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$\varphi-$Hilfer Fractional Cauchy Problems with Almost Sectorial and Lie Bracket Operators in Banach Algebras

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Abstract: In the theory of Banach algebras, we use the Schauder fixed-point theorem to obtain some results that concern the existence property for mild solutions of fractional Cauchy problems that involve the Lie bracket operator, the almost sectorial operator, and the $\varphi-$Hilfer derivative operator. For any Banach algebra and in two types of non-compact associated semigroups and compact associated semigroups, we prove some properties of the existence of these mild solutions using the Hausdorff measure of a non-compact associated semigroup in the collection of bounded sets. That is, we obtain the existence property of mild solutions when the semigroup associated with an almost sectorial operator is compact as well as non-compact. Some examples are introduced as applications for our results in commutative real Banach algebra $\mathbb{R}$ and commutative Banach algebra of the collection of continuous functions in $\mathbb{R}$.

Keywords: lie bracket operator; Banach algebra; compact; almost sectorial operator; Hilfer fractional derivative

MSC: Primary 11F22; 32A65; 08A45; 34K37

Citation: Damag, F.H.; Saif, A.; Kiliçman, A. $\varphi-$Hilfer Fractional Cauchy Problems with Almost Sectorial and Lie Bracket Operators in Banach Algebras. *Fractal Fract.* **2024**, *8*, 741. https://doi.org/10.3390/fractalfract8120741

Academic Editor: Riccardo Caponetto

Received: 7 November 2024
Revised: 29 November 2024
Accepted: 9 December 2024
Published: 16 December 2024

1. Introduction

In the 17th century, fractional calculus was introduced as an extension of ordinary calculus; that is, it gives us the derivatives and the integrals for functions of any real order. The Riemann–Liouville fractional differential and integral operators are the oldest of theses fractional differential operators. For these operators in fractional calculus, we have several integrals operators and fractional derivatives that have been developed by many researchers, such as the notions of Riemann-Liouville fractional differentials and integral operators, to introduce some fractional differentials and integral operators such as the Caputo derivative, Hilfer fractional derivative, Hilfer–Katugampola fractional derivative, Katugampola fractional integral, and Hadamard fractional integral. For more details about fractional differential and integral operators, see [1–6]. Recall that in [7], Hilfer introduced a fractional differential operator that involves the Riemann–Liouville fractional differential operator and Caputo fractional derivative opeartor. The Hilfer operator plays an important role in several applications such as in polymer science, rheological constitutive modeling, engineering, conceptual simulations of dielectric relaxation in crystal materials, and other fields. The theory of classical integer-order partial differential equations and fractional-order partial differential equations is considered a good representation for several models such as in describing issues in fluid flow, finance, fluid mechanics, engineering, polymer science, physics, and other areas of application [8–15]. Periago and B. Straub [16] introduced new spaces and developed a functional calculus concept; that is, they explored the properties of both the mild and classic semigroups, as well as possible explanations for their existence. Wang et al. [17] used functional calculus to construct two sets of operators and used the Caputo derivative to solve various fractional Cauchy problems; that is, they obtained the existence and uniqueness of mild solutions and classical solutions to the Cauchy problems. Shu and Shi [18] and Shu et al. [19] used the formula of mild solutions for impulsive fractional evolution equations and investigated the existence of mild solutions for fractional differential evolution systems with impulse employing sectorial operators.

In recent years, several operators and results have appeared supporting mathematical modeling using fractional calculus to describe the hereditary properties of various materials and processes. For the fractional Cauchy problem, many researchers solved this problem for different types depending on the used fractional differential operator in the theory of Banach spaces; for example, Karthikeyan et al. [20] introduced some solution of a $\varphi-$Hilfer fractional Cauchy problem (Equation (1)):

$$\begin{cases} \mathcal{H}_{0^+}^{\vartheta,\eta;\varphi}u(\iota) + \mathcal{A}u(\iota) = g(\iota, u(\iota)) \quad \iota \in \mathbb{V} = (0, a], \\ \Delta u|_{\iota=\iota_k} = h_k\big(u(\iota_k^-)\big), \quad k = 1, 2, 3, \cdots, m. \\ I_{0^+}^{(1-\eta)(1-\vartheta),\varphi}u(0) = u_0, \end{cases} \tag{1}$$

where $u(\iota_k^+)$ and $u(\iota_k^-)$ are the right and left limits of $u(\iota)$ at $\iota = \iota_k$, respectively; $\mathcal{H}_{0^+}^{\vartheta,\eta;\varphi}$ is the $\varphi-$Hilfer fractional derivative; $I_{0^+}^{(1-\eta)(1-\vartheta)}$ is the Riemann–Liouville fractional integral of order $(1-\eta)(1-\vartheta)$ and $\Delta u|_{\iota=\iota_k} = u(\iota_k^+) - u(\iota_k^-)$. In 2022, Zhou et al. [21] introduced new sufficient conditions for the existence of mild solutions for Hilfer fractional evolution, Equation (2), with an almost sectorial operator in the theory of Banach spaces:

$$\begin{cases} \mathcal{H}_{0^+}^{\vartheta,\eta;\varphi}u(\iota) = \mathcal{A}u(\iota) + g(\iota, u(\iota)) \quad \iota \in (0, T], \\ I_{0^+}^{(1-\eta)(1-\vartheta),\varphi}u(0) = u_0, \end{cases} \tag{2}$$

where $\mathcal{A}$ is an almost sectorial operator on Banach space $\mathcal{B}$, and $T \in (0, \infty)$. Zhou et al. [22] investigated the existence of attractive solutions of Hilfer fractional evolution, Equation (2). Their methods depend on Schauder's fixed-point theorem, the generalized Ascoli–Arzela theorem, Kuratowski's measure of non-compactness, and the Wright function. Varun Bose et al. [23] introduced the approximate controllability of a Hilfer fractional neutral Volterra integro-differential problem (Equation (3)), which involves almost sectorial operators, using the Leray–Schauder fixed-point theorem:

$$\begin{cases} \mathcal{D}_{0^+}^{\vartheta,\eta}[u(\iota) - \mathcal{N}(\iota, u(\iota))] \in \mathcal{A}u(\iota) + g\big(\iota, u(\iota), \int_0^\iota f(\iota, s, u(s))ds\big) \quad \iota \in (0, b], \\ I_{0^+}^{(1-\eta)(1-\vartheta)}u(0) = u_0, \end{cases} \tag{3}$$

where $b \in (0, \infty)$, and $\mathcal{D}_{0^+}^{\vartheta,\eta}$ is a Hilfer fractional differential operator. Varun Bose and Udhayakumar [23,24] studied the existence property of a mild solution for the Hilfer fractional neutral integro-differential problem (Equation (4)), which involves almost sectorial operators in Banach space $\mathcal{Y}$:

$$\begin{cases} \mathcal{D}_{0^+}^{\vartheta,\eta}[u(\iota) - \mathcal{K}(\iota, u(\iota))] \in \mathcal{A}u(\iota) + \mathcal{B}v(\iota) + g\big(\iota, u(\iota), \int_0^\iota f(\iota, s, u(s))ds\big) \quad \iota \in (0, b], \\ I_{0^+}^{(1-\eta)(1-\vartheta)}u(0) = u_0, \end{cases} \tag{4}$$

where $b \in (0, \infty)$, and $\mathcal{B} : \mathcal{Y} \to \mathcal{X}$ is an operator in the control term of two Banach spaces $\mathcal{Y}$ and $\mathcal{X}$. In the theory of Hilbert spaces, Sivasankar and Udhayakumar [25] studied the existence of a Hilfer fractional stochastic differential system (Equation (5)) via an almost sectorial operator $\mathcal{A}$:

$$\begin{cases} \mathcal{D}_{0^+}^{\vartheta,\eta}u(\iota) \in \mathcal{A}u(\iota) + g(\iota, u(\iota))\frac{dW(\iota)}{d\iota} \quad \iota \in (0, b], \\ I_{0^+}^{(1-\eta)(1-\vartheta)}u(0) = u_0, \end{cases} \tag{5}$$

where $b \in (0, \infty)$.

Banach algebra [26] is a Banach space $\mathcal{B}$ over a field $\mathbb{Q}$ with a norm $||\cdot||$ together with associative and distributive multiplication on $\mathcal{B}$ such that $r(ab) = (ra)b = a(rb)$ for all $a, b \in \mathcal{B}$ and $r \in \mathbb{K}$ and the operation $\mathcal{B} \times \mathcal{B} \to \mathcal{B} : (a, b) \to ab$ is a continuous function with respect to the mertizable topological space induced $||.||$. Note that the Banach algebra $\mathcal{B}$ is a topological semigroup and $||ab|| \leq ||a||\,||b||$ for all $a, b \in \mathcal{B}$. Consider the Hilfer fractional Cauchy problem

$$\begin{cases} \mathcal{H}_{0^+}^{\vartheta,\eta;\varphi} v(\iota) + \mathcal{G}v(\iota) = [u, v](\iota) + \int_0^\iota k(\iota, t) f(t, v(t))dt & \iota \in \mathbb{V} = (0, a], \\ I_{0^+}^{(1-\eta)(1-\vartheta);\varphi} v(0) = v_0, \end{cases} \tag{6}$$

where $f : \mathbb{V} \times \mathcal{B} \to \mathcal{B}$ and $u : \mathbb{V} \to \mathcal{B}$ are given functions; $\mathcal{H}_{0^+}^{\vartheta,\eta;\varphi}$ is the $\varphi-$Hilfer fractional derivative, $I_{0^+}^{(1-\eta)(1-\vartheta)}; \varphi$ is A $\varphi-$ Riemann–Liouville fractional integral of order $(1-\eta)(1-\vartheta)$; $\mathcal{G}$ is an almost sectorial operator on $\mathcal{B}$; and $[\cdot,\cdot] : C(\mathbb{V}, \mathcal{B}) \times C(\mathbb{V}, \mathcal{B}) \to C(\mathbb{V}, \mathcal{B})$ is a Lie bracket operator defined by $[u, v](\iota) = u(\iota)v(\iota) - v(\iota)u(\iota)$. For brevity, in Equation (6), take

$$\mathcal{F}v(\iota) = \int_0^\iota k(\iota, t) f(t, v(t))dt.$$

All existing studies have introduced some solutions for Hilfer fractional Cauchy problems with an almost sectorial operator in the theory of Banach spaces. Motivated by these studies, in this paper, we further develop these studies under the theory of Banach algebras as follows: Firstly, we construct the classification problem using a Lie bracket operator and almost sectorial operator to give the general $\varphi-$Hilfer fractional Cauchy problem. Secondly, we introduce and investigate the existence property of this problem. So, Section 2 of this paper presents some background and necessary information for Banach algebra, Lie bracket operators, almost sectorial operators, the $\varphi-$Hilfer derivative, measures of non-compactness, and some results from previous studies. Section 3 proves the existence of some mild solutions of the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)) in the cases that associated semigroups are compact or non-compact. Section 4 presents some examples as applications for our main results. The discussion and conclusions are presented in Section 5.

2. Preliminaries

In this section, we first recall the definitions of three operators that will be used in our work, such as the φ-Riemann–Liouville fractional integral operator $I_{0^+}^{\vartheta;\varphi}$, the φ-Hilfer fractional differential operator $\mathcal{H}_{0^+}^{\vartheta,\eta;\varphi}$, and the Hausdorff measure of non-compactness $\mathcal{M}$. Let φ be a positive increasing function on $(0, \infty)$ where the derivative $\varphi'(t) \neq 0$ is continuous on $(0, \infty)$. The φ-Riemann–Liouville operator $I_{0^+}^{\vartheta;\varphi}$ of order $\vartheta > 0$, for a function $v : [0, +\infty) \to \mathbb{R}$ [27], is given by

$$I_{0^+}^{\vartheta;\varphi} v(\iota) = \frac{1}{\Gamma(\vartheta)} \int_0^\iota \varphi'(t) \frac{v(t)}{(\varphi(\iota) - \varphi(t))^{1-\vartheta}} dt. \tag{7}$$

The φ -Hilfer fractional derivative of order $\vartheta > 0$ for a function $v : [0, +\infty) \to \mathbb{R}$ [28] is given by

$$\mathcal{H}_{0^+}^{\vartheta,\eta;\varphi} v(\iota) = I_{0^+}^{\eta(1-\vartheta);\varphi} \left(\frac{1}{\varphi'(\iota)} \frac{d}{d\iota} \right) I_{0^+}^{(1-\eta)(1-\vartheta);\varphi} v(\iota), \tag{8}$$

where $0 \leq \eta \leq 1$. The Hausdorff measures of non-compactness are used to obtain numbers associated with non-compact sets and compact sets where the compact sets have measure 0. That is, the main idea for this measure can be recalled as follows: All bounded sets can be covered by a single ball of some radius. Sometimes, these sets can be covered by many balls of a smaller radius. Since all compact sets are totally bounded sets, then they can be covered by finitely many balls of an arbitrarily small radius. In all cases, the Hausdorff measures of the compact set or non-compact set is the smallest radius that allows us to cover this set with finitely many balls. For a Banach space $\mathcal{B}$, the Hausdorff measure of the non-compactness $\mathcal{M}$ of bounded set $N \subseteq \mathcal{B}$ [29] is given by

$$\mathcal{M}(N) = \inf\{\epsilon > 0 : N \subseteq \cup_{k=1}^{n} B(\epsilon, o_k),\ o_k \in \mathcal{B},\ n \in \mathbb{N}\}$$

where $B(\epsilon, o_k)$ is a ball in $\mathcal{B}$ with centre o_k and radius ϵ. Let $C(\mathbb{V}, \mathcal{B})$ be the space of all continuous functions from $\mathbb{V}$ to $\mathcal{B}$. For all $\mathbb{Q} \subseteq C(\mathbb{V}, \mathcal{B})$ and for $\iota \in \mathbb{V}$, let

$$\mathbb{Q}_\iota := \{v(\iota) : v \in \mathbb{Q}\} \text{ and } {}_0^\iota\mathbb{Q} := \int_0^\iota \mathbb{Q}_t dt := \left\{\int_0^\iota v(t)dt : v \in \mathbb{Q}\right\}.$$

