fractal and fractional

Review

Review of the Fractional Black-Scholes Equations and Their Solution Techniques

Hongmei Zhang [1], Mengchen Zhang [1,*], Fawang Liu [2] and Ming Shen [1]

[1] School of Mathematics and Statistics, Fuzhou University, Fuzhou 350108, China
[2] School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, QLD 4001, Australia
* Correspondence: mengchen@fzu.edu.cn

Abstract: The pioneering work in finance by Black, Scholes and Merton during the 1970s led to the emergence of the Black-Scholes (B-S) equation, which offers a concise and transparent formula for determining the theoretical price of an option. The establishment of the B-S equation, however, relies on a set of rigorous assumptions that give rise to several limitations. The non-local property of the fractional derivative (FD) and the identification of fractal characteristics in financial markets have paved the way for the introduction and rapid development of fractional calculus in finance. In comparison to the classical B-S equation, the fractional B-S equations (FBSEs) offer a more flexible representation of market behavior by incorporating long-range dependence, heavy-tailed and leptokurtic distributions, as well as multifractality. This enables better modeling of extreme events and complex market phenomena, The fractional B-S equations can more accurately depict the price fluctuations in actual financial markets, thereby providing a more reliable basis for derivative pricing and risk management. This paper aims to offer a comprehensive review of various FBSEs for pricing European options, including associated solution techniques. It contributes to a deeper understanding of financial model development and its practical implications, thereby assisting researchers in making informed decisions about the most suitable approach for their needs.

Keywords: fractional derivative; fractional Black-Scholes equation; European option; analytic solution; numerical simulation

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1. Introduction

In the financial field, an option is a significant and popular financial derivative that grants the holder the right to purchase (call option) or sell (put option) an asset at a predetermined fixed price K (known as the strike price) within a specific period. Options trading can be traced back to the late 18th century in both American and European markets. However, it was not until 1973, when the Chicago Board Options Exchange introduced standardized options contracts, that this financial instrument witnessed significant advancements. Accordingly, determining the appropriate pricing for an option became a significant challenge. In the 1970s, Black and Scholes [1] along with Merton [2] developed an original option pricing model that governs the dynamic behavior of option prices over time.

When assuming that the random walk followed by the natural logarithm of the stock price S_t under the risk-neutral measure or equivalent martingale measure (EMM) is

$$d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dB_t^Q, \tag{1}$$

where $\sigma \geq 0$ is the volatility of the returns from holding S_t and r is the risk-free rate. dB_t^Q is the increment of a Brownian motion under the risk-neutral measure by using the superscript Q. It is assumed that the Brownian motion follows the Normal or Gaussian distribution.

Furthermore, by using Itō's lemma, the pricing of a European-style option $V(S,t)$, written on the underlying asset S_t, satisfies a partial differential equation (PDE) as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV, \tag{2}$$

where r is the risk-free rate. This is the well-known Black-Scholes (or Black-Scholes-Merton) equation (BSE). By means of the variable substitution $x_t = \ln S_t$, the above B-S equation can be rewritten as an advection-diffusion type equation

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial x} = rU, \tag{3}$$

where $U(x,t) = V(\ln S, t)$.

The Black-Scholes (B-S) model is a valuable tool in approximating the behavior of underlying assets and serves as a benchmark for comparing alternative models due to its simplicity and clarity in determining option prices. However, the B-S model was formulated based on a set of stringent assumptions, such as a frictionless and complete market which is not consistent with the behaviour observed in real financial markets. Furthermore, the assumption of constant volatility makes the equation unable to capture large moves or jumps over small intervals [3]. For a more realistic scenario, various alternative models have been proposed to improve the B-S model, such as the jump-diffusion model [3,4], stochastic volatility or time-changed models [5,6], a model with transaction costs [7], the regime-switching model [8]. Additionally, in the early 1960s, Mandelbrot [9] observed that the relative change of stock prices exhibits excess kurtosis and heavy tail phenomena, which cannot be adequately captured by assuming standard Brownian motion (BM) for underlying asset prices. Then, he proposed an exponential non-normal Lévy process to simulate this phenomenon. Subsequently, various improved Lévy processes have been presented. Notably, KoBoL process (Koponen [10] and Boyarchenko and Levendorskiǐ [11]), CGMY (Carr, Geman, Madan, and Yor [12]), and Finite Moment Log Stable (FMLS) process [13] have stood out and drawn extensive attentions among these Lévy processes. However, a systematic introduction of these modified Lévy processes has not been thoroughly reported.

