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Fractional Solitons in Optical Twin-Core Couplers with Kerr Law Nonlinearity and Local M-Derivative Using Modified Extended Mapping Method

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Abstract: This study focuses on optical twin-core couplers, which facilitate light transmission between two closely aligned optical fibers. These couplers operate based on the principle of coupling, allowing signals in one core to interact with those in the other. The Kerr effect, which describes how a material's refractive index changes in response to the intensity of light, induces the nonlinear behavior essential for generating solitons—self-sustaining wave packets that preserve their shape and speed. In our research, we employ fractional derivatives to investigate how fractional-order variations influence wave propagation and soliton dynamics. By utilizing the modified extended mapping method (MEMM), we derive solitary wave solutions for the equations governing the behavior of optical twin-core couplers under Kerr nonlinearity. This methodology produces novel fractional traveling wave solutions, including dark, bright, singular, and combined bright–dark solitons, as well as hyperbolic, Jacobi elliptic function (JEF), periodic, and singular periodic solutions. To enhance understanding, we present physical interpretations through contour plots and include both 2D and 3D graphical representations of the results.

Keywords: fractional system; solitons; couplers; modified extended mapping method

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1. Introduction

Traveling wave solutions are among the most fascinating areas of research in various engineering and physical science disciplines. Numerous equations have garnered significant attention from scientists, including the Klein-Gordon-Zakharov equation [1], the generalized nonlinear Schrödinger equation [2], the Drinfel'd-Sokolov-Wilson equation [3,4], the Sawada-Kotera equation [5], the Sasa-Satsuma equation [6], the Biswas-Milovic equation [7,8], the Gerdjikov-Ivanov equation [9], the Korteweg–de Vries–Zakharov–Kuznetsov equation [10], the Kadomtsev-Petviashvili equation [11], and the Biswas-Arshed equation [12]. These fundamental models hold immense importance in the sciences, finding diverse applications in areas such as fluid mechanics, physics, fiber optics, chemistry, biology, and numerous other fields of engineering and science. The study of the propagation of solitons has gained significant attention in nonlinear optics and the telecommunications industry in recent years, for example, Duran investigated the physical dynamics of a traveling wave solution [13]. Malik et al. studied the cubic-quartic optical solitons in fiber bragg gratings with dispersive reflectivity having parabolic law of nonlinear refractive index by lie symmetry [14]. Younas et al. discussed the exact soliton solutions and different wave structures to the double dispersive equation [15]. Sun studied the propagation of solitons in optical fibers with generalized Kudryashov's refractive index [16]. Faridi et al. constructed the exact solution and explicit propagating optical soliton waves of Kuralay equation by the new extended direct algebraic and Nucci's reduction techniques [17]. Rizvi et al. discussed the propagation of chirped periodic and solitary waves for the coupled nonlinear Schrödinger equation in two core optical fibers with parabolic law with weak non-local nonlinearity [18]. Seadawy et al. established solitons in magneto-optic waveguides with triple-power law nonlinearity [19].

Different optical devices that demonstrate the dynamics of soliton molecules include metamaterials, couplers, and optical fibers, which have been extensively studied by various authors, such as Asad, et al. studied the sensitive demonstration of the twin-core couplers including Kerr law non-linearity via beta derivative evolution [20]. Zayed et al. investigated cubic-quartic optical solitons in couplers with optical metamaterials having Kudryashov's law of arbitrary refractive index and parabolic non-local law nonlinearity [21,22]. Abbagari et al. established optical soliton to multi-core (coupling with all the neighbors) directional couplers and modulation instability [23]. Rabie and Ahmed constructed cubic-quartic solitons in optical metamaterials for the perturbed twin-core couplers with Kudryashov's sextic power law using extended F-expansion method [24]. Antikainen and Agrawal, discussed the soliton supermode transitions and total red shift suppression in multi-core fibers [25]. Islam and Atai studied the soliton interactions in a grating-assisted coupler with cubic-quintic nonlinearity [26].

The optical metamaterial is a type of electromagnetic metamaterial that interacts with light or visible wavelengths. An optical metamaterial is composed of elements that are smaller than the wavelength of light, but that can interact with light in interesting ways. Within the fiber-optic community, a subject that has received a great deal of study and attention is highly dispersive solitons, for example, Yadav et al. established the highly dispersive W–shaped and other optical solitons with quadratic–cubic nonlinearity [27]. Durmus et al. obtained the bright soliton of the third-order nonlinear Schrödinger equation with power law of self-phase modulation in the absence of chromatic dispersion [28]. Butt et al. introduced advanced non-linear effects on highly dispersive optical solitons with multiplicative white noise [29]. El-shamy et al. established new solitons in optical medium with higher-order dispersive and nonlinear effects [30]. Islam et al. investigated general solitons for eighth-order dispersive nonlinear Schrödinger equation with ninth-power law nonlinearity [31].

One of the most challenging aspects of recent nonlinear dynamics is the study of fractional partial differential equations, which has acquired significant attention from researchers, for example, Fahad et al. studied the soliton dynamics in the nonlinear fractional Kudryashov's equation [32]. Almatrafi established solitary wave solutions to a fractional model [33]. Razzaq et al. constructed solitons for fractional nonlinear Schrödinger equation with β-time derivative [34]. Murad et al. established exact solutions to the time-fractional nonlinear Schrödinger equation [35]. Islam and Ahmed obtained optical solitons and other wave solutions to fractional-order nonlinear Sasa–Satsuma equation in mono-mode optical fibers [36].

In this work, we introduce the fractional system in optical metamaterials that describes twin-core couplers with Kerr law nonlinearity in the following form:

$$\begin{aligned} i\,D^{\alpha,\mu}_{M,t}\mathcal{U} + i\varrho_{11}\mathcal{U}_x + \varrho_{12}\mathcal{U}_{xx} + i\varrho_{13}\mathcal{U}_{xxx} + \varrho_{14}\mathcal{U}_{xxxx} + i\varrho_{15}\mathcal{U}_{xxxxx} + \varrho_{16}\mathcal{U}_{xxxxxx} + \left(b_1|\mathcal{U}|^2\right)\mathcal{U} - \alpha_1\left(|\mathcal{U}|^2\mathcal{U}\right)_{xx} \\ - \beta_1|\mathcal{U}|^2\mathcal{U}_{xx} - \gamma_1\mathcal{U}^2(\mathcal{U}^*)_{xx} - \delta_1\mathcal{V} - i\left(\lambda_1\left(|\mathcal{U}|^2\mathcal{U}\right)_x + r_1\left(|\mathcal{U}|^2\right)_x\mathcal{U} + \theta_1|\mathcal{U}|^2\mathcal{U}_x\right) = 0, \end{aligned} \tag{1}$$

and

$$\begin{aligned} i\,D^{\alpha,\mu}_{M,t}\mathcal{V} + i\varrho_{21}\mathcal{V}_x + \varrho_{22}\mathcal{V}_{xx} + i\varrho_{23}\mathcal{V}_{xxx} + \varrho_{24}\mathcal{V}_{xxxx} + i\varrho_{25}\mathcal{V}_{xxxxx} + \varrho_{26}\mathcal{V}_{xxxxxx} + \left(b_2|\mathcal{V}|^2\right)\mathcal{V} - \alpha_2\left(|\mathcal{V}|^2\mathcal{V}\right)_{xx} \\ -\beta_2|\mathcal{V}|^2\mathcal{V}_{xx} - \gamma_2\mathcal{V}^2(\mathcal{V}^*)_{xx} - \delta_2\mathcal{U} - i\left(\lambda_2\left(|\mathcal{V}|^2\mathcal{V}\right)_x + r_2(|\mathcal{V}|^2)_x\mathcal{V} + \theta_2|\mathcal{V}|^2\mathcal{V}_x\right) = 0, \end{aligned} \tag{2}$$

where

$$D_M^{\alpha,\mu}\mathcal{U}(x,t) = \lim_{\varepsilon\to 0}\frac{\mathcal{U}(t\ E_\mu(\varepsilon t^{-\alpha}),x) - \mathcal{U}(t,x)}{\varepsilon}, \qquad \forall\, t > 0, \tag{3}$$

is the local M-derivative of order $\alpha \in (0,1)$ and $E_\mu(Z),\ \mu > 0$ is the Mittag–Leffler function (for more details, see [37]).