Theorem 1 ([30]). *Let $\mathbb{Q} \subseteq C(\mathbb{V}, \mathcal{B})$ be bounded and equicontinuous. Then, the function $\iota \to \mathcal{M}(\mathbb{Q}_\iota)$ is continuous from $\mathbb{V}$ into $\mathbb{R}$ and*

$$\mathcal{M}(\mathbb{Q}) = \max\{\mathcal{M}(\mathbb{Q}_\iota), \mathcal{M}({}_0^\iota\mathbb{Q})\} \leq \int_0^\iota \mathcal{M}(\mathbb{Q}_t)dt.$$

Theorem 2 ([31]). *If $\{v_n : n \in \mathbb{N}\}$ is a family of Bochner integrable functions from $\mathbb{V}$ to $\mathcal{B}$, then $||v_n|| \leq m(\iota)$ almost everywhere for all $n \in \mathbb{N}$ and*

$$\mathcal{M}\left(\left\{\int_0^\iota v_n(t)dt : n \in \mathbb{N}\right\}\right) \leq 2\int_0^\iota g(t)dt$$

where $m \in L^1(J, \mathbb{R}^+)$ and $g(\iota) = \mathcal{M}(\{v_n(\iota) : n \in \mathbb{N}\}) \in L^1(J, \mathbb{R}^+)$.

Theorem 3 ([31]). *For any bounded set $B \subseteq \mathcal{B}$ and for any $\epsilon > 0$, there is a sequence $\{v_n : n \in \mathbb{N}\}$ in B such that $\mathcal{M}(B) \leq \mathcal{M}(\{v_n(\iota) : n \in \mathbb{N}\}) + \epsilon$.*

Let $Z_{0\phi} := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}$ for $0 < \phi \leq \pi$. For any $0 < \phi \leq \frac{\pi}{2}$ and $-1 < \nu < 0$, $\mathbb{H}_{\phi\nu}$ denotes the collection of closed linear operators $\mathcal{G} : D(\mathcal{G}) \subseteq \mathcal{B} \to \mathcal{B}$ that have the spectrum $\sigma(\mathcal{G}) \subseteq Z_\phi$ [32], and for all $\mu \in (\phi, \pi)$, there is $C_\mu > 0$ such that $||P_z(\mathcal{G})||_{L(\mathcal{B})} \leq C_\mu |z|^\nu$, where $L(\mathcal{B})$ denotes the space of all bounded linear operators on $\mathcal{B}$, and $P_z(\mathcal{G}) = (zI - \mathcal{G})^{-1}$ is the resolvent operator. Every element in $\mathbb{H}_{\phi\nu}$ is called an almost sectorial operator on $\mathcal{B}$.

Theorem 4 ([32]). *For any $\mathcal{G} \in \mathbb{H}_{\phi\nu}$, $0 < \phi \leq \frac{\pi}{2}$ and $-1 < \nu < 0$, the following properties hold:*

1. *$\mathcal{J}(t)$ is analytic and $\frac{d^n}{dt^n}\mathcal{J}(t) = -\mathcal{G}^n\mathcal{J}(t),\ t \in Z_{0\frac{\pi}{2}}$;*
2. *$\mathcal{J}(t+s) = \mathcal{J}(t) + \mathcal{J}(s)$ for all $s, t \in Z_{0\frac{\pi}{2}}$;*
3. *$||\mathcal{J}(t)||_{L(\mathcal{B})} \leq c_0 t^{-\nu-1}$, where $c_0 := c(\nu) > 0$ is constant;*
4. *If $S_\mathcal{J} := \{\iota \in \mathcal{B} : \lim_{t\to 0^+} \mathcal{J}(t)\iota = \iota\}$, then $D(\mathcal{G}^\xi) \subset S_\mathcal{J}$ if $\xi > \nu + 1$;*
5. *$P_z(-\mathcal{G}) = \int_0^\infty e^{-zs}\mathcal{J}(s)ds$, where $z \in \mathbb{C}$ and $\Re(z) > 0$.*

For $\vartheta > 0$ and $0 < \phi \leq \pi$, define two operators $\mathcal{X}^{\phi\vartheta}, \mathbb{X}^{\phi\vartheta} : Z_{0(\frac{\pi}{2}-\phi)} \to \mathbb{R}$ by

$$\mathcal{X}^{\phi\vartheta}(\iota) = \int_0^\infty \Lambda_\vartheta(\iota)\mathcal{J}\left(\xi\iota^\vartheta\right)d\xi$$

and

$$\mathbb{X}^{\phi\vartheta}(\iota) = \int_0^\infty \vartheta\xi\Lambda_\vartheta(\xi)\mathcal{J}\left(\xi\iota^\vartheta\right)d\xi$$

where Λ_ϑ is a Wright-type function [33] given by

$$\Lambda_\vartheta(\xi) = \sum_{n\in\mathbb{N}} \frac{(-\xi)^{n-1}}{(n-1)!\Gamma(1-n\vartheta)}, \quad \xi \in \mathbb{C}$$

with the following properties:

1. $\Lambda_\vartheta(\xi) > 0$;
2. $\int_0^\infty \xi^r \Lambda_\vartheta(\xi) d\xi = \frac{\Gamma(1+r)}{\Gamma(1+r\vartheta)}$ for all $-1 < r < \infty$;
3. $\int_0^\infty \xi^r \vartheta \xi^{-\vartheta-1} e^{-r\xi} \Lambda_\vartheta(\xi^{-\vartheta}) d\xi = e^{-r^\vartheta}$ for all $r > 0$.

Theorem 5 ([32]). *The operators $\mathcal{X}^{\phi\vartheta}$ and $\mathbb{X}^{\phi\vartheta}$ have the following properties:*

1. *They are bounded linear operators on $\mathcal{B}$ with*

$$||\mathcal{X}^{\phi\vartheta}(\iota)|| \le c_1 \iota^{-\vartheta(1+\nu)}, \quad ||\mathbb{X}^{\phi\vartheta}(\iota)|| \le c_2 \iota^{-\vartheta(1+\nu)}, \quad \iota > 0$$

where c_1 and c_2 are constant, dependent on ϑ and ν;

2. *They have the continuity property with the uniform operator topology for $\iota > 0$, and this continuity is uniform on $[a, \infty]$ for $a > 0$.*