Fractional derivatives, as quasi-differential operators, exhibit non-local characteristics and thus serve as a powerful tool for describing the properties of non-locality and long memory observed in various physical phenomena. With the rapid development of the fractional calculus in the last several decades, fractional partial differential equations (FPDEs) have been applied in various fields including, but not limited to, physics, fluid mechanics, biology, and finance, engineering [14–29]. In the late 20th century, researchers discovered that financial markets exhibit fractal characteristics both domestically and internationally [9,30]. The self-similar and non-local properties of the fractional derivative allow the fractional B-S model to describe the fat-tailed and leptokurtic distributions of asset prices. Compared to the classical B-S model, it is preferable to simulate complex situations in the real market. The fractional B-S models have gained increasing attention due to notable contributions by Wys [31] and Cartea et al. [32]. Assuming that the dynamics of equity price follow Jump-diffusion processes or infinite activity Lévy processes, established price dynamics of financial derivatives satisfy FPDE. Since then, various fractional B-S equations (FBSEs) have been developed. For instance, a space fractional B-S equation [32] has been constructed by replacing the standard Brownian motion with fractional Brownian motion or specific Lévy processes. Chen et al. [33] derived a time-fractional B-S equation by treating the change in option price over time as a fractal transmission system. Moreover, by utilizing the Gaussian white noise and fractional Taylor series, a space-time fractional B-S equation has been formulated [34].

With the wide application of FBSEs in option pricing, an increasing interest has been attracted on the exploration of solution techniques. Since the beginning of this century, many studies have been devoted to finding solutions from both analytical and numerical perspectives. By employing the pure integral transform technique, an explicit analytic solution can be obtained in the form of a convolution involving some specific functions such as the Fox function and Mittag-Leffler (M-L) function [31,35–37]. However, it is challenging to derive the explicit analytic solution of FBSEs in most cases. Therefore, many improved methods have been proposed to obtain (approximate) analytic solutions in the form of infinite series, such as the Laplace homotopy perturbation method (LHPM) [38], the Laplace Legendre wavelet method [39], the Laplace homotopy analysis method (LHAM) [40], the Residual power series method (RPSM) [41], the Adomain decomposition method (ADM) [42], the Differential transform method (DTM) [43] and the Elzaki transform method (ETM) [25,44]. Regarding the numerical perspective, a variety of finite differential techniques are commonly employed [32,35,45–49]. In recent years, various finite difference coupling techniques have been developed to numerically approximate the fractional B-S model, such as combining the finite difference with the meshless method [50,51], coupling the finite difference with the collocation method [33,52] and incorporating the finite difference with the spectral method [53]. Additionally, other effective numerical methods have been developed by researchers, including a moving least-squares approach [54], a space-time spectral method [55], an operational matrix method [56] and the neural network technique [57]. Considerable efforts have been dedicated to studying fractional B-S equations and developing efficient methods for obtaining their analytical and numerical solutions. However, to the best of our knowledge, a comprehensive overview of various fractional B-S equations and their corresponding solution techniques has not been reported previously.

As we mentioned above, FBSEs play an important role in characterizing heavy-tailed phenomena of option prices. However, the availability of a detailed introduction of different fractional B-S equations is still limited. Furthermore, efficient solution techniques for obtaining analytical and numerical solutions provide a basis for further applications of fractional B-S equations. Although much work has been done to investigate effective methods, most of them are limited to a few specific techniques. A systematic and comprehensive overview of available literature concerning this topic is still lacking. Therefore, this review provides an introduction of many potential fractional B-S equations and their solution techniques. It aims to enrich the research methods in the field of finance. Moreover, it serves as a valuable resource for researchers in finance, enabling them to gain a deeper understanding of the practical applications of these equations in finance, which provides robust theoretical support for practical market analysis and risk management. Furthermore, considering the interdisciplinary nature of FBSEs, which involve mathematics, physics, and finance, this review also seeks to foster technological innovation in finance and promote interdisciplinary research collaboration.