Here, the functions $\mathcal{U} = \mathcal{U}(x,t)$ and $\mathcal{V} = \mathcal{V}(x,t)$ are complex-valued functions that describe the wave profiles in the respective cores of the optical fibers. The constants, ϱ_{jk} $(j = 1,2, k = 1,2,\ldots,6)$ represent the intermodal dispersion (D), chromatic D, third-order D, fourth-order D, fifth-order D, and sixth-order D, respectively. b_j yields Kerr law nonlinearity, while α_j , γ_j, and β_j give the optical metamaterial coefficients. δ_j comes from the coupling. λ_j arises from self-steepening terms, while θ_j and r_j stem from a self-frequency shift.

The proposed model was studied in [38] in classical derivative by using the unified Riccati equation expansion method and the enhanced Kudryashov's scheme. However, the proposed model with fractional local M-derivatives has not yet been considered in the literature.

In this work, the MEMM is employed to investigate the fractional traveling wave solutions for Equations (1) and (2). This method can derive different kinds of exact solutions such as (bright, dark, combined bright–dark, and singular) solitons, hyperbolic wave solutions, periodic solutions, and JEF solutions. These distinctive solutions demonstrate the method's efficacy and resilience, which are not documented elsewhere. Furthermore, the physical depiction of the found solutions is given through contour graphs and 2D and 3D graphics to highlight our findings further.

2. Description of the Suggested Approach

The following steps summarize the basic algorithm of MEMM (see [39,40]).

Consider the following nonlinear fractional partial differential equation:

$$\mathbb{F}\Big(\mathcal{Y}, i\,\mathcal{Y}_{M,t}^{\alpha,\mu}, \mathcal{Y}_x, \mathcal{Y}_{xx}, \mathcal{Y}_{xt}, \mathcal{Y}_{xxt}, \ldots\Big) = 0. \tag{4}$$

Step-(1): The following wave transformation will be used to convert (4) to be an ordinary differential equation (ODE):

$$\mathcal{Y}(x,t) = \mathbb{F}(\xi)\ e^{i\left(-\mathcal{K}\ x + \omega\ \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha} + \Delta\right)}, \qquad \xi = x - \eta\ \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}. \tag{5}$$

By plugging Equation (5) in Equation (4), we can rearrange to build up a nonlinear ODE as

$$\mathcal{R}(\mathbb{F}, \mathbb{F}', \mathbb{F}'', \mathbb{F}''' \ldots) = 0. \tag{6}$$

Step-(2): We can describe the solution to Equation (6) as follows by using the suggested method:

$$\mathbb{F}(\xi) = \sum_{j=0}^{N}\mathcal{A}_j\ \mathcal{Z}(\xi)^j + \sum_{j=-1}^{-N}\mathcal{B}_{-j}\ \mathcal{Z}(\xi)^j + \sum_{j=2}^{N}\mathcal{C}_j\ \mathcal{Z}(\xi)^{j-2}\ \mathcal{Z}'(\xi) + \sum_{j=1}^{N}R_j\ \mathcal{Z}(\xi)^{-j}\ \mathcal{Z}'(\xi), \tag{7}$$

where $\mathcal{A}_j, \mathcal{B}_{-j}, \mathcal{C}_j$ and R_j are real parameters and $\mathcal{Z}(\xi)$ must satisfy the following constraint:

$$\mathcal{Z}'(\xi) = \sqrt{q_0 + q_1\ \mathcal{Z}(\xi) + q_2\ \mathcal{Z}(\xi)^2 + q_3\ \mathcal{Z}(\xi)^3 + q_4\ \mathcal{Z}(\xi)^4 + q_6\ \mathcal{Z}(\xi)^6}. \tag{8}$$

Step-(3): Using the higher-order derivative to equate the largest nonlinear component in Equation (6), one may figure out the value of N in conformity with the balancing principle.

Equations (7) and (8) are inserted in Equation (6); hence, a polynomial in $\mathcal{Z}$ is formed.

Software such as Mathematica (11.3) packages may be used to solve a sequence of nonlinear algebraic equations (NLAEs) that result from equalizing the coefficients of the same powers to zero. After that, Equation (4) can have several precise solutions generated for it.

3. Exploring Fractional Wave Solutions

To explore exact solutions of Equations (1) and (2), we assume

$$\mathcal{U}(x,t) = \mathbb{F}_1(\xi)\ e^{i\left(-\mathcal{K}\ x+\omega\ \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}+\Delta\right)}, \tag{9}$$

and

$$\mathcal{V}(x,t) = \mathbb{F}_2(\xi)\ e^{i\left(-\mathcal{K}\ x+\omega\ \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}+\Delta\right)}, \tag{10}$$

where

$$\xi = x - \eta\ \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}, \tag{11}$$

where $\eta, \mathcal{K},\ \omega$, and Δ denote soliton velocity, soliton wave number, soliton frequency, and phase constant, respectively, and $\mathbb{F}_j\ (j=1,2)$ are amplitude components of the solitons.

Then, Equations (1) and (2) will be converted into ODEs using the transformation given by Equations (9)–(11). Then, the resultant NLODEs can be divided into real and imaginary components, as follows:

$$\begin{aligned}&\varrho_{j6}\mathbb{F}_j^{(6)} + (\varrho_{j4} + 5\varrho_{j5}\mathcal{K} - 15\varrho_{j6}\mathcal{K}^2)\mathbb{F}_j^{(4)} - (\beta_j + \gamma_j + 3\alpha_j)\mathbb{F}_j^2\mathbb{F}_j'' - 6\alpha_j\mathbb{F}_j(\mathbb{F}_j')^2 + (\varrho_{j2} + 3\varrho_{j3}\mathcal{K}\\&- 6\varrho_{j4}\mathcal{K}^2 - 10\varrho_{j5}\mathcal{K}^3 + 15\varrho_{j6}\mathcal{K}^4)\mathbb{F}_j'' + (-\omega + \varrho_{j1}\mathcal{K} - \varrho_{j2}\mathcal{K}^2 - \varrho_{j3}\mathcal{K}^3 + \varrho_{j4}\mathcal{K}^4 + \varrho_{j5}\mathcal{K}^5 - \varrho_{j6}\mathcal{K}^6)\mathbb{F}_j\\&- \delta_j\mathbb{F}_{\tilde{j}} + \Big(b_j - \mathcal{K}(\lambda_j + \theta_j) + \mathcal{K}^2(\beta_j + \alpha_j + \gamma_j)\Big)\mathbb{F}_j^3 = 0,\end{aligned} \tag{12}$$