Recall the statement of the Arzela–Ascoli theorem in [34] that if a sequence $(f_n)_1^\infty$ in $C(X)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence, where $C(X)$ denotes the space of all continuous functions on a space X with values in complex $\mathbb{C}$ or real $\mathbb{R}$. Recall the statement of Schauder's fixed-point theorem in [35] that if $(X, ||\cdot||)$ is a Banach space over $\mathbb{C}$ or $\mathbb{R}$ and $S \subseteq A$ is closed, bounded, convex, and nonempty, then any compact operator $A : S \to S$ has at least one fixed point. The family of all open balls in any metrizable topological space forms a sub-base [36]. Hence, the collection of all open balls $B(x, r)$ in a metrizable topological space $\mathcal{B}$ with the topology induced by metric function $d(x, y) = ||x - y||$ forms a sub-base of a space $\mathcal{B}$. Let $P \subseteq \mathcal{B}$ and $M[B(x, r), P] = \{u \in C(\mathbb{V}, \mathcal{B}) : u[B(x, r)] \subseteq P\}$. Since $\mathbb{V}$ is a Hausdorff subspace of the standard topological space $\mathbb{R}$, then by [37,38], the collection

$$\{M[B(x, r), P] : P \text{ is compact set in } \mathcal{B},\ x \in \mathcal{B},\ r > 0\}$$

forms a sub-base for the compact-open topology on $C(\mathbb{V}, \mathcal{B})$.

Theorem 6 ([39]). *Let X_1, X_2, and X_3 be three topological spaces. Then, any continuous function $u : X_1 \times X_2 \longrightarrow X_3$ implies a continuous function $x_1 \to u(x_1)(x_2)$. If X_2 is regular space and locally compact, then any continuous function $S : X_1 \longrightarrow C(X_2, X_3)$ implies a continuous function $(x_1, x_2) \to S(x_1)(x_2)$.*

The following considered hypotheses will be used in our work to help us in finding some solutions for Equation (6). These hypotheses involve the type of used topology with $C(X)$ in the Arzela–Ascoli theorem and the type of used continuous functions:

(C1) Consider a compact-open topology on $C(\mathbb{V}, \mathcal{B})$.
(C2) The functions $f(x, \cdot), h(x, \cdot, \cdot) : \mathcal{B} \to \mathcal{B}$, $f(\cdot, v), h(\cdot, u(x), \mathcal{F}v(x)) : \mathbb{V} \to \mathcal{B}$, and $u : \mathbb{V} \to \mathcal{B}$ are continuous for all $x \in \mathbb{V}$ and for all $v \in C(\mathbb{V}, \mathcal{B})$, where $h(x, u(x), \mathcal{F}v(x)) = [u, v](x) + \mathcal{F}v(x)$.

(**C3**) There is a continuous function $q \in L^1(\mathbb{V}, \mathbb{R}^+)$ such that $I_{0^+}^{-\vartheta\nu} q(x) \in C(\mathbb{V}, \mathcal{B})$,

$$\lim_{x \to 0^+} [\varphi(x)]^{(1+\vartheta\nu)(1-\eta)} q(x) = 0 \text{ and } 2||u(x)||\,||v(x)|| + ||\mathcal{F}v\,(x)|| \leq q(x)$$

for all $v \in \Psi_s(\mathbb{V})$ and for all $x \in \mathbb{V}$, where $\Psi_s(\mathbb{V}) = \{u \in C(\mathbb{V}, \mathcal{B}) :||u|| \leq s\}$.
(**C4**) For $\epsilon > 0$ and $v_0 \in D(\mathcal{G}^n)$ with $n > 1 + \nu$,

$$\sup_{x \in \mathbb{V} \cup \{0\}} \left[[\varphi(x)]^{(1+\vartheta\nu)(1-\eta)} ||v_0 \mathbb{M}_{\vartheta,\eta;\varphi}(x)|| + [\varphi(x)]^{(1+\vartheta\nu)(1-\eta)} \int_0^x \left|\left|\varphi'(t)\right|\right| \varphi_{xt}^{-\vartheta\nu-1} q(t) dt \right] \leq \epsilon$$

where $\mathbb{M}_{\vartheta,\eta;\varphi}(x) = \frac{1}{\Gamma(\vartheta+\eta(1-\vartheta))}(\varphi(x) - \varphi(0))^{(\eta-1)(1-\vartheta)}$ and $\varphi_{xt} = \varphi(x) - \varphi(t)$.

3. Results

By the mild solution of $\varphi-$Hilfer fractional Cauchy problems (Equation (6)), $v \in C(\mathbb{V}, \mathcal{B})$ satisfies the following:

$$v\,(\iota) = v_0 \mathbb{M}_{\vartheta,\eta;\varphi}(\iota) + \int_0^\iota \varphi'(t) \varphi_{\iota t}^{\vartheta-1} \mathbb{X}^{\phi\vartheta}(\varphi_{\iota t})[[u, v](t) + \mathcal{F}v\,(t)]dt. \tag{9}$$

Now, we will define the operator $\mathbb{O} : \Psi_s(\mathbb{V}) \to \Psi_s(\mathbb{V})$ on $\Psi_s(\mathbb{V})$, which will be used in Schauder's fixed-point theorem to obtain the existence property of the mild solutions of the problem (Equation (6)):

$$\mathbb{O}v\,(\iota) = \iota^{(1+\vartheta\nu)(1-\eta)} \overline{\mathbb{O}}v\,(\iota) \tag{10}$$

where

$$\left(\overline{\mathbb{O}}v\right)(\iota) = \begin{cases} v_0 \mathbb{M}_{\vartheta,\eta;\varphi}(\iota) + \int_0^\iota \varphi'(t) \varphi_{\iota t}^{\vartheta-1} \mathbb{X}^{\phi\vartheta}(\varphi_{\iota t})[[u, v](t) + \mathcal{F}v\,(t)]dt & \iota \in (0, a], \\ 0 \quad \iota = 0. \end{cases}$$

The following lemma proves that the operator $\Psi_s(\mathbb{V})$ is equicontinuous.

Lemma 1. *Let* $\mathcal{G} \in \mathbb{H}_{\phi\nu}$ *, where* $-1 < \nu < 0$ *and* $0 < \phi < \frac{\pi}{2}$ *. If* $(C1) - (C4)$ *hold, then the operator* $\mathbb{O} : \Psi_s(\mathbb{V}) \to \Psi_s(\mathbb{V})$ *is equicontinuous such that* $v_0 \in D(\mathcal{G}^\xi)$ *and* $\xi > 1 + \nu$ *.*

Proof. Let $\iota_1, \iota_2 \in \mathbb{V} \cup \{0\}$ with $\iota_1 < \iota_2$. If $\iota_1 = 0$, then

$$\begin{aligned}
&||\mathbb{O}v\,(\iota_2) - \mathbb{O}v\,(0)|| = \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)} \overline{\mathbb{O}}v\,(\iota_2)\right|\right| \\
&= \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)} \left\{ v_0 \mathbb{M}_{\vartheta,\eta;\varphi}(\iota_2) + \int_0^{\iota_2} \varphi'(t) \varphi_{\iota_2 t}^{\vartheta-1} \mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u, v](t) + \mathcal{F}v\,(t)]dt \right\}\right|\right| \\
&\leq \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)} \mathbb{M}_{\vartheta,\eta;\varphi}(\iota_2) v_0\right|\right| \\
&+ \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)} \int_0^{\iota_2} \varphi'(t) \varphi_{\iota_2 t}^{\vartheta-1} \mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u, v](t) + \mathcal{F}v\,(t)]dt\right|\right| \to 0, \quad \text{when } x_2 \to 0.
\end{aligned}$$