The rest of the paper is organized as follows: Section 2 introduces some preliminaries, including the definitions of fractional derivatives (FDs), the terminal and boundary conditions (TBCs) of European option and various approximation schemes for FDs. The space, space-time and time fractional B-S equations are outlined in Sections 3–5, respectively. Moreover, an extensive overview of various solution techniques is provided. Section 6 draws some conclusions.

2. Preliminaries

In this section, we initially present several commonly utilized definitions of FDs in the financial domain. Subsequently, the TBCs satisfied by some European options are listed. Finally, we also provide some approximation formulas for both the Riemann-Liouville derivative and Caputo derivative.

2.1. Definitions of Fractional Derivatives

Definition 1 ([14,15,58])**.** *The fractional integral ${}_aD_x^{-\alpha}(\alpha > 0)$ of $f(x)$ is defined as follows:*

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}}, \tag{4} $$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}$ is the gamma function.

Definition 2 ([14,15,59])**.** *If $f(x)$ is continuous on $[a,b]$, then the Grünwald-Letnikov (G-L) derivative $(\alpha > 0)$ and integral $(\alpha < 0)$ can be uniformly defined as follows:*

$$ {}_a^{GL}D_x^{\alpha} f(x) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh) \tag{5} $$

where $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-r+1)}{r!}$.

Definition 3 ([14,15,58])**.** *If $f(x)$ is integrable on $[a,b]$, then the left and right Riemann-Liouville (R-L) derivatives with order $\alpha(k-1 \le \alpha < k)$ on $[a,b]$ are, respectively, defined as*

$$ {}_a^{RL}D_x^{\alpha} f(x) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_a^x (x-\xi)^{k-\alpha-1} f(\xi) d\xi \tag{6} $$

and

$$ {}_x^{RL}D_b f(x) = \frac{(-1)^k}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_x^b f(\xi-x)^{k-\alpha-1}(\xi) d\xi. \tag{7} $$

Definition 4 ([14,15,60])**.** *If $f(x) \in C^k[a,b]$, then the left and right Caputo derivatives with order $\alpha(k-1 \le \alpha < k)$ on $[a,b]$ are, respectively, defined as*

$$ {}_a^{C}D_x^{\alpha} f(x) = \frac{1}{\Gamma(k-\alpha)} \int_a^x (x-\xi)^{k-\alpha-1} f^{(k)}(\xi) d\xi \tag{8} $$

and

$$ {}_x^{C}D_b f(x) = \frac{(-1)^k}{\Gamma(k-\alpha)} \int_x^b (\xi-x)^{k-\alpha-1} f^{(k)}(\xi) d\xi. \tag{9} $$

Remark 1. *The G-L, R-L, and Caputo FDs are among the most widely popularized. However, one of the drawbacks of the R-L derivative is its inconsistency with physical initial and boundary conditions for initial or boundary value problems. This issue can be overcome by employing the Caputo derivative. Moreover, there are the following relationships among them: If the function $f(x) \in C^{(n-1)}[a,b]$ and $f^{(n)}(x)$ is integrable in $[a,b]$, the R-L derivative is existent and aligns with the G-L derivative. Furthermore, for $m-1 \le \alpha < m$, one has*

$$ {}_a^{RL}D_x^{\alpha} f(x) = {}_a^{GL}D_x^{\alpha} f(x) = {}_a^{C}D_x^{\alpha} f(x) + \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^{j-\alpha}}{\Gamma(1+j-\alpha)}, \ a < x < b. \tag{10} $$

Definition 5 ([61,62])**.** *When some kind of fractional operators ${}_a\mathbf{D}_x^{\alpha}$ act upon $f(x)$, and the integration of ${}_a\mathbf{D}_x^{\alpha} f(x)$ is performed with respect to the order α, we obtain the following distributed-order derivative*

$$ {}_a\mathbf{D}_x^{w(\alpha)} f(x) = \int_{\gamma_1}^{\gamma_2} w(\alpha) {}_a\mathbf{D}_x^{\alpha} f(x) d\alpha \tag{11} $$

where $w(\alpha)$ represents the weight function associated with the distribution of order $\alpha \in [\gamma_1, \gamma_2]$.

