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Certain Novel Fixed-Point Theorems Applied to Fractional Differential Equations

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Abstract: In this paper, we introduce a new class of contractions in normed spaces, referred to as generalized enriched Kannan contractions. These contractions expand the familiar enriched Kannan contractions to three-point versions, broadening the scope of Kannan contractions. These mappings are typically discontinuous, except at the fixed points, where they exhibit continuity, similar to enriched Kannan mappings. However, through suitable examples, we demonstrate that these two classes of mappings are distinct from one another. We present new results for generalized enriched Kannan contractions. Additionally, by incorporating conditions of continuity and asymptotic regularity, we extend the class of operators to which fixed-point methods can be applied. Additionally, we derive two more results for generalized enriched Kannan contractions in normed spaces, without the requirement that they be Banach spaces. Finally, we use our main result to demonstrate the existence of solutions for a boundary value problem involving a fractional differential equation.

Keywords: perimeters of triangles; generalized Kannan operator; Krasnoselskii iterations; fixed point; *G*-enriched Kannan operators; boundary value problem; fractional differential equation

MSC: 47H10; 54H25

Citation: Din, M.; Ishtiaq, U.; Alnowibet, K.A.; Lazăr, T.A.; Lazăr, V.L.; Guran, L. Certain Novel Fixed-Point Theorems Applied to Fractional Differential Equations. *Fractal Fract.* **2024**, *8*, 701. https://doi.org/10.3390/fractalfract8120701

Academic Editor: Ivanka Stamova

Received: 29 September 2024
Revised: 18 November 2024
Accepted: 25 November 2024
Published: 27 November 2024

1. Introduction

In 2022, Argyros [1] worked on different iteration methods and presented several applications. Recently, Argyros et al. [2] presented recent advancements in theory and applications related to contemporary algorithms. The main result given by Banach is of profound significance in metric fixed-point theory. This theorem is essential in mathematics for solving both linear and nonlinear ordinary differential equations, as well as integral equations. Additionally, it has significant applications in related areas such as physics, fractal theory, biology, image processing and engineering. It is elegantly enumerated below:

Theorem 1. *Suppose a mapping $F^o : \mathfrak{Q} \to \mathfrak{Q}$ on a complete metric space $(\mathfrak{Q}, \rho)$ for which $\forall \varkappa^o, \mu \in \mathfrak{Q}$, and the inequality $\rho(F^o \varkappa^o, F^o \mu) \leq \beta \rho(\varkappa^o, \mu)$ holds, where $0 \leq \beta < 1$. Then, the fixed point of F^o is unique in $\mathfrak{Q}$.*

The Banach theorem has been studied extensively and expanded using a variety of approaches. These approaches include expanding the domain of the mapping and introducing more generalized criteria for contraction mappings. Nadler investigated multivalued contraction mappings, while Kannan pioneered results on fixed points with his renowned Kannan contractions. Dominguez et al. [3] investigated generalizations of the Kannan theorem in K-metric spaces. Jleli and Samet [4] developed a new extension of the Banach contraction principle. Ahmad et al. [5] examined fractals connected to generalized $\boldsymbol{\Theta}$-Hutchinson operators. Shioji et al. [6] explored Kannan mappings with metric completeness. Abbas et al. [7] investigated generalized enriched cyclic contractions applied to iterated function systems (IFSs). Almalki et al. [8] examined the novel generalizations of Perov type results. Shukla and Radenović [9] investigated Prešić–Boyd–Wong theorems. Tarafdar [10] presented a new result in uniform spaces. Ilić et al. [11] offered iterative approximations of fixed points for Prešić mappings in partial metric spaces. Wardowski and Dung [12] established fixed points for F-weak contractions. Din et al. [13] concentrated on Perov results for F-contraction mappings. Berinde and Păcurar [14] approximated the fixed points of enriched contractions in Banach spaces. Anjum et al. [15] explored fractals associated with two types of enriched Hutchinson–Barnsley transformations. Din et al. [16] also examined generalized Sehgal–Guseman-like contractions and their applications. Additionally, Anjum et al. [17] investigated applications to activation functions using symmetric contractions.

In [18], Kannan established a result regarding the fixed point for discontinuous mappings. Let $\digamma^o : \mathfrak{Q} \to \mathfrak{Q}$ be a mapping on a complete metric space $(\mathfrak{Q}, \rho)$ such that

$$\rho(\digamma^o \varkappa^o, \digamma^o \mu) \leq \theta[\rho(\varkappa^o, \digamma^o \varkappa^o) + \rho(\mu, \digamma^o \mu)], \quad (1)$$

where $0 \leq \theta < \frac{1}{2}$ and $\varkappa^o, \mu \in \mathfrak{Q}$. In this case, $\digamma^o$ has a unique fixed point. If $\exists 0 \leq \theta < \frac{1}{2}$, for which the inequality (1) holds for all $\varkappa^o, \mu \in \mathfrak{Q}$, then $\digamma^o$ is referred to as a Kannan mapping.

Subrahmanyam [19] demonstrated that Kannan's result offers a characterization of metric completeness. Every Kannan-type mapping on $\mathfrak{Q}$ possesses a fixed point if and only if metric space $\mathfrak{Q}$ is complete. It is crucial to observe that Banach contractions do not provide a characterization of completeness. For example, there are metric spaces $\mathfrak{Q}$ that are incomplete, yet every contraction mapping on $\mathfrak{Q}$ still possesses a fixed point, as demonstrated in [20]. For a thorough comparison between Banach contractions and Kannan-type mappings, consult the following studies. Suzuki [21] investigated the differences between contractive and Kannan contractions, while Kikkawa and Suzuki [22] explored the similarities between these two types of contractions. Berinde [23,24] approximated the fixed points of weak contractions using the Picard iteration method. In [25], Petrov introduced a novel type of mapping characterized by its ability to contract the perimeters of triangles. In [26], he applied this concept to construct generalized Kannan-type mappings. Berinde and Păcurar [27] explored enriched Kannan operators and their applications to split feasibility and variational inequality problems.

In [25], an innovative mapping approach was proposed, distinguished by its ability to contract the perimeters of triangles.

Definition 1. *Let $(\mathfrak{Q}, \rho)$ with $|\mathfrak{Q}| \geq 3$ be a metric space. We define a mapping $\digamma^o : \mathfrak{Q} \to \mathfrak{Q}$ as one that contracts the distances between the vertices of any triangle, provided $\exists \beta \in [0, 1)$, for which the following inequality is valid for all distinct points $\varkappa^o$, μ, and ϖ in $\mathfrak{Q}$:*

$$\rho(\digamma^o \varkappa^o, \digamma^o \mu) + \rho(\digamma^o \mu, \digamma^o \varpi) + \rho(\digamma^o \varkappa^o, \digamma^o \varpi) \leq \beta(\rho(\varkappa^o, \mu) + \rho(\mu, \varpi) + \rho(\varkappa^o, \varpi)). \quad (2)$$

Remark 1. *It is important to note that requiring $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$ to be pairwise distinct is crucial. Without this condition, the definition would be analogous to that of a standard contraction mapping.*

A theorem on fixed points has been proven for these mappings. Although the proof utilizes ideas from Banach's classic theorem, it differs fundamentally in that it deals with three points in the space rather than two. Furthermore, an additional assumption is imposed to prevent these operators from having periodic points with prime period 2. It is important to highlight that ordinary contraction mappings are a notable subset of these mappings.

Building on [25], Petrov [26] proposed the extended version of Kannan-type mappings, defined as follows:

Definition 2. *Suppose that* $(\mathfrak{Q}, \rho)$ *with* $|\mathfrak{Q}| \geq 3$ *be a metric space. A mapping* $F^o : \mathfrak{Q} \to \mathfrak{Q}$ *is called a generalized Kannan mapping if* $\exists 0 \leq \theta < \frac{2}{3}$*, for which the following inequality holds:*

$$\rho(F^o \varkappa^o, F^o \mu) + \rho(F^o \mu, F^o \varpi) + \rho(F^o \varkappa^o, F^o \varpi) \leq \theta(\rho(\varkappa^o, F^o \varkappa^o) + \rho(\mu, F^o \mu) + \rho(\varpi, F^o \varpi)) \quad (3)$$

for all three distinct points $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$.

In recent studies, many researchers have focused on extending classical contractions to a three-point analogue, which involves contracting the perimeters of triangles formed by three distinct points. In [28], Pacurar and Popescu transformed the Chatterjea contraction into a three-point analogue, while Bisht and Petrov, in [29], explored a three-point extension of Chatterjea's fixed-point theorem, proving the existence of at most two fixed points. Recently, Zhou and Petrov [30] extended classical Banach and Kannan contractions to k-polygons, further generalizing the concept of three-point analogues. They examined the connections between Banach contractions, generalized Kannan-type mappings, and mappings that contract the perimeters of k-polygons.

In 2020, Berinde and Păcurar [14] presented a new and expansive category of contraction operators termed enriched contractions. Such a category of operators includes not only Banach contractions but also a variety of other non-expansive operators described in the literature. Their research showed that every enriched contraction operator owns a unique fixed point, that can be approximated through a suitable Krasnoselskii iteration method in normed spaces.

Definition 3. *Suppose a normed space* $(\mathfrak{Q}, |\cdot|)$ *and operator* $F^o : \mathfrak{Q} \to \mathfrak{Q}$*. The operator* F^o *is said to be an enriched operator if* $\exists \mathfrak{s} \geq 0$ *and* $\kappa \in [0, \mathfrak{s}+1)$*, for which the following holds:*

$$|\mathfrak{s}(\varkappa^o - \mu) + F^o \varkappa^o - F^o \mu| \leq \kappa |\varkappa^o - \mu|, \quad \varkappa^o, \mu \in \mathfrak{Q}. \quad (4)$$

Enriched contraction mappings are highly significant because they include both non-expansive mappings and the contraction operators. Non-expansive mappings do not always guarantee fixed points, as their existence is not assured for every mapping. However, enriched contraction mappings uniquely ensure the existence of a single fixed point for each mapping. This assurance of uniqueness distinguishes enriched contraction mappings from non-expansive ones. It is noteworthy that the sets of fixed points for F^o and ${F^o}_\kappa$ are identical, if $\kappa \in (0, 1]$.

Using the enriched technique, Berinde and Păcurar [27] proposed the novel concept of enriched Kannan operators as follows:

Definition 4. *Consider a normed space* $(\mathfrak{Q}, ||\cdot||)$ *and operator* $F^o : \mathfrak{Q} \to \mathfrak{Q}$*. The operator* F^o *is referred to as an enriched Kannan operator if there exist parameters* $\theta \in [0, \frac{1}{2})$ *and* $\mathfrak{s} \in [0, \infty)$*, for which the following inequality is satisfied:*

$$||\mathfrak{s}(\varkappa^o - \mu) + F^o \varkappa^o - F^o \mu|| \leq \theta(\|\varkappa^o - F^o \varkappa^o\| + \|\mu - F^o \mu\|), \quad \forall \varkappa^o, \mu \in \mathfrak{Q}. \quad (5)$$

Inspired by Petrov's work [25,26] on mappings that contract the perimeters of triangles formed by three distinct points, and by Berinde and Păcurar's study [14,27] on enriched contractions, we introduced the concept of generalized enriched Kannan mappings. This new category extends the idea of enriched Kannan mappings [27] to a three-point analogue, merging these foundational studies to advance the theory of fixed-point mappings. The generalized enriched Kannan mappings represent a notable advancement in fixed-point theory by extending the established concept of enriched Kannan mappings to a three-point analogue and also extending the generalized Kannan contractions [26]. This new class of mappings offers a deeper insight into the relationship between continuity and discontinuity, as they generally exhibit discontinuity but preserve continuity at fixed points, much like enriched Kannan mappings. By developing a fixed-point theorem for these generalized operators and incorporating conditions of continuity and asymptotic regularity, we significantly expand the scope of mappings to which fixed-point theorems can be applied. Building on the foundational research of Berinde and Păcurar, our results introduce two additional fixed-point theorems that extend the theory to normed spaces, going beyond the limitations of Banach spaces. At the end, we apply these results to establish the existence of solutions for a boundary value problem involving a fractional differential equation. This research presents a novel class of mappings that greatly broadens the scope of fixed-point theory, offering robust tools for investigating problems across diverse mathematical and applied domains. By broadening fixed-point results to more general settings, this work enhances the theoretical foundation while also increasing the practical applicability of fixed-point theorems. These developments create new opportunities for addressing a variety of equations and systems, emphasizing their significance in both theoretical progress and practical applications.

2. Main Results

In the following sections, we present our main definitions and results, which offer new insights into the properties of generalized enriched Kannan operators.

Definition 5. *Suppose that* $(\mathfrak{Q}, ||\cdot||)$, *with* $|\mathfrak{Q}| \geq 3$ *being a normed space. A self mapping* $\digamma^o$ *on* $\mathfrak{Q}$ *is referred to as a generalized enriched Kannan operator if there are parameters* $\mathfrak{s} \geq 0$ *and* $\beta \in [0, \frac{2}{3})$ *such that*

$$||\mathfrak{s}(\varkappa^o - \mu) + \digamma^o\varkappa^o - \digamma^o\mu|| + ||\mathfrak{s}(\mu - \varpi) + \digamma^o\mu - \digamma^o\varpi|| + ||\mathfrak{s}(\varkappa^o - \varpi) + \digamma^o\varkappa^o - \digamma^o\varpi|| \\ \leq \beta(||\varkappa^o - \digamma^o\varkappa^o|| + ||\mu - \digamma^o\mu|| + ||\varpi - \digamma^o\varpi||) \quad (6)$$

meets for all the mutually distinct points $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$.

To emphasize the constants in (6), we will refer to it as a generalized $(\mathfrak{s}, \beta)$ enriched Kannan mapping. To illustrate the relationship between the generalized Kannan mappings [26] and generalized enriched Kannan mappings (Definition 5), consider the following example. Note that any generalized Kannan mapping is a generalized $(0, \beta)$ enriched Kannan mapping, as it meets the condition with $\mathfrak{s} = 0$.

Example 1. *Any generalized Kannan mapping qualifies as a generalized* $(0, \beta)$ *enriched Kannan mapping. Specifically, it satisfies the condition* (3) *with* $\mathfrak{s} = 0$. *This indicates that every generalized Kannan mapping is a special case within the broader category of generalized enriched Kannan mappings, where* $\mathfrak{s}$ *takes on the value of zero.*

However, the converse of the above example is not generally true; that is, not every generalized enriched Kannan mapping is necessarily a generalized Kannan mapping, as can be seen in the following example.