If $x_1 \neq 0$, then

$$\begin{aligned}
||\mathbb{O}v\,(\iota_2) - \mathbb{O}v\,(\iota_1)|| &= \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\overline{\mathbb{O}}v\,(\iota_2) - \iota_1^{(1+\vartheta\nu)(1-\eta)}\overline{\mathbb{O}}v\,(\iota_1)\right|\right| \\
&\leq \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota_2)v_0 - \iota_1^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota_1)v_0\right|\right| \\
&+ \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_2}\varphi'(t)\varphi_{\iota_2 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right.\right. \\
&- \left.\left.\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_1 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\
&\leq \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota_2)v_0 - \iota_1^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota_1)v_0\right|\right| \\
&+ \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_{\iota_1}^{\iota_2}\varphi'(t)\varphi_{\iota_2 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\
&+ \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_2 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right.\right. \\
&- \left.\left.\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\
&+ \left|\left|\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right.\right. \\
&- \left.\left.\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_1 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right|.
\end{aligned}$$

Since $\mathbb{M}_{\vartheta,\eta;\varphi}$ is strongly continuous, then when $\iota_2 \to \iota_1$, we have

$$\Delta_1 := \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota_2)v_0 - \iota_1^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota_1)v_0\right|\right| \to 0.$$

Hence, by $(C3)$ and Theorem (5), we have

$$\begin{aligned}
\Delta_2 &:= \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_{\iota_1}^{\iota_2}\varphi'(t)\varphi_{\iota_2 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\
&\leq C\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_{\iota_1}^{\iota_2}\varphi'(t)\varphi_{\iota_2 t}^{-\vartheta\nu-1}[2||u(t)||\,||v(t)|| + ||\mathcal{F}v\,(t)\,||]dt \\
&\leq C\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_{\iota_1}^{\iota_2}\varphi'(t)\varphi_{\iota_2 t}^{-\vartheta\nu-1}q(t)dt \to 0 \quad \text{when } \iota_2 \to \iota_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Delta_3 &:= \left|\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_2 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right.\right. \\
&- \left.\left.\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\
&\leq C\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_2 t}^{\vartheta-1}\left|\iota_2^{(1+\vartheta\nu)(1-\eta)}\varphi_{\iota_2 t}^{\vartheta-1} - \iota_1^{(1+\vartheta\nu)(1-\eta)}\varphi_{\iota_1 t}^{\vartheta-1}\right|q(t)dt \\
&\to 0 \quad \text{when } x_2 \to x_1.
\end{aligned}$$

Since $\mathbb{X}^{\phi\vartheta}$ is uniformly continuous and $\Delta_2 \to 0$ when $\iota_2 \to \iota_1$, then for $\gamma > 0$,

$$\begin{aligned}
\Delta_4 &:= \left|\left|\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right.\right. \\
&- \left.\left.\iota_1^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_1 t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\
&\leq \int_0^{\iota_1-\gamma}\varphi'(t)\iota_1^{(1+\vartheta\nu)(1-\eta)}\left|\left|\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t}) - \mathbb{X}^{\phi\vartheta}(\varphi_{\iota_1 t})\right|\right|_{L(\mathcal{B})}\varphi_{\iota_1 t}^{\vartheta-1}q(t)dt \\
&+ \int_{\iota_1-\gamma}^{\iota_1}\varphi'(t)\iota_1^{(1+\vartheta\nu)(1-\eta)}\left|\left|\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t}) - \mathbb{X}^{\phi\vartheta}(\varphi_{\iota_1 t})\right|\right|_{L(\mathcal{B})}\varphi_{\iota_1 t}^{\vartheta-1}q(t)dt \\
&\leq \iota_1^{(1+\vartheta\nu)(1-\eta)}\sup_{[0,\iota_1-\gamma]}\left|\left|\mathbb{X}^{\phi\vartheta}(\varphi_{\iota_2 t}) - \mathbb{X}^{\phi\vartheta}(\varphi_{\iota_1 t})\right|\right|_{L(\mathcal{B})}\int_0^{\iota_1}\varphi'(t)\varphi_{\iota_1 t}^{\vartheta-1}q(t)dt \\
&+ C\int_{\iota_1-\gamma}^{\iota_1}\varphi'(t)\iota_1^{(1+\vartheta\nu)(1-\eta)}\left[\varphi_{\iota_2 t}^{\vartheta-\vartheta\nu} - \varphi_{\iota_1 t}^{\vartheta-\vartheta\nu}\right]\varphi_{\iota_1 t}^{\vartheta-1}q(t)dt \\
&\to 0 \quad \text{when } \iota_2 \to \iota_1.
\end{aligned}$$

Hence, $||\mathbb{O}v\,(\iota_2) - \mathbb{O}v\,(\iota_1)|| \to 0$ when $\iota_2 \to \iota_1$; that is, $\mathbb{O}$ is equicontinuous. □

The following theorem proves that the operator $\Psi_s(\mathbb{V})$ is bounded and continuous.

Theorem 7. *Let $\mathcal{G} \in \mathbb{H}_{\phi\nu}$, where $-1 < \nu < 0$ and $0 < \phi < \frac{\pi}{2}$. If (C1) − (C4) hold, then the operator $\mathbb{O} : \Psi_s(\mathbb{V}) \to \Psi_s(\mathbb{V})$ is bounded and continuous such that $v_0 \in D(\mathcal{G}^{\xi})$ and $\xi > 1 + \nu$.*

Proof. For the boundedness property for $\mathbb{O}$, we have that for all $v \in \Psi_s(\mathbb{V})$,

$$\begin{aligned} & ||\mathbb{O}v\,(\iota)|| \\ \leq & \left|\left|[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0\right|\right| \\ + & \left|\left|[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota t})[[u,v](t) + \mathcal{F}v\,(t)]dt\right|\right| \\ \leq & [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\left|\left|\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0\right|\right| + [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota\left|\left|\varphi'(t)\right|\right|\varphi_{\iota t}^{-\vartheta\nu-1}q(t)dt \\ \leq & \sup_{[0,a]}\left[[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\left|\left|\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0\right|\right| + [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota\left|\left|\varphi'(t)\right|\right|\varphi_{\iota t}^{-\vartheta\nu-1}q(t)dt\right] < \epsilon. \end{aligned}$$

