&\vartheta_1^1(t) = \frac{1}{t^2 + 20}, \quad \vartheta_2^1(t) = \frac{1}{e^t(3t^3 + 30)}, \quad \vartheta_3^1(t) = \frac{1}{e^{t^5}(5t^4 + 40)}, \quad \vartheta_4^1(t) = \frac{1}{2t^2 + 28}, \\
&\vartheta_5^1(t) = \frac{1}{6t^3 + 30}, \quad \vartheta_6^1(t) = \frac{1}{5t^4 + 45}.
\end{aligned}$$

The associated supremum values are:

$$\eta_1^1 = \frac{1}{20}, \quad \eta_2^1 = \frac{1}{30}, \quad \eta_3^1 = \frac{1}{40}, \quad \eta_4^1 = \frac{1}{28}, \quad \eta_5^1 = \frac{1}{30}, \quad \eta_6^1 = \frac{1}{45}.$$

From

$$|h_2(t,v_1,v_2,v_3,v_4,v_5,v_6) - h_2(t,w_1,w_2,w_3,w_4,w_5,w_6)| \leq \frac{e^{-t}}{3\sqrt{60+t^3}}|v_1 - w_1| + \frac{e^{-t}}{7\sqrt{60+t^3}}|v_2 - w_2|$$
$$+ \frac{e^{-t}}{5\sqrt{60+t^3}}|v_3 - w_3| + \frac{e^{-t}}{\sqrt{60+t^3}}|v_4 - w_4| + \frac{e^{-t}}{\sqrt{60+t^3}}|v_5 - w_5| + \frac{e^{-t}}{\sqrt{60+t^3}}|v_6 - w_6|,$$

we have

$$\vartheta_1^2(t) = \frac{e^{-t}}{3\sqrt{60+t^3}}, \quad \vartheta_2^2(t) = \frac{e^{-t}}{7\sqrt{60+t^3}}, \quad \vartheta_3^2(t) = \frac{e^{-t}}{5\sqrt{60+t^3}}, \quad \vartheta_4^2(t) = \vartheta_5^2(t) = \vartheta_6^2(t) = \frac{e^{-t}}{\sqrt{60+t^3}}.$$

The associated supremum values are:

$$\eta_1^2 = \frac{1}{6\sqrt{15}}, \quad \eta_2^2 = \frac{1}{14\sqrt{15}}, \quad \eta_3^2 = \frac{1}{10\sqrt{15}}, \quad \eta_4^2 = \eta_5^2 = \eta_6^2 = \frac{1}{2\sqrt{15}}.$$

It follows that

$$\max\left(\mathbf{C}_1 S_1, \mathbf{C}_2 S_2, \mathbf{C}_3 S_3, \mathbf{C}_4 S_1, \mathbf{C}_5 S_2, \mathbf{C}_6 S_3\right) < 1.$$

Therefore, by Theorem 2, we conclude that the solution to the tripled system (73) is a unique one. Also, the requirements of Theorem 4 are fulfilled. Therefore, it is H-U stable.

7. Conclusions

In this research, we investigated a tripled system of n-term NFDEs. We explored this novel class of differential equations, focusing on existence and stability results. We determined sufficient conditions for the existence of at least one unique solution by applying Schaefer's and Banach's fixed point theorems, respectively. Furthermore, by employing Hyers–Ulam stability analysis, we established criteria for the system's stability. The applicability of these main results is illustrated through a self-explanatory example. Tripled systems of n-term NFDEs have a wide range of applications. Notably, they can be applied to gene regulatory networks, epidemiology, the dynamics of three hormones in an endocrine system, three-species food chains, three-stage life cycles, microbial community dynamics, and so on.

Author Contributions: Conceptualization, M.S.A.; Software, O.O.; Formal analysis, A.M. and A.A. (Amer Alsulami); Investigation, M.S.A.; Resources, M.S.A.; writing—original draft preparation, A.A. (Arshad Ali); writing—review and editing, K.A. and A.A. (Amer Alsulami); Visualization, A.M.; Project administration, K.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for the financial support (QU-APC-2024-9/1).