Example 2. *Let $\mathfrak{Q} = \{\varkappa^o, \mu, \varpi\}$ with $||\varkappa^o - \mu|| = 1, ||\mu - \varpi|| = 4, ||\varkappa^o - \varpi|| = 4$. Define $\digamma^o : \mathfrak{Q} \to \mathbb{R}$ such that $\digamma^o\varkappa^o = \varkappa^o, \digamma^o\mu = \mu$ and $\digamma^o\varpi = 2\varkappa^o - \varpi$. Then, being non-self mapping, $\digamma^o$ cannot be a generalized Kannan mapping, but for $\mathfrak{s} = 1$, we obtain $\kappa = \frac{1}{2}$ and the mapping $\digamma^o{}_\kappa : \mathfrak{Q} \to \mathfrak{Q}$ given by $\digamma^o{}_\kappa\varkappa^o = \varkappa^o, \digamma^o{}_\kappa\mu = \mu$, and $\digamma^o{}_\kappa\varpi = \varkappa^o$. It is straightforward to observe that inequality (6) holds with $\beta = \frac{1}{2}$. Thus, $\digamma^o$ is a generalized $(1, \frac{1}{2})$ enriched Kannan operator.*

In the following, we demonstrate that the families of enriched Kannan operators [27] and generalized enriched Kannan operators are different from each other.

Example 3. *Consider $\mathfrak{Q} = [0,1]$ and $||\cdot||$ as a usual norm on $\mathfrak{Q}$. Take an operator $\digamma^o : \mathfrak{Q} \to \mathfrak{Q}$ by $\digamma^o(\varkappa^o) = \frac{2\varkappa^o - m\varkappa^o}{m}$, for some $m > 1$. If we choose $\mathfrak{s} = 1$, then $\kappa = \frac{1}{2}$, so we obtain $\digamma^o{}_{\frac{1}{2}}(\varkappa^o) = \frac{\varkappa^o}{m}$. Without restricting generality, consider $\varkappa^o \geq \mu$; using (5) for this mapping, we obtain*

$$\begin{aligned} \frac{\varkappa^o}{m} - \frac{\mu}{m} &\leq \beta(\varkappa^o - \tfrac{\varkappa^o}{m} + \mu - \tfrac{\mu}{m}) \\ \Longrightarrow \varkappa^o - \mu &\leq \beta(m-1)(\varkappa^o + \mu). \end{aligned} \tag{7}$$

It is noted that (7) only holds for $\varkappa^o \geq \mu$ if $\beta(m-1) \geq 1$. Let us consider the system of inequalities:

$$\begin{cases} 0 \leq \beta < \frac{1}{2}, \\ \beta(m-1) \geq 1. \end{cases}$$

Then, we obtain

$$\frac{1}{m-1} \leq \beta < \frac{1}{2}.$$

So that $\digamma^o$ is an enriched Kannan contraction if $m > 3$.

Next, suppose, without restricting generality, that $\varkappa^o > \mu > \varpi \in \mathfrak{Q}$, and using the inequality (6) for all mutually distinct points $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$:

$$\begin{aligned} \frac{1}{m}(\varkappa^o - \mu + \varkappa^o - \varpi + \mu - \varpi) &\leq \beta(1 - \tfrac{1}{m})(\varkappa^o + \mu + \varpi) \\ \Longrightarrow (2\varkappa^o - 2\varpi) &\leq \beta(m-1)(\varkappa^o + \mu + \varpi) \\ \Longrightarrow (\varkappa^o - \varpi) &\leq \tfrac{\beta}{2}(m-1)(\varkappa^o + \mu + \varpi) \end{aligned} \tag{8}$$

It is worth noting that (8) is applicable only for $\varkappa^o > \mu > \varpi$ iff $\frac{\beta}{2}(m-1) \geq 1$. If we take the system of inequalities

$$\begin{cases} 0 \leq \beta < \frac{2}{3}, \\ \beta(m-1) \geq 2. \end{cases}$$

Then, we obtain

$$\frac{2}{m-1} \leq \beta < \frac{2}{3}. \tag{9}$$

Hence, $\digamma^o$ is a generalized enriched Kannan operator if $m > 4$. Thus, for $m \in (3,4]$, the operator $\digamma^o$ is an enriched Kannan operator and not a generalized enriched Kannan operator.

In the subsequent discussion, we investigate the connection between generalized enriched Kannan operators and enriched Kannan operators that contract the perimeters of triangles. Furthermore, we present an example of a discontinuous generalized enriched Kannan operator.

Proposition 1. *Every enriched Kannan operator for $\beta \in [0, \frac{1}{3})$ on a normed space $(\mathfrak{Q}, ||\cdot||)$ with $|\mathfrak{Q}| \geq 3$ is also a generalized enriched Kannan operator.*

Proof. By using the inequality (5), for the pairs μ, ϖ and $\varkappa^o, \varpi$, we obtain

$$||\mathfrak{s}(\mu - \varpi) + F^o\mu - F^o\varpi|| \leq \beta(||\mu - F^o\mu|| + ||\varpi - F^o\varpi||), \tag{10}$$

$$||\mathfrak{s}(\varkappa^o - \varpi) + F^o\varkappa^o - F^o\varpi|| \leq \beta(||\varkappa^o - F^o\varkappa^o|| + ||\varpi - F^o\varpi||). \tag{11}$$

Adding the inequalities (5), (10) and (11), we obtain

$$\begin{aligned} &||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| + ||\mathfrak{s}(\mu - \varpi) + F^o\mu - F^o\varpi|| + ||\mathfrak{s}(\varkappa^o - \varpi) + F^o\varkappa^o - F^o\varpi|| \\ &\leq 2\beta(||\varkappa^o - F^o\varkappa^o|| + ||\mu - F^o\mu|| + ||\varpi - F^o\varpi||). \end{aligned}$$

Thus, the proof is concluded. □

Proposition 2. *Let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be a generalized enriched Kannan operator with $\beta \in [0, \frac{2}{3})$. If F^o is continuous at $\varkappa^o$ and $\varkappa^o$ is an accumulation point in $\mathfrak{Q}$, then the following inequality must be fulfilled:*

$$||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| \leq \beta\left(||\varkappa^o - F^o\varkappa^o|| + \frac{||\mu - F^o\mu||}{2}\right), \quad \forall \mu \in \mathfrak{Q}. \tag{12}$$

Proof. Suppose $\varkappa^o \in \mathfrak{Q}$ is an accumulation point of $\mathfrak{Q}$ and let $\mu \in \mathfrak{Q}$ be any other element. If $\varkappa^o = \mu$, then (12) obviously holds. So, let $\varkappa^o \neq \mu$. Due to the accumulation point of $\varkappa^o$, there exists a sequence of distinct points $\varpi_n \in \mathfrak{Q}$ such that $\varpi_n \to \varkappa^o$, where $\varpi_n \neq \varkappa^o$ and $\varpi_n \neq \mu$. Hence, using inequality (6), we have

$$\begin{aligned} &||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| + ||\mathfrak{s}(\mu - \varpi_n) + F^o\mu - F^o\varpi_n|| + ||\mathfrak{s}(\varkappa^o - \varpi_n) + F^o\varkappa^o - F^o\varpi_n|| \\ &\leq \beta(||\varkappa^o - F^o\varkappa^o|| + ||\mu - F^o\mu|| + ||\varpi_n - F^o\varpi_n||), \quad \forall n \in \mathbf{N}. \end{aligned} \tag{13}$$

Given that $\varpi_n \to \varkappa^o$ and F^o is continuous at $\varkappa^o$, so $F^o\varpi_n \to F^o\varkappa^o$. Therefore, letting $n \to \infty$ in (13), we obtain (12). □

In the next corollary, we provide a sufficient condition for a generalized enriched Kannan operator to be an enriched Kannan operator, which implies the connection between both operators.

Corollary 1. *Consider a normed space $(\mathfrak{Q}, ||\cdot||)$ and operator $F^o : \mathfrak{Q} \to \mathfrak{Q}$, which is a continuous generalized enriched Kannan operator with all points in $\mathfrak{Q}$ being accumulation points. Then, F^o is an enriched Kannan operator.*

Proof. Based on Proposition (2), inequality (12) is satisfied. Thus, the following inequality also holds:

$$||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| \leq \beta\left(||\mu - F^o\mu|| + \frac{||\varkappa^o - F^o\varkappa^o||}{2}\right). \tag{14}$$

From inequalities (12) and (14), we obtain

$$||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| \leq \frac{3\beta}{4}(||\varkappa^o - F^o\varkappa^o|| + ||\mu - F^o\mu||).$$

where $\frac{3\beta}{4}$ belongs to the interval $[0, \frac{1}{2})$ given that β is within $[0, \frac{2}{3})$. This concludes the proof. □

Proposition 3. *Consider a normed space $(\mathfrak{Q}, ||\cdot||)$ with $|\mathfrak{Q}| \geq 3$ and the operator $\digamma^o : \mathfrak{Q} \to \mathfrak{Q}$ as an enriched mapping that contracts the perimeters of triangles for any $0 \leq \beta < \frac{1}{4}$. Then, $\digamma^o$ is a generalized enriched Kannan operator for the norm $\|\cdot\|$.*

Proof. By iteratively using the triangle inequality on the right-hand side of (4), we derive

$$
\begin{aligned}
&||\digamma^o{}_\kappa \varkappa^o - \digamma^o{}_\kappa \mu|| + ||\digamma^o{}_\kappa \mu - \digamma^o{}_\kappa \varpi|| + ||\digamma^o{}_\kappa \varkappa^o - \digamma^o{}_\kappa \varpi|| \\
&\leq \beta(||\varkappa^o - \digamma^o{}_\kappa \varkappa^o|| + ||\digamma^o{}_\kappa \varkappa^o - \mu|| + ||\mu - \digamma^o{}_\kappa \mu|| + ||\varpi - \digamma^o{}_\kappa \mu|| + ||\varpi - \digamma^o{}_\kappa \varpi|| + ||\varkappa^o - \digamma^o{}_\kappa \varpi||) \\
&\leq \beta(||\varkappa^o - \digamma^o{}_\kappa \varkappa^o|| + ||\digamma^o{}_\kappa \varkappa^o - \digamma^o{}_\kappa \mu|| + ||\digamma^o{}_\kappa \mu - \mu|| + ||\mu - \digamma^o{}_\kappa \mu|| + ||\digamma^o{}_\kappa \varpi - \digamma^o{}_\kappa \mu|| \\
&\quad + ||\digamma^o{}_\kappa \varpi - \varpi|| + ||\varpi - \digamma^o{}_\kappa \varpi|| + ||\digamma^o{}_\kappa \varkappa^o - \digamma^o{}_\kappa \varpi|| + ||\digamma^o{}_\kappa \varkappa^o - \varkappa^o||).
\end{aligned}
$$

By rearranging the inequality, we obtain

$$
\begin{aligned}
&||\digamma^o{}_\kappa \varkappa^o - \digamma^o{}_\kappa \mu|| + ||\digamma^o{}_\kappa \mu - \digamma^o{}_\kappa \varpi|| + ||\digamma^o{}_\kappa \varkappa^o - \digamma^o{}_\kappa \varpi|| \\
&\leq \tfrac{2\beta}{1-\beta}(||\varkappa^o - \digamma^o{}_\kappa \varkappa^o|| + ||\mu - \digamma^o{}_\kappa \mu|| + ||\varpi - \digamma^o{}_\kappa \varpi||) \\
\Longrightarrow\ &||\mathfrak{s}(\varkappa^o - \mu) + \digamma^o \varkappa^o - \digamma^o \mu|| + ||\mathfrak{s}(\mu - \varpi) + \digamma^o \mu - \digamma^o \varpi|| + ||\mathfrak{s}(\varkappa^o - \varpi) + \digamma^o \varkappa^o - \digamma^o \varpi|| \\
&\leq \tfrac{2\beta}{1-\beta}(||\varkappa^o - \digamma^o{}_\kappa \varkappa^o|| + ||\mu - \digamma^o{}_\kappa \mu|| + ||\varpi - \digamma^o{}_\kappa \varpi||),
\end{aligned}
$$

for all mutually distinct elements $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$. Since $\beta' = \frac{2\beta}{1-\beta} \in (0, \frac{2}{3})$, it follows that $\digamma^o$ qualifies as a generalized enriched Kannan operator. □

In the example below, we establish that, in general, the generalized enriched Kannan operators are discontinuous, as with enriched Kannan operators.

Example 4. *Let $\mathfrak{Q} = [0,1]$ with the standard norm $\|\cdot\|$. Consider a mapping $\digamma^o : \mathfrak{Q} \to \mathfrak{Q}$ defined by*

$$
\digamma^o(\varkappa^o) = \begin{cases} \frac{2\varkappa^o - m_1 \varkappa^o}{m_1}, & \varkappa^o \in [0, \frac{1}{2}] \\ \frac{2\varkappa^o - m_2 \varkappa^o}{m_2}, & \varkappa^o \in (\frac{1}{2}, 1], \end{cases}
$$

where $m_1, m_2 > 1$ and $m_1 \neq m_2$. It is evident that $\varkappa^o = \frac{1}{2}$ is a discontinuity point for the mapping $\digamma^o$. If we set $\mathfrak{s} = 1$, then $\kappa = \frac{1}{2}$ and we obtain the following operator:

$$
\digamma^o{}_{\frac{1}{2}}(\varkappa^o) = \begin{cases} \frac{\varkappa^o}{m_1}, & \varkappa^o \in [0, \frac{1}{2}] \\ \frac{\varkappa^o}{m_2}, & \varkappa^o \in (\frac{1}{2}, 1], \end{cases} .
$$

Next, we demonstrate that there exist values of m_1 and m_2 such that inequality (6) is satisfied for all distinct elements $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$, for some β where $0 \leq \beta < \frac{2}{3}$. This implies that $\digamma^o$ qualifies as a generalized enriched Kannan operator. To address this, assume $\varkappa^o > \mu > \varpi$ without restricting generality. Clearly, we only need to examine the following four scenarios:

1. $\varkappa^o, \mu, \varpi \in [0, \frac{1}{2}]$;
2. $\varkappa^o, \mu, \varpi \in (\frac{1}{2}, 1]$;
3. $\mu, \varpi \in [0, \frac{1}{2}], \varkappa^o \in (\frac{1}{2}, 1]$;
4. $\varpi \in [0, \frac{1}{2}], \mu, \varkappa^o \in (\frac{1}{2}, 1]$.

