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Approximate and Exact Controllability for Hilfer Fractional Stochastic Evolution Equations

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Abstract: This paper investigates the controllability of Hilfer fractional stochastic evolution equations (HFSEEs). Initially, we obtain a conclusion regarding the approximate controllability of HFSEEs by employing the Tikhonov-type regularization method and Schauder′s fixed-point theorem. Additionally, the conditions for the exact controllability of HFSEEs are explored, utilizing the Mönch′s fixed-point theorem and measure of noncompactness. Finally, the proposed method is validated through an example, thereby demonstrating its effectiveness.

Keywords: controllability; Hilfer fractional derivative; stochastic evolution equations

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1. Introduction

Control theory plays a vital role in mathematical exploration, serving as a foundation for system optimization and stability analysis [1–3]. In the past few years, numerous academics have conducted research on the controllability of diverse dynamical systems utilizing a range of methodologies [4–8]. Exact controllability means that the system can accurately reach the target state through deterministic control, while approximate controllability means that the system can approach the target state through appropriate random control. The control theory of stochastic differential equations plays an important role in risk management, stock trading, weather forecasting, disease control, etc., which can improve the quality and effectiveness of decision-making and reduce risks and costs [9–11].

Compared with integer derivatives, fractional derivatives have wider applicability, more complete descriptive power, better disclosure of non-local properties, and more mathematical and physical applications. Therefore, the control theory of fractional stochastic differential equations has been garnering increasing attention from researchers. In [12], Sakthivel et al. studied the approximate controllability of the Caputo FSEEs via the fixed point theorem. In [13], Shu et al. studied the approximate controllability of the Riemann–Liouville FSEEs with order $1 < \alpha < 2$ by using the concepts related to sectorial operators and Mönch′s fixed point theorem. For further research on the approximate controllability of fractional differential equations, we recommend consulting [14–18]. Ding and Li in [19], studied the exact controllability of the Caputo FSEEs with order $0 < \alpha < 1$ by using measure of noncompactness and Mönch′s fixed-point theorem. For research achievements related to the exact controllability of fractional differential equations, we recommend readers refer to [20–22].

The Hilfer fractional derivative can be regarded as a synthesis or extension of the Riemann–Liouville fractional derivative and the Caputo fractional derivative [23]. When studying Hilfer fractional systems, we face a problem: their equivalent integral equations make sense only on open intervals. This limits our analysis, especially when trying to use the fixed-point theorem and Ascoli-Arzelà theorem to study the properties of systems. It is worth noting that, compared with reference [24], the hypothesis conditions of this paper are weaker.

In order to avoid confusion, we will first introduce some basic notations and concepts. Let $\mathbb{H}$, $\mathbb{K}$, and $\mathbb{U}$ be separable Hilbert spaces with norm $\|\cdot\|$. Moreover, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where Ω is a nonempty sample space, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. The stochastic process $\{w(t)\}_{t\geq 0}$ is a $\mathbb{K}$-value Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Moreover, this Wiener process $\{w(t)\}_{t\geq 0}$ has a nonnegative covariance operator $\mathcal{Q}$ with a finite trace, $Tr(\mathcal{Q}) = \sum_{k=1}^{\infty} \lambda_k < \infty$, where $\{\lambda_k, e_k\}_{k\geq 1}$ is orthogonal system satisfying $\mathcal{Q}e_k = \lambda_k e_k$.

We explore the HFSEEs:

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\mu,\nu} y(t) = Ay(t) + \mathcal{G}(t, y(t)) + Bu(t) + \mathcal{T}(t, y(t)) \frac{dw(t)}{dt}, \quad t \in (0, h], \\ \left(\mathcal{I}_{0^+}^{2-\beta} y\right)(0) = y_0, \quad \left(\mathcal{I}_{0^+}^{2-\beta} y\right)'(0) = y_1. \end{cases} \tag{1}$$

In this equation, ${}^{H}\mathcal{D}_{0^+}^{\mu,\nu}$ represents the Hilfer fractional derivative with order $1 < \mu < 2$ and type $0 \leq \nu \leq 1$. The Riemann–Liouville integral operator $\mathcal{I}_{0^+}^{2-\beta}$ with order $2-\beta$, $\beta = \mu + \nu(2-\mu)$. $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a cosine family $\{\mathcal{C}(t)\}_{t\geq 0}$ consisting of strongly continuous and uniformly bounded linear operators. The stochastic process $\{w(t)\}_{t\geq 0}$ is a $\mathbb{K}$-value Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The control function $u \in L^2_{\mathcal{F}}([0, h], \mathbb{U})$. $B : \mathbb{U} \to \mathbb{H}$ is a bounded linear operator and $\|B\|_{L(\mathbb{U},\mathbb{H})} \leq M_B$. $\mathcal{G} : [0, h] \times \mathbb{H} \to \mathbb{H}$ and $\mathcal{T} : [0, h] \times \mathbb{H} \to L(\mathbb{K}, \mathbb{H})$ are given. $y_0, y_1 \in L^2_0(\Omega, \mathbb{H})$.

To ensure a clear structure, the paper is divided into several parts. Section 2 introduces fundamental information essential for our analysis. Following that, Section 3 presents an approximate controllability result for problem (1), while Section 4 provides an exact controllability result for the same problem. In Section 5, we validate the effectiveness of our findings with an example. Finally, Section 6 summarizes the content discussed throughout the paper.

2. Preliminaries

$L(\mathbb{K}, \mathbb{H})$ represents the set of bounded linear operators mapping from $\mathbb{K}$ to $\mathbb{H}$, where the norm is denoted as $\|\cdot\|_{L(\mathbb{K},\mathbb{H})}$. In particular, we use $L(\mathbb{H})$ to denote $L(\mathbb{H}, \mathbb{H})$. $L^2(\Omega, \mathbb{H})$ represent a Banach space comprising square-integrable, strongly-measurable random variables. The norm $\|y(\cdot)\|_{L^2(\Omega,\mathbb{H})} = \left(E\|y(\cdot, w)\|^2\right)^{\frac{1}{2}}$, where $E(y(\cdot)) = \int_\Omega y(\cdot, w) d\mathbb{P}$. $C\big([0, h], L^2(\Omega, \mathbb{H})\big)$ denote the Banach space consisting of continuous mappings from $[0, h]$ into $L^2(\Omega, \mathbb{H})$. Let

$$L^2_0(\Omega, \mathbb{H}) := \Big\{ y \in L^2(\Omega, \mathbb{H}), y \text{ is } \mathcal{F}_0 - \text{measurable} \Big\},$$

$$L^2_{\mathcal{F}}([0, h], \mathbb{U}) := \{ y : [0, h] \times \Omega \to \mathbb{U} \text{ is a square integrable and } \mathcal{F}_t - \text{adapted process} \},$$

$$C_{[0,h]} := \left\{ y \in C\Big([0, h], L^2(\Omega, \mathbb{H})\Big) : \|y(\cdot)\|_{C_{[0,h]}} = \left(\sup_{t\in[0,h]} E\|y(t)\|^2 \right)^{\frac{1}{2}} < \infty \right\},$$

$$C_{(0,h]} := \left\{ y \in C\Big((0, h], L^2(\Omega, \mathbb{H})\Big) : \lim_{t\to 0^+} t^{2-\beta} y(t) \text{ exists}, \|y(\cdot)\|_{C_{(0,h]}} = \left(\sup_{t\in(0,h]} E\|t^{2-\beta} y(t)\|^2 \right)^{\frac{1}{2}} \right\}.$$

Lemma 1. *(see [25]) If $\mathcal{T}(t) \in L(\mathbb{K}, \mathbb{H})$ satisfies*
(i) For $t \in [0, h]$, $\mathcal{T}(t)$ is $\mathcal{F}_t$−measurable ,
(ii) $\int_0^t E\|\mathcal{T}(s)\|^2 ds < \infty$, then

$$E\left\| \int_0^t \mathcal{T}(s) dw(s) \right\|^2 \leq Tr(\mathcal{Q}) \int_0^t E\|\mathcal{T}(s)\|^2 ds. \tag{2}$$

Definition 1. *(see [26]) The Riemann–Liouville fractional integral is defined as follows:*

$$\mathcal{I}_{0^+}^{\mu} y(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} y(s) ds, \ t > 0, \ \mu > 0.$$

Definition 2. *(see [26]) The Riemann–Liouville fractional derivative is defined as follows:*

$${}^{RL}\mathcal{D}_{0^+}^{\mu} y(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \left(\int_0^t (t-s)^{n-\mu-1} y(s) ds \right), \ \ t > 0, \ n-1 < \mu < n.$$

Definition 3. *(see [26]) The Caputo fractional derivative is defined as follows:*

$${}^{C}\mathcal{D}_{0^+}^{\mu} y(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-s)^{n-\mu-1} y^{(n)}(s) ds, \ t > 0, \ n-1 < \mu < n,$$

where the function $y(t)$ is absolutely continuous and $d^{n-1}y(t)/dt^{n-1}$ is continuous.

Definition 4. *(see [23]) The Hilfer fractional derivative is defined as follows:*

$${}^{H}\mathcal{D}_{0^+}^{\mu,\nu} y(t) = \mathcal{I}_{0^+}^{\nu(n-\mu)} \frac{d^n}{dt^n} \mathcal{I}_{0^+}^{(1-\nu)(n-\mu)} y(t), \ t > 0,$$

where $n-1 < \mu < n, 0 \leq \nu \leq 1$.

Remark 1. *(i) Especially, if $\nu = 0$, $n-1 < \mu < n$, then*

$${}^{H}\mathcal{D}_{0^+}^{\mu,0} y(t) = \frac{d^n}{dt^n} \mathcal{I}_{0^+}^{n-\mu} y(t) = {}^{RL}\mathcal{D}_{0^+}^{\mu} y(t).$$

(ii) If $\nu = 1$, $n-1 < \mu < n$, then

$${}^{H}\mathcal{D}_{0^+}^{\mu,1} y(t) = \mathcal{I}_{0^+}^{n-\mu} \frac{d^n}{dt^n} y(t) = {}^{C}\mathcal{D}_{0^+}^{\mu} y(t).$$

Let D be the bounded subset of Banach space X with the norm $\|\cdot\|_X$. The definition of the Kuratowski measure of noncompactness χ is as follows:

$$\chi(D) = inf \left\{ d > 0 : D \subset \bigcup_{j=1}^{n} V_j \text{ and } diam(V_j) \leq d \right\},$$

where $diam(V_j) = sup\{\|x_1 - x_2\|_X : x_1, x_2 \in V_j\}, j = 1, 2, \cdots, n$.