$$\begin{aligned}&(\varrho_{j5} - 6\varrho_{j6}\mathcal{K})\mathbb{F}_j^{(5)} + (\varrho_{j3} - 4\varrho_{j4}\mathcal{K} - 10\varrho_{j5}\mathcal{K}^2 + 20\varrho_{j6}\mathcal{K}^3)\mathbb{F}_j''' - \big(2\mathcal{K}(\beta_j - \gamma_j + 3\alpha_j) - (3\lambda_j + 2\,r_j + \theta_j)\big)\\&\mathbb{F}_j^2\mathbb{F}_j' + (-\eta + \varrho_{j1} - 2\varrho_{j2}\mathcal{K} - 3\varrho_{j3}\mathcal{K}^2 + 4\varrho_{j4}\mathcal{K}^3 + 5\varrho_{j5}\mathcal{K}^4 - 6\varrho_{j6}\mathcal{K}^5)\mathbb{F}_j' = 0,\end{aligned} \tag{13}$$

where $j = 1,2$ and $\tilde{j} = 1 - j = 1,2$. Now, if we can integrate Equation (13) once while substituting into Equation (11) and setting the integration constant to zero, we obtain

$$\varrho_{j6}\ \mathbb{F}_j^{(6)} - (3\alpha_j + \beta_j + \gamma_j)\mathbb{F}_j^2\ \mathbb{F}_j'' + 6\alpha_j\ \mathbb{F}_j\left(\mathbb{F}_j'\right)^2 + \mathfrak{R}_1\ \mathbb{F}_j'' + \mathfrak{R}_2\ \mathbb{F}_j^3 + \mathfrak{R}_3\ \mathbb{F}_j - \delta_j\ \mathbb{F}_{\tilde{j}} = 0, \tag{14}$$

where

$$\mathfrak{R}_1 = \mathcal{K}(\mathcal{K}(5\mathcal{K}(3\mathcal{K}\varrho_{j6} - 2\varrho_{j5}) - 6\varrho_{j4}) + 3\varrho_{j3}) + \varrho_{j2} + \frac{(5\mathcal{K}(\varrho_{j5} - 3\mathcal{K}\varrho_{j6}) + \varrho_{j4})(2\mathcal{K}(5\mathcal{K}(\varrho_{j5} - 2\mathcal{K}\varrho_{j6}) + 2\varrho_{j4}) - \varrho_{j3})}{\varrho_{j5} - 6\mathcal{K}\varrho_{j6}},$$

$$\mathfrak{R}_2 = b_j + \mathcal{K}(-\theta_j - \lambda_j + \mathcal{K}(\alpha_j + \beta_j + \gamma_j)) + \frac{(5\mathcal{K}(\varrho_{j5} - 3\mathcal{K}\varrho_{j6}) + \varrho_{j4})(-\theta_j - 3\lambda_j + 2\mathcal{K}(3\alpha_j + \beta_j - \gamma_j) - 2r_j)}{3(\varrho_{j5} - 6\mathcal{K}\varrho_{j6})}$$

$$\mathfrak{R}_3 = \frac{(5\mathcal{K}(\varrho_{j5} - 3\mathcal{K}\varrho_{j6}) + \varrho_{j4})(\eta + 6\mathcal{K}^5\varrho_{j6} - 5\mathcal{K}^4\varrho_{j5} - 4\mathcal{K}^3\varrho_{j4} + 3\mathcal{K}^2\varrho_{j3} + 2\mathcal{K}\varrho_{j2} - \varrho_{j1})}{\varrho_{j5} - 6\mathcal{K}\varrho_{j6}} - (\mathcal{K}^2(\mathcal{K}(\mathcal{K}^3\varrho_{j6} - \mathcal{K}(\mathcal{K}\varrho_{j5} + \varrho_{j4}) + \varrho_{j3}) + \varrho_{j2})) + \mathcal{K}\varrho_{j1} - \omega.$$

Then, we impose the constraint

$$\mathbb{F}_{\tilde{j}} = \hbar\ \mathbb{F}_j, \qquad \hbar \neq 0,1. \tag{15}$$

Therefore, Equation (14) can be represented as follows:

$$\varrho_{j6}\ \mathbb{F}_j^{(6)} - (3\alpha_j + \beta_j + \gamma_j)\mathbb{F}_j^2\ \mathbb{F}_j'' + 6\alpha_j\ \mathbb{F}_j\left(\mathbb{F}_j'\right)^2 + \mathfrak{R}_1\ \mathbb{F}_j'' + \mathfrak{R}_2\ \mathbb{F}_j^3 + (\mathfrak{R}_3 - \hbar\ \delta_j)\ \mathbb{F}_j = 0. \tag{16}$$

By applying the method proposed in Section 2, we can write the general solution of Equation (16) as follows:

$$\mathbb{F}_j = \mathcal{A}_0 + \mathcal{A}_1\, \mathcal{Z}(\xi) + \mathcal{A}_2\, \mathcal{Z}(\xi)^2 + \mathcal{A}_3\, \mathcal{Z}(\xi)^3 + \frac{\mathcal{B}_1}{\mathcal{Z}(\xi)} + \frac{\mathcal{B}_2}{\mathcal{Z}(\xi)^2} + \frac{\mathcal{B}_3}{\mathcal{Z}(\xi)^3} + \mathcal{C}_2\, \mathcal{Z}'(\xi) + \mathcal{C}_3\, \mathcal{Z}(\xi)\, \mathcal{Z}'(\xi) + \frac{R_1 \mathcal{Z}'(\xi)}{\mathcal{Z}(\xi)} + \frac{R_2 \mathcal{Z}'(\xi)}{\mathcal{Z}(\xi)^2} + \frac{R_3 \mathcal{Z}'(\xi)}{\mathcal{Z}(\xi)^3}, \tag{17}$$

where $\mathcal{A}_i$, $\mathcal{B}_j$, $\mathcal{C}_r$, and $R_j(i = 0,1,2,3,\ j = 1,2,3,\ r = 2,3)$ are constants that will be calculated under the restrictions that $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{C}_3$ or $R_3 \neq 0$. In order to generate system NLAEs, we substitute Equations (8) and (12) into Equation (17). We might be able to create a system of NLAEs by setting all of the terms of the same powers to zero. Using the Mathematica program, the foregoing system is solved under constraint $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{C}_2 = R_2 = R_1 = 0$ to obtain the following useful results:
Case-(1): If $q_0 = q_1 = q_3 = q_6 = 0$, the next set of solutions are obtained as follows:

$$\mathcal{A}_3 = R_3 = 0,\ \mathcal{C}_3 = \pm\frac{2q_4\sqrt{15\mathfrak{R}_1}}{q_2\sqrt{\mathfrak{R}_2}},\ \beta_j = -\gamma_j,\ \delta_j = \frac{80q_2\,\mathfrak{R}_1 + 21\,\mathfrak{R}_3}{21\,\hbar},\ \alpha_j = 0,\ \varrho_{j6} = -\frac{\mathfrak{R}_1}{336q_2^2}.$$

The analytical solutions of Equations (1) and (2) can be obtained from the obtained set of solutions in the following manner:

(1.1) If $q_2 > 0$, $q_4 < 0$ and $\mathfrak{R}_1\,\mathfrak{R}_2 > 0$, then, dark solitons can be obtained as

$$\mathcal{U}_{1.1} = \pm 2\sqrt{\frac{15\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\left[1 - \tanh^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{q_2}\right)\right]\tanh\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{18}$$

$$\mathcal{V}_{1.1} = \pm 2\,\hbar\,\sqrt{\frac{15\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\left[1 - \tanh^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{q_2}\right)\right]\tanh\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ t^\alpha}{\alpha}\ \omega + \Delta\right)}. \tag{19}$$