From the previous Example (3), we observe that cases 1 and 2 lead to the restrictions given by inequality (9) and

$$
\frac{2}{m_2 - 1} \leq \beta < \frac{2}{3}. \tag{15}
$$

Next, for simplicity, let us assume $m_1 > m_2$ and use the inequality (6) for case 3:

$$
\left(\frac{\varkappa^o}{m_2} - \frac{\mu}{m_1} + \frac{\varkappa^o}{m_2} - \frac{\varpi}{m_2} + \frac{\mu}{m_1} - \frac{\varpi}{m_1} \right) \leq \beta \left(\varkappa^o - \frac{\varkappa^o}{m_2} + \mu - \frac{\mu}{m_1} + \varpi - \frac{\varpi}{m_1} \right).
$$

So that,

$$0 \leq \varkappa^o\left[\beta - \frac{\beta}{m_2} - \frac{2}{m_2}\right] + \mu\left[\beta - \frac{\beta}{m_1}\right] + \varpi\left[\beta - \frac{\beta}{m_1} + \frac{2}{m_1}\right]. \quad (16)$$

For case 4, using the inequality (6), we have

$$\left(\frac{\varkappa^o}{m_2} - \frac{\mu}{m_2} + \frac{\varkappa^o}{m_2} - \frac{\varpi}{m_1} + \frac{\mu}{m_2} - \frac{\varpi}{m_1}\right) \leq \beta\left(\varkappa^o - \frac{\varkappa^o}{m_2} + \mu - \frac{\mu}{m_2} + \varpi - \frac{\varpi}{m_1}\right).$$

Hence,

$$0 \leq \varkappa^o\left[\beta - \frac{\beta}{m_2} - \frac{2}{m_2}\right] + \mu\left[\beta - \frac{\beta}{m_2}\right] + \varpi\left[\beta - \frac{\beta}{m_1} - \frac{2}{m_1}\right]. \quad (17)$$

Clearly, for every $0 \leq \beta < \frac{2}{3}$, there exist very large values of m_1 and m_2 with $m_1 \neq m_2$, such that inequalities (15), (16), and (17) are satisfied simultaneously. This shows that $\digamma^o$ is a discontinuous generalized enriched Kannan operator.

Recall that a point $\varkappa^o \in \mathfrak{Q}$ is referred to as a periodic point of period $\mathfrak{q}$ for a mapping $\digamma^o$ if $\digamma^{o\mathfrak{q}}(\varkappa^o)$ returns $\varkappa^o$. The smallest positive integer $\mathfrak{q}$ for which $\digamma^{o\mathfrak{q}}(\varkappa^o) = \varkappa^o$ is known as the prime period of the point $\varkappa^o$. Indeed, a point $\varkappa^o$ has a prime period of 2 if $\digamma^{o2}(\varkappa^o) = \digamma^o(\digamma^o(\varkappa^o)) = \varkappa^o$ but $\digamma^o(\varkappa^o) \neq \varkappa^o$.

Remark 2. *For a generalized enriched Kannan operator $\digamma^o$, and any $\kappa \in [0,1]$, the operator $\digamma^o{}_\kappa$ cannot possess periodic points with a prime period of 3. In particular, if for some $\varkappa^o \in \mathfrak{Q}$, $\digamma^o{}_\kappa(\varkappa^o) = \mu, \mu \neq \varkappa^o, \digamma^o{}_\kappa(\mu) = \varpi, \varpi \neq \mu \neq \varkappa^o, \digamma^o{}_\kappa(\varpi) = \varkappa^o$. Then, we obtain the following equation:*

$$||\digamma^o{}_\kappa\varkappa^o - \digamma^o{}_\kappa\mu|| + ||\digamma^o{}_\kappa\mu - \digamma^o{}_\kappa\varpi|| + ||\digamma^o{}_\kappa\varkappa^o - \digamma^o{}_\kappa\varpi||$$
$$= ||\varkappa^o - \digamma^o{}_\kappa\varkappa^o|| + ||\mu - \digamma^o{}_\kappa\mu|| + ||\varpi - \digamma^o{}_\kappa\varpi||,$$

which contradicts inequality (6).

Next, we stated our first fixed point theorem as follows:

Theorem 2. *Let $(\mathfrak{Q}, ||\cdot||)$ be a Banach space with $|\mathfrak{Q}| \geq 3$. Suppose the mapping $\digamma^o : \mathfrak{Q} \to \mathfrak{Q}$ fulfills the following assumptions:*

(i) *$\digamma^o{}_\kappa$ lacks periodic points with a prime period of 2, where $\kappa = \frac{1}{\mathfrak{s}+1}$;*

(ii) *$\digamma^o$ is a generalized enriched Kannan operator on $\mathfrak{Q}$.*

In this case, $\digamma^o$ must have a fixed point, and there can be no more than two fixed points.

Proof. From assumption (ii), we can write for all mutually distinct elements $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$

$$||\digamma^o{}_\kappa\varkappa^o - \digamma^o{}_\kappa\mu|| + ||\digamma^o{}_\kappa\mu - \digamma^o{}_\kappa\varpi|| + ||\digamma^o{}_\kappa\varkappa^o - \digamma^o{}_\kappa\varpi||$$
$$\leq \beta(||\varkappa^o - \digamma^o{}_\kappa\varkappa^o|| + ||\mu - \digamma^o{}_\kappa\mu|| + ||\varpi - \digamma^o{}_\kappa\varpi||). \quad (18)$$

Let $\varkappa^o{}_0 \in \mathfrak{Q}$ and consider the iterative sequence defined by $\varkappa^o{}_{\mathfrak{q}+1} = \digamma^o\varkappa^o{}_{\mathfrak{q}} = (1-\kappa)\varkappa^o{}_{\mathfrak{q}} + \kappa\digamma^o\varkappa^o{}_{\mathfrak{q}} = \digamma^o{}_\kappa\varkappa^o{}_{\mathfrak{q}}, \quad \forall \mathfrak{q} \in \mathbf{N}$. Let $\varkappa^o{}_{\mathfrak{q}}$ be such that it is not a fixed point of $\digamma^o$ for all $\mathfrak{q} = 0, 1, \ldots$. Since $\varkappa^o{}_{\mathfrak{q}-1}$ is not a fixed point, it implies that $\varkappa^o{}_{\mathfrak{q}-1} \neq \varkappa^o{}_{\mathfrak{q}}$ because $\varkappa^o{}_{\mathfrak{q}} = \digamma^o{}_\kappa(\varkappa^o{}_{\mathfrak{q}-1})$. Based on assumption (i), $\varkappa^o{}_{\mathfrak{q}+1} = \digamma^o{}_\kappa(\digamma^o{}_\kappa(\varkappa^o{}_{\mathfrak{q}-1})) \neq \varkappa^o{}_{\mathfrak{q}-1}$. Furthermore, because $\varkappa^o{}_{\mathfrak{q}}$ is not a fixed point of operator $\digamma^o$, this yields $\varkappa^o{}_{\mathfrak{q}} \neq \varkappa^o{}_{\mathfrak{q}+1}$, where $\varkappa^o{}_{\mathfrak{q}+1} = \digamma^o{}_\kappa(\varkappa^o{}_{\mathfrak{q}})$. Therefore, $\varkappa^o{}_{\mathfrak{q}-1}$, $\varkappa^o{}_{\mathfrak{q}}$, and $\varkappa^o{}_{\mathfrak{q}+1}$ are all distinct from one another. Put $x = x_{\mathfrak{q}-1}, y = x_{\mathfrak{q}}, z = x_{\mathfrak{q}+1}$ in (6). Then,

$$||F^o{}_\kappa \varkappa^o{}_{q-1} - F^o{}_\kappa \varkappa^o{}_q|| + ||F^o{}_\kappa \varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_{q+1}|| + ||F^o{}_\kappa \varkappa^o{}_{q-1} - F^o{}_\kappa \varkappa^o{}_{q+1}||$$
$$\leqslant \beta\big(||\varkappa^o{}_{q-1} - F^o{}_\kappa \varkappa^o{}_{q-1}|| + ||\varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_q|| + ||\varkappa^o{}_{q+1} - F^o{}_\kappa \varkappa^o{}_{q+1}||\big)$$

and

$$||\varkappa^o{}_q - \varkappa^o{}_{q+1}|| + ||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| + ||\varkappa^o{}_{q+2} - \varkappa^o{}_q||$$
$$\leqslant \beta\big(||\varkappa^o{}_{q-1} - \varkappa^o{}_q|| + ||\varkappa^o{}_q - \varkappa^o{}_{q+1}|| + ||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}||\big).$$

Hence,

$$(1-\beta)||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}||$$
$$\leqslant \beta\big(||\varkappa^o{}_{q-1} - \varkappa^o{}_q|| + ||\varkappa^o{}_q - \varkappa^o{}_{q+1}||\big) - ||\varkappa^o{}_q - \varkappa^o{}_{q+1}|| - ||\varkappa^o{}_{q+2} - \varkappa^o{}_q||.$$

Using the triangle inequality $||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| \leqslant ||\varkappa^o{}_q - \varkappa^o{}_{q+1}|| + ||\varkappa^o{}_{q+2} - \varkappa^o{}_q||$, we obtain

$$(1-\beta)||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| \leqslant \beta\big(||\varkappa^o{}_{q-1} - \varkappa^o{}_q|| + ||\varkappa^o{}_q - \varkappa^o{}_{q+1}||\big) - ||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}||.$$

In addition,

$$\begin{aligned}(2-\beta)||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| &\leqslant \beta\big(||\varkappa^o{}_{q-1} - \varkappa^o{}_q|| + ||\varkappa^o{}_q - \varkappa^o{}_{q+1}||\big),\\ \Longrightarrow ||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| &\leqslant \tfrac{\beta}{2-\beta}\big(||\varkappa^o{}_{q-1} - \varkappa^o{}_q|| + ||\varkappa^o{}_q - \varkappa^o{}_{q+1}||\big)\end{aligned}$$

and

$$||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| \leqslant \frac{2\beta}{2-\beta}\max\big\{||\varkappa^o{}_{q-1} - \varkappa^o{}_q||, ||\varkappa^o{}_q - \varkappa^o{}_{q+1}||\big\}.$$

Let $\theta = \frac{2\beta}{2-\beta}$. Since $\beta \in \big[0, \frac{2}{3}\big)$, we therefore obtain $\theta \in [0,1)$. Additionally,

$$||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| \leqslant \theta\max\big\{||\varkappa^o{}_{q-1} - \varkappa^o{}_q||, ||\varkappa^o{}_q - \varkappa^o{}_{q+1}||\big\}. \quad (19)$$

Consider $b_q = ||\varkappa^o{}_{q-1} - \varkappa^o{}_q||, q = 1,2,\ldots$, and let $b = \max\{b_1, b_2\}$. So, by (19), we have

$$b_1 \leqslant b, b_2 \leqslant b, b_3 \leqslant \theta b, b_4 \leqslant \theta b, b_5 \leqslant \theta^2 b, b_6 \leqslant \theta^2 b, b_7 \leqslant \theta^3 b, \ldots$$

Since $\theta < 1$, it is obvious that the inequalities

$$b_1 \leqslant b, b_2 \leqslant b, b_3 \leqslant \theta^{\frac{1}{2}} b, b_4 \leqslant \theta b, b_5 \leqslant \theta^{\frac{3}{2}} b, b_6 \leqslant \theta^2 b, b_7 \leqslant \theta^{\frac{5}{2}} b, \ldots$$

also hold. That is,

$$b_q \leqslant \theta^{\frac{\pi}{2}-1} b \quad (20)$$

for $q = 3,4,\ldots$ Let $p \in \mathbb{N}, p \geqslant 2$. By the triangle inequality, for $q \geqslant 3$ we have

$$\begin{aligned}||\varkappa^o{}_q - \varkappa^o{}_{q+p}|| &\leqslant ||\varkappa^o{}_q - \varkappa^o{}_{q+1}|| + ||\varkappa^o{}_{q+1} - \varkappa^o{}_{q+2}|| + \ldots + ||\varkappa^o{}_{q+p-1} - \varkappa^o{}_{q+p}||\\ &= b_{q+1} + b_{q+2} + \cdots + b_{q+p} \leqslant b\Big(\theta^{\frac{q+1}{2}-1} + \theta^{\frac{q+2}{2}-1} + \cdots + \theta^{\frac{q+p}{2}-1}\Big)\\ &= b\theta^{\frac{q+1}{2}-1}\Big(1 + \theta^{\frac{1}{2}} + \cdots + \theta^{\frac{q-1}{2}}\Big) = b\theta^{\frac{q-1}{2}}\frac{1-\sqrt{\theta^q}}{1-\sqrt{\theta}}.\end{aligned}$$

Since, by assumption, $0 \leqslant \theta < 1$, then $0 \leqslant \sqrt{\theta^q} < 1$ and $||\varkappa^o{}_q - \varkappa^o{}_{q+p}|| \leqslant b\theta^{\frac{q-1}{2}}\frac{1}{1-\sqrt{\theta}}$. Hence, $||\varkappa^o{}_q - \varkappa^o{}_{q+p}|| \to 0$ as $q \to \infty$ for all $p > 0$. Therefore, $\{\varkappa^o{}_q\}$ forms a Cauchy sequence. Given the completeness of $\mathfrak{Q}$, the sequence $(\varkappa^o{}_q)$ converges to $\varkappa^{o*} \in \mathfrak{Q}$.

Remember that any three successive elements in $(\varkappa^o{}_q)$ are different from one another. If $\varkappa^{o*} \neq \varkappa^o{}_k$ for all $k \in \{1,2,\ldots\}$, then inequality (6) is satisfied for the distinct points $\varkappa^{o*}, \varkappa^o{}_{q-1}$, and $\varkappa^o{}_q$. Assume there is a smallest possible integer $k \in \{1,2,\ldots\}$ for which $\varkappa^{o*} = \varkappa^o{}_k$. Let $m > k$ be such that $\varkappa^{o*} = \varkappa^o{}_m$. In this case, the sequence $(\varkappa^o{}_q)$ becomes cyclic initiating from index k and, consequently, it is not a Cauchy sequence. Therefore, the points $\varkappa^{o*}$, $\varkappa^{o}{}_{\mathrm{q}-1}$, and $\varkappa^{o}{}_{\mathrm{q}}$ remain mutually distinct, at least for $\mathrm{q}-1>k$.