Lemma 2. *(see [27]) Let $\{\psi_n(t)\}_{n=1}^{\infty} : [0,h] \to X$ be Bochner integrable. If there exists $\phi \in L^1([0,h], \mathbb{R}^+)$ such that $\|\psi_n(t)\|_X \leq \phi(t)$ for $t \in [0,h]$. Then*

$$\chi\left(\left\{ \int_0^t \psi_n(s) ds \right\}_{n=1}^{\infty} \right) \leq 2 \int_0^t \chi(\{\psi_n(s)\}_{n=1}^{\infty}) ds.$$

Definition 5. *(see [28]) The definition of Wright function W_α is given by the following:*

$$W_\alpha(\xi) = \sum_{m=0}^{\infty} \frac{(-\xi)^m}{\Gamma(1-\alpha(m+1))m!}, \ 0 < \alpha < 1, \ \xi \in \mathbb{C},$$

which satisfies

$$\int_0^{\infty} \xi^{\delta} W_\alpha(\xi) d\xi = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \ \text{for } \delta \geq 0.$$

Definition 6. *(see [29]) If a bounded linear operator maps $\{\mathcal{C}(t)\}_{t\in\mathbb{R}} : \mathbb{H} \to \mathbb{H}$, it is referred to as a strongly continuous cosine family if and only if*
(i) $\mathcal{C}(t+s) + \mathcal{C}(t-s) = 2\mathcal{C}(t)\mathcal{C}(s)$ for all $t, s \in \mathbb{R}$,
(ii) $\mathcal{C}(0) = I$,
(iii) $\mathcal{C}(t)y$ is continuous for $t \in \mathbb{R}$ and $y \in \mathbb{H}$.

The family of operators $\{\mathcal{S}(t)\}_{t\in\mathbb{R}}$ is defined as follows:

$$\mathcal{S}(t)y = \int_0^t \mathcal{C}(s)yds, \ t \in \mathbb{R}, \ y \in \mathbb{H}.$$

The operator $A : \mathbb{H} \to \mathbb{H}$ is defined as the generator of a cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$, which is strongly continuous. It satisfies the following equation:

$$Ay = \frac{d^2\mathcal{C}(t)}{dt^2}y\bigg|_{t=0}, \ y \in D(A),$$

where $D(A) = \{y \in \mathbb{H} : \mathcal{C}(t)y$ is a twice continuously differentiable function with respect to $t\}$.

This paper discusses a strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\geq 0}$ in $\mathbb{H}$ which consists of uniformly bounded linear operators. Consequently, there exists a constant $M > 1$ satisfying $\|\mathcal{C}(t)\|_{L(\mathbb{H})} \leq M$ for $t \geq 0$.

Definition 7. *(see [30]) $y \in C\big((0,h], L^2(\Omega,\mathbb{H})\big)$ is an $\mathcal{F}_t-$adapted stochastic process, $y_0, y_1 \in L_0^2(\Omega,\mathbb{H})$, the mild solution of problem (1) is defined as follows:*

$$\begin{aligned} y(t) = {} & J(t)y_0 + K(t)y_1 + \int_0^t N(t-s)[\mathcal{G}(s,y(s)) + Bu(s)]ds \\ & + \int_0^t N(t-s)\mathcal{T}(s,y(s))dw(s), \ t \in (0,h]. \end{aligned} \tag{3}$$

where

$$J(t) = {}^{RL}\mathcal{D}_{0^+}^{1-\nu(2-\mu)}\Big(t^{\frac{\mu}{2}-1}Q(t)\Big), \ K(t) = \mathcal{I}_{0^+}^{\nu(2-\mu)}\Big(t^{\frac{\mu}{2}-1}Q(t)\Big), \ N(t) = t^{\frac{\mu}{2}-1}Q(t),$$
$$Q(t) = \int_0^\infty \frac{\mu}{2}\xi W_{\frac{\mu}{2}}(\xi)\mathcal{S}\Big(t^{\frac{\mu}{2}}\xi\Big)d\xi.$$

Lemma 3. *(see [30]) The following inequality holds for any $y \in \mathbb{H}$ and $t > 0$.*

$$\|N(t)y\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|y\|, \ \|J(t)y\| \leq \frac{Mt^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\|y\|, \ \|K(t)y\| \leq \frac{Mt^{\beta-1}}{\Gamma(\beta)}\|y\|.$$

Lemma 4. *(see [30]) The following formula is true for $y \in \mathbb{H}$ and any $t > 0$.*

$$\frac{d}{dt}(N(t)y) = \Big(\frac{\mu}{2}-1\Big)t^{-1}N(t)y + t^{\mu-2}\int_0^\infty \Big(\frac{\mu}{2}\Big)^2\xi^2 M_{\frac{1}{2}}(\xi)\mathcal{C}\Big(t^{\frac{\mu}{2}}\xi\Big)yd\xi.$$

Moreover,

$$\left\|\frac{d}{dt}(N(t)y)\right\| \leq \frac{Mt^{\mu-2}}{\Gamma(\mu)}\|y\|, \ \ t > 0.$$

Lemma 5. *(Schauder's fixed point theorem, see [31]) Let V be a closed, convex, and nonempty subset of a Banach space X. Let $\Phi : V \to V$ be a continuous mapping such that ΦV is a relatively compact subset of X. Then, Φ has at least one fixed point in V.*

Lemma 6. *(Mönch's fixed point theorem, see [32]) Let V be a closed convex subset of a Banach space X and $0 \in V$. Assume that $\Phi : V \to V$ is a continuous map that satisfies Mönch's condition, i.e., for $D \subset V$ is countable and $D \subset \overline{co}(\{0\} \cup \Phi(D)) \Rightarrow \bar{D}$ is compact. Then, Φ has at least one fixed point in V.*

Let $y(h;u)$ be the state value of system (1) at time h with control u and reachable set $R(h) = \{y(h;u) : u \in L^2_{\mathcal{F}}([0,h],\mathbb{U})\}$.

Definition 8. *(see [33]) The fractional stochastic control system (1) is said to be*
(i) approximate controllability on the interval $[0,h]$ if $\overline{R(h)} = L^2(\Omega,\mathbb{H})$;
(ii) exact controllability on the interval $[0,h]$ if $R(h) = L^2(\Omega,\mathbb{H})$.

Lemma 7. *(see [33]) For any $\varrho \in L^2(\Omega,\mathbb{H})$, there exists an $\mathcal{F}_t$−adapted stochastic process $\varphi : [0,h] \to L(\mathbb{K},\mathbb{H})$ such that $\int_0^h E\|\varphi(s)\|^2 ds < \infty$ and $\varrho = E\varrho + \int_0^h \varphi(s)dw(s)$.*

In order to present the main result of this paper, the following assumption is required:
(A_1): $\mathcal{G}(\cdot,\cdot)$ satisfies the Caristi condition: for $t \in [0,h]$, $\mathcal{G}(t,\cdot)$ is Lebesgue measurable and for each $y \in \mathbb{H}$, $\mathcal{G}(\cdot,y)$ is continuous.
(A_2): $\mathcal{T}(\cdot,\cdot)$ satisfies the Caristi condition: for $t \in [0,h]$, $\mathcal{T}(t,\cdot)$ is $\mathcal{F}_t$-measurable and $\int_0^t E\|\mathcal{T}(s,\cdot)\|^2 ds < \infty$, for each $y \in \mathbb{H}$, $\mathcal{T}(\cdot,y)$ is continuous .
(A_3): For $t \in [0,h]$ and each $y \in \mathbb{H}$, there exists $g \in L^1([0,h];\mathbb{R}^+)$ that satisfies

$$E\|\mathcal{G}(t,y)\|^2 \vee E\|\mathcal{T}(t,y)\|^2 \le g(t),$$

where $\vee$ means the maximum of the two.

Define mapping Φ :

$$(\Phi y)(t) = (\Phi_1 y)(t) + (\Phi_2 y)(t),\ y \in C_{(0,h]}, \tag{4}$$

where

$$(\Phi_1 y)(t) = J(t)y_0 + K(t)y_1,\ for\ t \in (0,h],$$
$$(\Phi_2 y)(t) = \int_0^t N(t-s)[\mathcal{G}(s,y(s)) + Bu(s)]ds + \int_0^t N(t-s)\mathcal{T}(s,y(s))dw(s),\ for\ t \in (0,h].$$

If Φ has a fixed-point $y^* \in C_{(0,h]}$, then y^* is a mild solution for problem (1).

As the Ascoli–Arzelà theorem is applicable only to finite closed intervals. Hence, it is necessary to transform Equation (4).

Let $\forall z \in C_{[0,h]}$, we define $y(t) = t^{\beta-2}z(t)$ for $t \in (0,h]$. It can be easily seen that $y \in C_{(0,h]}$.

Introduce the operator Ψ as follows:

$$(\Psi z)(t) = (\Psi_1 z)(t) + (\Psi_2 z)(t),\ for\ t \in [0,h],$$

where

$$(\Psi_1 z)(t) = \begin{cases} t^{2-\beta}(\Phi_1 y)(t), & for\ t \in (0,h], \\ \frac{y_0}{\Gamma(\beta-1)}, & for\ t = 0, \end{cases}$$

$$(\Psi_2 z)(t) = \begin{cases} t^{2-\beta}(\Phi_2 y)(t), & for\ t \in (0,h], \\ 0, & for\ t = 0. \end{cases} \tag{5}$$

So, Ψ has a fixed point that is equivalent to Φ's fixed point.

3. Approximate Controllability

We introduce a controllability matrix:

$$\Gamma_0^h = \int_0^h N(h-s)BB^*N^*(h-s)ds,$$

B^* and $N^*(t)$ denote the adjoint of B and $N(t)$, respectively. According to Lemma 3, it becomes apparent that Γ_0^h is linear and bounded.