Remark 2. *When $w(\alpha)$ only takes a discrete value in $[\gamma_1, \gamma_2]$, the distributed-order derivative becomes FD. The operator has been demonstrated to be a more effective tool for quantifying and characterizing various physical phenomena [63–69].*

Definition 6 ([70])**.** *Suppose that $f(x) \in C_\alpha(a, b)$. The local FD of order α is characterized by*

$$^{LF}D_x^\alpha f(x_0) := \frac{\Delta^\alpha[f(x) - f(x_0)]}{(x - x_0)^\alpha}, 0 < \alpha \leq 1, \tag{12}$$

where $\Delta^\alpha[f(x) - f(x_0)] \cong \Gamma(1+\alpha)[f(x) - f(x_0)]$, and $C_\alpha(a, b)$ denotes a class of locally fractional continuous functions defined as follows: for arbitrary $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \epsilon^\alpha,\ 0 < \alpha \leq 1.$$

Remark 3. *Local FD provides a powerful tool to analyze pointwise behavior of irregular signals.*

Definition 7 ([34])**.** *If $f(x)$ is integrable on $[a, b]$, then the modified left and right R-L derivatives of order $\alpha(k - 1 \leq \alpha < k)$ are defined by the following expression:*

$$^{MR}_{\ a}D_x^\alpha f(x) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_a^x (x - \xi)^{k-\alpha-1}(f(\xi) - f(a))d\xi, \tag{13}$$

and

$$^{MR}_{\ x}D_b^\alpha f(x) = \frac{(-1)^k}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_x^b (\xi - x)^{k-\alpha-1}(f(b) - f(\xi))d\xi, \tag{14}$$

Remark 4. *The modified R-L derivatives can mitigate the side effects of non-zero initial conditions under the R-L derivative definition. Furthermore, they yield identical results to the Caputo definition when the function exhibits differentiability.*

Definition 8 ([71])**.** *Let $Re(\alpha) \geq 0$ and $k = [Re(\alpha)] + 1$. If $f(x) \in AC_\delta^k[a, b]$, then the Caputo-type modification of left- and right-sided Hadamard FDs (referred to as the C-H derivative) of order $\alpha(k - 1 < \alpha < k)$ can be, respectively, expressed by*

$$^{CH}_{\ a}D_x^\alpha f(x) = \frac{1}{\Gamma(k-\alpha)} \int_a^x (\log\frac{x}{\xi})^{k-\alpha-1} \delta^k f(\xi) \frac{d\xi}{\xi}, \tag{15}$$

and

$$^{CH}_{\ x}D_b^\alpha f(x) = \frac{(-1)^k}{\Gamma(k-\alpha)} \int_x^b (\log\frac{\xi}{x})^{k-\alpha-1} \delta^k f(\xi) \frac{d\xi}{\xi}, \tag{16}$$

where $AC_\delta^k[a, b] = \{f | f : [a, b] \to \mathbb{C}, \delta^{k-1} f(x) \in AC[a, b], \delta = x\frac{d}{dx}\}$ and $AC[a, b]$ is the set of functions that are absolute continuous on $[a, b]$.

Remark 5. *The C-H derivative exhibits physically interpretable initial conditions, similar to those observed in Caputo derivatives, as the derivative of a constant is zero. The logarithmic kernel renders it suitable for characterizing ultra-slow processes rising from super-heavy tailed distributions of waiting time in particle motion.*

Definition 9 ([72,73])**.** *Let $Re(\alpha) \geq 0$ and $k = [Re(\alpha)] + 1$. If $f(x) \in AC^k_\gamma[a,b]$, then, using the Katugampola's fractional integral, the generalized Caputo derivative (referred to as the K-C derivative) of f with order α is defined by*

$$ {}^{KC}_{a}D^{\alpha,\rho}_{x} f(x) = \frac{1}{\Gamma(k-\alpha)} \int_a^x (\frac{x^\rho - \xi^\rho}{\rho})^{k-\alpha-1} \gamma^k f(\xi) \frac{d\xi}{\xi^{1-\rho}} \tag{17}$$

and

$$ {}^{KC}_{x}D^{\alpha,\rho}_{b} f(x) = \frac{1}{\Gamma(k-\alpha)} \int_x^b (\frac{\xi^\rho - x^\rho}{\rho})^{k-\alpha-1} (-\gamma)^k f(\xi) \frac{d\xi}{\xi^{1-\rho}}, \tag{18}$$

where $AC^k_\gamma[a,b] = \{f | f : [a,b] \to \mathbb{C}, \gamma^{k-1} f(x) \in AC[a,b], \gamma = x^{1-\rho}\frac{d}{dx}, \rho > 0\}$.