(1.2) If $q_2 < 0$, $q_4 > 0$ and $\mathfrak{R}_1\,\mathfrak{R}_2 > 0$, then, we can obtain singular periodic solutions represented as follows:

$$\mathcal{U}_{1.2} = \pm 2\sqrt{-\frac{15\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\left[1 - \tan^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{-q_2}\right)\right]\tan\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{20}$$

$$\mathcal{V}_{1.2} = \pm 2\,\hbar\,\sqrt{-\frac{15\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\left[1 - \tan^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{-q_2}\right)\right]\tan\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^\alpha}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{21}$$

$$\mathcal{U}_{1.3} = \pm 2\sqrt{-\frac{15\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\left[1-\cot^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right)\right]\cot\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{22}$$

$$\mathcal{V}_{1.3} = \pm 2\,\hbar\sqrt{-\frac{15\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\left[1-\cot^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right)\right]\cot\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}. \tag{23}$$

Case-(2): If $q_1 = q_3 = q_6 = 0$ and $q_0 = \frac{q_2^2}{4\, q_4}$, the next set of solutions are obtained:

(2.1) $\mathcal{A}_3 = \mathcal{C}_3 = 0,\ R_3 = \pm\frac{q_2\,\sqrt{15\,\mathfrak{R}_1}}{q_4\,\sqrt{\mathfrak{R}_2}},\ \beta_j = -\gamma_j,\ \delta_j = \frac{21\,\mathfrak{R}_3 - 40\, q_2\,\mathfrak{R}_1}{21\,\hbar},\ \alpha_j = 0,\ \varrho_{j6} = -\frac{\mathfrak{R}_1}{84 q_2^2}$.

(2.2) $\mathcal{A}_3 = R_3 = 0,\ \mathcal{C}_3 = \pm\frac{4\, q_4\,\sqrt{15\,\mathfrak{R}_1}}{q_2\sqrt{\mathfrak{R}_2}},\ \beta_j = -\gamma_j,\ \delta_j = \frac{21\,\mathfrak{R}_3 - 40\, q_2\,\mathfrak{R}_1}{21\,\hbar},\ \alpha_j = 0,\ \varrho_{j6} = -\frac{\mathfrak{R}_1}{84 q_2^2}$.

(2.3) $\mathcal{A}_3 = 0,\ \mathcal{C}_3 = \pm\frac{q_4\,\sqrt{15\,\mathfrak{R}_1}}{q_2\sqrt{\mathfrak{R}_2}},\ R_3 = \mp\frac{q_2\,\sqrt{15\,\mathfrak{R}_1}}{4\, q_4\,\sqrt{\mathfrak{R}_2}},\ \beta_j = -\gamma_j,\ \delta_j = \frac{21\,\mathfrak{R}_3 - 160\, q_2\,\mathfrak{R}_1}{21\,\hbar},\ \alpha_j = 0,$ $\varrho_{j6} = -\frac{\mathfrak{R}_1}{1344\, q_2^2}$.

The analytical solutions of Equations (1) and (2) can be obtained from (2.1) in the following way:

(2.1, 1) If $q_2 < 0$, $q_4 > 0$ and $\mathfrak{R}_1\,\mathfrak{R}_2 > 0$, then, singular solitons can obtained as

$$\mathcal{U}_{2.1,1} = \pm\sqrt{-\frac{30\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\,\text{csch}^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right)\ \coth\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{24}$$

$$\mathcal{V}_{2.1,1} = \pm\,\hbar\sqrt{-\frac{30\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\,\text{csch}^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right)\ \coth\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{25}$$

(2.1, 2) Singular periodic solutions:

$$\mathcal{U}_{2.1,2} = \pm\sqrt{\frac{30\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\ \csc^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{\frac{q_2}{2}}\right)\ \cot\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{\frac{q_2}{2}}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{26}$$

$$\mathcal{V}_{2.1,2} = \pm\,\hbar\,\sqrt{\frac{30\,q_2\,\Re_1}{\Re_2}}\,\csc^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{\frac{q_2}{2}}\right)\,\cot\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{\frac{q_2}{2}}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{27}$$

when $q_2 > 0$, $q_4 > 0$ and $\Re_1\,\Re_2 > 0$.

From (2.2), we can obtain the analytical solutions of Equations (1) and (2) as follows:

(2.2, 1) Dark solitons:

$$\mathcal{U}_{2.2,1} = \mp 2\sqrt{-\frac{15\,q_2\,\Re_1}{2\,\Re_2}}\left[1-\tanh^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right)\right] \times \tanh\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right)\,e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{28}$$

$$\mathcal{V}_{2.2,1} = \mp 2\,\hbar\,\sqrt{-\frac{15\,q_2\,\Re_1}{2\,\Re_2}}\left[1-\tanh^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right)\right] \times \tanh\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-\frac{q_2}{2}}\right)\,e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{29}$$

when $q_2 < 0$, $q_4 > 0$ and $\Re_1\,\Re_2 > 0$.

(2.2, 2) Singular periodic solutions:

$$\mathcal{U}_{2.2,2} = \pm 8\sqrt{\frac{30\,q_2\,\Re_1}{\Re_2}}\,\csc^3\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{2\,q_2}\right)\,\sin\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{\frac{q_2}{2}}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{30}$$

$$\mathcal{V}_{2.2,2} = \pm 8\,\hbar\,\sqrt{\frac{30\,q_2\,\Re_1}{\Re_2}}\,\csc^3\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{2\,q_2}\right)\,\sin\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{\frac{q_2}{2}}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{31}$$

where $q_2 > 0$, $q_4 > 0$ and $\Re_1\,\Re_2 > 0$.

From (2.3), we can obtain the analytical solutions of Equations (1) and (2) as follows:

(2.3, 1) If $q_2 < 0$, $q_4 > 0$ and $\Re_1\,\Re_2 > 0$, then, we can obtain hyperbolic wave solutions represented as follows:

$$\mathcal{U}_{2.3,1} = \pm\frac{1}{4}\sqrt{-\frac{15\,q_2\,\Re_1}{2\,\Re_2}}\,\mathrm{csch}^4\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-2\,q_2}\right) \times \sinh\left(2\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-2\,q_2}\right)e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{32}$$

$$\mathcal{V}_{2.3,1} = \pm\frac{\hbar}{4}\sqrt{-\frac{15\,q_2\,\mathfrak{R}_1}{2\,\mathfrak{R}_2}}\;\operatorname{csch}^4\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-2\,q_2}\right)$$
$$\times\sinh\left(2\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{-2\,q_2}\right)e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\,t^{\alpha}}{\alpha}\,\omega+\Delta\right)}. \quad (33)$$

(2.2, 2) Singular periodic solutions:

$$\mathcal{U}_{2.3,2} = \mp 2\sqrt{-\frac{30\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\;\csc^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{2\,q_2}\right)$$
$$\times\cot\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{2\,q_2}\right)e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\,t^{\alpha}}{\alpha}\,\omega+\Delta\right)}, \quad (34)$$

$$\mathcal{V}_{2.3,2} = \mp 2\hbar\sqrt{-\frac{30\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\;\csc^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{2\,q_2}\right)$$
$$\times\cot\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{2\,q_2}\right)e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\,t^{\alpha}}{\alpha}\,\omega+\Delta\right)}, \quad (35)$$

when $q_2 > 0$, $q_4 > 0$ and $\mathfrak{R}_1\,\mathfrak{R}_2 > 0$.