Let $R(a,\Gamma_0^h) = (aI+\Gamma_0^h)^{-1}$, $a > 0$. We define the control function $u(t) = u^a(t;y)$ as follows:

$$u^a(t;y) = B^*N^*(h-t)R(a,\Gamma_0^h)\widetilde{S}(y), \tag{6}$$

where

$$\widetilde{S}(y) = E\varrho - J(h)y_0 - K(h)y_1 - \int_0^h N(h-s)\mathcal{G}(s,y(s))ds - \int_0^h N(h-s)(\mathcal{T}(s,y(s)) - \varphi(s))dw(s).$$

By Definition 8, we can establish that the system (1) is approximate controllability on the interval $[0,h]$ if and only if there exists $E\|y^*(h)-\varrho\|^2 \to 0$, where y^* represents the mild solution to system (1) corresponding to $u(t) = u^a(t;y)$. To prove this, our initial step is to demonstrate the existence of a mild solution for system (1) under the condition $u(t) = u^a(t;y)$.

Because $u(t) = u^a(t;y)$, then, operator Ψ_2 in (5) becomes

$$(\Psi_2 z)(t) = t^{2-\beta}\int_0^t N(t-s)[\mathcal{G}(s,y(s)) + Bu^a(s;y)]ds + t^{2-\beta}\int_0^t N(t-s)\mathcal{T}(s,y(s))dw(s),\ for\ t\in(0,h].$$

To demonstrate the approximate controllability outcome, the subsequent assumption is necessary:

(B_1): $\{\mathcal{S}(t), t \geq 0\}$ is a compact semigroup and $\|aR(a,\Gamma_0^h)\| \leq 1$ for any $a > 0$.

(B_2): There exists a constant $N > 0$, such that

$$\|\mathcal{G}(t,y(t))\| + \|\mathcal{T}(t,y(t))\| \leq N,\ \ \forall y \in C_{(0,h]},\ \forall t \in (0,h].$$

(B_3): $aR(a,\Gamma_0^h) \to 0$ as $a \to 0^+$ in the strong operator topology.

By the fact form (A_3), we have

$$\sup_{t\in[0,h]}\left\{5\left(\frac{M}{(\mu-1)\Gamma(\beta-1)}\right)^2 E\|y_0\|^2 + 5\left(\frac{t}{\Gamma(\beta)}\right)^2 E\|y_1\|^2 + 5\left(\frac{Mt^{2-\beta+\frac{\mu}{2}}}{\Gamma(\mu)}\right)^2\frac{1}{\mu}\int_0^t (t-s)^{\mu-1}g(s)ds + 5\left(\frac{M_B M t^{2-\beta+\mu}}{\Gamma(\mu+1)}\right)^2 L_u + 5\left(\frac{Mt^{2-\beta}}{\Gamma(\mu)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}g(s)ds\right\} \leq r, \tag{7}$$

where L_u is defined in Lemma 8 of this article and $r > 0$ is a constant.

Let

$$D_r = \left\{z : z \in C_{[0,h]}, \|z\|_{C_{[0,h]}} \leq r\right\},\ \ \tilde{D}_r = \left\{y : y \in C_{(0,h]}, \|y\|_{C_{(0,h]}} \leq r\right\}.$$

Obviously, $D_r \subseteq C_{[0,h]}$ and $\tilde{D}_r \subseteq C_{(0,h]}$ are convex, nonempty and closed.

Next, we will establish several lemmas that are pertinent to main result.

Lemma 8. *Suppose that* $(A_1)-(A_3)$ *and* (B_1) *are satisfied for* $t\in(0,h]$*. Then*

$$\begin{aligned}E\|u^a(t;y)\|^2 \le\, & 6\Big(\frac{M_BMh^{\mu-1}}{a\Gamma(\mu)}\Big)^2\Big\{E\|\varrho\|^2+\Big(\frac{Mh^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\Big)^2E\|y_0\|^2+\Big(\frac{Mh^{\beta-1}}{\Gamma(\beta)}\Big)^2E\|y_1\|^2\\&+\Big(\frac{M}{\Gamma(\mu)}\Big)^2\frac{1}{\mu}h^{2\mu-1}\|g\|_{L^1}+Tr(\mathcal{Q})\int_0^h E\|\varphi(s)\|^2ds+\Big(\frac{Mh^{\mu-1}}{\Gamma(\mu)}\Big)^2Tr(\mathcal{Q})\|g\|_{L^1}\Big\}\\=:\,& L_u\end{aligned}$$

Proof. By Lemma 3, (2), Hölder's inequality and assumption (A_3), (B_1), we have

$$\begin{aligned}E\|u^a(t;y)\|^2 =\, & E\|B^*N^*(h-t)R(a,\Gamma_0^h)\widetilde{S}(y)\|^2\\ \le\, & \Big(\frac{M_BMh^{\mu-1}}{a\Gamma(\mu)}\Big)^2E\|\widetilde{S}(y)\|^2\\ \le\, & 6\Big(\frac{M_BMh^{\mu-1}}{a\Gamma(\mu)}\Big)^2\Big\{E\|\varrho\|^2+E\|J(h)y_0\|^2+E\|K(h)y_1\|^2+E\Big\|\int_0^h\varphi(s)dw(s)\Big\|^2\\&+E\Big\|\int_0^hN(h-s)\mathcal{G}(s,y(s))ds\Big\|^2+E\Big\|\int_0^hN(h-s)\mathcal{T}(s,y(s))dw(s)\Big\|^2\Big\}\\ \le\, & 6\Big(\frac{M_BMh^{\mu-1}}{a\Gamma(\mu)}\Big)^2\Big\{E\|\varrho\|^2+\Big(\frac{Mh^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\Big)^2E\|y_0\|^2+\Big(\frac{Mh^{\beta-1}}{\Gamma(\beta)}\Big)^2E\|y_1\|^2\\&+\Big(\frac{M}{\Gamma(\mu)}\Big)^2\frac{1}{\mu}h^{\mu}\int_0^h(h-s)^{\mu-1}E\|\mathcal{G}(s,y(s))\|^2ds+Tr(\mathcal{Q})\int_0^hE\|\varphi(s)\|^2ds\\&+\Big(\frac{M}{\Gamma(\mu)}\Big)^2Tr(\mathcal{Q})\int_0^h(h-s)^{2(\mu-1)}E\|\mathcal{T}(s,y(s))\|^2ds\Big\}\\ \le\, & 6\Big(\frac{M_BMh^{\mu-1}}{a\Gamma(\mu)}\Big)^2\Big\{E\|\varrho\|^2+\Big(\frac{Mh^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\Big)^2E\|y_0\|^2+\Big(\frac{Mh^{\beta-1}}{\Gamma(\beta)}\Big)^2E\|y_1\|^2\\&+\Big(\frac{M}{\Gamma(\mu)}\Big)^2\frac{1}{\mu}h^{2\mu-1}\|g\|_{L^1}+Tr(\mathcal{Q})\int_0^hE\|\varphi(s)\|^2ds+\Big(\frac{Mh^{\mu-1}}{\Gamma(\mu)}\Big)^2Tr(\mathcal{Q})\|g\|_{L^1}\Big\}\\=:\,& L_u.\end{aligned}$$

□

Theorem 1. *If* $(A_1)-(A_3)$ *and* (B_1) *hold. Then, there is at least one mild solution to problem* (1) *in* $\tilde{D}_r$*.*

Proof. Now, we divide this part of the proof into the following steps:

Step 1: Ψ is equicontinuous for $z\in D_r$.

From [30], we obtain that Ψ_1 is equicontinuous. Next, we prove that Ψ_2 is equicontinuous.

When $t_1=0, 0<t_2\le h$, by Lemma 3, Lemma 8, (2), (A_3) and Hölder's inequality, we can obtain

$$\begin{aligned}&E\big\|(\Psi_2z)(t_2)-(\Psi_2z)(0)\big\|^2\\&\le 3E\Big\|t_2^{2-\beta}\int_0^{t_2}N(t_2-s)\mathcal{G}(s,y(s))ds\Big\|^2+3E\Big\|t_2^{2-\beta}\int_0^{t_2}N(t_2-s)Bu^{\alpha}(s;y)ds\Big\|^2\\&\quad+3E\Big\|t_2^{2-\beta}\int_0^{t_2}N(t_2-s)\mathcal{T}(s,y(s))dw(s)\Big\|^2\\&\le 3\Big(\frac{Mt_2^{2-\beta+\frac{\mu}{2}}}{\Gamma(\mu)}\Big)^2\frac{1}{\mu}\int_0^{t_2}(t_2-s)^{\mu-1}g(s)ds+3\Big(\frac{Mt_2^{2-\beta+\frac{\mu}{2}}}{\Gamma(\mu)}\Big)^2\frac{1}{\mu}M_B^2L_u\int_0^{t_2}(t_2-s)^{\mu-1}ds\\&\quad+3\Big(\frac{Mt_2^{2-\beta}}{\Gamma(\mu)}\Big)^2Tr(\mathcal{Q})\int_0^{t_2}(t_2-s)^{2(\mu-1)}g(s)ds\\&\to 0,\ as\ t_2\to 0.\end{aligned}$$

When $0 < t_1 < t_2 \le h$, by C_r inequality, we can obtain

$$\begin{aligned}
&E\Big\|(\Psi_2 z)(t_2) - (\Psi_2 z)(t_1)\Big\|^2 \\
&\le 3E\Big\|t_2^{2-\beta}\int_0^{t_2} N(t_2-s)\mathcal{G}(s,y(s))ds - t_1^{2-\beta}\int_0^{t_1} N(t_1-s)\mathcal{G}(s,y(s))ds\Big\|^2 \\
&\quad + 3E\Big\|t_2^{2-\beta}\int_0^{t_2} N(t_2-s)u^{\alpha}(s;y)ds - t_1^{2-\beta}\int_0^{t_1} N(t_1-s)u^{\alpha}(s;y)ds\Big\|^2 \\
&\quad + 3E\Big\|t_2^{2-\beta}\int_0^{t_2} N(t_2-s)\mathcal{T}(s,y(s))dw(s) - t_1^{2-\beta}\int_0^{t_1} N(t_1-s)\mathcal{T}(s,y(s))dw(s)\Big\|^2 \\
&=: 3J_1 + 3J_2 + 3J_3.
\end{aligned}$$