That is, $\mathbb{O}$ is bounded. For the continuity property, we use Theorem (6). Since $\mathbb{V}$ is regular and locally compact, to prove that the operator $\mathbb{O}$ is continuous, it is enough to prove that the functions $\mathbb{O}v$ are continuous for all $v \in \Psi_s(\mathbb{V}) \subseteq C(\mathbb{V}, \mathcal{B})$. Firstly, we see that t $\mathbb{O}$ is well defined, that is, $\mathbb{O}v \in \Psi_s(\mathbb{V})$ for all $v \in \Psi_s(\mathbb{V})$. Let $v \in \Psi_s(\mathbb{V})$ be any element in $\Psi_s(\mathbb{V})$. Let $\iota_n, \iota \in \mathbb{V}$ for all $n = 1, 2, 3, \cdots$ such that $\iota_n \to \iota$ as $n \to +\infty$. By Lemma (1), we obtain that for all $n = 1, 2, 3, \cdots$,

$$||\mathbb{O}v\,(\iota_n) - \mathbb{O}v\,(\iota)|| \to 0 \quad \text{when } \iota_n \to \iota.$$

Hence, $||\mathbb{O}v\,(\iota_n) - \mathbb{O}v\,(\iota)|| \to 0$ when $n \to +\infty$; that is, $\mathbb{O}v$ is continuous. Hence, by Theorem (6), $\mathbb{O}$ is continuous. □

The following theorem proves the existence of the mild solution of the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)) when $\mathcal{J}(\iota)$ $(\iota > 0)$ has the compactness property.

Theorem 8. *Let $\mathcal{G} \in \mathbb{H}_{\phi\nu}$ where $-1 < \nu < 0$ and $0 < \phi < \frac{\pi}{2}$. If (C1) − (C4) hold and $\mathcal{J}(\iota)$ has the compactness property for all $\iota > 0$, then there exists a mild solution of the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)) such that $v_0 \in D(\mathcal{G}^{\xi})$ and $\xi > 1 + \nu$.*

Proof. By the compactness of $\mathcal{J}(\iota)$ $(\iota > 0)$, we obtain that $\mathcal{J}(\iota)$ $(\iota > 0)$ is equicontinuous. Note that

$$\begin{aligned} \mathbb{O}v\,(\iota) &= [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0 \\ &+ [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\mathbb{X}^{\phi\vartheta}(\varphi_{\iota t})[[u,v](t) + \mathcal{F}v\,(t)]dt \\ &= [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0 \\ &+ [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\int_0^\infty \xi\vartheta\Lambda_\vartheta(\xi)\mathcal{J}\left(\varphi_{xt}^{\vartheta}\xi\right)[[u,v](t) + \mathcal{F}v\,(t)]d\xi dt \\ &= [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0 \\ &+ \vartheta[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota\int_0^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\xi\Lambda_\vartheta(\xi)\mathcal{J}\left(\varphi_{\iota t}^{\vartheta}\xi\right)[[u,v](t) + \mathcal{F}v\,(t)]d\xi dt. \end{aligned}$$

For $\iota \in \mathbb{V}$, $0 < \xi < \iota$ and $\theta > 0$, define the operator $\mathbb{O}^{\xi\theta} : \Psi_s(\mathbb{V}) \to \Psi_s(\mathbb{V})$ by

$$\begin{aligned} \mathbb{O}^{\xi\theta}v\,(\iota) &= [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0 \\ &+ \vartheta[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^{\iota-\xi}\int_\theta^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\xi\Lambda_\vartheta(\xi)\mathcal{J}\left(\varphi_{\iota t}^{\vartheta}\xi\right)[[u,v](t) + \mathcal{F}v\,(t)]d\xi dt \\ &= [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\Big\{\mathbb{M}_{\vartheta,\eta;\varphi}(\iota)v_0 \\ &+ \vartheta\mathcal{J}\left(\theta\xi^{\vartheta}\right)\int_0^{\iota-\xi}\int_\theta^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\xi\Lambda_\vartheta(\xi)\mathcal{J}\left(\varphi_{\iota t}^{\vartheta}\xi - \theta\xi^{\vartheta}\right)[[u,v](t) + \mathcal{F}v\,(t)]d\xi dt\Big\}. \end{aligned}$$

Hence, for $\iota \in \mathbb{V}$, $0 < \xi < \iota$, $\theta > 0$ and $v \in \Psi_s(\mathbb{V})$, we have

$$
\begin{aligned}
& ||\mathbb{O}_2 v(\iota) - \mathbb{O}_2^{\xi\theta} v(\iota)|| \\
\leq\ & \left|\left| \vartheta[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \int_0^\iota \int_0^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1} \xi \Lambda_\vartheta(\xi) \mathcal{J}\left(\varphi_{\iota t}^\vartheta \xi\right)[[u,v](t) + \mathcal{F}v(t)] d\xi dt \right|\right| \\
+\ & \left|\left| \vartheta[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \mathcal{J}\left(\theta\xi^\vartheta\right) \int_0^{\iota-\xi} \int_\theta^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1} \xi \Lambda_\vartheta(\xi) \mathcal{J}\left(\varphi_{\iota t}^\vartheta \xi - \theta\xi^\vartheta\right)[[u,v](t) + \mathcal{F}v(t)] d\xi dt \right|\right| \\
\leq\ & \left|\left| \vartheta[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \int_0^\iota \int_0^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1} \xi \Lambda_\vartheta(\xi) \mathcal{J}\left(\varphi_{\iota t}^\vartheta \xi\right)[[u,v](t) + \mathcal{F}v(t)] d\xi dt \right|\right| \\
+\ & \left|\left| \vartheta[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \int_0^{\iota-\xi} \int_\theta^\infty \varphi'(t)\varphi_{\iota t}^{\vartheta-1} \xi \Lambda_\vartheta(\xi) \mathcal{J}\left(\varphi_{\iota t}^\vartheta \xi - \theta\xi^\vartheta\right)[[u,v](t) + \mathcal{F}v(t)] d\xi dt \right|\right| \\
\leq\ & \vartheta C[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \left[\int_0^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1} q(t) dt \int_0^\theta \xi^{-\nu} \Lambda_\vartheta(\xi) d\xi \right. \\
+\ & \left. \int_{\iota-\xi}^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1} q(t) dt \int_0^\infty \xi^{-\nu} \Lambda_\vartheta(\xi) d\xi \right]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \vartheta C[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \left[\int_0^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1} q(t) dt \int_0^\theta \xi^{-\nu} \Lambda_\vartheta(\xi) d\xi \right. \\
+\ & \left. \frac{\Gamma(1-\nu)}{\Gamma(1-\vartheta\nu)} \int_{\iota-\xi}^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1} q(t) dt \right] \\
\to\ & 0 \quad \text{when } \theta \to 0, \xi \to 0.
\end{aligned}
$$

Therefore, $\{\mathbb{O}v(\iota) : v \in \Psi_s(\mathbb{V})\}$ has the relative compactness in $\mathcal{B}$ for all $\iota \in \mathbb{V} \cup \{0\}$. By the Arzela–Ascoli theorem, $\{\mathbb{O}v : v \in \Psi_s(\mathbb{V})\}$ has the relative compactness in $\mathcal{B}$, and by Theorem (7), it is continuous. Then, by Schauder's fixed-point theorem, there is a fixed point $v \in \Psi_s(\mathbb{V})$ such that $\mathbb{O}v = v$, which is considered a mild solution of the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)). □

The following theorem shows the existence of the mild solution of the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)) when $\mathcal{J}(\iota)$ $(\iota > 0)$ has no compactness property.