We aim to prove that $F^{o}\varkappa^{o*}=\varkappa^{o*}$. Assume there is an index $k\in\{1,2,\ldots\}$ for which $\varkappa^{o}{}_{k}=\varkappa^{o*}$. If $\mathrm{q}-1>k$, then applying the triangle inequality along with inequality (6), we obtain

$$\begin{aligned}||\varkappa^{o*}-F^{o}\varkappa^{o*}|| &\leqslant ||\varkappa^{o*}-\varkappa^{o}{}_{\mathrm{q}}||+||\varkappa^{o}{}_{\mathrm{q}}-F^{o}\varkappa^{o*}||=||\varkappa^{o*}-\varkappa^{o}{}_{\mathrm{q}}||+||F^{o}\varkappa^{o}{}_{\mathrm{q}-1}-F^{o}\varkappa^{o*}||\\ &\leqslant ||\varkappa^{o*}-\varkappa^{o}{}_{\mathrm{q}}||+||F^{o}\varkappa^{o}{}_{\mathrm{q}-1}-F^{o}\varkappa^{o*}||+||F^{o}\varkappa^{o}{}_{\mathrm{q}-1}-F^{o}\varkappa^{o}{}_{\mathrm{q}}||+||F^{o}\varkappa^{o}{}_{\mathrm{q}}-F^{o}\varkappa^{o*}||\\ &\leqslant ||\varkappa^{o*}-\varkappa^{o}{}_{\mathrm{q}}||+\beta\big(||\varkappa^{o}{}_{\mathrm{q}-1}-F^{o}\varkappa^{o}{}_{\mathrm{q}-1}||+||\varkappa^{o}{}_{\mathrm{q}}-F^{o}\varkappa^{o}{}_{\mathrm{q}}||+||\varkappa^{o*}-F^{o}\varkappa^{o*}||\big).\end{aligned}$$

Hence,

$$||\varkappa^{o*}-F^{o}\varkappa^{o*}||(1-\beta)\leqslant ||\varkappa^{o*}-\varkappa^{o}{}_{\mathrm{q}}||+\beta\big(||\varkappa^{o}{}_{\mathrm{q}-1}-\varkappa^{o}{}_{\mathrm{q}}||+||\varkappa^{o}{}_{\mathrm{q}}-\varkappa^{o}{}_{\mathrm{q}+1}||\big)$$

and

$$||\varkappa^{o*}-F^{o}{}_{\kappa}\varkappa^{o*}||\leqslant \frac{1}{1-\beta}\big(||\varkappa^{o*}-\varkappa^{o}{}_{\mathrm{q}}||+\beta\big(||\varkappa^{o}{}_{\mathrm{q}-1}-\varkappa^{o}{}_{\mathrm{q}}||+||\varkappa^{o}{}_{\mathrm{q}}-\varkappa^{o}{}_{\mathrm{q}+1}||\big)\big). \quad (21)$$

As the distances on the right-hand side approach zero as $\mathrm{q}\to\infty$, we find that $\|\varkappa^{o*}-F^{o}{}_{\kappa}\varkappa^{o*}\|=0$. Assume there are no fewer than three distinct fixed points: $\varkappa^{o}$, μ, and ϖ. Then, $F^{o}{}_{\kappa}\varkappa^{o}=\varkappa^{o}, F^{o}{}_{\kappa}\mu=\mu$, and $F^{o}{}_{\kappa}\varpi=\varpi$, which contradicts (6). Hence, there can be at most two fixed points of $F^{o}{}_{\kappa}$, and thus of F^{o} as well. □

Remark 3. *Theorem 2 is not applicable for any $\beta>\frac{2}{3}$. Specifically, take $\beta>\frac{2}{3}$ as an example. Let $\mathfrak{Q}=\{\varkappa^{o},\mu,\varpi,\mathfrak{t}\}$ with the following distances: $\|\varkappa^{o}-\mu\|=\|\mu-\varpi\|=\|\varpi-\mathfrak{t}\|=\|\mathfrak{t}-\varkappa^{o}\|=p$ and $\|\varkappa^{o}-\varpi\|=\|\mu-\mathfrak{t}\|=r$. It is simple to check that $(\mathfrak{Q},\|\cdot\|)$ forms a Banach space for $0<r\leq 2p$. Define $F^{o}:\mathfrak{Q}\to\mathbb{R}$ by $F^{o}\varkappa^{o}=2\mu-\varkappa^{o}, F^{o}\mu=2\varpi-\mu, F^{o}\varpi=2\mathfrak{t}-\varpi$ and $F^{o}\mathfrak{t}=2\varkappa^{o}-\mathfrak{t}$. Then, for $\mathfrak{s}=1$, we have $\kappa=\frac{1}{2}$ and $F^{o}{}_{\kappa}\varkappa^{o}=\mu, F^{o}{}_{\kappa}\mu=\varpi, F^{o}{}_{\kappa}\varpi=\mathfrak{t}$, and $F^{o}{}_{\kappa}\mathfrak{t}=\varkappa^{o}$. Consider inequality (6) for the set of points $\{\varkappa^{o},\mu,\varpi\}$:*

$$p+p+r\leqslant\beta(p+p+p).$$

Hence,

$$\frac{2p+r}{3p}\leqslant\beta. \quad (22)$$

Clearly, for every $\beta>\frac{2}{3}$, there exists a sufficiently small r for which inequality (22) is satisfied. Note that applying inequality (22) to the point triplets $\{\mu,\varpi,\mathfrak{t}\}$, $\{\varpi,\mathfrak{t},\varkappa^{o}\}$, and $\{\mathfrak{t},\varkappa^{o},\mu\}$ yields inequality (22). So, for each $\beta>\frac{2}{3}$, there must exist a Banach space $\mathfrak{Q}$ and a mapping $F^{o}:\mathfrak{Q}\to\mathfrak{Q}$ for which the following holds:

1. *The inequality (6) is satisfied with the coefficient β for all sets of three distinct points in the space $\mathfrak{Q}$;*
2. *$F^{o}{}_{\kappa}$ lacks periodic points with a prime period of two;*
3. *$F^{o}{}_{\kappa}$ and F^{o} do not have any fixed points.*

Remark 4. *Considering all the conditions of Theorem 2, the operator F^{o} has the fixed point $\varkappa^{o*}$, that is the limit of sequence defined iteratively by $\varkappa^{o}{}_{\mathrm{q}+1}=(1-\kappa)\varkappa^{o}{}_{\mathrm{q}}+\kappa F^{o}\varkappa^{o}{}_{\mathrm{q}}=F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}$, where $\varkappa^{o}{}_{\mathrm{q}}\neq\varkappa^{o*}$ for all $\mathrm{q}=1,2,\ldots$. In this situation, $\varkappa^{o*}$ must be the unique fixed point. To demonstrate this, assume that F^{o} has an additional fixed point $\varkappa^{o**}\neq\varkappa^{o*}$. It is evident that $\varkappa^{o}{}_{\mathrm{q}}\neq\varkappa^{o**}$ for all $\mathrm{q}=1,2,\ldots$. Therefore, the points $\varkappa^{o*}$, $\varkappa^{o**}$, and $\varkappa^{o}{}_{\mathrm{q}}$ are distinct from each*

other for all $\mathrm{q} = 1, 2, \ldots$.
Evaluate the expression,

$$\begin{aligned} \mathrm{R}_{\mathrm{q}} &= \frac{||F^{o}{}_{\kappa}\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o**}|| + ||F^{o}{}_{\kappa}\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||F^{o}{}_{\kappa}\varkappa^{o**} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}||}{||\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o*}|| + ||\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||\varkappa^{o**} - F^{o}{}_{\kappa}\varkappa^{o**}||} \\ &= \frac{||\varkappa^{o*} - \varkappa^{o**}|| + ||\varkappa^{o*} - \varkappa^{o}{}_{\mathrm{q}+1}|| + ||\varkappa^{o**} - \varkappa^{o}{}_{\mathrm{q}+1}||}{||\varkappa^{o}{}_{\mathrm{q}} - \varkappa^{o}{}_{\mathrm{q}+1}||}. \end{aligned}$$

Since $||\varkappa^{o*} - \varkappa^{o}{}_{\mathrm{q}+1}|| \to 0$, $||\varkappa^{o**} - \varkappa^{o}{}_{\mathrm{q}+1}|| \to ||\varkappa^{o**} - \varkappa^{o*}||$, *and* $||\varkappa^{o}{}_{\mathrm{q}} - \varkappa^{o}{}_{\mathrm{q}+1}|| \to 0$ *as* $n \to \infty$. *Therefore, we find that* $\mathrm{R}_{\mathrm{q}} \to \infty$ *as* $\mathrm{q} \to \infty$. *This contradicts the condition stated in* (6).

Example 5. *Let us illustrate with an example of a generalized enriched Kannan operator,* F^{o}, *that has exactly two fixed points. Let* $\mathfrak{Q} = \{\varkappa^{o}, \mu, \varpi\}$ *with* $||\varkappa^{o} - \mu|| = 1, ||\mu - \varpi|| = 4, ||\varkappa^{o} - \varpi|| = 4$. *Define* $F^{o} : \mathfrak{Q} \to \mathbb{R}$ *be such that* $F^{o}\varkappa^{o} = \varkappa^{o}, F^{o}\mu = \mu$ *and* $F^{o}\varpi = 2\varkappa^{o} - \varpi$. *Then, for* $\mathfrak{s} = 1$, *we obtain* $\kappa = \frac{1}{2}$ *and the mapping* $F^{o}{}_{\kappa} : \mathfrak{Q} \to \mathfrak{Q}$ *given by* $F^{o}{}_{\kappa}\varkappa^{o} = \varkappa^{o}, F^{o}{}_{\kappa}\mu = \mu$, *and* $F^{o}{}_{\kappa}\varpi = \varkappa^{o}$. *It is straightforward to observe that condition (i) of Theorem 2 is fulfilled and inequality* (6) *holds with* $\beta = \frac{1}{2}$. *Note also that* F^{o} *is not an enriched Kannan-type mapping since inequality* (5) *fails to hold for any* $0 \leq \beta < \frac{1}{2}$.

Example 6. *Let us demonstrate that condition (i) of Theorem 2 is essential. Let the space* $(\mathfrak{Q}, ||\cdot||)$ *be the same as in the previous example and define* $F^{o} : \mathfrak{Q} \to \mathbb{R}$ *such that* $F^{o}\varkappa^{o} = 2\mu - \varkappa^{o}, F^{o}\mu = 2\varkappa^{o} - \mu$ *and* $F^{o}\varpi = 2\varkappa^{o} - \varpi$. *Then, for* $\mathfrak{s} = 1$, *we obtain* $\kappa = \frac{1}{2}$ *and the mapping* $F^{o}{}_{\kappa} : \mathfrak{Q} \to \mathfrak{Q}$ *given by* $F^{o}{}_{\kappa}\varkappa^{o} = \mu, F^{o}{}_{\kappa}\mu = \varkappa^{o}$ *and* $F^{o}{}_{\kappa}\varpi = \varkappa^{o}$. *It is straightforward to verify that inequality* (6) *is satisfied for any value of* $\frac{1}{3} \leqslant \beta < \frac{2}{3}$ *but* F^{o} *does not have any fixed point.*

Proposition 4. *The generalized enriched Kannan operators are always continuous at their fixed points.*

Proof. Suppose $(\mathfrak{Q}, ||\cdot||)$ be a Banach space with at least three elements. Suppose $F^{o} : \mathfrak{Q} \to \mathfrak{Q}$ is a generalized enriched Kannan operator, and let $\varkappa^{o*}$ be a fixed point of F^{o}. Consider a sequence $(\varkappa^{o}{}_{\mathrm{q}})$ where $\varkappa^{o}{}_{\mathrm{q}} \to \varkappa^{o*}$, $\varkappa^{o}{}_{\mathrm{q}} \neq \varkappa^{o}{}_{\mathrm{q}+1}$, and $\varkappa^{o}{}_{\mathrm{q}} \neq \varkappa^{o*}$ for every q. To prove the claim, first we show that $F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}} \to F^{o}{}_{\kappa}\varkappa^{o*}$. By (6), we have

$$\begin{aligned} &||F^{o}{}_{\kappa}\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1} - F^{o}{}_{\kappa}\varkappa^{o*}|| \\ &\leqslant \beta\big(||\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o*}|| + ||\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||\varkappa^{o}{}_{\mathrm{q}+1} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}||\big). \end{aligned}$$

Hence,

$$\begin{aligned} &||F^{o}{}_{\kappa}\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1} - F^{o}{}_{\kappa}\varkappa^{o*}|| \\ &\leqslant \beta\big(||\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||\varkappa^{o}{}_{\mathrm{q}+1} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}||\big). \end{aligned}$$

By applying the triangle inequality, we obtain

$$\begin{aligned} &||F^{o}{}_{\kappa}\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1} - F^{o}{}_{\kappa}\varkappa^{o*}|| \\ &\leqslant \beta\big(||\varkappa^{o}{}_{\mathrm{q}} - \varkappa^{o*}|| + ||\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + ||\varkappa^{o}{}_{\mathrm{q}+1} - \varkappa^{o*}|| + ||\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}||\big). \end{aligned}$$

Further,

$$\begin{aligned} &||F^{o}{}_{\kappa}\varkappa^{o*} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}}|| + \frac{1}{1-\beta}||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}} - F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1}|| + ||F^{o}{}_{\kappa}\varkappa^{o}{}_{\mathrm{q}+1} - F^{o}{}_{\kappa}\varkappa^{o*}|| \\ &\leqslant \frac{\beta}{1-\beta}\big(||\varkappa^{o}{}_{\mathrm{q}} - \varkappa^{o*}|| + ||\varkappa^{o}{}_{\mathrm{q}+1} - \varkappa^{o*}||\big). \end{aligned}$$

Since $||\varkappa^o{}_{\mathrm{q}} - \varkappa^{o*}|| \to 0$ and $||\varkappa^o{}_{\mathrm{q}+1} - \varkappa^{o*}|| \to 0$ as $\mathrm{q} \to \infty$,

$$||F^o{}_\kappa \varkappa^{o*} - F^o{}_\kappa \varkappa^o{}_{\mathrm{q}}|| + \frac{1}{1-\beta}||F^o{}_\kappa \varkappa^o{}_{\mathrm{q}} - F^o{}_\kappa \varkappa^o{}_{\mathrm{q}+1}|| + ||F^o{}_\kappa \varkappa^o{}_{\mathrm{q}+1} - F^o{}_\kappa \varkappa^{o*}|| \to 0.$$

Hence, $||F^o{}_\kappa \varkappa^{o*} - F^o{}_\kappa \varkappa^o{}_{\mathrm{q}}|| \to 0$. Consider a sequence $(\varkappa^o{}_{\mathrm{q}})$ where $\varkappa^o{}_{\mathrm{q}} \to \varkappa^{o*}$ and $\varkappa^o{}_{\mathrm{q}} \neq \varkappa^{o*}$ for all n. In this case, $\varkappa^o{}_{\mathrm{q}}$ may be equal to $\varkappa^o{}_{\mathrm{q}+1}$. Let $\left(\varkappa^o{}_{\mathrm{q}_k}\right)$ be a subsequence of $(\varkappa^o{}_{\mathrm{q}})$ formed by removing repeated elements, ensuring $\varkappa^o{}_{\mathrm{q}_k} \neq \varkappa^o{}_{\mathrm{q}_{k+1}}$ for all k. It is evident that $\varkappa^o{}_{\mathrm{q}_k} \to \varkappa^{o*}$. As demonstrated, $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}_k} \to F^o{}_\kappa \varkappa^{o*} = \varkappa^{o*}$. The difference between $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}_k}$ and $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}}$ is that $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}}$ can be deduced from $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}_k}$ by adding the appropriate repeating consecutive points. Thus, it follows that $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}} \to F^o{}_\kappa \varkappa^{o*}$. Next, consider a sequence $(\varkappa^o{}_{\mathrm{q}})$ where $\varkappa^o{}_{\mathrm{q}} = \varkappa^{o*}$ for all $\mathrm{q} > N$, with N being a natural number. In this case, it is evident that $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}} \to F^o{}_\kappa \varkappa^{o*}$. Let $(\varkappa^o{}_{\mathrm{q}})$ represent any arbitrary sequence such that $\varkappa^o{}_{\mathrm{q}} \to \varkappa^{o*}$, but not in the manner described previously. Consider a subsequence $\left(\varkappa^o{}_{\mathrm{q}_k}\right)$ derived from $(\varkappa^o{}_{\mathrm{q}})$ by removing any occurrences of $\varkappa^{o*}$ (if present). Clearly, $\varkappa^o{}_{\mathrm{q}_k} \to \varkappa^{o*}$. It has been demonstrated that $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}_k} \to F^o{}_\kappa \varkappa^{o*}$. Similarly, $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}}$ can be derived from $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}_k}$ by adding elements $F^o{}_\kappa \varkappa^{o*} = \varkappa^{o*}$ in certain positions. Thus, it is evident that $F^o{}_\kappa \varkappa^o{}_{\mathrm{q}} \to F^o{}_\kappa \varkappa^{o*}$. Since F^o is just a scaling and translation of $F^o{}_\kappa$, it follows that F^o is also continuous at $\varkappa^{o*}$. □

3. Asymptotic Regularity

Asymptotic regularity provides a way to expand the range of mappings to which fixed point techniques can be applied.