Now, we prove $\lim_{t_2\to t_1} J_1 \to 0$. By C_r inequality, we have

$$\begin{aligned}
J_1 &\le 3E\Big\|t_1^{2-\beta}\int_{t_1}^{t_2} N(t_2-s)\mathcal{G}(s,y(s))ds\Big\|^2 \\
&\quad + 3E\Big\|t_1^{2-\beta}\int_0^{t_1}\Big(N(t_2-s)-N(t_1-s)\Big)\mathcal{G}(s,y(s))ds\Big\|^2 \\
&\quad + 3\Big(t_2^{2-\beta}-t_1^{2-\beta}\Big)^2 E\Big\|\int_0^{t_2} N(t_2-s)\mathcal{G}(s,y(s))ds\Big\|^2 \\
&\le 3\sum_{i=1}^{3} J_{1i},
\end{aligned}$$

where

$$\begin{aligned}
J_{11} &= \left(\frac{Mt_1^{2-\beta}}{\Gamma(\mu)}\right)^2 E\Big\|\int_{t_1}^{t_2}(t_2-s)^{\mu-1}\mathcal{G}(s,y(s))ds\Big\|^2, \\
J_{12} &= E\Big\|t_1^{2-\beta}\int_0^{t_1}\Big(N(t_2-s)-N(t_1-s)\Big)\mathcal{G}(s,y(s))ds\Big\|^2, \\
J_{13} &= \left(\frac{M}{\Gamma(\mu)}\right)^2\Big(t_2^{2-\beta}-t_1^{2-\beta}\Big)^2 E\Big\|\int_0^{t_2}(t_2-s)^{\mu-1}\mathcal{G}(s,y(s))ds\Big\|^2.
\end{aligned}$$

By Hölder′s inequality and (A_3), we have

$$\begin{aligned}
J_{11} &\le \left(\frac{Mt_1^{2-\beta}}{\Gamma(\mu)}\right)^2 \int_{t_1}^{t_2}\big(t_2-s\big)^{2\mu-2}ds\int_{t_1}^{t_2} E\|\mathcal{G}(s,y(s))\|^2 ds \\
&\le \left(\frac{Mt_1^{2-\beta}}{\Gamma(\mu)}\right)^2 \frac{1}{2\mu-1}(t_2-t_1)^{2\mu-1}\int_{t_1}^{t_2} g(s)ds \\
&\to 0,\ as\ t_2 \to t_1.
\end{aligned}$$

Because

$$J_{12} = t_1^{2(2-\beta)} E\Big\|\int_0^{t_1}\int_{t_1-s}^{t_2-s}\frac{d}{dt}\Big\{N(t)\mathcal{G}(s,y(s))\Big\}dtds\Big\|^2,$$

so, by Lemma 4 and Hölder′s inequality, we have

$$\begin{aligned}J_{12} &\le t_1^{2(2-\beta)}E\left\|\frac{M}{\Gamma(\mu)}\int_0^{t_1}\int_{t_1-s}^{t_2-s}t^{\mu-2}\mathcal{G}(s,y(s))dtds\right\|^2\\&\le\left(\frac{Mt_1^{2-\beta}}{(\mu-1)\Gamma(\mu)}\right)^2\int_0^{t_1}\left((t_2-s)^{\mu-1}-(t_1-s)^{\mu-1}\right)^2ds\int_0^{t_1}E\|\mathcal{G}(s,y(s))ds\|^2ds\\&\le\left(\frac{Mt_1^{2-\beta}}{(\mu-1)\Gamma(\mu)}\right)^2\frac{1}{2\mu-1}(t_2^{2\mu-1}+(t_2-t_1)^{2\mu-1}-t_1^{2\mu-1})\|g\|_{L^1}\\&\to 0,\ as\ t_2\to t_1.\end{aligned}$$

It is obvious that $\lim_{t_2\to t_1}J_{13}\to 0$. Hence $\lim_{t_2\to t_1}J_1\to 0$.

Next, we prove the $\lim_{t_2\to t_1}J_2\to 0$. By C_r inequality, we have

$$\begin{aligned}J_2 &\le 3E\left\|t_1^{2-\beta}\int_{t_1}^{t_2}N(t_2-s)u^a(s;y)ds\right\|^2\\&\quad+3E\left\|t_1^{2-\beta}\int_0^{t_1}\Big(N(t_2-s)-N(t_1-s)\Big)u^a(s;y)ds\right\|^2\\&\quad+3\left(t_2^{2-\beta}-t_1^{2-\beta}\right)^2E\left\|\int_0^{t_2}N(t_2-s)u^a(s;y)ds\right\|^2\\&\le 3\sum_{i=1}^{3}J_{2i},\end{aligned}$$

where

$$\begin{aligned}J_{21}&=\left(\frac{Mt_1^{2-\beta}}{\Gamma(\mu)}\right)^2E\left\|\int_{t_1}^{t_2}(t_2-s)^{\mu-1}u^a(s;y)ds\right\|^2,\\J_{22}&=E\left\|t_1^{2-\beta}\int_0^{t_1}\Big(N(t_2-s)-N(t_1-s)\Big)u^a(s;y)ds\right\|^2,\\J_{23}&=\left(\frac{M}{\Gamma(\mu)}\right)^2\left(t_2^{2-\beta}-t_1^{2-\beta}\right)^2E\left\|\int_0^{t_2}(t_2-s)^{\mu-1}u^a(s;y)ds\right\|^2.\end{aligned}$$

By Lemma 8 and Hölder's inequality, we have

$$\begin{aligned}J_{21}&\le\left(\frac{Mt_1^{2-\beta}}{\Gamma(\mu)}\right)^2\int_{t_1}^{t_2}(t_2-s)^{2\mu-2}ds\int_{t_1}^{t_2}E\|u^a(s;y)\|^2ds\\&\le\left(\frac{Mt_1^{2-\beta}}{\Gamma(\mu)}\right)^2\frac{1}{2\mu-1}(t_2-t_1)^{2\mu}L_\mu\\&\to 0,\ as\ t_2\to t_1.\end{aligned}$$

Because

$$J_{22}=t_1^{2(2-\beta)}E\left\|\int_0^{t_1}\int_{t_1-s}^{t_2-s}\frac{d}{dt}\Big\{N(t)u^a(s;y)\Big\}dtds\right\|^2,$$

so, by Hölder's inequality and Lemma 4, we have

$$\begin{aligned}J_{22}&\le t_1^{2(2-\beta)}E\left\|\frac{M}{\Gamma(\mu)}\int_0^{t_1}\int_{t_1-s}^{t_2-s}t^{\mu-2}u^a(s;y)dtds\right\|^2\\&\le\left(\frac{Mt_1^{2-\beta}}{(\mu-1)\Gamma(\mu)}\right)^2\int_0^{t_1}\left((t_2-s)^{\mu-1}-(t_1-s)^{\mu-1}\right)^2ds\int_0^{t_1}E\|u^a(s;y)ds\|^2ds\\&\le\left(\frac{Mt_1^{2-\beta}}{(\mu-1)\Gamma(\mu)}\right)^2\frac{1}{2\mu-1}(t_2^{2\mu-1}+(t_2-t_1)^{2\mu-1}-t_1^{2\mu-1})t_1L_\mu\\&\to 0,\ as\ t_2\to t_1.\end{aligned}$$

It's obvious that $J_{23} \to 0$. Hence $J_2 \to 0$ as $t_2 \to t_1$.

From [30], we can obtain $J_3 \to 0$ as $t_2 \to t_1$.

Consequently,

$$E\big\|(\Psi_2 z)(t_2) - (\Psi_2 z)(t_1)\big\|^2 \to 0,\ as\ t_2 \to t_1.$$

Through the above analysis, $\lim_{t_2 \to t_1} \|(\Psi_2 z)(t_2) - (\Psi_2 z)(t_1)\|_{C_{[0,h]}} \to 0$, for $t_1, t_2 \in [0,h]$. To sum up, Ψ is equicontinuous for $z \in D_r$.

Step 2: Ψ is continuous.

Let $\{z_n\}$ be a sequence, which is convergent to z in D_r, then

$$\lim_{n\to\infty} z_n(t) = z(t)\ and\ \lim_{n\to\infty} t^{\beta-2} z_n(t) = t^{\beta-2} z(t),\ for\ t \in (0,h].$$

Because $y(t) = t^{\beta-2} z(t)$, $t \in (0,h]$, by (A_1) and (A_2), we have

$$\lim_{n\to\infty} E\big\|\mathcal{G}(t, y_n(t))\big\|^2 = \lim_{n\to\infty} E\big\|\mathcal{G}(t, t^{\beta-2} z_n(t))\big\|^2 = E\big\|\mathcal{G}(t, t^{\beta-2} z(t))\big\|^2 = E\big\|\mathcal{G}(t, y(t))\big\|^2,$$
$$\lim_{n\to\infty} E\big\|\mathcal{T}(t, y_n(t))\big\|^2 = \lim_{n\to\infty} E\big\|\mathcal{T}(t, t^{\beta-2} z_n(t))\big\|^2 = E\big\|\mathcal{T}(t, t^{\beta-2} z(t))\big\|^2 = E\big\|\mathcal{T}(t, y(t))\big\|^2.$$

Using (A_3), we can obtain

$$(t-s)^{\mu-1} E\big\|\mathcal{G}(s, y_n(s)) - \mathcal{G}(s, y(s))\big\|^2 \le 4(t-s)^{\mu-1} g(s),\ t \in (0,h].$$

As $s \to 4(t-s)^{\mu-1} g(s)$ is integrable for $s \in [0,t]$, we can use the Lebesgue dominated convergence theorem to derive

$$E\bigg\|\int_0^t (t-s)^{\mu-1} [\mathcal{G}(s, y_n(s)) - \mathcal{G}(s, y(s))] ds\bigg\|^2 \to 0,\ as\ n \to \infty. \tag{8}$$

Similarly, we have

$$E\bigg\|\int_0^t (t-s)^{\mu-1} [\mathcal{T}(s, y_n(s)) - \mathcal{T}(s, y(s))] dw(s)\bigg\|^2 \to 0,\ as\ n \to \infty. \tag{9}$$