Case-(3): If $q_3 = q_4 = q_6 = 0$, the next set of solutions are obtained:

$$\mathcal{A}_3 = \mathcal{C}_3 = 0,\ R_3 = \pm\frac{2\,q_0\sqrt{15\,\mathfrak{R}_1}}{q_2\,\sqrt{\mathfrak{R}_2}},\ \gamma_j = -\beta_j,\ \delta_j = \frac{80\,q_2\,\mathfrak{R}_1 + 21\,\mathfrak{R}_3}{21\,\hbar},\ \alpha_j = 0,\ \varrho_{j6} = -\frac{\mathfrak{R}_1}{336\,q_2^2},\ q_1 = 0.$$

(3.1) Singular solitons:

$$\mathcal{U}_{3.1} = \pm 2\sqrt{\frac{15\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\;\operatorname{csch}^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right)\coth\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right)$$
$$\times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\,t^{\alpha}}{\alpha}\,\omega+\Delta\right)}, \quad (36)$$

$$\mathcal{V}_{3.1} = \pm 2\,\hbar\sqrt{\frac{15\,q_2\,\mathfrak{R}_1}{\mathfrak{R}_2}}\;\operatorname{csch}^2\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right)\coth\left(\left[x-\frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right)$$
$$\times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\,t^{\alpha}}{\alpha}\,\omega+\Delta\right)}, \quad (37)$$

when $q_0 > 0$, $q_2 > 0$ and $\mathfrak{R}_1\,\mathfrak{R}_2 > 0$.

(3.2) Singular periodic solutions:

$$\mathcal{U}_{3.2} = \mp 2\sqrt{-\frac{15\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\, \csc^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right)\cot\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{38}$$

$$\mathcal{V}_{3.2} = \mp 2\,\hbar\sqrt{-\frac{15\, q_2\, \mathfrak{R}_1}{\mathfrak{R}_2}}\, \csc^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right)\cot\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{39}$$

when $q_0 > 0$, $q_2 < 0$ and $\mathfrak{R}_1\, \mathfrak{R}_2 > 0$.

Case-(4): If $q_0 = q_1 = q_6 = 0$, the next set of solutions are obtained:

$$\mathcal{A}_3 = R_3 = \alpha_j = q_3 = 0,\ \mathcal{C}_3 = \pm\frac{24\, q_4\, \sqrt{-35\, \varrho_{j6}}}{\sqrt{\mathfrak{R}_2}},\ \beta_j = -\gamma_j,\ \delta_j = \frac{\mathfrak{R}_3 - 1280\, q_2^3\, \varrho_{j6}}{\hbar},\ \mathfrak{R}_1 = -336\, q_2^2 \varrho_{j6}.$$

(4.1) Singular solitons:

$$\mathcal{U}_{4.1} = \mp 24\, q_2\sqrt{\frac{-35\, q_2\, \varrho_{j6}}{\mathfrak{R}_2}}\, \operatorname{csch}^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right)\coth\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{40}$$

$$\mathcal{V}_{4.1} = \mp 24\, q_2\, \hbar\sqrt{\frac{-35\, q_2\, \varrho_{j6}}{\mathfrak{R}_2}}\, \operatorname{csch}^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right)\coth\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{41}$$

when $q_2 > 0$, $q_4 > 0$, $q_3^2 \neq 4\, q_2\, q_4$ and $\mathfrak{R}_2\, \varrho_{j6} < 0$.

(4.2) Singular periodic solutions:

$$\mathcal{U}_{4.2} = \pm 24\, q_2\sqrt{\frac{35\, q_2\, \varrho_{j6}}{\mathfrak{R}_2}}\, \csc^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right)\cot\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{42}$$

$$\mathcal{V}_{4.2} = \pm 24\, q_2\sqrt{\frac{35\, q_2\, \varrho_{j6}}{\mathfrak{R}_2}}\, \csc^2\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right)\cot\left(\left[x - \frac{\Gamma(\mu+1)\,\eta\, t^{\alpha}}{\alpha}\right]\sqrt{-q_2}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{43}$$

when $q_2 < 0$, $q_4 > 0$, $q_3^2 \neq 4\, q_2\, q_4$ and $\mathfrak{R}_2\, \varrho_{j6} < 0$.

Case-(5): If $q_1 = q_3 = q_6 = 0$, the next set of solutions are obtained:

(5.1) $\mathcal{A}_3 = \mathcal{C}_3 = 0, R_3 = \pm\frac{8q_2^2\sqrt{-35\varrho_{j6}}}{q_4\sqrt{\mathfrak{R}_2}}, \gamma_j = \frac{3\alpha_j}{2} - \beta_j, \delta_j = \frac{640\, q_2^3\varrho_{j6}+\mathfrak{R}_3}{\hbar}, q_0 = \frac{q_2^2}{3q_4},$
$\mathfrak{R}_1 = \frac{3360q_2^3\alpha_j\varrho_{j6}}{\mathfrak{R}_2}.$

(5.2) $\mathcal{A}_3 = R_3 = 0, \mathcal{C}_3 = \pm\frac{24q_4\sqrt{-35\varrho_{j6}}}{\sqrt{\mathfrak{R}_2}}$, and

(5.2, 1) $\gamma_j = \frac{3\alpha_j}{2} - \beta_j, \delta_j = \frac{640q_2^3\varrho_{j6}+\mathfrak{R}_3}{\hbar}, q_0 = \frac{q_2^2}{3q_4}, \mathfrak{R}_1 = \frac{3360q_2^3\alpha_j\varrho_{j6}}{\mathfrak{R}_2}.$

(5.2, 2) $\gamma_j = -\beta_j, \delta_j = \frac{640q_2\left(9q_0q_4-2q_2^2\right)\varrho_{j6}+\mathfrak{R}_3}{\hbar}, \alpha_j = 0, \mathfrak{R}_1 = -336\left(q_2^2 - 3q_0q_4\right)\varrho_{j6}.$

(5.2, 3) $\gamma_j = -\beta_j, \alpha_j = 0, \delta_j = \frac{\mathfrak{R}_3-1280q_2^3\varrho_{j6}}{\hbar}, q_0 = 0; \mathfrak{R}_1 = -336q_2^2\varrho_{j6}.$

(5.3) $\mathcal{A}_3 = \pm 3q_4\sqrt{-\frac{4730q_4\varrho_{j6}}{\mathfrak{R}_2}}, \mathcal{C}_3 = R_3 = 0,\ \gamma_j = -\beta_j - \frac{83\mathfrak{R}_2}{1419q_2}, \alpha_j = -\frac{166\mathfrak{R}_2}{4257q_2},$
$\delta_j = \frac{\mathfrak{R}_3-5610q_2^3\varrho_{j6}}{\hbar}, q_0 = -\frac{2q_2^2}{3q_4}, \mathfrak{R}_1 = 77q_2^2\varrho_{j6}.$

From the set (5.1) of solutions, we obtain the analytical solutions of Equations (1) and (2) as follows:

(5.1,1) If $q_0 = 1,\ q_2 = -m^2 - 1, q_4 = m^2$, $\mathfrak{R}_2\, \varrho_{j6} < 0$ and $0 < m \leq 1$, then, JEF solutions can be obtained as

$$\mathcal{U}_{5.1,1} = \pm\frac{8\left(m^2+1\right)^2}{m^2}\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\left(\frac{\mathrm{cn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]\mathrm{dn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}{\mathrm{sn}^3\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{44}$$

$$\mathcal{V}_{5.1,1} = \pm\frac{8\left(m^2+1\right)^2\hbar}{m^2}\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\left(\frac{\mathrm{cn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]\mathrm{dn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}{\mathrm{sn}^3\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}. \tag{45}$$

Through these equations, when $m = 1$, we can obtain singular solitons in the following forms:

$$\mathcal{U}_{5.1,1.1} = \pm 32\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\ \mathrm{csch}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\coth\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right] \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{46}$$

$$\mathcal{V}_{5.1,1.1} = \pm 32\hbar\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\ \mathrm{csch}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\coth\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right] \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}. \tag{47}$$