Definition 6. *Consider a metric space $(\mathfrak{Q}, ||\cdot||)$. A mapping $F^o : \mathfrak{Q} \to \mathfrak{Q}$ is said to be asymptotically regular if it satisfies the following condition:*

$$\lim_{\mathrm{q}\to\infty} ||F^{o\,\mathrm{q}+1}(\varkappa^o) - F^{o\,\mathrm{q}}(\varkappa^o)|| = 0, \tag{23}$$

for all $\varkappa^o \in \mathfrak{Q}$.

Remark 5. *Let $(\mathfrak{Q}, ||\cdot||)$ be a metric space and consider a self-mapping F^o defined on $\mathfrak{Q}$. Consider a sequence $\{\varkappa^o{}_{\mathrm{q}}\}$ defined by $\varkappa^o{}_{\mathrm{q}+1} = (1-\kappa)\varkappa^o{}_{\mathrm{q}} + \kappa F^o \varkappa^o{}_{\mathrm{q}} = F^o{}_\kappa(\varkappa^o{}_{\mathrm{q}})$ for all $\mathrm{q} \in \mathbb{N}$, where $\kappa \in (0,1]$ and $\varkappa^o{}_0 \in \mathfrak{Q}$. If $F^o{}_\kappa$ exhibits asymptotic regularity and considering the sequence $\{\varkappa^o{}_{\mathrm{q}}\}$ has no fixed point of F^o, then all points $\varkappa^o{}_{\mathrm{q}}$ for $\mathrm{q} \geq 0$ are distinct. If the points $\varkappa^o{}_{\mathrm{q}}$ were not distinct, then the sequence $\{\varkappa^o{}_{\mathrm{q}}\}$ would eventually become cyclic starting from some index, which would contradict the condition (23).*

Theorem 3. *Consider a Banach space $(\mathfrak{Q}, ||\cdot||)$ with at least three elements, and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be a mapping that qualifies as a generalized enriched Kannan operator. Assume there is a $\kappa \in (0,1]$ for which the mapping $F^o{}_\kappa : \mathfrak{Q} \to \mathfrak{Q}$ exhibits asymptotic regularity. In this situation, F^o must have at least one fixed point, and the number of fixed points cannot exceed two.*

Proof. Define a sequence from initial point $\varkappa^o{}_0 \in \mathfrak{Q}$ by $\varkappa^o{}_{\mathrm{q}+1} = (1-\kappa)\varkappa^o{}_{\mathrm{q}} + \kappa F^o \varkappa^o{}_{\mathrm{q}} = F^o{}_\kappa(\varkappa^o{}_{\mathrm{q}}), \quad \forall \mathrm{q} \in \mathbb{N}$, where $\kappa \in (0,1]$. Suppose that the sequence $(\varkappa^o{}_{\mathrm{q}})$ lacks a fixed point for F^o. We will show that $(\varkappa^o{}_{\mathrm{q}})$ is a Cauchy sequence. It is enough to demonstrate that $||\varkappa^o{}_{\mathrm{q}} - \varkappa^o{}_{\mathrm{q}+p}|| \to 0$ as $\mathrm{q} \to \infty$ for any $p > 0$. When $p = 1$, this is directly derived based on the definition of asymptotic regularity. For $p \geq 2$, Remark 5 indicates that the points $\varkappa^o{}_{\mathrm{q}}$, $\varkappa^o{}_{\mathrm{q}+p-1}$, and $\varkappa^o{}_{\mathrm{q}+p}$ are distinct from each other. Applying the triangle inequality repeatedly, along with inequality (6) and the property of asymptotic regularity, we obtain

$$
\begin{aligned}
&||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+p}|| \leqslant ||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + ||\varkappa^o{}_{\mathsf{q}+1} - \varkappa^o{}_{\mathsf{q}+p+1}|| + ||\mathfrak{r}_{\mathsf{q}+p+1} - \varkappa^o{}_{\mathsf{q}+p}|| \\
\leqslant &||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + ||\varkappa^o{}_{\mathsf{q}+1} - \varkappa^o{}_{\mathsf{q}+p+1}|| + ||\varkappa^o{}_{\mathsf{q}+p+1} - \varkappa^o{}_{\mathsf{q}+p}|| + ||\varkappa^o{}_{\mathsf{q}+1} - \varkappa^o{}_{\mathsf{q}+p}|| \\
\leqslant &||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + \beta\big(||\varkappa^o{}_{\mathsf{q}} - F^o{}_{\kappa}\varkappa^o{}_{\mathsf{q}}|| + ||\varkappa^o{}_{\mathsf{q}+p} - F^o{}_{\kappa}\varkappa^o{}_{\mathsf{q}+p}|| + ||\varkappa^o{}_{\mathsf{q}+p-1} - F^o{}_{\kappa}\varkappa^o{}_{\mathsf{q}+p-1}||\big) \\
= &||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + \beta\big(||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + ||\varkappa^o{}_{\mathsf{q}+p} - \varkappa^o{}_{\mathsf{q}+p+1}|| + ||\varkappa^o{}_{\mathsf{q}+p-1} - \varkappa^o{}_{\mathsf{q}+p}||\big) \to 0
\end{aligned}
$$

as $\mathsf{q} \to \infty$, this implies that $(\varkappa^o{}_{\mathsf{q}})$ is a Cauchy sequence. The remaining part of the proof is analogous to the argument presented in Theorem 2, as demonstrated in (21). □

We will show that by assuming continuity for the mappings F^o, we can extend fixed-point theorems to a wider range of mappings beyond generalized enriched Kannan mappings. This holds even when the coefficient factor β falls within the interval $[0, 1)$.

We will now introduce a broader class of mappings, extending the concept of enriched Kannan mappings. The following definitions will outline this more generalized form. We start by introducing the class $\mathcal{G}$, which consists of mappings $G : \mathbb{R}^3_+ \to \mathbb{R}_+$ that meet the following criteria, where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers:
$(\mathfrak{g}_1)$: $G(\mathring{\imath}, \mathring{\imath}, \mathring{\imath}) = 0$ if $\mathring{\imath} = 0$;
$(\mathfrak{g}_2)$: G is continuous at $(0, 0, 0)$;
$(\mathfrak{g}_3)$: G satisfies the condition $G(\kappa\varkappa^o, \kappa\mu, \kappa\varpi) \leq \kappa G(\varkappa^o, \mu, \varpi)$.

Definition 7. *Consider a normed space $(\mathfrak{Q}, ||\cdot||)$ with $|\mathfrak{Q}| \geq 3$. We call $F^o : \mathfrak{Q} \to \mathfrak{Q}$ a generalized G-enriched Kannan mapping if there exists $\mathfrak{s} \in [0, \infty)$ and a function $G \in \mathcal{G}$ such that*

$$
\begin{aligned}
&||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| + ||\mathfrak{s}(\mu - \varpi) + F^o\mu - F^o\varpi|| + ||\mathfrak{s}(\varkappa^o - \varpi) + F^o\varkappa^o - F^o\varpi|| \\
&\qquad \leqslant G(||\varkappa^o - F^o\varkappa^o||, ||\mu - F^o\mu||, ||\varpi - F^o\varpi||) \qquad (24)
\end{aligned}
$$

is satisfied for all three pairwise distinct points $\varkappa^o$, μ, and ϖ in $\mathfrak{Q}$.

Theorem 4. *Let $(\mathfrak{Q}, ||\cdot||)$ be a Banach space with $|\mathfrak{Q}| \geq 3$, and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be a continuous mapping that is generalized G-enriched Kannan. Assume there exists $\kappa \in (0, 1]$ such that the mapping $F^o{}_{\kappa} : \mathfrak{Q} \to \mathfrak{Q}$ is asymptotically regular. Then, F^o must have at least one fixed point, and there can be no more than two fixed points in total.*

Proof. Consider a sequence $(\varkappa^o{}_{\mathsf{q}})$ defined by $\varkappa^o{}_{\mathsf{q}+1} = (1-\kappa)\varkappa^o{}_{\mathsf{q}} + \kappa F^o\varkappa^o{}_{\mathsf{q}} = F^o{}_{\kappa}(\varkappa^o{}_{\mathsf{q}})$, where $\kappa \in (0, 1]$ and $\varkappa^o{}_0 \in \mathfrak{Q}$. Suppose that this sequence does not have a fixed point of F^o. To demonstrate that $(\varkappa^o{}_{\mathsf{q}})$ is a Cauchy sequence, it is enough to show that $||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+p}|| \to 0$ as $\mathsf{q} \to \infty$ for any $p > 0$. When $p = 1$, this result is directly obtained based on the definition of asymptotic regularity. For $p \geq 2$, Remark 5 indicates that the points $\varkappa^o{}_{\mathsf{q}}$, $\varkappa^o{}_{\mathsf{q}+p-1}$, and $\varkappa^o{}_{\mathsf{q}+p}$ are not equal to each other. By utilizing the repeated triangle inequality, inequality (24), and the concept of asymptotic regularity, one can deduce that

$$
\begin{aligned}
&||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+p}|| \leqslant ||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + ||\varkappa^o{}_{\mathsf{q}+1} - \varkappa^o{}_{\mathsf{q}+p+1}|| + ||\varkappa^o{}_{\mathsf{q}+p+1} - \varkappa^o{}_{\mathsf{q}+p}|| \\
\leqslant &||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + ||\varkappa^o{}_{\mathsf{q}+1} - \varkappa^o{}_{\mathsf{q}+p+1}|| + ||\varkappa^o{}_{\mathsf{q}+p+1} - \varkappa^o{}_{\mathsf{q}+p}|| + ||\varkappa^o{}_{\mathsf{q}+1} - \varkappa^o{}_{\mathsf{q}+p}|| \\
\leqslant &||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + G\big(||\varkappa^o{}_{\mathsf{q}} - F^o{}_{\kappa}\varkappa^o{}_{\mathsf{q}}||, ||\varkappa^o{}_{\mathsf{q}+p} - F^o{}_{\kappa}\varkappa^o{}_{\mathsf{q}+p}||, ||\varkappa^o{}_{\mathsf{q}+p-1} - F^o{}_{\kappa}\varkappa^o{}_{\mathsf{q}+p-1}||\big) \\
= &||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}|| + G\big(||\varkappa^o{}_{\mathsf{q}} - \varkappa^o{}_{\mathsf{q}+1}||, ||\varkappa^o{}_{\mathsf{q}+p} - \varkappa^o{}_{\mathsf{q}+p+1}||, ||\varkappa^o{}_{\mathsf{q}+p-1} - \varkappa^o{}_{\mathsf{q}+p}||\big) \to 0
\end{aligned}
$$

as $\mathsf{q} \to \infty$, it follows that $(\varkappa^o{}_{\mathsf{q}})$ is a Cauchy sequence. Since $\mathfrak{Q}$ is complete, this sequence must converge to some limit $\varkappa^{o*} \in \mathfrak{Q}$. Therefore, we have

$$||F^o{}_\kappa \varkappa^{o*} - \varkappa^{o*}|| \leqslant ||F^o{}_\kappa \varkappa^{o*} - \varkappa^o{}_{\mathtt{q}}|| + ||\varkappa^o{}_{\mathtt{q}} - \varkappa^{o*}|| = ||F^o{}_\kappa \varkappa^{o*} - F^o{}_\kappa \varkappa^o{}_{\mathtt{q}-1}|| + ||\varkappa^o{}_{\mathtt{q}} - \varkappa^{o*}||.$$

Since F^o is continuous, its translated and scaled mapping $F^o{}_\kappa$ is also continuous. Therefore, taking $\mathtt{q} \to \infty$, we obtain $F^o{}_\kappa \varkappa^{o*} = \varkappa^{o*}$. Hence, $\varkappa^o \in fix_{F^o{}_\kappa}(\mathfrak{Q}) = fix_{F^o}(\mathfrak{Q})$. The subsequent steps of the proof align with the approach outlined in the proof for Theorem 2. □

Let $\mathfrak{S}$ be the set of functions $\alpha : [0, \infty) \to [0, \infty)$ such that $\limsup \alpha(t) < \infty$ and $\alpha(\kappa t) \leq \kappa\alpha(t)$ for all $\kappa > 0$.

Definition 8. *Let $(\mathfrak{Q}, ||\cdot||)$ be a Banach space with $|\mathfrak{Q}| \geq 3$. A mapping $F^o : \mathfrak{Q} \to \mathfrak{Q}$ is called a generalized $\mathfrak{s}$-enriched Kannan mapping if there exists a non-negative constant $\mathfrak{s}$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{S}$ such that*

$$\begin{aligned} &||\mathfrak{s}(\varkappa^o - \mu) + F^o\varkappa^o - F^o\mu|| + ||\mathfrak{s}(\mu - \varpi) + F^o\mu - F^o\varpi|| + ||\mathfrak{s}(\varkappa^o - \varpi) + F^o\varkappa^o - F^o\varpi|| \\ &\quad\leqslant \alpha_1(||\varkappa^o - F^o\varkappa^o||)||\varkappa^o - F^o\varkappa^o|| + \alpha_2(||\mu - F^o\mu||)||\mu - F^o\mu|| \\ &\qquad + \alpha_3(||\varpi - F^o\varpi||)||\varpi - F^o\varpi|| \end{aligned} \tag{25}$$

holds for all three pairwise distinct points $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$.