By use (6), (8) and (9), we can obtain

$$\begin{aligned}
& E\Big\|u^a(t; y_n) - u^a(t; y)\Big\|^2 \\
& \le 2\left(\frac{M_B M h^{\mu-1}}{a\Gamma(\mu)}\right)^2 E\bigg\|\int_0^h N(h-s)[\mathcal{G}(s, y_n(s)) - \mathcal{G}(s, y(s))] ds\bigg\|^2 \\
& \quad + 2\left(\frac{M_B M h^{\mu-1}}{a\Gamma(\mu)}\right)^2 E\bigg\|\int_0^h N(h-s)[\mathcal{T}(s, y_n(s)) - \mathcal{T}(s, y(s))] dw(s)\bigg\|^2 \\
& \le 2\left(\frac{M_B M^2 h^{\mu-1}}{a\Gamma^2(\mu)}\right)^2 E\bigg\|\int_0^h (h-s)^{\mu-1}[\mathcal{G}(s, y_n(s)) - \mathcal{G}(s, y(s))] ds\bigg\|^2 \\
& \quad + 2\left(\frac{M_B M^2 h^{\mu-1}}{a\Gamma^2(\mu)}\right)^2 E\bigg\|\int_0^h (h-s)^{\mu-1}[\mathcal{T}(s, y_n(s)) - \mathcal{T}(s, y(s))] dw(s)\bigg\|^2 \\
& \to 0,\ as\ n \to \infty.
\end{aligned}$$

Because, from Lemma 8, we have

$$(t-s)^{\mu-1} E\|u^a(s; y_n) - u^a(s; y)\|^2 \le 4(t-s)^{\mu-1} L_u.$$

By using the Lebesgue dominated convergence theorem, we can obtain

$$E\left\| \int_0^t (t-s)^{\mu-1}[u^a(s;y_n) - u^a(s;y)] \right\|^2 \to 0, \ as \ n \to \infty. \tag{10}$$

So, by using (8)–(10), for each $t \in [0,h]$, we obtain

$$\begin{aligned}
&E\left\|(\Psi_2 z_n)(t) - (\Psi_2 z)(t)\right\|^2 \\
&\le 3(\frac{Mt^{2-\beta}}{\Gamma(\mu)})^2 E\left\| \int_0^t (t-s)^{\mu-1}(\mathcal{G}(s,y_n(s)) - \mathcal{G}(s,y(s)))ds \right\|^2 \\
&\quad + 3(\frac{Mt^{2-\beta}}{\Gamma(\mu)})^2 E\left\| \int_0^t (t-s)^{\mu-1}(u^a(s;y_n) - u^a(s;y))ds \right\|^2 \\
&\quad + 3(\frac{Mt^{2-\beta}}{\Gamma(\mu)})^2 E\left\| \int_0^t (t-s)^{\mu-1}(\mathcal{T}(s,y_n(s)) - \mathcal{T}(s,y(s)))dw(s) \right\|^2 \\
&\to 0, \ as \ n \to \infty.
\end{aligned}$$

Therefore, Ψ is continuous.

Step 3: $\Psi(D_r) \subset D_r$.

For $t \in (0,h]$, by (A_3), Lemma 4 and (7), we have

$$\begin{aligned}
E\left\|(\Psi z)(t)\right\|^2 &= E\left\|t^{2-\beta}(\Phi y)(t)\right\|^2 \\
&= t^{2(2-\beta)} E\left\| J(t)y_0 + K(t)y_1 + \int_0^t N(t-s)[\mathcal{G}(s,y(s)) + Bu^a(s;y)]ds \right. \\
&\quad \left. + \int_0^t N(t-s)\mathcal{T}(s,y(s))dw(s) \right\|^2 \\
&\le 5t^{2(2-\beta)} E\left\|J(t)y_0\right\|^2 + 5t^{2(2-\beta)} E\left\|K(t)y_1\right\|^2 \\
&\quad + 5t^{2(2-\beta)} E\left\| \int_0^t N(t-s)(\mathcal{G}(s,y(s)) + Bu^a(s;y))ds \right\|^2 \\
&\quad + 5t^{2(2-\beta)} E\left\| \int_0^t N(t-s)\mathcal{T}(s,y(s))dw(s) \right\|^2 \\
&\le 5\left(\frac{M}{(\mu-1)\Gamma(\beta-1)}\right)^2 E\|y_0\|^2 + 5\left(\frac{t}{\Gamma(\beta)}\right)^2 E\|y_1\|^2 \\
&\quad + 5\left(\frac{Mt^{2-\beta+\frac{\mu}{2}}}{\Gamma(\mu)}\right)^2 \frac{1}{\mu} \int_0^t (t-s)^{\mu-1}(E\|\mathcal{G}(s,y(s))\|^2 + E\|Bu^a(s;y)\|^2)ds \\
&\quad + 5\left(\frac{Mt^{2-\beta}}{\Gamma(\mu)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{2(\mu-1)} E\|\mathcal{T}(s,y(s))\|^2 ds \\
&\le \sup_{t\in(0,h]} \left\{ 5\left(\frac{M}{(\mu-1)\Gamma(\beta-1)}\right)^2 E\|y_0\|^2 + 5\left(\frac{t}{\Gamma(\beta)}\right)^2 E\|y_1\|^2 \right. \\
&\quad + 5\left(\frac{Mt^{2-\beta+\frac{\mu}{2}}}{\Gamma(\mu)}\right)^2 \frac{1}{\mu} \int_0^t (t-s)^{\mu-1} g(s)ds + 5\left(\frac{M_B M t^{2-\beta+\mu}}{\Gamma(\mu+1)}\right)^2 L_u \\
&\quad \left. + 5\left(\frac{Mt^{2-\beta}}{\Gamma(\mu)}\right)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{2(\mu-1)} g(s)ds \right\} \\
&\le r.
\end{aligned}$$

For $t = 0$, since $M > 1$, we have

$$E\left\|(\Psi z)(0)\right\|^2 = E\left\|\frac{y_0}{\Gamma(\beta-1)}\right\|^2 \leq \left(\frac{M}{(\mu-1)\Gamma(\beta-1)}\right)^2 E\|y_0\|^2 < r.$$

Therefore, we have $\Psi(D_r) \subset D_r$.

Step 4: $\Psi : D_r \to D_r$ is completely continuous.

It is evident that problem (1) has a mild solution $y \in \tilde{D}_r$ if and only if Ψ has a fixed-point $z \in D_r$. Based on Step 2 and Step 3, it can be concluded that the operator $\Psi : D_r \to D_r$ is continuous. It is clear that $\Psi : D_r \to D_r$ is completely continuous if $\Psi(D_r)$ is relatively compact in $L^2(\Omega, \mathbb{H})$. From Step 1, Ψ is equicontinuous. According to the Ascoli-Azelà theorem, to prove that Ψ is completely continuous, we need to show that $(\Psi D_r)(t)$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for $0 \leq t \leq h$. However, it is clear that $(\Psi D_r)(0)$ is relatively compact. Now, we will demonstrate that $(\Psi D_r)(t)$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for $t > 0$.

When $\eta \in (0, t)$ and $\sigma > 0$, we have the following definition for $\Psi_{\eta,\sigma}$ on D_r:

$$\begin{aligned}
&(\Psi_{\eta,\sigma} z)(t) \\
&:= t^{2-\beta}(\Phi_{\eta,\sigma} y)(t) \\
&= t^{2-\beta} J(t) y_0 + t^{2-\beta} K(t) y_1 \\
&\quad + t^{2-\beta} \frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma} \int_0^{t-\eta} \int_\sigma^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi - \eta^{\frac{\mu}{2}}\sigma\big) \mathcal{G}(s, y(s)) d\xi ds \\
&\quad + t^{2-\beta} \frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma} \int_0^{t-\eta} \int_\sigma^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi - \eta^{\frac{\mu}{2}}\sigma\big) B u^a(s; y) d\xi ds \\
&\quad + t^{2-\beta} \frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma} \int_0^{t-\eta} \int_\sigma^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi - \eta^{\frac{\mu}{2}}\sigma\big) \mathcal{T}(s, y(s)) d\xi dw(s).
\end{aligned}$$

As $\{\mathcal{S}(t)\}_{t>0}$ is compact, then $\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)/(\eta^{\frac{\mu}{2}}\sigma)$ is also compact. Therefore, for any $\sigma > 0$ and any $\eta \in (0, t)$, it follows that $(\Psi_{\eta,\sigma} z)(t)$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for $z \in D_r$. Additionally, for any $z \in D_r$, we can conclude that:

$$\begin{aligned}
&E\left\|(\Psi z)(t) - (\Psi_{\eta,\sigma} z)(t)\right\|^2 \\
&= t^{2(2-\beta)} 3E\bigg\| \int_0^t \int_0^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big) \mathcal{G}(s, y(s)) d\xi ds \\
&\quad - \frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma} \int_0^{t-\eta} \int_\sigma^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi - \eta^{\frac{\mu}{2}}\sigma\big) \mathcal{G}(s, y(s)) d\xi ds \bigg\|^2 \\
&\quad + t^{2(2-\beta)} 3E\bigg\| \int_0^t \int_0^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big) B u^a(s; y) d\xi ds \\
&\quad - \frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma} \int_0^{t-\eta} \int_\sigma^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi - \eta^{\frac{\mu}{2}}\sigma\big) B u^a(s; y) d\xi ds \bigg\|^2 \\
&\quad + t^{2(2-\beta)} 3E\bigg\| \int_0^t \int_0^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big) \mathcal{T}(s, y(s)) d\xi dw(s) \\
&\quad - \frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma} \int_0^{t-\eta} \int_\sigma^\infty \frac{\mu}{2} \xi (t-s)^{\frac{\mu}{2}-1} W_{\frac{\mu}{2}}(\xi) \mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi - \eta^{\frac{\mu}{2}}\sigma\big) \mathcal{T}(s, y(s)) d\xi dw(s) \bigg\|^2 \\
&=: d_1 + d_2 + d_3.
\end{aligned}$$

In order to prove that $E\left\|(\Psi z)(t)-(\Psi_{\eta,\sigma}z)(t)\right\|^2\to 0$, we first need to establish that $d_1\to 0$,

$$\begin{aligned}
d_1 &= t^{2(2-\beta)}3E\bigg\|\int_0^t\int_0^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big)\mathcal{G}(s,y(s))d\xi ds \\
&\quad -\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}\int_0^{t-\eta}\int_\sigma^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi-\eta^{\frac{\mu}{2}}\sigma\big)\mathcal{G}(s,y(s))d\xi ds\bigg\|^2 \\
&\le t^{2(2-\beta)}9E\bigg\|\int_0^t\int_0^\sigma \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big)\mathcal{G}(s,y(s))d\xi ds\bigg\|^2 \\
&\quad + t^{2(2-\beta)}9E\bigg\|\int_{t-\eta}^t\int_\sigma^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big)\mathcal{G}(s,y(s))d\xi ds\bigg\|^2 \\
&\quad + t^{2(2-\beta)}9E\bigg\|\int_0^{t-\eta}\int_\sigma^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\Big(\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big)-\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi-\eta^{\frac{\mu}{2}}\sigma\big)\Big) \\
&\quad \times\mathcal{G}(s,y(s))d\xi ds\bigg\|^2 \\
&=: d_{11}+d_{12}+d_{13},
\end{aligned}$$