Theorem 9. *Let* $\mathcal{G} \in \mathbb{H}_{\phi\nu}$ *, where* $-1 < \nu < 0$ *and* $0 < \phi < \frac{\pi}{2}$ *. If* $(C1)-(C4)$ *hold and there is* $\mu > 0$ *such that*

$$\mathcal{M}(h(\iota, B_1, B_2)) \leq \mu\ \ \mathcal{M}(B_1, B_2), \quad \text{for all } \iota \in \mathbb{V} \cup \{0\}$$

for all $B_1, B_2 \subset \mathcal{B}$ *, and then there exists a mild solution to the* $\varphi-$*Hilfer fractional Cauchy problem (Equation (6)) such that* $v_0 \in D\left(\mathcal{G}^\xi\right)$ *and* $\xi > 1+\nu$ *.*

Proof. Let B be any bounded set in $\Psi_s(\mathbb{V})$. Let

$$\mathbb{O}^{(1)}(B) = \mathbb{O}(B) \ \text{ and } \ \mathbb{O}^{(n)}(B) = \mathbb{O}\left[\overline{co}\left(\mathbb{O}^{(n-1)}(B)\right)\right] \quad n = 2,3,4,\cdots$$

where $\overline{co}\left(\mathbb{O}^{(n-1)}(B)\right)$ is the convex hull of $\mathbb{O}^{(n-1)}(B)$. By Theorem (8), we obtain that for all $\epsilon > 0$, there is a subsequence $v_n^{(1)}$ in B such that

$$
\begin{aligned}
& \mathcal{M}\left(\mathbb{O}^{(1)}(B(\iota))\right) = \mathcal{M}(\mathbb{O}(B(\iota))) \\
\leq\ & 2\ \mathcal{M}\left[[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \int_0^\iota \varphi'(t)\varphi_{\iota t}^{\vartheta-1} \mathbb{X}^{\phi\xi}(\varphi_{\iota t}) h\left[t, t^{-(1+\vartheta\nu)(1-\eta)}\left(v_n^{(1)}(t), \mathcal{F}v_n^{(1)}(t)\right)\right] dt\right] \\
\leq\ & 4C\ [\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \int_0^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1} \mathcal{M}\left[h\left[t, t^{-(1+\vartheta\nu)(1-\eta)}\left(v_n^{(1)}(t), \mathcal{F}v_n^{(1)}(t)\right)\right]\right] dt \\
\leq\ & 4C\mu\ \mathcal{M}(B)[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)} \int_0^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1} \varphi(t)^{-(1+\vartheta\nu)(1-\eta)} dt \\
=\ & 4C\mu\ \mathcal{M}(B)[\varphi(\iota)]^{-\vartheta\nu}\ \frac{\Gamma(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{\Gamma(-2\vartheta\nu+\eta(1+\vartheta\nu))}.
\end{aligned}
$$

Similarly, for all $\epsilon > 0$, there is a subsequence $v_n^{(2)}$ in B such that

$$\begin{aligned} &\mathcal{M}\left(\mathbb{O}^{(2)}(B(\iota))\right)=\mathcal{M}\left[\mathbb{O}\left[\overline{co}\left(\mathbb{O}^{(1)}(B)\right)\right]\right] \\ \leq\ & 2\,\mathcal{M}\left[[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota \varphi'(t)\varphi_{\iota t}^{\vartheta-1}\mathbb{X}^{\phi\xi}(\varphi_{\iota t})h\left[t,t^{-(1+\vartheta\nu)(1-\eta)}\left(v_n^{(2)}(t),\mathcal{F}v_n^{(2)}(t)\right)\right]dt\right] \\ \leq\ & 4C\,[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\int_0^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1}\mathcal{M}\left[h\left[t,t^{-(1+\vartheta\nu)(1-\eta)}\left(v_n^{(2)}(t),\mathcal{F}v_n^{(2)}(t)\right)\right]\right]dt \end{aligned}$$

$$\begin{aligned} &\leq (4C\mu)^2\,\mathcal{M}(B)[\varphi(\iota)]^{(1+\vartheta\nu)(1-\eta)}\frac{\Gamma(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{\Gamma(-2\vartheta\nu+\eta(1+\vartheta\nu))} \\ &\times \int_0^\iota \varphi'(t)\varphi_{\iota t}^{-\vartheta\nu-1}\varphi(t)^{-(1+\vartheta\nu)(1-\eta)-\vartheta\nu}dt \\ &= (4C\mu)^2\,\mathcal{M}(B)[\varphi(\iota)]^{-2\vartheta\nu}\frac{\Gamma^2(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{\Gamma(-3\vartheta\nu+\eta(1+\vartheta\nu))}. \end{aligned}$$

By mathematical indication, we obtain that

$$\mathcal{M}\left(\mathbb{O}^{(n)}(B(\iota))\right)\leq(4C\mu)^n\,\mathcal{M}(B)[\varphi(\iota)]^{-n\vartheta\nu}\,\frac{\Gamma^n(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{\Gamma(-(n+1)\vartheta\nu+\eta(1+\vartheta\nu))}\quad n=1,2,3,\cdots$$

So, we can choose $m,i\in\mathbb{N}$ to be large enough such that $1<m\vartheta\nu<\frac{m}{m-1}$ and $n+a>2m$ for all $n>i\Gamma(-(n+1)\vartheta\nu+\eta(1+\vartheta\nu))>\Gamma\left(\frac{m}{m+1}\right)$. That is,

$$\frac{\Gamma^n(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{\Gamma(-(n+1)\vartheta\nu+\eta(1+\vartheta\nu))}<\frac{\Gamma^n(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{i!}\to 0$$

when $i\to 0$. Hence, there is $n_0\in\mathbb{N}$ such that $\mathcal{M}\left(\mathbb{O}^{(n)}(B(\iota))\right)\leq\Gamma_0\mathcal{M}(B(\iota))$, where

$$\Gamma_0:=\frac{\Gamma^{n_0}(-\vartheta\nu)\Gamma(-\vartheta\nu+\eta(1+\vartheta\nu))}{\Gamma(-(n_0+1)\vartheta\nu+\eta(1+\vartheta\nu))}.$$

By Theorem (7) and Lemma (1), we have that $\mathbb{O}$ is continuous and bounded and $\{\mathbb{O}v : v\in\Psi_s(\mathbb{V})\}$ is equicontinuous. So, since $\mathcal{M}\left(\mathbb{O}^{(n_0)}(B(\iota))\right)$ is bounded and equicontinuous, from Theorem (1), $\mathcal{M}\left(\mathbb{O}^{(n_0)}(B)\right)=\max_{\iota\in\mathbb{V}\cup\{0\}}\mathcal{M}\left(\mathbb{O}^{(n_0)}(B(\iota))\right)$. That is, $\mathcal{M}\left(\mathbb{O}^{(n)}(B)\right)\leq\Gamma_0\mathcal{M}(B)$. Then, by Schauder's fixed-point theorem, there is a fixed point $v\in\Psi_s(\mathbb{V})$ such that $\mathbb{O}v=v$, which is a mild solution of the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)). □