(5.1,2) If $q_0 = m^2 - 1,\ q_2 = 2 - m^2, q_4 = -1, \Re_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we can obtain JEF solutions represented as follows:

$$\mathcal{U}_{5.1,2} = \pm 8\ m\left(2 - m^2\right)^2 \sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}} \left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right] \text{sn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}{\text{dn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{48}$$

$$\mathcal{V}_{5.1,2} = \pm 8\ m\ \hbar\left(2 - m^2\right)^2 \sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}} \left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right] \text{sn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}{\text{dn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}. \tag{49}$$

When $m = 1$, we obtain hyperbolic wave solutions:

$$\mathcal{U}_{5.1,2.1} = \pm 4\sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}}\ \sinh\left[2\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{50}$$

$$\mathcal{V}_{5.1,2.1} = \pm 4\hbar\sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}}\ \sinh\left[2\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}. \tag{51}$$

(5.1,3) If $q_0 = -m^2,\ q_2 = -1 + 2m^2, q_4 = 1 - m^2, \Re_2\ \varrho_{j6} < 0$ and $0 \leq m < 1$, then, we can obtain JEF solutions:

$$\mathcal{U}_{5.1,3} = \mp \frac{8m\left(1 - 2m^2\right)^2}{(m^2 - 1)} \sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}} \left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right] \text{sn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}{\text{nd}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{52}$$

$$\mathcal{V}_{5.1,3} = \pm \frac{8m\ \hbar\left(1 - 2m^2\right)^2}{(m^2 - 1)} \sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}} \left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right] \text{sn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}{\text{nd}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}. \tag{53}$$

When $m = 0$, we obtain periodic waves solutions:

$$\mathcal{U}_{5.1,3.1} = \pm 4\sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}}\ \sin\left[2\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}, \tag{54}$$

$$\mathcal{V}_{5.1,3.1} = \pm 4\hbar\sqrt{\frac{-35\ \varrho_{j6}}{\Re_2}}\ \sin\left[2\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^\alpha}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^\alpha}{\alpha}\ \omega + \Delta\right)}. \tag{55}$$

(5.1,4) If $q_0 = \frac{1}{4},\ q_2 = \frac{m^2-2}{2}, q_4 = \frac{m^4}{4}$, $\mathfrak{R}_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we can obtain JEF solutions:

$$\mathcal{U}_{5.1,4} = \pm\frac{8\left(m^2-2\right)^2}{m^4}\sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\left(1+\text{dn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\right)^2}{\text{sn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{56}$$

$$\mathcal{V}_{5.1,4} = \pm\frac{8\ \hbar\left(m^2-2\right)^2}{m^4}\sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\left(1+\text{dn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\right)^2}{\text{sn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}. \tag{57}$$

When $m = 1$, we obtain hyperbolic wave solutions:

$$\mathcal{U}_{5.1,4.1} = \pm 2\ \sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \cosh^4\left[\frac{1}{2}\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\right]\ \sinh\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right] \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{58}$$

$$\mathcal{V}_{5.1,4.1} = \pm 4\ \hbar\ \sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \cosh^4\left[\frac{1}{2}\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\right]\ \sinh\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right] \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}. \tag{59}$$

From (5.2), we obtain the analytical solutions of Equations (1) and (2) as follows:

(5.2,1) If $q_0 = 1,\ q_2 = -m^2 - 1, q_4 = m^2$, $\mathfrak{R}_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we obtain JEF solutions:

$$\mathcal{U}_{5.2,1} = \pm 24m^2\sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \left(\frac{\text{cn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]\text{dn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}{\text{ns}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{60}$$

$$\mathcal{V}_{5.2,1} = \pm 24m^2\ \hbar\sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \left(\frac{\text{cn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]\text{dn}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}{\text{ns}\left[\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\sqrt{q_2}\right]}\right) \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}. \tag{61}$$

When $m = 1$, we obtain combined bright–dark solitons:

$$\mathcal{U}_{5.2,1.1} = \pm 24\sqrt{\frac{-35\ \varrho_{j6}}{\mathfrak{R}_2}}\ \text{sech}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\ \tanh\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right] \times e^{i\left(-\mathcal{K}x+\frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega+\Delta\right)}, \tag{62}$$

$$\mathcal{V}_{5.2,1.1} = \pm 24\hbar\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\ \text{sech}^2\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\tanh\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right] \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{63}$$

(5.2,2) If $q_0 = m^2 - 1,\ q_2 = 2 - m^2, q_4 = -1$, $\mathfrak{R}_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we obtain JEF solutions:

$$\mathcal{U}_{5.2,2} = \pm 24\ m\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\text{dn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\text{ns}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{64}$$

$$\mathcal{V}_{5.2,2} = \pm 24\ m\ \hbar\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\left(\frac{\text{cn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\text{dn}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\text{ns}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{65}$$

(5.2,3) If $q_0 = -m^2,\ q_2 = -1 + 2m^2, q_4 = 1 - m^2$, $\mathfrak{R}_2\ \varrho_{j6} < 0$ and $0 \leq m < 1$, then, we obtain JEF solutions:

$$\mathcal{U}_{5.2,3} = \mp 24\left(-1 + m^2\right)\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\left(\frac{\text{dc}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\text{nc}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\text{cs}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{66}$$

$$\mathcal{V}_{5.2,3} = \mp 24\hbar\left(-1 + m^2\right)\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\left(\frac{\text{dc}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\text{nc}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\text{cs}\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{67}$$

When $m = 0$, we obtain singular periodic solutions:

$$\mathcal{U}_{5.2,3.1} = \pm 24\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\ \sec^2\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\ \tan\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\ e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{68}$$

$$\mathcal{V}_{5.2,3.1} = \pm 24\hbar\sqrt{\frac{-35\,\varrho_{j6}}{\mathfrak{R}_2}}\ \sec^2\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\ \tan\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right]\ e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{69}$$

From (5.3), we obtain the analytical solutions of Equations (1) and (2) as follows:

(5.3,1) If $q_0 = 1,\ q_2 = -m^2 - 1, q_4 = m^2$, $\mathfrak{R}_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we can obtain JEF solutions represented as follows:

$$\mathcal{U}_{5.3,1} = \pm 3\, m^3 \sqrt{\frac{-4730\; q_4\; \varrho_{j6}}{\Re_2}}\;\; \text{sn}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}, \tag{70}$$

$$\mathcal{V}_{5.3,1} = \pm 3\, m^3\, \hbar \sqrt{\frac{-4730\; q_4\; \varrho_{j6}}{\Re_2}}\;\; \text{sn}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}. \tag{71}$$

When $m = 1$, we obtain dark solitons:

$$\mathcal{U}_{5.3,1.1} = \pm 3 \sqrt{\frac{-4730\; \varrho_{j6}}{\Re_2}}\;\; \tanh^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}, \tag{72}$$

$$\mathcal{V}_{5.3,1.1} = \pm 3\hbar \sqrt{\frac{-4730\; \varrho_{j6}}{\Re_2}}\;\; \tanh^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}. \tag{73}$$

(5.3,2) If $q_0 = m^2 - 1,\; q_2 = 2 - m^2, q_4 = -1, \Re_2\; \varrho_{j6} < 0$ and $0 \leq m \leq 1$, then, we obtain JEF solutions:

$$\mathcal{U}_{5.3,2} = \mp 3 \sqrt{\frac{4730\; \varrho_{j6}}{\Re_2}}\;\; \text{dn}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}, \tag{74}$$

$$\mathcal{V}_{5.3,2} = \mp 3\, \hbar \sqrt{\frac{4730\; \varrho_{j6}}{\Re_2}}\;\; \text{dn}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}. \tag{75}$$