Because $\|\mathcal{S}(t)\|\le Mt$ for any $t\ge 0$ and (7), we have

$$\begin{aligned}
d_{11} &= t^{2(2-\beta)}9E\bigg\|\int_0^t\int_0^\sigma \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\mathcal{S}((t-s)^{\frac{\mu}{2}}\xi)\mathcal{G}(s,y(s))d\xi ds\bigg\|^2 \\
&\le 9M^2t^{2(2-\beta)}\frac{t^\mu}{\mu}\int_0^t(t-s)^{\mu-1}g(s)ds\Big(\int_0^\sigma\frac{\mu}{2}\xi^2W_{\frac{\mu}{2}}(\xi)d\xi\Big)^2 \\
&\le \frac{9}{5}r(\Gamma(\mu))^2\Big(\int_0^\sigma\frac{\mu}{2}\xi^2W_{\frac{\mu}{2}}(\xi)d\xi\Big)^2 \\
&\to 0,\ as\ \sigma\to 0.
\end{aligned}$$

By utilizing Definition 5, we obtain

$$\begin{aligned}
d_{12} &= t^{2(2-\beta)}9E\bigg\|\int_{t-\eta}^t\int_\sigma^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\mathcal{S}((t-s)^{\frac{\mu}{2}}\xi)\mathcal{G}(s,y(s))d\xi ds\bigg\|^2 \\
&\le 9M^2t^{2(2-\beta)}\frac{\eta^\mu}{\mu}\int_{t-\eta}^t(t-s)^{\mu-1}g(s)ds\Big(\int_0^\infty\frac{\mu}{2}\xi^2W_{\frac{\mu}{2}}(\xi)d\xi\Big)^2 \\
&\le \frac{9}{4}M^2h^{2(2-\beta)}\frac{\eta^\mu}{\mu}(\frac{1}{\Gamma(\mu)})^2\int_{t-\eta}^t(t-s)^{\mu-1}g(s)ds \\
&\to 0,\ as\ \eta\to 0.
\end{aligned}$$

Considering d_{23}, by utilizing $\lim\limits_{t\to 0}\|\frac{\mathcal{S}(t)y}{t}-y\|=0$ and $\|\mathcal{S}(t)-\mathcal{S}(k)\|_{L(\mathbb{H})}\le M|t-k|$ for any $y\in\mathbb{H}$, we can deduce that

$$\begin{aligned}
&\bigg\|\mathcal{S}\Big((t-s)^{\frac{\mu}{2}}\xi\Big)y-\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}\mathcal{S}\Big((t-s)^{\frac{\mu}{2}}\xi-\eta^{\frac{\mu}{2}}\sigma\Big)y\bigg\| \\
&\le \bigg\|\Big(\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}-I\Big)\mathcal{S}\Big((t-s)^{\frac{\mu}{2}}\xi\Big)y\bigg\|+\bigg\|\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}\Big(\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi-\eta^{\frac{\mu}{2}}\sigma\big)-\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big)\Big)y\bigg\| \\
&\le M(t-s)^{\frac{\mu}{2}}\xi\bigg\|\Big(\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}-I\Big)y\bigg\|+M\bigg\|\Big(\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi-\eta^{\frac{\mu}{2}}\sigma\big)-\mathcal{S}\big((t-s)^{\frac{\mu}{2}}\xi\big)\Big)y\bigg\| \\
&\to 0,\ as\ \eta,\ \sigma\to 0,
\end{aligned}$$

so, we can get

$$t^{2(2-\beta)}\int_0^{t-\eta}\left(\int_\sigma^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)\left(\mathcal{S}((t-s)^{\frac{\mu}{2}}\xi)-\frac{\mathcal{S}(\eta^{\frac{\mu}{2}}\sigma)}{\eta^{\frac{\mu}{2}}\sigma}\mathcal{S}\left((t-s)^{\frac{\mu}{2}}\xi-\eta^{\frac{\mu}{2}}\sigma\right)d\xi\right)\right)^2 ds$$
$$\times\int_0^{t-\eta}E\|\mathcal{G}(s,y(s))\|^2 ds$$
$$\leq t^{2(2-\beta)}\int_0^{t-\eta}\left(\int_\sigma^\infty \frac{\mu}{2}\xi(t-s)^{\frac{\mu}{2}-1}W_{\frac{\mu}{2}}(\xi)(M(t-s)^{\frac{\mu}{2}}\xi+M^2(t-s)^{\frac{\mu}{2}}\xi)d\xi\right)^2 ds\int_0^{t-\eta}g(s)ds$$
$$\leq t^{2(2-\beta)}M^2(M+1)^2\int_0^{t-\eta}(t-s)^{2(\mu-1)}ds\left(\int_0^\infty \frac{\mu}{2}\xi^2 W_{\frac{\mu}{2}}(\xi)d\xi\right)^2\|g(s)\|_{L^1}$$
$$\leq (M+1)^2\left(\frac{Mh^{2-\beta}}{2\Gamma(\mu)}\right)^2\frac{\eta^{2\mu-1}}{2\mu-1}\|g(s)\|_{L^1}.$$

By the Lebegue dominated convergence theorem, we derive that $d_{13}\to 0$ as $\eta\to 0$ or $\sigma\to 0$. Thus, $d_1\to 0$. Similar, we can get $d_2\to 0$ and $d_3\to 0$. So, $\Psi(D_r)$ is relatively compact in $L^2(\Omega,\mathbb{H})$. Thus, $\Psi: D_r\to D_r$ is completely continuous.

By using Schauder′s fixed-point theorem, It can be inferred that Ψ possesses at least one fixed point $z^*\in D_r$. Let $y^*=t^{\beta-2}z^*$ for $t\in(0,h]$, thus

$$y^*=J(t)y_0+K(t)y_1+\int_0^t N(t-s)\mathcal{G}(s,y^*(s))ds$$
$$+\int_0^t N(t-s)Bu^a(s;y^*)ds+\int_0^t N(t-s)\mathcal{T}(s,y^*(s))dw(s),\ t\in(0,h].$$

□

The following theorem justifies the approximate controllability results of system (1).

Theorem 2. *Assume that* $(A_1)-(A_3)$ *and* $(B_1)-(B_3)$ *are fulfilled. Then, the system* (1) *is approximately controllable on* $[0,h]$.

Proof. For $\forall a>0,\forall\varrho\in L^2(\Omega,\mathbb{H})$, according to Theorem 1, it follows that Φ has a fixed point in $C_{(0,h]}$ when the control function $u(t)=u^a(t;y)$. Let y^a be the fixed-point of Φ. Then

$$y^a(t)=J(t)y_0+K(t)y_1+\int_0^t N(t-s)[\mathcal{G}(s,y^a(s))+Bu^a(s;y^a)]ds$$
$$+\int_0^t N(t-s)\mathcal{T}(s,y^a(s))dw(s),\ t\in(0,h]. \tag{11}$$

where

$$u^a(s;y^a)=B^*N^*(h-s)R(a,\Gamma_0^h)\widetilde{S}(y^a),$$

$$\widetilde{S}(y^a)=E\varrho-J(h)y_0-K(h)y_1-\int_0^h N(h-s)\mathcal{G}(s,y^a(s))ds$$
$$-\int_0^h N(h-s)(\mathcal{T}(s,y^a(s))-\varphi(s))dw(s).$$

Taking into consideration $I-\Gamma_0^hR(a,\Gamma_0^h)=aR(a,\Gamma_0^h)$ and Lemma 7, simple calculation yields

$$
\begin{aligned}
y^a(h) &= J(h)y_0 + K(h)y_1 + \int_0^h N(h-s)\mathcal{G}(s,y^a(s))ds \\
&\quad + \int_0^h N(h-s)BB^*N^*(h-s)R(a,\Gamma_0^h)\widetilde{S}(y^a)ds \\
&\quad + \int_0^h N(h-s)\mathcal{T}(s,y^a(s))dw(s) \\
&= \varrho - \widetilde{S}(y^a) + \Gamma_0^h R(a,\Gamma_0^h)\widetilde{S}(y^a) \\
&= \varrho - \left(I - \Gamma_0^h R(a,\Gamma_0^h)\right)\widetilde{S}(y^a) \\
&= \varrho - aR(a,\Gamma_0^h)\widetilde{S}(y^a).
\end{aligned}
$$

From (B_2), it follows that there exist two subsequences, which we will still denote by $\{\mathcal{G}(s,y^a(s))\}$ and $\{\mathcal{T}(s,y^a(s))\}$, and these subsequences weakly converge to $\mathcal{G}(s)$ and $\mathcal{T}(s)$, respectively. Therefore

$$
\begin{aligned}
&E\|y^a(h)-\varrho\|^2 \\
&= E\|aR(a,\Gamma_0^h)\widetilde{S}(y^a)\|^2 \\
&= E\left\|aR(a,\Gamma_0^h)\left\{E\varrho - J(h)y_0 - K(h)y_1 - \int_0^h N(h-s)\mathcal{G}(s,y^a(s))ds\right.\right. \\
&\quad \left.\left. - \int_0^h N(h-s)(\mathcal{T}(s,y^a(s)) - \varphi(s))dw(s)\right\}\right\|^2 \\
&\leq 6E\left\|aR(a,\Gamma_0^h)\left\{\varrho - J(h)y_0 - K(h)y_1\right\}\right\|^2 + 6E\left\|\int_0^h aR(a,\Gamma_0^h)N(h-s)[\mathcal{G}(s,y^a(s)) - \mathcal{G}(s)]ds\right\|^2 \\
&\quad + 6E\left\|\int_0^h aR(a,\Gamma_0^h)N(h-s)\mathcal{G}(s)ds\right\|^2 \\
&\quad + 6E\left\|\int_0^h aR(a,\Gamma_0^h)N(h-s)[\mathcal{T}(s,y^a(s)) - \mathcal{T}(s)]dw(s)\right\|^2 + 6E\left\|\int_0^h aR(a,\Gamma_0^h)N(h-s)\mathcal{T}(s)dw(s)\right\|^2 \\
&\quad + 6E\left\|\int_0^h aR(a,\Gamma_0^h)N(h-s)\varphi(s)dw(s)\right\|^2.
\end{aligned}
$$

By using $(B_1), (B_3)$ and the Lebesgue dominated convergence theorem, it follows that

$$
E\|y^a(h)-\varrho\|^2 \to 0^+, \ as\ a \to 0^+.
$$

This proves that the system (1) is approximately controllable on the interval $[0,h]$. □

4. Exact Controllability

To establish the exact controllability of the system (1), the following hypotheses are necessary:

$\mathbf{H_1}$. *The linear operator*

$$
\Upsilon u = \int_0^h N(h-s)Bu(s)ds,
$$

it is bounded and invertible, $\|\Upsilon^{-1}\| \leq L_\gamma$.