4. Some Examples

Example 1. *It is clear that if the Banach algebra $\mathcal{B}$ in the $\varphi-$Hilfer fractional Cauchy problem (Equation (6)) is commutative, then $[u,v]=0$. The Banach space $\mathbb{R}$ is a commutative algebra with the usual multiplication and with the norm $||\iota||=|\iota|$. Consider the $\varphi-$Hilfer fractional Cauchy problem*

$$\begin{cases} \mathcal{H}_{0^+}^{\vartheta,\eta;\varphi}v(\iota)+v''(\iota)=\int_0^\iota e^{t-\iota}\cos v(t)dt & \iota\in\mathbb{V}, \\ I_{0^+}^{(1-\eta)(1-\vartheta);\varphi}v(0)=0. \end{cases} \tag{11}$$

In the problem (Equation (11)) above, we have the following:

1. *Take $\vartheta=\frac{3}{4}$, $\eta=\frac{1}{2}$, and $\varphi(\cdot)=\cdot$ on $(0,\infty)$.*
2. *Compared to the general problem (Equation (6)), $k(\iota,t):=e^{t-\iota}$ and $f(t,v(t)):=\cos v(t)$ are continuous functions.*
3. *Take the operator $\mathcal{J}$ as*

$$\mathcal{J}(v(\iota))=\begin{cases} \int_0^\pi \frac{v(t)e^{-\frac{t^2}{4\iota}}}{2\sqrt{\pi\iota}}dt & \iota\in\mathbb{V}=(0,1], \\ 0, & \iota=0. \end{cases}$$

4. *Note that $\mathcal{J}$ has a compactness property since $|\mathcal{J}(v(\iota))|\leq 1$.*

5. *Take the almost sectorial operator* $\mathcal{G} : D(\mathcal{G}) \to \mathbb{R}$ *as* $\mathcal{G} = \frac{d^2}{d\iota^2}$ *, where*

$$D(\mathcal{G}) = \left\{ v \in C(\mathbb{V}, \mathbb{R}) : \frac{d}{d\iota}v, \frac{d^2}{d\iota^2}v \in C(\mathbb{V}, \mathbb{R}) \text{ and } v(0) = 0 \right\}.$$

Note that operator $\mathcal{G}$ *is an infinitesimal generator of a differentiable semigroup* $\mathcal{J}$ *. So, we obtain that all of our conditions* $(C1) - (C4)$ *hold, and so our results introduce a mild solution of the* $\varphi-$*Hilfer fractional Cauchy problem (Equation (11)).*

Example 2. *The space* $C(\mathbb{E}, \mathbb{R})$ *is a commutative Banach algebra with the usual functional multiplication and the sup-norm* $||v|| = \sup_{(s,t)\in\mathbb{E}} |v(s,t)|$ *, where* $\mathbb{E} = (0,1] \times [0, \pi]$ *. Consider the* $\varphi-$*Hilfer fractional Cauchy problem*

$$\begin{cases} \mathcal{H}_{0^+}^{\vartheta,\eta;\varphi} v(s,t) + \frac{\partial^2}{\partial t^2} v(s,t) = \int_0^s e^{\iota - s} \cos v(\iota, t) d\iota & (s,t) \in \mathbb{E}, \\ v(s,0) = v(s,\pi) = 0, \\ I_{0^+}^{(1-\eta)(1-\vartheta);\varphi} v(0,0) = 0, \end{cases} \tag{12}$$

where $\vartheta = \frac{3}{4}$ *,* $\eta = \frac{1}{2}$ *, and* $\varphi(\cdot) = \cdot$ *on* $(0, \infty)$ *. Similarly, compared to the general problem (Equation (6)),* $k(\iota, s) = e^{\iota - s}$ *and* $f(t, v(\iota, t)) = \cos v(\iota, t)$ *are taken as continuous functions. For the almost sectorial operator* $\mathcal{G} : D(\mathcal{G}) \to \mathbb{R}$ *is given by* $\mathcal{G} = \frac{\partial^2}{\partial t^2}$ *, where*

$$D(\mathcal{G}) = \left\{ v \in C(\mathbb{E}, \mathbb{R}) : \frac{\partial}{\partial t}v, \frac{\partial^2}{\partial t^2}v \in C(\mathbb{E}, \mathbb{R}) \text{ and } v(0,0) = 0 \right\}.$$

The operator $\mathcal{G}$ *is an infinitesimal generator of a differentiable semigroup* $\mathcal{J}$ *, where*

$$\mathcal{J}(v)(s,t) = \begin{cases} \int_0^\pi \frac{v(\iota,t) e^{-\frac{(t-\iota)^2}{4s}}}{2\sqrt{\pi s}} dt & (s,t) \in \mathbb{E}, \\ 0, & (s,t) = (0,0). \end{cases}$$

Note that $|\mathcal{J}(v)(s,t)| \leq 1$ *, and so* $\mathcal{J}$ *has a compactness property. Conditions* $(C1) - (C4)$ *hold, and hence, our results give us the mild solution of the* $\varphi-$*Hilfer fractional Cauchy problem (Equation (11)).*

5. Conclusions

It is clear that the $\varphi-$Hilfer fractional problem in the theory of Banach algebras is the extension of the $\varphi-$Hilfer fractional problem in the theory of Banach spaces. In this work, we introduced some extensions in Banach algebra theory for the existence of some mild solutions of the $\varphi-$Hilfer fractional Cauchy problem that involves a Lie bracket operator and almost sectorial operators. This extension was fpr the cases of a non-compact associated semigroup and the measurement of a compact associated semigroup. We used the Arzela–Ascoli theorem to satisfy the desired conditions in Schauder's fixed-point theorem, which is used to examine the possibility of mild solutions for the $\varphi-$Hilfer fractional problem via almost sectorial operators. For future work, we suggest studying the $\varphi-$Hilfer fractional problem under the theory of Banach algebras in the class of weak topologies and the Hausdorff measure of weak non-compactness. In this case, the Leray–Schauder-type fixed-point theorem can be used in the examination of the possibility of mild solutions for the $\varphi-$Hilfer fractional problem via almost sectorial operators.

Author Contributions: Conceptualization, A.S.; Methodology, F.H.D. and A.K.; Validation, F.H.D., A.S. and A.K.; Formal analysis, F.H.D., A.S. and A.K.; Investigation, F.H.D., A.S. and A.K.; Writing—original draft, F.H.D. and A.S.; Writing—review & editing, A.K.; Visualization, F.H.D.; Supervision, A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors thank the reviewers for their constructive comments.

Conflicts of Interest: The authors declare no conflicts of interest.

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