Corollary 2. *Consider a normed space $(\mathfrak{Q}, ||\cdot||)$ with $|\mathfrak{Q}| \geq 3$ where operator $F^o : \mathfrak{Q} \to \mathfrak{Q}$ is a continuous generalized $\mathfrak{s}$-enriched Kannan operator. Assume there exists $\kappa \in (0, 1]$ such that $F^o{}_\kappa : \mathfrak{Q} \to \mathfrak{Q}$ is an asymptotically regular mapping. In this case, F^o has at most two fixed points.*

Proof. Set $G(\varkappa^o, \mu, \varpi) = \alpha_1(\varkappa^o)\varkappa^o + \alpha_2(\mu)\mu + \alpha_3(\varpi)\varpi$. Then, $G(0, 0, 0) = 0$, $\lim_{\varkappa^o,\mu,\varpi\to 0} G(\varkappa^o, \mu, \varpi) = 0$ and $G(\kappa\varkappa^o, \kappa\mathfrak{y},) \leq \kappa G(\varkappa^o, \mu, \varpi)$, since $\alpha_i(t) \in \mathfrak{S}$ for $i = 1, 2, 3$. Thus, this result is a direct consequence of Theorem 4. □

By choosing $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \beta \geq 0$ in Equation (25), we obtain a generalized enriched Kannan operator with the parameter $\beta \in [0, \infty)$. Consequently, we can directly derive from the result given below.

Corollary 3. *Let $(\mathfrak{Q}, ||\cdot||)$ be a Banach space with $|\mathfrak{Q}| \geq 3$, and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be a continuous generalized enriched Kannan mapping with a coefficient $\beta \in [0, \infty)$. Assume there is a $\kappa \in (0, 1]$ such that the mapping $F^o{}_\kappa : \mathfrak{Q} \to \mathfrak{Q}$ is asymptotically regular. Then, F^o has a fixed point, with at most two fixed points in total.*

Remark 6. *It should be emphasized that the continuity assumption for the mapping F^o in Corollary 3 is not strictly necessary. Corollary 3 remains valid if we replace the continuity condition with any of the following: orbital continuity, $\varkappa^o{}_0$-orbital continuity, almost orbital continuity, weak orbital continuity, F^o-orbital lower semi-continuity, or k-continuity.*

The proposition given below is mostly straightforward.

Proposition 5. *Consider a Banach space $(\mathfrak{Q}, ||\cdot||)$ with $|\mathfrak{Q}| \geqslant 3$, and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be a generalized enriched Kannan mapping. Then, for some $\kappa \in (0, 1]$, the mapping $F^o{}_\kappa : \mathfrak{Q} \to \mathfrak{Q}$ is an asymptotically regular mapping if and only if $F^o{}_\kappa$ lacks periodic points with a prime period of $n \geq 2$.*

Proposition 5 and Corollary 3 lead to the following result.

Corollary 4. *Consider a finite Banach space $\mathfrak{Q}$, and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be an operator. If, for some $\kappa \in (0,1]$, the mapping ${F^o}_\kappa$ lacks periodic points with prime period $n \geq 2$, then F^o must possess a fixed point.*

Proof. Since $\mathfrak{Q}$ is complete and F^o is continuous, and given that ${F^o}_\kappa$ is an asymptotically regular mapping by Proposition 5, if F^o has no fixed points, then ${F^o}_\kappa$ will also have no fixed points. Thus, $||\varkappa^o - {F^o}_\kappa \varkappa^o|| + ||\mu - {F^o}_\kappa \mu|| + ||\varpi - {F^o}_\kappa \varpi|| \neq 0$ for all $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$. It is straightforward to verify that F^o is a generalized enriched Kannan mapping with the factor

$$\beta = \max_{\varkappa^o,\mu,\varpi \in \mathfrak{Q}} \frac{||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \varpi||}{||\varkappa^o - {F^o}_\kappa \varkappa^o|| + ||\mu - {F^o}_\kappa \mu|| + ||\varpi - {F^o}_\kappa \varpi||},$$

where the maximum is considered over all distinct triplets $\varkappa^o, \mu, \varpi \in \mathfrak{Q}$. Applying Corollary 3 leads to a contradiction. □

Since a fixed point corresponds to a periodic point with period $n = 1$, we can directly deduce the following.

Corollary 5. *Consider a finite normed space $\mathfrak{Q}$ and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be an operator. Then, for some $\kappa \in (0,1]$, the mapping ${F^o}_\kappa$ will have a periodic point. This assertion appears to be well known and is also quite elementary.*

Proof. Define a sequence from initial point ${\varkappa^o}_0 \in \mathfrak{Q}$ by ${\varkappa^o}_{\mathtt{q}+1} = (1-\kappa){\varkappa^o}_{\mathtt{q}} + \kappa F^o {\varkappa^o}_{\mathtt{q}} = {F^o}_\kappa({\varkappa^o}_{\mathtt{q}}), \quad \forall \mathtt{q} \in \mathbb{N}$, where $\kappa \in (0,1]$. By the pigeonhole principle, there is at least one integer $\mathtt{q}$, where $1 \leq \mathtt{q} \leq |\mathfrak{Q}|$, such that ${\varkappa^o}_{\mathtt{q}}$ matches a previous term in the sequence. Let k be the smallest index for which this holds, and let $i < k$ be an index such that ${\varkappa^o}_i = {\varkappa^o}_k$. Then, ${\varkappa^o}_i$ is a periodic point with a prime period of $k - i$. □

4. Sequences of Approximate Fixed Points and Generalized *G*-Enriched Kannan Mappings

Let $(\mathfrak{Q}, ||\cdot||)$ be a normed space and $F^o : \mathfrak{Q} \to \mathfrak{Q}$. A sequence $\{{\varkappa^o}_n\} \subset \mathfrak{Q}$ is referred to as an approximate fixed point sequence for F^o if $||{\varkappa^o}_n - F^o {\varkappa^o}_n|| \to 0$ as n approaches infinity.

In the example below, we notice that although ${F^o}_\kappa$ for $\kappa \in (0,1]$, does not meet the condition of asymptotic regularity, ${F^o}_\kappa$ still has an approximate sequence of fixed points.

Example 7. *Let $\mathfrak{Q} = [0,1]$ be equipped with the usual norm $||\cdot||$, and let $F^o : \mathfrak{Q} \to \mathfrak{Q}$ be defined by $F^o(\varkappa^o) = 3 - 5\varkappa^o$ for all $\varkappa^o \in [0,1]$. Choosing $\mathfrak{s} = 2$, we set $\kappa = \frac{1}{3}$, resulting in the mapping ${F^o}_{\frac{1}{3}} = 1 - \varkappa^o$. It is crucial to highlight that ${F^o}_{\frac{1}{3}}$ is not asymptotically regular. This is because every point in $\mathfrak{Q}$ is a periodic point with a prime period of two, except for the point $\varkappa^o = \frac{1}{2}$, which is a fixed point of F^o. Nevertheless, the sequence ${\varkappa^o}_{\mathtt{q}} = \frac{1}{2} + \frac{1}{\mathtt{q}}$ for $\mathtt{q} \geq 2$ acts as an approximate fixed-point sequence for ${F^o}_{\frac{1}{3}}$. In particular,*

$$||{\varkappa^o}_{\mathtt{q}} - {F^o}_{\frac{1}{3}} {\varkappa^o}_{\mathtt{q}}|| = \left| \frac{1}{2} + \frac{1}{\mathtt{q}} - \left(1 - \left(\frac{1}{2} + \frac{1}{\mathtt{q}}\right)\right) \right| = \frac{2}{\mathtt{q}} \to 0$$

as $\mathtt{q} \to \infty$.

In the subsequent discussion, we explore mappings that might not demonstrate asymptotic regularity but still satisfy the condition of being a generalized ***G***-enriched Kannan mapping.

Theorem 5. *Consider a Banach space* $(\mathfrak{Q}, ||\cdot||)$ *with* $|\mathfrak{Q}| \geq 3$*, and a continuous generalized* $\boldsymbol{G}$*-enriched Kannan operator* $F^o : \mathfrak{Q} \to \mathfrak{Q}$*. Furthermore, assume that* $F^o{}_\kappa$ *has an approximate fixed-point sequence, meaning there exists a sequence* $\{\varkappa^o{}_q\} \subset \mathfrak{Q}$ *such that* $||\varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_q|| \to 0$ *as* $q \to \infty$*. In that case,* F^o *will have at least one fixed point, and the total number of fixed points will be at most two.*

Proof. Utilizing the triangle inequality along with the definition of a generalized $\boldsymbol{G}$-enriched Kannan mapping, we derive the following:

$$
\begin{aligned}
||\varkappa^o{}_q - \varkappa^o{}_{q+p}|| &\leq ||\varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_q|| + ||F^o{}_\kappa \varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_{q+p}|| + ||F^o{}_\kappa \varkappa^o{}_{q+p} - \varkappa^o{}_{q+p}|| \\
&\leq ||\varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_q|| + ||F^o{}_\kappa \varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_{q+p}|| + ||F^o{}_\kappa \varkappa^o{}_{q+p} - F^o{}_\kappa \varkappa^o{}_{q+p-1}|| \\
&+ ||F^o{}_\kappa \varkappa^o{}_{q+p-1} - F^o{}_\kappa \varkappa^o{}_q|| + ||F^o{}_\kappa \varkappa^o{}_{q+p} - \varkappa^o{}_{q+p}|| \\
&\leq ||\varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_q|| + \boldsymbol{G}(||\varkappa^o{}_q - F^o{}_\kappa \varkappa^o{}_q||, ||\varkappa^o{}_{q+p} - F^o{}_\kappa \varkappa^o{}_{q+p}||, ||\varkappa^o{}_{q+p-1} \\
&- F^o{}_\kappa \varkappa^o{}_{q+p-1}||) + ||F^o{}_\kappa \varkappa^o{}_{q+p} - \varkappa^o{}_{q+p}||
\end{aligned}
$$

which implies $||\varkappa^o{}_q - \varkappa^o{}_{q+p}|| \to 0$ as $q \to \infty$. Therefore, $(\varkappa^o{}_q)$ is a Cauchy sequence. The remainder of the proof follows directly. □

5. Fixed-Point Theorems in Incomplete Normed Spaces

In the following result, we remove the necessity for the normed space to be complete and instead introduce two new conditions, (iii) and (iv).

Theorem 6. *Let* $(\mathfrak{Q}, \|\cdot\|)$ *be a normed space where* $|\mathfrak{Q}| \geq 3$*. Consider a mapping* $F^o : \mathfrak{Q} \to \mathfrak{Q}$ *that fulfills these four conditions:*

(i) F^o *is a generalized enriched Kannan operator on* $\mathfrak{Q}$*;*

(ii) $F^o{}_\kappa$ *lacks periodic points with a prime period of 2;*

(iii) F^o *is continuous at* $\varkappa^{o*} \in \mathfrak{Q}$*;*

(iv) There is a point $\varkappa^o{}_0 \in \mathfrak{Q}$ *such that the sequence of iterates* $\varkappa^o{}_q = (1-\kappa)\varkappa^o{}_{q-1} + \kappa F^o \varkappa^o{}_{q-1} = F^o{}_\kappa(\varkappa^o{}_{q-1})$ *for* $q = 1, 2, \ldots$*, which has a subsequence* $\varkappa^o{}_{q_k}$ *that converges to* $\varkappa^{o*}$*.*

Thus, $\varkappa^{o*}$ *is a fixed point of* F^o*. Moreover, the maximum number of fixed points for* F^o *is two.*

Proof. Given that F^o is continuous at $\varkappa^{o*}$, it follows that $F^o{}_\kappa$ is also continuous at $\varkappa^{o*}$. Given that $\varkappa^o{}_{q_k} \to \varkappa^{o*}$, we have $F^o{}_\kappa \varkappa^o{}_{q_k} = \varkappa^o{}_{q_k+1} \to F^o{}_\kappa \varkappa^{o*}$. Observe that $\varkappa^o{}_{q_k+1}$ is a subsequence of $\varkappa^o{}_q$, even though it does not have to be a subsequence of $\varkappa^o{}_{q_k}$. Assume that $\varkappa^{o*}$ is not equal to $F^o{}_\kappa \varkappa^{o*}$. Consider two balls $A_1 = A_1(\varkappa^{o*}, s)$ and $A_2 = A_2(F^o{}_\kappa \varkappa^{o*}, s)$, where $s < \frac{1}{3}\|\varkappa^{o*} - F^o{}_\kappa \varkappa^{o*}\|$. Therefore, there exists a positive integer N such that for $i > N$, it follows that

$$\varkappa^o{}_{q_i} \in A_1 \text{ and } \varkappa^o{}_{q_i+1} \in A_2.$$

Hence,

$$||\varkappa^o{}_{q_i} - \varkappa^o{}_{q_i+1}|| > s \tag{26}$$

for $i > N$, if the sequence $\varkappa^o{}_q$ does not include a fixed point of the mapping $F^o{}_\kappa$ (or F^o), then the analysis provided in Theorem 2 can be applied. Based on (20), for $q = 3, 4, \ldots$, it follows that

$$||\varkappa^o{}_{q-1} - \varkappa^o{}_q|| \leqslant \theta^{\frac{q}{2}-1} b,$$

where $b = \max\{||\varkappa^o{}_0 - \varkappa^o{}_1||, ||\varkappa^o{}_1 - \varkappa^o{}_2||\}$ and $\theta = 2\beta/(2-\beta) \in [0, 1)$. Hence,

$$||\varkappa^o{}_{q_i} - \varkappa^o{}_{q_i+1}|| \leqslant \beta^{\frac{q_i+1}{2}-1} b.$$

However, since the the previous expression tends to 0 as $i \to \infty$, this contradicts (26). Therefore, it must be that $F^o{}_\kappa \varkappa^{o*} = \varkappa^{o*}$. The fact that there can be at most two fixed points is a consequence of the last paragraph of Theorem 2. □

Theorem 7. *Consider a normed space* $(\mathfrak{Q}, \|\cdot\|)$ *with* $|\mathfrak{Q}| \geq 3$, *and let* $F^o : \mathfrak{Q} \to \mathfrak{Q}$ *be a mapping. Furthermore, assume that*

(i) F^o *is a generalized enriched Kannan mapping on* N, *where* $N \subseteq \mathfrak{Q}$ *is a set that is dense everywhere in* $\mathfrak{Q}$;
(ii) F^o *is continuous mapping;*
(iii) ${F^o}_\kappa$ *lacks periodic points with a prime period of 2;*
(iv) *There exists an element* ${\varkappa^o}_0 \in \mathfrak{Q}$ *such that the iterated sequence* ${\varkappa^o}_\mathrm{q} = {F^o}_\kappa {\varkappa^o}_{\mathrm{q}-1}$ *for* $\mathrm{q} = 1, 2, \ldots$ *contains a subsequence* ${\varkappa^o}_{\mathrm{q}_k}$ *that converges to* ${\varkappa^o}^*$.