$\mathbf{H_2}$. *For any bounded set* $V \in \mathbb{H}$, *there exists contant* $l > 0$, *such that*

$$
\chi(\mathcal{G}(s,V)) \vee \chi(\mathcal{T}(s,V)) \leq lt^{2-\beta}\chi(V), \text{ for a.e. } t \in [0,h].
$$

$\mathbf{H_3}$. *Assume that the following inequality holds,*

$$
\left(h^{2-\beta}\frac{2Ml}{\Gamma(\mu)}\frac{h^\mu}{\mu} + h^{2-\beta}\frac{2M_B M\bar{M}}{\Gamma(\mu)}\frac{h^\mu}{\mu} + h^{2-\beta}\frac{M}{\Gamma(\mu)}\left(2Tr(\mathcal{Q})\frac{h^{2\mu-1}}{2\mu-1}\right)^{\frac{1}{2}}\right) =: \rho < 1,
$$

where

$$\bar{M} := L_\gamma\left(\frac{2Ml}{\Gamma(\mu)}\frac{h^\mu}{\mu} + \frac{Ml}{\Gamma(\mu)}\left(2Tr(\mathcal{Q})\frac{h^{2\mu-1}}{2\mu-1}\right)^{\frac{1}{2}}\right).$$

For any $\varrho \in L^2(\Omega, \mathbb{H})$, we define the control function $u(t) = u(t; y)$ as follows:

$$u(t;y) = \Upsilon^{-1}\bigg(\varrho - J(h)y_0 - K(h)y_1 - \int_0^h N(h-s)\mathcal{G}(s, y(s))ds - \int_0^h N(h-s)\mathcal{T}(s, y(s))dw(s)\bigg)(t).$$

By Definition 8, we can conclude that the system (1) is exactly controllable on $[0, h]$ if and only if there exists $y^*(h) = \varrho$, where y^* is the mild solution to system (1) corresponding to $u(t) = u(t; y)$. To prove this, we only need to prove that system (1) has a mild solution when $u(t) = u(t; y)$. Next, let us make some preparations for applying the Mönch′s fixed point theorem.

As $u(t) = u(t; y)$, the operator Ψ_2 in (5) becomes

$$(\Psi_2 z)(t) = t^{2-\beta}\int_0^t N(t-s)[\mathcal{G}(s, y(s)) + Bu(s; y)]ds + t^{2-\beta}\int_0^t N(t-s)\mathcal{T}(s, y(s))dw(s), \ for\ t \in (0, h].$$

From form (A_3), we have

$$\sup_{t\in[0,h]}\bigg\{5\left(\frac{M}{(\mu-1)\Gamma(\beta-1)}\right)^2 E\|y_0\|^2 + 5\left(\frac{t}{\Gamma(\beta)}\right)^2 E\|y_1\|^2 + 5\left(\frac{Mt^{2-\beta+\frac{\mu}{2}}}{\Gamma(\mu)}\right)^2 \frac{1}{\mu}\int_0^t (t-s)^{\mu-1}g(s)ds + 5\left(\frac{M_B M t^{2-\beta+\mu}}{\Gamma(\mu+1)}\right)^2 L_\nu + 5\left(\frac{Mt^{2-\beta}}{\Gamma(\mu)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}g(s)ds\bigg\} \le r_1, \tag{12}$$

where L_ν is defined in Lemma 9 of this article and $r_1 > 0$ is a constants.

Let

$$\mathcal{D}_{r_1} = \left\{z : z \in C_{[0,h]}, \|z\|_{C_{[0,h]}} \le r_1\right\},\ \ \tilde{\mathcal{D}}_{r_1} = \left\{y : y \in C_{(0,h]}, \|y\|_{C_{(0,h]}} \le r_1\right\}.$$

Obviously, $\mathcal{D}_{r_1} \subset C_{[0,h]}$ and $\tilde{\mathcal{D}}_{r_1} \subset C_{(0,h]}$ are convex, nonempty and closed.

Lemma 9. *Suppose that (A_1)–(A_3) are satisfied. Then*

$$E\|u(t;y)\|^2 \le 5L_\gamma^2\bigg\{E\|\varrho\|^2 + \left(\frac{Mh^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\right)^2 E\|y_0\|^2 + \left(\frac{Mh^{\beta-1}}{\Gamma(\beta)}\right)^2 E\|y_1\|^2 + \left(\frac{M}{\Gamma(\mu)}\right)^2 \frac{h^{2\mu-1}}{\mu}\|g\|_{L^1} + \left(\frac{Mh^{\mu-1}}{\Gamma(\mu)}\right)^2 Tr(\mathcal{Q})\|g\|_{L^1}\bigg\} =: L_\nu.$$

Proof. By Lemma 3, Hölder′s inequality and assumption (A_3), (H_1), we have

$$
\begin{aligned}
E\|u(t;y)\|^2 \le\, & 5L_\gamma^2\Big\{E\|\varrho\|^2 + E\|J(h)y_0\|^2 + E\|K(h)y_1\|^2 \\
& + E\Big\|\int_0^h N(h-s)\mathcal{G}(s,y(s))ds\Big\|^2 + E\Big\|\int_0^h N(h-s)\mathcal{T}(s,y(s))dw(s)\Big\|^2\Big\} \\
\le\, & 5L_\gamma^2\Big\{E\|\varrho\|^2 + \Big(\frac{Mh^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\Big)^2 E\|y_0\|^2 + \Big(\frac{Mh^{\beta-1}}{\Gamma(\beta)}\Big)^2 E\|y_1\|^2 \\
& + \Big(\frac{M}{\Gamma(\mu)}\Big)^2 \frac{h^\mu}{\mu}\int_0^h (h-s)^{\mu-1}E\|\mathcal{G}(s,y(s))\|^2 ds \\
& + \Big(\frac{M}{\Gamma(\mu)}\Big)^2 Tr(\mathcal{Q})\int_0^h (h-s)^{2(\mu-1)}E\|\mathcal{T}(s,y(s))\|^2 ds\Big\} \\
\le\, & 5L_\gamma^2\Big\{E\|\varrho\|^2 + \Big(\frac{Mh^{\beta-2}}{(\mu-1)\Gamma(\beta-1)}\Big)^2 E\|y_0\|^2 + \Big(\frac{Mh^{\beta-1}}{\Gamma(\beta)}\Big)^2 E\|y_1\|^2 \\
& + \Big(\frac{M}{\Gamma(\mu)}\Big)^2 \frac{h^{2\mu-1}}{\mu}\|g\|_{L^1} + \Big(\frac{Mh^{\mu-1}}{\Gamma(\mu)}\Big)^2 Tr(\mathcal{Q})\|g\|_{L^1}\Big\} \\
=\, & L_\nu.
\end{aligned}
$$

□

Theorem 3. *Suppose that* $(A_1)-(A_3)$ *and* $(H_1)-(H_3)$ *are satisfied, and* $\{\mathcal{S}(t)\}_{t>0}$ *is noncompact. Then, the system* (1) *is exact controllability on* $[0,h]$.

Proof. Similar to Step 1, 2, and 3 in Theorem 1, by applying Lemma 9, we can verify that Ψ is equicontinuous, continuous, and that $\Psi(\mathcal{D}_{r_1}) \subset \mathcal{D}_{r_1}$. Next, we will prove that Ψ satisfies Mönch's condition. Suppose that $V_1 = \{y_n\}_{n=1}^\infty \subset \tilde{\mathcal{D}}_{r_1}$ and $V_1 \subset \overline{co}(\{0\} \cup \Phi(V_1))$. Suppose that $V_2 = \{z_n\}_{n=1}^\infty \subset \mathcal{D}_{r_1}$ and $V_2 \subset \overline{co}(\{0\} \cup \Psi(V_2))$. Then, we have $V_2(t) = t^{2-\beta}V_1(t)$ for $t \in [0,h]$.