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

1. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993.
2. ALazopoulos, K.A. Non-local continuum mechanics and fractional calculus. *Mech. Res. Commun.* **2006**, *33*, 753–757.
3. Carpinteri, A.; Cornetti, P.; Sapora, A. A fractional calculus approach to nonlocal elasticity. *Eur. Phys. J. Spec. Top.* **2011**, *193*, 193–204.
4. Riewe, F. Mechanics with fractional derivatives. *Phys. Rev. E* **1997**, *55*, 3581.
5. Rossikhin, Y.A.; Shitikova, M.V. Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. *Appl. Mech. Rev.*, **2010**, *63*, 010801.
6. Yang, X.J.; Machado, J.T. A new fractional operator of variable order: Application in the description of anomalous diffusion. *Phys. Stat. Mech. Its Appl.* **2017**, *481*, 276–283.
7. Ahmed, S.; Azar, A.T.; Tounsi, M. Design of adaptive fractional-order fixed-time sliding mode control for robotic manipulators. *Entropy* **2022**, *24*, 1838.
8. Hedrih, K.R. (Stevanović); Milovanović, G.V. Elements of mathematical phenomenology and analogies of electrical and mechanical oscillators of the fractional type with finite number of degrees of freedom of oscillations: Linear and nonlinear modes. *Commun. Anal. Mech.* **2024** *16*, 738–785.
9. Shah, K.; Naz, H.; Sarwar, M.; Abdeljawad, T. On spectral numerical method for variable-order partial differential equations. *Aims Math.* **2022**, *7*, 10422–10438.
10. Arik, İ. A.; Araz, S. İ. Delay differential equations with fractional differential operators: Existence, uniqueness and applications to chaos. *Commun. Anal. Mech.* **2024**, *16*, 169–192.
11. Hamza, A.; Osman, O.; Ali, A.; Alsulami, A.; Aldwoah, K.; Mustafa, A.; Saber, H. Fractal-Fractional-Order Modeling of Liver Fibrosis Disease and Its Mathematical Results with Subinterval Transitions. *Fractal Fract.* **2024**, *8*, 638.
12. Khan, H.; Ahmed, S.; Alzabut, J.; Azar, A.T. A generalized coupled system of fractional differential equations with application to finite time sliding mode control for Leukemia therapy. *Chaos, Solitons Fractals* **2023**, *174*, 113901.
13. Wang, Z.; Sun, L. The Allen-Cahn equation with a time Caputo-Hadamard derivative: Mathematical and Numerical Analysis. *Commun. Anal. Mech.* **2023** *15*, 611–637.
14. Ansari, K.J.; Asma; Ilyas, F.; Shah, K.; Khan, A.; Abdeljawad, T. On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative. *Waves Random Complex Media* **2023**, *2023*, 1–20.
15. Tian, Y.; Bai, Z. Existence results for the three-point impulsive boundary value problem involving fractional differential equations. *Comput. Math. Appl.* **2010**, *59*, 2601–2609.
16. Abdeljawad, T.; Mlaiki, N.; Abdo, M.S. Caputo-type fractional systems with variable order depending on the impulses and changing the kernel. *Fractals* **2022**, *30*, 2240219.
17. Ali, A.; Ansari, K.J.; Alrabaiah, H.; Aloqaily, A.; Mlaiki, N. Coupled system of fractional impulsive problem involving power-law kernel with piecewise order. *Fractal Fract.* **2023**, *7*, Artical ID 436.
18. Baghani, H.; Alzabut, J.; Farokhi-Ostad, J.; Nieto, J.J. Existence and uniqueness of solutions for a coupled system of sequential fractional differential equations with initial conditions. *J. Pseudo-Differ. Oper. Appl.* **2020**, *11*, 1731–1741.
19. Alraqad, T.; Almalahi, M.A.; Mohammed, N.; Alahmade, A.; Aldwoah, K.A.; Saber, H. Modeling Ebola Dynamics with a Piecewise Hybrid Fractional Derivative Approach. *Fractal Fract.* **2024**, *8*, 596.
20. Taieb, A.; Dahmani, Z. A coupled system of nonlinear differential equations involving m nonlinear terms. *Georgian Math. J.* **2016**, *23*, 447–458.
21. Madani, Y.A.; Rabih, M.N.A.; Alqarni, F.A.; Ali, Z.; Aldwoah, K.A.; Hleili, M. Existence, Uniqueness, and Stability of a Nonlinear Tripled Fractional Order Differential System. *Fractal Fract.* **2024**, *8*, 416. https://doi.org/10.3390/fractalfract8070416
22. Podlubny, L. Fractional Differential Equations. In *An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press: San Diego, CA, USA, 1999; Volume 198.
23. Bainov, D.D.; Simenonv, P.S. *Impulsive Differential Equations: Periodic Solutions and Applications*; Longman: Harlow, UK, 1993.
24. Ali, A.; Rabiei, F.; Shah, K. On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4760–4775
25. Pietsch, A. *History of Banach Spaces and Linear Operators*; Springer Science & Business Media: Cham, Switzerland, 2007.
26. Schaefer, H. Über die methode der a priori-schranken. *Math. Ann.* **1955**, *129*, 415–416.
27. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpath. J. Math.* **2010**, *26*, 103–107.