When $m = 1$, we obtain bright solitons:

$$\mathcal{U}_{5.3,2.1} = \mp 3 \sqrt{\frac{4730\; \varrho_{j6}}{\Re_2}}\;\; \text{sech}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}, \tag{76}$$

$$\mathcal{V}_{5.3,2.1} = \mp 3\hbar \sqrt{\frac{4730\; \varrho_{j6}}{\Re_2}}\;\; \text{sech}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}. \tag{77}$$

(5.3,3) If $q_0 = -m^2,\; q_2 = -1 + 2m^2, q_4 = 1 - m^2,\; \Re_2\; \varrho_{j6} < 0$ and $0 \leq m < 1$, then, we obtain JEF solutions:

$$\mathcal{U}_{5.3,3} = \mp 3\left(m^2 - 1\right) \sqrt{\frac{4730(m^2-1)\; \varrho_{j6}}{\Re_2}}\;\; \text{nc}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}, \tag{78}$$

$$\mathcal{V}_{5.3,3} = \mp 3\hbar\left(m^2 - 1\right) \sqrt{\frac{4730(m^2-1)\; \varrho_{j6}}{\Re_2}}\;\; \text{nc}^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}. \tag{79}$$

When $m = 0$, we obtain singular periodic solutions:

$$\mathcal{U}_{5.3,3.1} = \pm 3 \sqrt{\frac{-4730\; \varrho_{j6}}{\Re_2}}\;\; \sec^3\left[x - \frac{\Gamma(\mu+1)\;\eta\; t^{\alpha}}{\alpha}\right]\; e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\;\; t^{\alpha}}{\alpha}\;\omega + \Delta\right)}, \tag{80}$$

$$\mathcal{V}_{5.3,3.1} = \pm 3\,\hbar\sqrt{\frac{-4730\,\varrho_{j6}}{\Re_2}}\ \sec^3\left[x - \frac{\Gamma(\mu+1)\,\eta\,t^{\alpha}}{\alpha}\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{81}$$

(5.3,4) If $q_0 = m^4 - 2m^3 + m^2,\ q_2 = -\frac{4}{m}, q_4 = -m^2 + 6m - 1$, $\Re_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we obtain JEF solutions:

$$\mathcal{U}_{5.3,4} = \pm 3\ m^3[1 + m(m-6)]\sqrt{\frac{4730\ [1 + m(m-6)]\ \varrho_{j6}}{\Re_2}}\left(\frac{\mathrm{cn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\ \mathrm{dn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\left(-2 + \mathrm{dn}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\right)^3}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{82}$$

$$\mathcal{V}_{5.3,4} = \pm 3\ m^3\ \hbar[1 + m(m-6)]\sqrt{\frac{4730\ [1 + m(m-6)]\ \varrho_{j6}}{\Re_2}}\left(\frac{\mathrm{cn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\ \mathrm{dn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\left(-2 + \mathrm{dn}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\right)^3}\right) \times e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{83}$$

When $m = 1$, we obtain bright solitons:

$$\mathcal{U}_{5.3,4.1} = \pm 24\ \sqrt{\frac{-4730\ \varrho_{j6}}{\Re_2}}\ \mathrm{sech}^3\left[\frac{1}{2}\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{84}$$

$$\mathcal{V}_{5.3,4.1} = \pm 24\hbar\ \sqrt{\frac{-4730\ \varrho_{j6}}{\Re_2}}\ \mathrm{sech}^3\left[\frac{1}{2}\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{85}$$

(5.3,5) If $q_0 = \frac{1}{4},\ q_2 = \frac{m^2-2}{2}, q_4 = \frac{m^4}{4}$, $\Re_2\ \varrho_{j6} < 0$ and $0 < m \leq 1$, then, we can obtain JEF solutions:

$$\mathcal{U}_{5.3,5} = \pm 3m^6\ \sqrt{\frac{-2365\ \varrho_{j6}}{2\Re_2}}\ \left(\frac{\mathrm{sn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\left(1 + \mathrm{dn}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\right)^3}\right) e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{86}$$

$$\mathcal{V}_{5.3,5} = \pm 3m^6\ \sqrt{\frac{-2365\ \varrho_{j6}}{2\Re_2}}\ \left(\frac{\mathrm{sn}^3\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]}{\left(1 + \mathrm{dn}^2\left[x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right]\right)^3}\right) e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{87}$$

When $m = 1$, we obtain dark solitons:

$$\mathcal{U}_{5.3,5.1} = \pm\frac{3}{4}\ \sqrt{\frac{-2365\ \varrho_{j6}}{2\ \Re_2}}\ \tanh^3\left[\frac{1}{2}\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}, \tag{88}$$

$$\mathcal{V}_{5.3,5.1} = \pm\frac{3\hbar}{4}\ \sqrt{\frac{-2365\ \varrho_{j6}}{2\ \Re_2}}\ \tanh^3\left[\frac{1}{2}\left(x - \frac{\Gamma(\mu+1)\ \eta\ t^{\alpha}}{\alpha}\right)\right] e^{i\left(-\mathcal{K}x + \frac{\Gamma(\mu+1)\ \ t^{\alpha}}{\alpha}\ \omega + \Delta\right)}. \tag{89}$$

4. Graphical Illustration

In this section, some numerical simulations of some of the extracted solutions are explained by giving specific values for the variables and showing the physical properties of the extracted solutions. Three- and two-dimensional diagrams of some special solutions are also presented as follows.

Figure 1 presents the dark soliton for Equation (70) with parameters $b_1 = 0.7$, $\mathcal{K} = 1.9$, $\alpha_1 = 0.7$, $\beta_1 = 0.7$, $\gamma_1 = -1.7$, $\theta_1 = -1.7$, $\lambda_1 = -1.73$, $r_1 = 0.8$, $\varrho_{14} = 0.6$, $\varrho_{15} = -0.8$, $\varrho_{16} = -0.5$, $\Delta = 0.9$, $\eta = -1$, $\omega = 0.8$, $\hbar = 2$, and $\mu = 2$. Figure 2 presents a bright soliton of Equation (76) when $b_1 = 0.71$, $\mathcal{K} = 0.9$, $\alpha_1 = -1.7$, $\beta_1 = 0.7$, $\gamma_1 = 0.7$, $\theta_1 = -0.7$, $\lambda_1 = 0.53$, $r_1 = 0.82$, $\varrho_{14} = 0.63$, $\varrho_{15} = 1.8$, $\varrho_{16} = 0.5$, $\Delta = 0.89$, $\eta = -1.2$, $\omega = 0.83$, $\hbar = 2$, and $\mu = 2$. Figure 3 shows a singular periodic solution to Equation (80) when $b_1 = 1.71$, $\mathcal{K} = 1.49$, $\alpha_1 = 3.7$, $\beta_1 = 3.72$, $\gamma_1 = 0.73$, $\theta_1 = -1.76$, $\lambda_1 = 0.56$, $r_1 = -2.72$, $\varrho_{14} = 1.63$, $\varrho_{15} = 1.8$, $\varrho_{16} = -0.45$, $\Delta = 0.89$, $\eta = -1.2$, $\omega = 0.83$, $\hbar = 4$, and $\mu = 2$.