Remark 6. *Note that when $\rho = 1$, the K-C derivative becomes the Caputo derivative, while it converges to the C-H derivative as ρ approaches 0. The K-C derivative serves as a generalization of both the Caputo and C-H derivatives.*

Definition 10 ([74])**.** *For $x > 0$, the α-order conformable derivative is defined as*

$$ {}^{CFM}D^{\alpha}_{x}(f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \ \alpha \in (0,1], \tag{19}$$

when $f : [0,\infty) \to \mathbb{R}$, and

$$ {}^{CFM}D^{\alpha}_{x}(f)(x) = \lim_{\varepsilon \to 0} \frac{f^{\lceil\alpha\rceil-1}(x + \varepsilon x^{(\lceil\alpha\rceil-\alpha)}) - f^{\lceil\alpha\rceil-1}(x)}{\varepsilon}, \ \alpha \in (n, n+1], \tag{20}$$

when f is differentiable with respect to x up to the n-th order. Here, $\lceil\alpha\rceil$ denotes the smallest integer that is greater than or equal to α.

Remark 7. *The conformable derivative aligns with the established FDs on polynomials. Moreover, it adheres to the same formulas for differentiating the product and quotient of two functions, as well as the chain rule, just like its integer-order counterpart.*

Definition 11 ([75])**.** *For $0 < \alpha < 1$, the Caputo-Fabrizio (C-F) derivative is definied by*

$$ {}^{CF}_{a}D^{\alpha}_{x} f(x) = \frac{M(\alpha)}{(1-\alpha)} \int_a^x \exp\{-\alpha\frac{x-\xi}{1-\alpha}\} f'(\xi) d\xi, \tag{21}$$

when $f(x) \in H^1(a,b)$ and

$$ {}^{CF}_{a}D^{\alpha}_{x} f(x) = \frac{M(\alpha)}{(1-\alpha)} \int_a^x (f(x) - f(\xi)) \exp\{-\alpha\frac{x-\xi}{1-\alpha}\} d\xi, \tag{22}$$

when $f(x) \in L^1(a,b)$. Here, $M(\alpha)$ is a normalization function, satisfying the conditions $M(0) = M(1) = 1$.

Definition 12 ([76])**.** *The Atangana and Baleanu (A-B) derivative in the Caputo sense for $f \in H^1(a,b)$ can be expressed as follows:*

$$ {}^{ABC}_{a}D^{\alpha}_{x} f(x) = \frac{M(\alpha)}{(1-\alpha)} \int_a^x E_\alpha\{-\alpha\frac{(x-\xi)^\alpha}{1-\alpha}\} f'(\xi) d\xi, \tag{23}$$

where $0 < \alpha < 1$ and $E_\alpha(-x^\alpha) = \sum\limits_{k=0}^{\infty} \frac{(-x)^{\alpha k}}{\Gamma(\alpha k+1)}$ is the Mettag-Leffler(M-L) function.

Remark 8. *Unlike the definitions of FDs with integral form, neither the C-F derivative nor the A-B derivative possesses a singular kernel.*

Remark 9. *If the order of FD α is permitted to vary as a function, the resulting derivative is denominated as a variable-order derivative.*

Definition 13 ([77])**.** *Assume that $\alpha > 0$ and $\psi \in C^1([a,b])$ is a positive and monotonically increasing function. The left-sided fractional integral of $f \in AC[a,b]$ with respect to ψ is defined as follows:*

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} f(t)dt. \tag{24}$$

If $n-1 < \alpha < n, \beta \in [0,1]$ and $f, \psi \in C^n([a,b])$, then the left ψ-Hilfer FD of order α and type β is specified by

$$^{H}D_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi} (\frac{1}{\psi'(x)} \frac{d}{dx})^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x), \tag{25}$$

where ψ is positive and increasing.

Remark 10. *Let $\psi(x) = x$, ψ-Hilfer FD become R-L FD when $\beta \to 0$, and transforms into a Caputo FD when $\beta \to 1$.*

2.2. The Terminal and Boundary Conditions Satisfied by Different European Options

Let $V = V(S,t)$ represent the price of a European option on the underlying asset price S at time t, and let $U = U(x,t) = V(\ln S, t)$. These two notations are consistently used throughout this paper.