Thus, ${\varkappa^o}^*$ *is a fixed point of* F^o. *Furthermore, there can be at most two fixed points.*

Proof. The proof will be completed using Theorem 6 if we can show that F^o is a generalized enriched Kannan mapping on $\mathfrak{Q}$. Let $\varkappa^o$, μ, and ϖ be three distinct points in $\mathfrak{Q}$ where $\varkappa^o$ and μ are in M and ϖ is in $\mathfrak{Q} \setminus N$. Consider a sequence (d_q) in N such that $d_\mathrm{q} \to \varpi$, $d_\mathrm{q} \neq \varkappa^o$ and $d_\mathrm{q} \neq \mu$ for all q, and $d_i \neq d_j$ for $i \neq j$. Then,

$$\begin{aligned}
&||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \varpi|| \\
&\leqslant ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa d_\mathrm{q}|| + ||{F^o}_\kappa d_\mathrm{q} - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa d_\mathrm{q}|| \\
&\quad + ||{F^o}_\kappa d_\mathrm{q} - {F^o}_\kappa \varpi|| \\
&\leqslant \beta(||\varkappa^o - {F^o}_\kappa \varkappa^o|| + ||\mu - {F^o}_\kappa \mu|| + ||d_\mathrm{q} - {F^o}_\kappa d_\mathrm{q}||) + 2||{F^o}_\kappa d_\mathrm{q} - {F^o}_\kappa \varpi||.
\end{aligned}$$

Using the inequality

$$||d_\mathrm{q} - {F^o}_\kappa d_\mathrm{q}|| \leqslant ||d_\mathrm{q} - \varpi|| + ||\varpi - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varpi - {F^o}_\kappa d_\mathrm{q}|| \tag{27}$$

in the above inequality, we obtain

$$\begin{aligned}
&||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \varpi|| \leqslant \beta(||\varkappa^o - {F^o}_\kappa \varkappa^o|| \\
&+ ||\mu - {F^o}_\kappa \mu|| + ||\varpi - {F^o}_\kappa \varpi||) + \beta||d_\mathrm{q} - \varpi|| + \beta||{F^o}_\kappa \varpi - {F^o}_\kappa d_\mathrm{q}|| + 2||{F^o}_\kappa d_\mathrm{q} - {F^o}_\kappa \varpi||.
\end{aligned}$$

Letting $n \to \infty$ we obtain $||d_\mathrm{q} - \varpi|| \to 0$ and $|{F^o}_\kappa d_\mathrm{q} - {F^o}_\kappa \varpi|| \to 0$. Hence, inequality (6) follows.

Let $\varkappa^o \in N$, and $\mu, \varpi \in \mathfrak{Q} \setminus N$. Consider sequences (b_q) and (c_q) in N such that $b_\mathrm{q} \to \mu$ and $c_\mathrm{q} \to \varpi$. (Throughout, it is assumed that $\varkappa^o$, μ, ϖ, and all elements of the sequences converging to these points are pairwise distinct.) Then,

$$\begin{aligned}
&||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \varpi|| \\
&\leqslant ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa b_\mathrm{q}|| + ||{F^o}_\kappa b_\mathrm{q} - {F^o}_\kappa \mu|| \\
&+ ||{F^o}_\kappa \mu - {F^o}_\kappa b_\mathrm{q}|| + ||{F^o}_\kappa b_\mathrm{q} - {F^o}_\kappa c_\mathrm{q}|| + ||{F^o}_\kappa c_\mathrm{q} - {F^o}_\kappa \varpi|| \\
&+ ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa c_\mathrm{q}|| + ||{F^o}_\kappa c_\mathrm{q} - {F^o}_\kappa \varpi|| \\
&\leqslant \beta(||\varkappa^o - {F^o}_\kappa \varkappa^o|| + ||b_\mathrm{q} - {F^o}_\kappa b_\mathrm{q}|| + ||c_\mathrm{q} - {F^o}_\kappa c_\mathrm{q}||) + 2||{F^o}_\kappa b_\mathrm{q} - {F^o}_\kappa \mu|| + 2||{F^o}_\kappa c_\mathrm{q} - {F^o}_\kappa \varpi||.
\end{aligned}$$

Using the following inequality

$$||b_\mathrm{q} - {F^o}_\kappa b_\mathrm{q}|| \leqslant ||b_\mathrm{q} - \mu|| + ||\mu - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa b_\mathrm{q}|| \tag{28}$$

and inequality (27) in the above expression, we obtain

$$\begin{aligned}
&||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa \varpi|| + ||{F^o}_\kappa \varkappa^o - {F^o}_\kappa \varpi|| \\
&\leqslant \beta(||\varkappa^o - {F^o}_\kappa \varkappa^o|| + ||\mu - {F^o}_\kappa \mu|| + ||\varpi - {F^o}_\kappa \varpi||) + 2||{F^o}_\kappa b_\mathrm{q} - {F^o}_\kappa \mu|| + 2||{F^o}_\kappa c_\mathrm{q} - {F^o}_\kappa \varpi|| \\
&+ \beta(||b_\mathrm{q} - \mu|| + ||{F^o}_\kappa \mu - {F^o}_\kappa b_n|| + ||c_\mathrm{q} - \varpi|| + ||{F^o}_\kappa \varpi - {F^o}_\kappa c_\mathrm{q}||).
\end{aligned}$$

Taking the limit as $\mathrm{q} \to \infty$, we obtain inequality (6).

Now, let $\varkappa^o$, μ, and ϖ be points in $\mathfrak{Q} \setminus N$, and let (a_{q}), (b_{q}), and (c_{q}) be sequences in N such that $a_{\mathrm{q}} \to \varkappa^o$, $b_{\mathrm{q}} \to \mu$, and $c_{\mathrm{q}} \to \varpi$. Then,

$$\begin{aligned}
&||F^o{}_\kappa\varkappa^o - F^o{}_\kappa\mu|| + ||F^o{}_\kappa\mu - F^o{}_\kappa\varpi|| + ||F^o{}_\kappa\varkappa^o - F^o{}_\kappa\varpi|| \\
&\leqslant ||F^o{}_\kappa\varkappa^o - F^o{}_\kappa a_{\mathrm{q}}|| + ||F^o{}_\kappa a_{\mathrm{q}} - F^o{}_\kappa b_{\mathrm{q}}|| + ||F^o{}_\kappa b_{\mathrm{q}} - F^o{}_\kappa\mu|| + ||F^o{}_\kappa\mu - F^o{}_\kappa b_{\mathrm{q}}|| \\
&+||F^o{}_\kappa b_{\mathrm{q}} - F^o{}_\kappa c_{\mathrm{q}}|| + ||F^o{}_\kappa c_{\mathrm{q}} - F^o{}_\kappa\varpi|| + ||F^o{}_\kappa\varkappa^o - F^o{}_\kappa a_{\mathrm{q}}|| \\
&+||F^o{}_\kappa a_{\mathrm{q}} - F^o{}_\kappa c_{\mathrm{q}}|| + ||F^o{}_\kappa c_{\mathrm{q}} - F^o{}_\kappa\varpi|| \\
&\leqslant \beta(||a_{\mathrm{q}} - F^o{}_\kappa a_{\mathrm{q}}|| + ||b_{\mathrm{q}} - F^o{}_\kappa b_{\mathrm{q}}|| + ||c_{\mathrm{q}} - F^o{}_\kappa c_{\mathrm{q}}||) \\
&+2||F^o{}_\kappa a_{\mathrm{q}} - F^o{}_\kappa\varkappa^o|| + 2||F^o{}_\kappa b_{\mathrm{q}} - F^o{}_\kappa\mu|| + 2||F^o{}_\kappa c_{\mathrm{q}} - F^o{}_\kappa\varpi||.
\end{aligned}$$

Using the following inequality

$$||a_{\mathrm{q}} - F^o{}_\kappa a_{\mathrm{q}}|| \leqslant ||a_{\mathrm{q}} - \varkappa^o|| + ||\varkappa^o - F^o{}_\kappa\varkappa^o|| + ||F^o{}_\kappa\varkappa^o - F^o{}_\kappa a_{\mathrm{q}}||, \tag{29}$$

and inequalities (27) and (28) in the above expression, we obtain

$$\begin{aligned}
&||F^o{}_\kappa\varkappa^o - F^o{}_\kappa\mu|| + ||F^o{}_\kappa\mu - F^o{}_\kappa\varpi|| + ||F^o{}_\kappa\varkappa^o - F^o{}_\kappa\varpi|| \\
&\leqslant \beta(||\varkappa^o - F^o{}_\kappa\varkappa^o|| + ||\mu - F^o{}_\kappa\mu|| + ||\varpi - F^o{}_\kappa\varpi||) \\
&+2||F^o{}_\kappa a_{\mathrm{q}} - F^o{}_\kappa\varkappa^o|| + 2||F^o{}_\kappa b_{\mathrm{q}} - F^o{}_\kappa\mu|| + 2||F^o{}_\kappa c_{\mathrm{q}} - F^o{}_\kappa\varpi|| \\
&+\beta(||a_{\mathrm{q}} - \varkappa^o|| + ||F^o{}_\kappa\varkappa^o - F^o{}_\kappa a_{\mathrm{q}}|| + ||b_{\mathrm{q}} - \mu|| + ||F^o{}_\kappa\mu - F^o{}_\kappa b_{\mathrm{q}}|| \\
&+||c_{\mathrm{q}} - \varpi|| + ||F^o{}_\kappa\varpi - F^o{}_\kappa c_{\mathrm{q}}||).
\end{aligned}$$

Taking the limit as $\mathrm{q} \to \infty$, we derive inequality (6). Therefore, since F^o is a generalized enriched Kannan mapping on $\mathfrak{Q}$, the proof is concluded with the help of Theorem 6. □

6. Application to Fractional Differential Equations

In this section, we use our main results to explore the existence of solutions for boundary value problems concerning fractional differential equations that incorporate the Caputo fractional derivative.

Suppose $\mathfrak{Q} = \mathfrak{C}([0,1], \mathbb{R})$ denotes the Banach space of all continuous function mappings from $[0,1]$ to $\mathbb{R}$ equipped with the norm

$$\|\varkappa^o\| = \max_{\eta \in [0,1]} |\varkappa^o(\eta)|.$$

Now, we will review the following fundamental concepts that will be required later.

Definition 9 ([31])**.** *For a function v defined on the interval $[a,b]$, the Caputo fractional derivative of order $\zeta > 0$ is expressed as follows:*

$$\left({}^c\mathfrak{D}^{\zeta}_{a^+}\right)v(\eta) = \frac{1}{\Gamma(m-\zeta)}\int_a^\eta (\eta - s)^{m-\zeta-1} v^{(m)}(s)ds, \quad (m-1 \le \zeta < m, m = [\zeta]+1), \tag{30}$$

where $[\zeta]$ indicates the integer part of the positive real number ζ, and Γ refers to the gamma function. Suppose the boundary value problem for a fractional order differential equation is defined as follows:

$$\begin{aligned}
&{}^c\mathfrak{D}^{\zeta}_{0^+}(\varkappa^o(\eta)) = h(\eta, \varkappa^o(\eta)), \qquad (\eta \in [0,1], 2 < \zeta \le 3) \\
&\varkappa^o(0) = c_0, \varkappa^{o\prime}(0) = c, \varkappa^{o\prime\prime}(1) = c_1,
\end{aligned} \tag{31}$$

where ${}^c\mathfrak{D}^{\zeta}_{0^+}$ signifies the Caputo fractional derivative of order ζ, $h : [0,1] \to \mathbb{R}$ is a continuous function, and c_0, c, c_1 are real constants.

Definition 10 ([32]). *A function $\varkappa^o \in \mathfrak{C}^3([0,1],\mathbb{R})$ for which the ζ-derivative exists on $[0,1]$ is considered a solution of* (31) *if it satisfies the equation ${}^c\mathfrak{D}^{\beta}_{0^+}(\varkappa^o(\eta)) = h(\eta, \varkappa^o(\eta))$ on $[0,1]$ along with the conditions $\varkappa^o(0) = c_0$, $\varkappa^{o\prime}(0) = c$, and $\varkappa^{o\prime\prime}(1) = c_1$.*

The following lemma will be essential for the subsequent discussion.

Lemma 1 ([32]). *Let $2 < \zeta \leq 3$ and let $v : [0,1] \to \mathbb{R}$ be a continuous function. A function $\varkappa^o$ is considered a solution of the fractional integral equation*

$$\varkappa^o(\eta) = \frac{1}{\Gamma(\zeta)} \int_0^{\eta} (\eta - s)^{\zeta-1} v(s) ds - \frac{\eta^2}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} v(s) ds + c_0 + c_0\eta + \frac{c_1}{2}\eta^2 \quad (32)$$

if $\varkappa^o$ is a solution to the fractional boundary value problem

$$\begin{aligned} {}^c\mathfrak{D}^{\zeta}_{0^+}(\varkappa^o(\eta)) &= v(\eta) \\ \varkappa^o(0) = c_0, \varkappa^{o\prime}(0) &= c, \varkappa^{o\prime\prime}(1) = c_1, \end{aligned} \quad (33)$$

where

$$\varkappa^{o\prime\prime}(1) = 2c_2 + \frac{1}{\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} v(s) ds = c_1,$$

and c, c_0, c_1 and c_2 are constants in $\mathbb{R}$.

Next, we define the mapping $F^o : \mathfrak{Q} \to \mathfrak{Q}$ by

$$\begin{aligned} F^o\varkappa^o(\eta) \quad = \quad & \frac{1}{\Gamma(\zeta)} \int_0^{\eta} (\eta - s)^{\zeta-1} h(s, \varkappa^o(s)) ds \\ & - \frac{\eta^2}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} h(s, \varkappa^o(s)) ds + c_0 + c_0^*\eta + \frac{c_1}{2}\eta^2 \end{aligned} \quad (34)$$

where

$$c_1 = 2c_2 + \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\zeta-3} h(s, \varkappa^o(s)) ds, c_i, c_0^* \in \mathbb{R}, (i = 0,1,2) \text{ are constant.}$$

Then, for some $\kappa \in [0,1]$, we obtain

$$\begin{aligned} F^o{}_{\kappa}\varkappa^o(\eta) = & \frac{\kappa}{\Gamma(\zeta)} \int_0^{\eta} (\eta - s)^{\zeta-1} h(s, \varkappa^o(s)) d\, s \\ & - \frac{\kappa\eta^2}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} h(s, \varkappa^o(s)) ds + \kappa(c_0 + c_0^*\eta + \frac{c_1}{2}\eta^2) + (1-\kappa)\varkappa^o(\eta). \end{aligned}$$

Clearly, the fixed points of mapping F^o, as defined in (34), are the solutions of a boundary value problem for a fractional order differential Equation (31).

Now, we state and prove our main result in this section.