By Lemmas 2 and 3 and (H_2), we have

$$
\begin{aligned}
& \chi\Big\{\int_0^t N(t-s)\mathcal{G}(s,V_1(s))ds\Big\} \\
\le\, & \frac{2M}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\chi(\mathcal{G}(s,\{s^{\beta-2}V_2(s)\}))ds \\
\le\, & \frac{2Ml}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}s^{2-\beta}\chi(s^{\beta-2}V_2(s))ds \\
\le\, & \frac{2Ml}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\chi(V_2(s))ds.
\end{aligned}
$$

For any $y_1, y_2 \in \mathbb{H}$, by using Lemma 3 and (2), we can derive that

$$
\begin{aligned}
& \Big(E\Big\|\int_0^t N(t-s)\mathcal{T}(s,y_1(s))dw(s) - \int_0^t N(t-s)\mathcal{T}(s,y_2(s))dw(s)\Big\|^2\Big)^{\frac{1}{2}} \\
\le\, & \frac{M}{\Gamma(\mu)}\Big(E\Big\|\int_0^t (t-s)^{\mu-1}(\mathcal{T}(s,y_1(s)) - \mathcal{T}(s,y_2(s)))dw(s)\Big\|^2\Big)^{\frac{1}{2}} \\
\le\, & \frac{M}{\Gamma(\mu)}\Big(Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}E\|\mathcal{T}(s,y_1(s)) - \mathcal{T}(s,y_2(s))\|^2 ds\Big)^{\frac{1}{2}}. \qquad (13)
\end{aligned}
$$

Thus, by (13), (H_2) and referring to [13], we have

$$\begin{aligned}
&\chi\bigg(\int_0^t N(t-s)\mathcal{T}(s,V_1(s))dw(s)\bigg)\\
&\leq \frac{M}{\Gamma(\mu)}\bigg(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}\big[\chi(\mathcal{T}(s,s^{\beta-2}V_2(s)))\big]^2 ds\bigg)^{\frac{1}{2}}\\
&\leq \frac{Ml}{\Gamma(\mu)}\bigg(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}s^{2(2-\beta)}\big[\chi(s^{\beta-2}V_2(s))\big]^2 ds\bigg)^{\frac{1}{2}}\\
&\leq \frac{Ml}{\Gamma(\mu)}\bigg(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}\big[\chi(V_2(s))\big]^2 ds\bigg)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}
&\chi(u(t,V_1))\\
&\leq L_\gamma\chi\bigg\{\varrho-J(h)y_0-K(h)y_1-\int_0^h N(h-s)\mathcal{G}(s,V_1(s))ds-\int_0^h N(h-s)\mathcal{T}(s,V_1(s))dw(s)\bigg\}\\
&\leq L_\gamma\bigg(\chi\bigg\{\int_0^h N(h-s)\mathcal{G}(s,V_1(s))ds\bigg\}+\chi\bigg\{\int_0^h N(h-s)\mathcal{T}(s,V_1(s))dw(s)\bigg\}\bigg)\\
&\leq L_\gamma\bigg(\frac{2Ml}{\Gamma(\mu)}\int_0^h (h-s)^{\mu-1}\chi(V_2(s))ds+\frac{Ml}{\Gamma(\mu)}\bigg[2Tr(\mathcal{Q})\int_0^h (h-s)^{2(\mu-1)}(\chi\{V_2(s)\})^2 ds\bigg]^{\frac{1}{2}}\bigg)\\
&\leq L_\gamma\bigg(\frac{2Ml}{\Gamma(\mu)}\int_0^h (h-s)^{\mu-1}ds+\frac{Ml}{\Gamma(\mu)}\bigg[2Tr(\mathcal{Q})\int_0^h (h-s)^{2(\mu-1)}ds\bigg]^{\frac{1}{2}}\bigg)\sup_{t\in[0,h]}\chi\{V_2(t)\}\\
&\leq L_\gamma\bigg(\frac{2Ml}{\Gamma(\mu)}\frac{h^\mu}{\mu}+\frac{Ml}{\Gamma(\mu)}\bigg(2Tr(\mathcal{Q})\frac{h^{2\mu-1}}{2\mu-1}\bigg)^{\frac{1}{2}}\bigg)\sup_{t\in[0,h]}\chi\{V_2(t)\}\\
&=\bar{M}\sup_{t\in[0,h]}\chi\{V_2(t)\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\chi\{(\Psi V_2)\}\\
&=\chi\{(\Psi_1V_2)(t)+(\Psi_2V_2)(t)\}\\
&\leq\chi\bigg\{t^{2-\beta}\int_0^t N(t-s)[\mathcal{G}(s,V_1(s))+Bu(s,V_1(s))]ds+t^{2-\beta}\int_0^t N(t-s)\mathcal{T}(s,V_1(s))dw(s)\bigg\}\\
&\leq\chi\bigg\{t^{2-\beta}\int_0^t N(t-s)\mathcal{G}(s,V_1(s))ds\bigg\}+\chi\bigg\{t^{2-\beta}\int_0^t N(t-s)Bu(s,V_1(s))ds\bigg\}\\
&\quad+\chi\bigg\{t^{2-\beta}\int_0^t N(t-s)\mathcal{T}(s,V_1(s))dw(s)\bigg\}\\
&\leq t^{2-\beta}\frac{2M}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\chi\{\mathcal{G}(s,V_1(s))\}ds+t^{2-\beta}\frac{2M_BM}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\chi\{u(s,V_1(s))\}ds\\
&\quad+t^{2-\beta}\frac{M}{\Gamma(\mu)}\bigg(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}\big(\chi\{\mathcal{T}(s,V_1(s))\}\big)^2 ds\bigg)^{\frac{1}{2}}\\
&\leq t^{2-\beta}\frac{2Ml}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}s^{2-\beta}\chi\{s^{\beta-2}V_2(s)\}ds\\
&\quad+t^{2-\beta}\frac{2M_BM}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\bar{M}ds\sup_{t\in[0,h]}\chi(V_2(t))\\
&\quad+t^{2-\beta}\frac{Ml}{\Gamma(\mu)}\bigg(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}s^{2(2-\beta)}\big(\chi\{s^{(\beta-2)}V_2(s)\}\big)^2 ds\bigg)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{ t^{2-\beta}\frac{2Ml}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}ds + t^{2-\beta}\frac{2M_B M\bar{M}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}ds \right. \\
&\left. + t^{2-\beta}\frac{Ml}{\Gamma(\mu)}\left(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(\mu-1)}ds\right)^{\frac{1}{2}} \right\} \sup_{t\in[0,h]} \chi(V_2(t)) \\
&\leq \left\{ h^{2-\beta}\frac{2Ml}{\Gamma(\mu)}\frac{h^\mu}{\mu} + h^{2-\beta}\frac{2M_B M\bar{M}}{\Gamma(\mu)}\frac{h^\mu}{\mu} + h^{2-\beta}\frac{M}{\Gamma(\mu)}\left(2Tr(\mathcal{Q})\frac{h^{2\mu-1}}{2\mu-1}\right)^{\frac{1}{2}} \right\} \sup_{t\in[0,h]} \chi(V_2(t)) \\
&=: \rho \sup_{t\in[0,h]} \chi(V_2(t)).
\end{aligned}
$$

Combined with the above calculations, we have

$$
\chi(V_2) \leq \chi\Big(\overline{co}\big(\{0\} \cup \Psi(V_2)\big)\Big) = \chi\big(\Psi(V_2)\big) \leq \rho \sup_{t\in[0,h]} \chi\big(V_2(t)\big) \leq \rho\chi(V_2) < \chi(V_2).
$$

By (H_3), we can conclude that $\chi(V_2) = 0$ and that V_2 is relatively compact. According to the Mönch′s fixed point theorem, Ψ has at least one fixed point $y^* \in \tilde{\mathcal{D}}_{r_1}$. This fixed point y^* is a mild solution of the fractional stochastic control system (1) when the control function is taken as $u(t) = u(t;y)$. Furthermore, it satisfies $y^*(h) = \varrho$ for any $\varrho \in L^2(\Omega,\mathbb{H})$. Therefore, we can conclude that system (1) has exact controllability on $[0,h]$. □

Theorem 4. *Suppose that $(A_1) - (A_3)$ and $(H_1) - (H_2)$ are satisfied, and $\{\mathcal{S}(t)\}_{t>0}$ is compact. Then, the system (1) is exact controllability on $[0,h]$.*

Proof. The proof follows a similar approach to that of Theorem 1. □

5. An Application

Example 1. *Consider following equation:*

$$
\begin{cases}
\partial_t^{\mu,\nu} z(t,v) = \partial_v^2 z(t,v) + e^t cos(z(t,v)) + Bu(t,v) + e^t sin(z(t,v))\frac{dw(t)}{dt}, \;\; t\in(0,1],\; v\in[0,\pi], \\
z(t,0) = z(t,\pi) = 0, \;\; t\in(0,1], \\
(\mathcal{I}_{0^+}^{2-\beta}z)(0,v) = z_0(v), (\mathcal{I}_{0^+}^{2-\beta}z)'(0,v) = z_1(v), \;\; v\in[0,\pi],
\end{cases} \tag{14}
$$

where $\partial_t^{\mu,\nu}$ is a Hilfer fractional partial derivative with order $1<\mu<2$ and type $0\leq\nu\leq1$, $\beta = \mu + \nu(2-\mu)$. $\mathbb{H} = \mathbb{K} = \mathbb{U} = L^2([0,\pi])$.

Let $Az = \frac{d^2}{dt^2}z$, $D(A) = \{z\in\mathbb{H} : z(0) = z(\pi) = 0; z''\in\mathbb{H}; z', z''$ are absolutely continuous$\}$. Then, A is infinitesimal generator of uniformly bounded strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\geq0}$.

Let $\psi_n(v) = \sqrt{\frac{2}{\pi}}\sin(n\pi v)$, whiich implies that $\{-n^2\pi^2, n\in N\}$ is eigenvalues of A and $\{\psi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathbb{H}$. Then

$$
Az = -\sum_{n=1}^{\infty} n^2 < z,\psi_n > \psi_n, z\in D(A),
$$

$< \cdot,\cdot >$ is the inner product in $\mathbb{H}$.

See [34], we can obtain

$$
\mathcal{C}(t)z = \sum_{n=1}^{\infty}\cos(n\pi t) < z,\psi_n > \psi_n, \quad \mathcal{S}(t)z = \sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t) < z,\psi_n > \psi_n,\; z\in\mathbb{H},
$$

and

$$
Q(t)z = \sum_{n=1}^{\infty} t^{\frac{\mu}{2}} E_{\mu,\mu}(-n^2 t^\mu) < z,\psi_n > \psi_n,
$$

where $E_{\mu,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu(n+1))}$ *is the Mittag-Leffler function.*

Let $y(t)v = z(t,v)$*, then the problem* (14) *can be reformulated as the problem* (1) *in* $\mathbb{H}$ *for* $\mathcal{G}(t,y(t)) = e^t cos(y(t))$ *and* $\mathcal{T}(t,y(t)) = e^t sin(y(t))$*. Clearly, the assumtions* (A_1)*–*(A_3) *and* (H_2) *are satisfied.*

Based on [34], it can be deduced that (H_1) *is valid. Therefore, Theorem 3 implies that the problem* (14) *is exact controllability.*

6. Conclusions

In this paper, we investigate the approximate and exact controllability for the HFSEEs. To accomplish this, we use stochastic analysis techniques, fractional calculus, measure of noncompactness and the fixed point theorem. Our findings indicate that the conditions for both approximate and exact controllability do not necessitate the Lipschitz condition being satisfied by $\mathcal{G}(t,y(t))$ and $\mathcal{T}(t,y(t))$. Additionally, we demonstrate the exact controllability for both cases: when the semigroup is compact or noncompact.

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