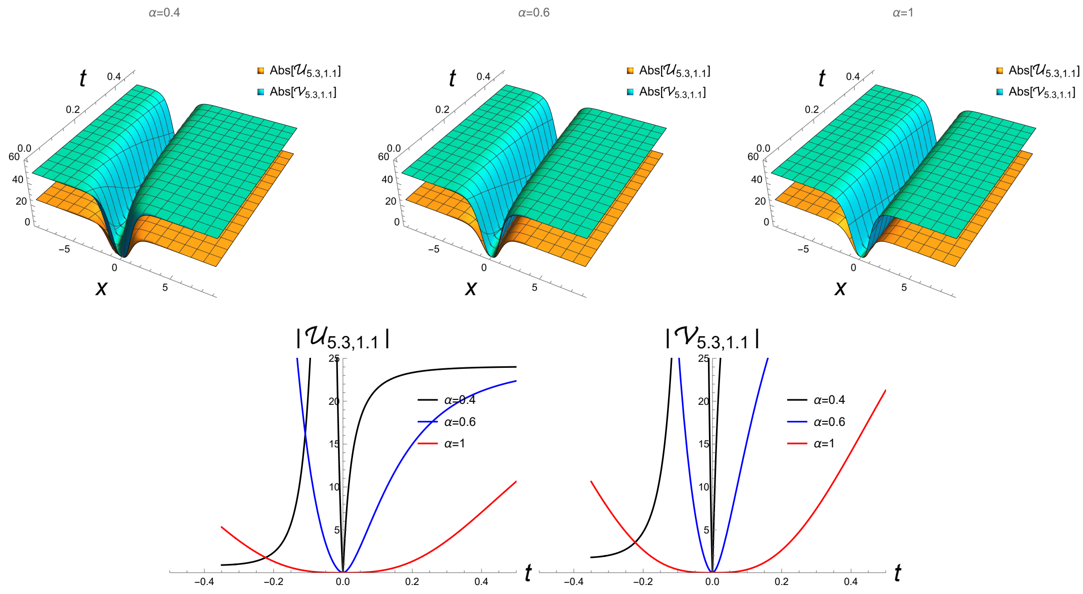

Figure 1. Dark soliton solution of Equation (72).

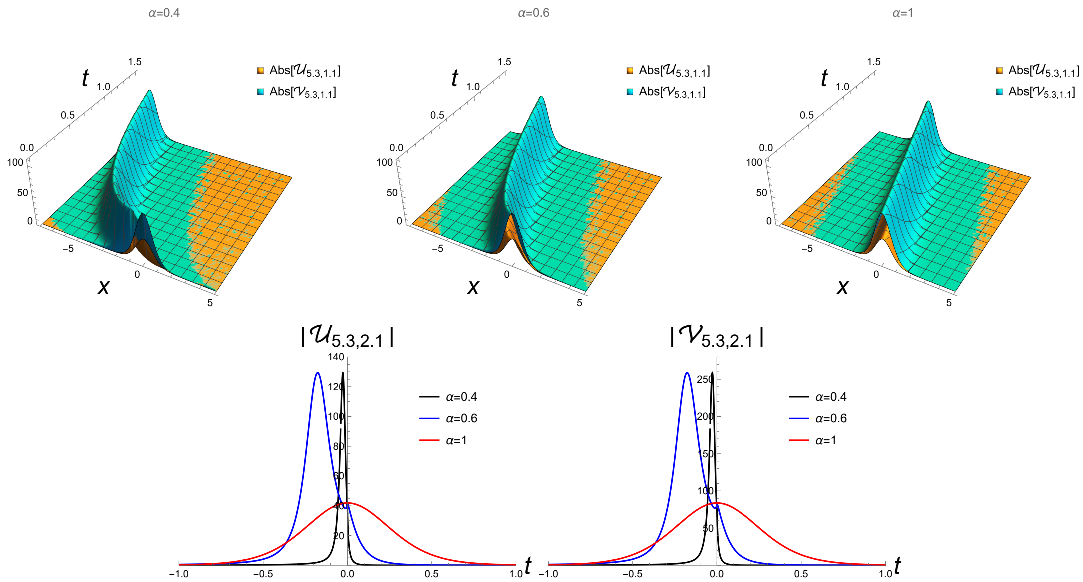

Figure 2. Bright soliton solution of Equation (76).

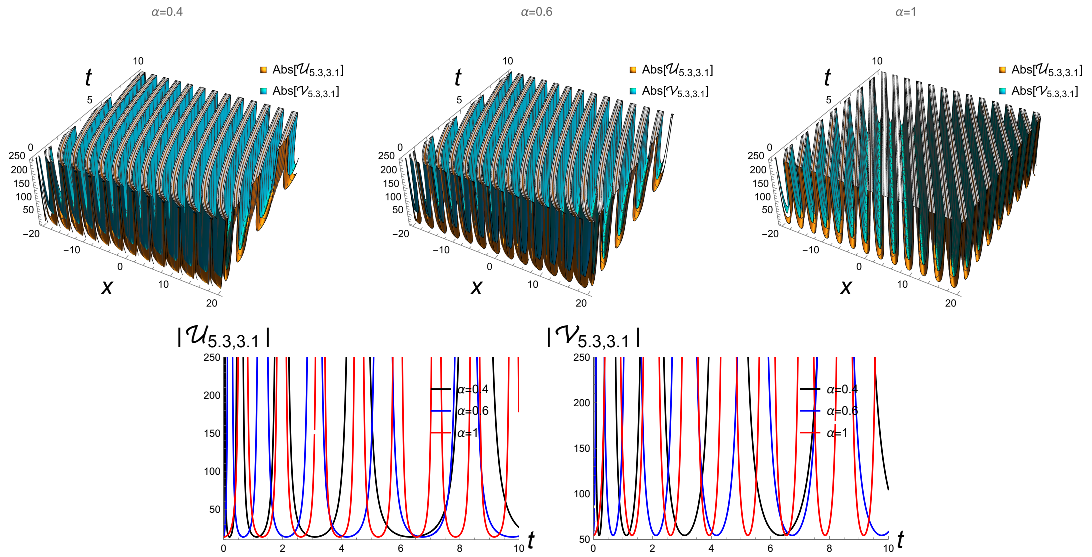

Figure 3. Singular periodic solution of Equation (80).

5. Conclusions

In this study, we investigated the impact of fractional order on wave propagation in optical twin-core couplers with Kerr law nonlinearity. We discovered a range of novel fractional traveling wave solutions using the modified extended mapping method. These solutions can model various forms of wave behavior, including bright solitons that feature localized intensity peaks, dark solitons that represent regions of minimal intensity against a continuous background, combined bright–dark solitons that exhibit the characteristics of both bright and dark solitons through the precise manipulation of nonlinear effects and structural parameters in the waveguide, and singular solitons that are identified by their distinctive properties, emerging under specific nonlinear conditions and maintaining bounded behavior within an overall field.

Additionally, we identified periodic and singular periodic solutions, which are vital for modeling light wave behavior in metamaterials. In addition to Jacobi elliptic function solutions, these solutions represent a special set of periodic functions that remain crucial throughout the modeling of wave dynamics in nonlinear optical couplers.

To effectively present our findings, we employed 2D and 3D visualizations, which vividly illustrated the properties and behaviors of the identified soliton solutions.

The fractional derivative provides a new layer of complexity that captures various physical phenomena that normal derivatives may overlook, including memory effects and non-local behavior in optics.

When $\alpha = 1$, the proposed model was studied in [38] using the unified Riccati equation expansion method and the enhanced Kudryashov's scheme. By comparing our results for $\alpha = 1$ with those in [38], we observe that the current work presents a wider variety of solutions, including dark solitons, bright solitons, singular solitons, combined bright–dark solitons, hyperbolic solutions, Jacobi elliptic function solutions, periodic solutions, and singular periodic solutions, all derived using the modified extended mapping method.

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