Theorem 8. *Suppose that for all $\varkappa^o{}_1, \varkappa^o{}_2 \in I$ there exists $0 < \lambda < \frac{1}{4}$ such that $\forall \eta \in [0,1]$,*

$$\begin{aligned} & |h(\eta, \varkappa^o{}_1(\eta)) - h(\eta, \varkappa^o{}_2(\eta))| \\ & \leq \lambda[|\varkappa^o{}_1(\eta) - F^o\varkappa^o{}_1(\eta)| + |\varkappa^o{}_2(\eta) - F^o\varkappa^o{}_2(\eta)|] - \frac{1}{2}(1-\kappa)(\varkappa^o{}_1(\eta) - \varkappa^o{}_2(\eta)) \end{aligned} \quad (35)$$

where

$$0 < \sigma := 2\lambda\left(\frac{1}{\Gamma(\zeta+1)} + \frac{1}{2\Gamma(\zeta-1)}\right) < 1. \quad (36)$$

Then, Equation (31) *has at most two solutions in I.*

Proof. Inequality (6) can be equivalently expressed for $\kappa = \frac{1}{1+\mathfrak{s}} \leq 1$ and $0 \leq \beta < 1$ as follows:

$$||F^o{}_\kappa \varkappa^o - F^o{}_\kappa \mu|| + ||F^o{}_\kappa \mu - F^o{}_\kappa \varpi|| + ||F^o{}_\kappa \varkappa^o - F^o{}_\kappa \varpi|| \\ \leq \beta(||\varkappa^o - F^o{}_\kappa \varkappa^o|| + ||\mu - F^o{}_\kappa \mu|| + ||\varpi - F^o{}_\kappa \varpi||), \quad \forall \varkappa^o, \mu, \varpi \in \mathfrak{Q}.$$

Therefore, considering the following and using the given assumption in (35) and (36), we obtain

$$\begin{aligned}
& |F^o{}_\kappa \varkappa^o(\eta) - F^o{}_\kappa \mu(\eta)| + |F^o{}_\kappa \mu(\eta) - F^o{}_\kappa \varpi(\eta)| + |F^o{}_\kappa \varkappa^o(\eta) - F^o{}_\kappa \varpi(\eta)| \\
= \; & \frac{\kappa}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} |h(s, \varkappa^o(s)) - h(s, \mu(s))| d\, s \\
& + \frac{\kappa \eta^2}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} |h(s, \varkappa^o(s)) - h(s, \mu(s))| ds + (1-\kappa)(\varkappa^o(\eta) - \mu(\eta)) \\
& + \frac{\kappa}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} |h(s, \mu(s)) - h(s, \varpi(s))| ds \\
& + \frac{\kappa \eta^2}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} |h(s, \mu(s)) - h(s, \varpi(s))| ds + (1-\kappa)(\mu(\eta) - \varpi(\eta)) \\
& + \frac{\kappa}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} |h(s, \varkappa^o(s)) - h(s, \varpi(s))| ds \\
& + \frac{\kappa \eta^2}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} |h(s, \varkappa^o(s)) - h(s, \varpi(s))| ds + (1-\kappa)(\varkappa^o(\eta) - \varpi(\eta)) \\
\leq \; & \frac{\kappa}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} \lambda[|\varkappa^o(\eta) - F^o \varkappa^o(\eta)| + |\mu(\eta) - F^o \mu(\eta)|] ds \\
& + \frac{\kappa}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} \lambda[|\varkappa^o(\eta) - F^o \varkappa^o(\eta)| + |\mu(\eta) - F^o \mu(\eta)|] d\, s \\
& + \frac{\kappa}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\zeta-1} \lambda[|\mu(\eta) - F^o \mu(\eta)| + |\varpi(\eta) - F^o \varpi(\eta)|] ds \\
& + \frac{\kappa}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} \lambda[|\mu(\eta) - F^o \mu(\eta)| + |\varpi(\eta) - F^o \varpi(\eta)|] ds \\
& + \frac{\kappa}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} \lambda[|\varkappa^o(\eta) - F^o \varkappa^o(\eta)| + |\varpi(\eta) - F^o \varpi(\eta)|] ds \\
& + \frac{\kappa}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} \lambda[|\varkappa^o(\eta) - F^o \varkappa^o(\eta)| + |\varpi(\eta) - F^o \varpi(\eta)|] ds \\
= \; & \frac{2\kappa\lambda}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} [|\varkappa^o(\eta) - F^o \varkappa^o(\eta)| + |\mu(\eta) - F^o \mu(\eta)| + |\varpi(\eta) - F^o \varpi(\eta)|] ds \\
& + \frac{2\kappa\lambda}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} [|\varkappa^o(\eta) - F^o \varkappa^o(\eta)| + |\mu(\eta) - F^o \mu(\eta)| + |\varpi(\eta) - F^o \varpi(\eta)|] ds \\
\leq \; & \frac{2\kappa\lambda}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} [\|\varkappa^o(\eta) - F^o \varkappa^o(\eta)\| + \|\mu(\eta) - F^o \mu(\eta)\| + \|\varpi(\eta) - F^o \varpi(\eta)\|] ds \\
& + \frac{2\kappa\lambda}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} [\|\varkappa^o(\eta) - F^o \varkappa^o(\eta)\| + \|\mu(\eta) - F^o \mu(\eta)\| + \|\varpi(\eta) - F^o \varpi(\eta)\|] d\, s \\
= \; & \left[\frac{2\lambda}{\Gamma(\zeta)} \int_0^\eta (\eta - s)^{\zeta-1} ds + \frac{2\lambda}{2\Gamma(\zeta-2)} \int_0^1 (1-s)^{\zeta-3} ds \right] \\
& \times [\|\varkappa^o(\eta) - F^o \varkappa^o(\eta)\| + \|\mu(\eta) - F^o \mu(\eta)\| + \|\varpi(\eta) - F^o \varpi(\eta)\|] \\
\leq \; & 2\lambda \left(\frac{1}{\Gamma(\zeta+1)} + \frac{1}{2\Gamma(\zeta-1)} \right) [\|\varkappa^o(\eta) - F^o \varkappa^o(\eta)\| + \|\mu(\eta) - F^o \mu(\eta)\| + \|\varpi(\eta) - F^o \varpi(\eta)\|] \\
= \; & \sigma[\|\varkappa^o(\eta) - F^o \varkappa^o(\eta)\| + \|\mu(\eta) - F^o \mu(\eta)\| + \|\varpi(\eta) - F^o \varpi(\eta)\|].
\end{aligned}$$

Hence, this implies that

$$\|F^o{}_\kappa \varkappa^o - F^o{}_\kappa \mu\| + \|F^o{}_\kappa \mu - F^o{}_\kappa \varpi\| + \|F^o{}_\kappa \varkappa^o - F^o{}_\kappa \varpi\|$$
$$\leq \sigma[\|\varkappa^o - F^o{}_\kappa \varkappa^o\| + \|\mu - F^o{}_\kappa \mu\| + \|\varpi - F^o{}_\kappa \varpi\|].$$

So, it follows that F^o is a generalized $(\mathfrak{s}, \sigma)$-enriched Kannan operator. Also, $F^o{}_\kappa$ does not possess periodic points with a prime period of 2. Hence, by Theorem 2, the problem (31) has at most two solutions. □

7. Conclusions

In this research, we introduced a novel class of mappings in linear normed spaces, termed generalized enriched Kannan mappings. These mappings extend enriched Kannan mappings to three-point analogues and simultaneously expand the framework of generalized Kannan mappings. They are generally discontinuous, except at fixed points, where they maintain a continuity similar to enriched Kannan mappings. Although similar, generalized enriched Kannan mappings and enriched Kannan mappings represent distinct classes. We formulated a fixed-point theorem for these generalized mappings and broadened the applicability of fixed-point theorems by incorporating conditions of asymptotic regularity and continuity. Furthermore, we established two additional fixed-point theorems applicable to normed spaces, without the need for them to be Banach spaces. In this work, we addressed a boundary value problem for fractional differential equations and used our main results to establish the existence of solutions. Although the primary focus was on fractional differential equations, the methods developed can also be extended to other types of equations, including ordinary and partial differential equations, as well as integral equations. The underlying principles of fixed-point theory provide a powerful framework, demonstrating its broad applicability and potential for solving diverse mathematical problems.

A promising direction for future research involves utilizing three-point analogue techniques to investigate how generalized enriched contractions can characterize the completeness of normed spaces. This approach could provide valuable insights into the structural properties of normed spaces and deepen the theoretical understanding of fixed-point theorems. Furthermore, the concept of cyclic representation can be applied to three-point analogues to further generalize this work, making it a promising direction for future research.

Author Contributions: Conceptualization, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; methodology, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; software, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; validation, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; formal analysis, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; investigation, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; resources, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; data curation, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; writing—original draft preparation, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; writing—review and editing, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; visualization, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; supervision, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; project administration, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G.; funding acquisition, M.D., U.I., K.A.A., T.A.L., V.L.L. and L.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data will be available on demand from the corresponding author.

Acknowledgments: The authors extend their appreciation to King Saud University, Saudi Arabia, for funding this work through Researchers Supporting Project number (RSP2024R305), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Argyros, I.K. *The Theory and Applications of Iteration Methods*, 2nd ed.; Taylor and Francis and CRC Press: Boca Raton, FL, USA, 2022.
2. Argyros, I.K.; George,S.; Samundra R.; Argyros, G. *Contemporary Algorithms: Theory and Applications*; Nova Science Publishers: New York, NY, USA, 2024; Volume IV.

3. Domínguez Benavides, T.; Lorenzo Ramírez, J.; Gatica Araus, I. Some generalizations of Kannan's fixed point theorem in K-metric spaces. *Fixed Point Theory* **2012**, *13*, 73–83.
4. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequalities Appl.* **2014**, *2014*, 38. [CrossRef]
5. Ahmad, J.; Al-Mazrooei, A.E.; Rassias, T.M. Fractals of generalized Θ-Hutchinson operator. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 1–12.
6. Shioji, N.; Suzuki, T.; Takahashi, W. Contractive mappings, Kannan mappings and metric completeness. *Proc. Am. Math. Soc.* **1998**, *126*, 3117–3124. [CrossRef]
7. Abbas, M.; Anjum, R.; Iqbal, H. Generalized enriched cyclic contractions with application to generalized iterated function system. *Chaos Solitons Fractals* **2022**, *154*, 111591. [CrossRef]
8. Almalki, Y.; Din, F.U.; Din, M.; Ali, M.U.; Jan, N. Perov-fixed point theorems on a metric space equipped with ordered theoretic relation. *AIMS Math.* **2022**, *11*, 20199–20212. [CrossRef]
9. Shukla, S.; Radenovic, S. Presic-Boyd-Wong Type Results in Ordered Metric Spaces. *Int. J. Anal. Appl.* **2014**, *5*, 154–166.
10. Tarafdar, E. An approach to fixed-point theorems on uniform spaces. *Trans. Am. Math. Soc.* **1974**, *191*, 209–225. [CrossRef]
11. Ilić, D.; Abbas, M.; Nazir, T. Iterative approximation of fixed points of Prešić operators on partial metric spaces. *Math. Nachrichten* **2015**, *288*, 1634–1646. [CrossRef]
12. Wardowski, D.; Dung, N.V. Fixed points of F-weak contractions on complete metric spaces. *Demonstr. Math.* **2014**, *47*, 146–155. [CrossRef]
13. Din, F.U.; Din, M.; Ishtiaq, U.; Sessa, S. Perov fixed-point results on F-contraction mappings equipped with binary relation. *Mathematics* **2023**, *11*, 238. [CrossRef]
14. Berinde, V.; Păcurar, M. Approximating fixed points of enriched contractions in Banach spaces. *J. Fixed Point Theory Appl.* **2020**, *22*, 1–10. [CrossRef]
15. Anjum, R.; Din, M.; Zhou, M. Fractals of two types of enriched (q, θ)-Hutchinson–Barnsley operators. *Chaos Solitons Fractals* **2024**, *181*, 114589. [CrossRef]
16. Din, M.; Ishtiaq, U.; Mukhtar, M.; Sessa, S.; Ghazwani, H.A. On Generalized Sehgal–Guseman-Like Contractions and Their Fixed-Point Results with Applications to Nonlinear Fractional Differential Equations and Boundary Value Problems for Homogeneous Transverse Bars. *Mathematics* **2024**, *12*, 541. [CrossRef]
17. Anjum, R.; Abbas, M.; Safdar, H.; Din, M.; Zhou, M.; Radenović, S. Application to Activation Functions through Fixed-Circle Problems with Symmetric Contractions. *Symmetry* **2024**, *16*, 69. [CrossRef]
18. Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
19. Subrahmanyam, P.V. Completeness and fixed-points. *Monatshefte Math.* **1975**, *80*, 325–330. [CrossRef]
20. Connell, E.H. Properties of fixed point spaces. *Proc. Am. Math. Soc.* **1959**, *10*, 974–979. [CrossRef]
21. Suzuki, T. Contractive mappings are Kannan mappings, and Kannan mappings are contractive mappings in some sense. *Comment. Math.* **2005**, *45*, 45–58.
22. Kikkawa, M.; Suzuki, T. Some similarity between contractions and Kannan mappings II. *Bull. Kyushu Inst. Technol. Pure Appl. Math.* **2008**, *55*, 1–13.
23. Berinde, V. Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **2004**, *9*, 43–54.
24. Berinde, V.; Takens, F. *Iterative Approximation of Fixed Points*; Springer: Berlin/Heidelberg, Germany, 2007; Volume 1912, pp. xvi+-322.
25. Petrov, E. Fixed point theorem for mappings contracting perimeters of triangles. *J. Fixed Point Theory Appl.* **2023**, *25*, 74. [CrossRef]
26. Petrov, E.; Bisht, R.K. Fixed point theorem for generalized Kannan type mappings. In *Rendiconti del Circolo Matematico di Palermo Series 2*; Springer: Berlin/Heidelberg, Germany, 2024; pp. 1–18. . [CrossRef]
27. Berinde, V.; Păcurar, M. Fixed point theorems for Kannan type mappings with applications to split feasibility and variational inequality problems. *arXiv* **2019**, arXiv:1909.02379.
28. Păcurar, C.M.; Popescu, O. Fixed point theorem for generalized Chatterjea type mappings. *Acta Math. Hung.* **2024**, *173*, 500–509. [CrossRef]
29. Bisht, R.K.; Petrov, E. A three point extension of Chatterjea's fixed point theorem with at most two fixed points. *arXiv* **2024**, arXiv:2403.07906.
30. Zhou, M.; Petrov, E. Perimetric Contraction on Polygons and Related Fixed Point Theorems. *arXiv* **2024**, arXiv:2410.20449.
31. Kilbas, A.A.; Marzan, S.A. Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions. *Differ. Equ.* **2005**, *41*, 84–89. [CrossRef]
32. Agarwal, R.P.; Benchohra, M.; Hamani, S. Boundary value problems for fractional differential equations. *Georgian Math. J.* **2009**, *16*, 401–411. [CrossRef]