The terminal and boundary conditions (TBCs) of European option $V(S,t)$ depend on the relationship between the price of the underlying asset S and the exercise price K of the option. Let T denote the expiry date and r be the risk-free interest rate. The most prevalent European options are call and put options, which adhere to the following TBCs:

Call option:

$$\begin{array}{l} V(S,T) = max\{S-K, 0\} \\ V(0,t) = 0, \ \lim\limits_{S \to +\infty} [V(S,t) - (S - Ke^{-r(T-t)})] = 0, t \in [0,T). \end{array} \tag{26}$$

Put option:

$$\begin{cases} V(S,T) = max\{K-S, 0\} \\ V(0,t) = Ke^{-r(T-t)}, \ \lim\limits_{S \to +\infty} V(S,t) = 0, t \in [0,T). \end{cases} \tag{27}$$

It should be noted that different European options may have different TBCs, which mainly depend on the specific terms of the options and market conditions, for example, for certain options with special exercise conditions, such as barrier option and butterfly option. Here we list the TBCs for the two options relative to the call option.

Barrier option (there are three different cases) [32]:

Case 1. European up and out call option with barrier located at $S = S_u$:

$$V(S,t) = \begin{cases} max\{S-K, 0\}, \ 0 < S < S_u, \ t = T \\ 0, \quad S \geq S_u, \ 0 \leq t < T. \end{cases} \tag{28}$$

Case 2. European down and out call option with barrier located at $S = S_d$:

$$V(S,t) = \begin{cases} max\{S-K, 0\}, \ S > S_d, \ t = T \\ 0, \quad S \leq S_d, \ 0 \leq t < T. \end{cases} \tag{29}$$

Case 3. European double-knock-out call option with barrier located at $S = S_u$ and $S = S_d$:

$$V(S,t) = \begin{cases} max\{S-K,0\},\ S_d < S < S_u,\ t = T \\ 0, \quad S \le S_d \text{ and } S \ge S_u,\ 0 \le t < T. \end{cases} \tag{30}$$

Butterfly call option [78]:

$$\begin{cases} V(S,T) = max\{S-K_1,0\} - 2max\{S-K_2,0\} + max\{S-K_3,0\} \\ V(0,t) = 0,\ \lim\limits_{S\to+\infty}[V(S,t) - (S - Ke^{-r(T-t)})] = 0,\ t \in [0,T), \end{cases} \tag{31}$$

where $K_i (i = 1,2,3)$ are the exercise price and $K_2 = \frac{K_1+K_2}{2}$.

2.3. Several Classical Approximation Schemes for Fractional Derivatives

Here, we only present several widely employed approximation schemes for the R-L derivative and the Caputo derivative. Additionally, numerous other approximation formulas can be found in relevant books [79,80].

2.3.1. The G-L Approximation of the R-L Fractional Derivative

Denote $A^{\alpha}_{h,p}f$ as the shifted G-L formula over the function f, which is expressed by

$$A^{\alpha}_{h,p}f(x) = h^{-\alpha}\sum_{k=0}^{\infty} g^{(\alpha)}_k f(x-(k-p)h),\ \ 0 \le n-1 \le \alpha < n, \tag{32}$$

where p is a constant, referred to as displacement, and

$$g^{(\alpha)}_k = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix},\ \begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Let

$$\mathfrak{L}^{n+\alpha}(R) = \{f|f \in L^1(R), \int_{-\infty}^{\infty}(1+|\omega|)^{n+\alpha}|F(\omega|d\omega < \infty)\},$$

here, $F(\omega)$ is the Fourier transform of function $f(x)$. Then, we have the following approximations: a first-order approximation formula [81]:

$${}^{RL}_{-\infty}D^{\alpha}_x f(x) = A^{\alpha}_{h,p}f(x) + O(h),\ f \in \mathfrak{L}^{1+\alpha}(R) \tag{33}$$

and a second order approximation formula [82]:

$${}^{RL}_{-\infty}D^{\alpha}_x f(x) = \lambda_1 A^{\alpha}_{h,p}f(x) + \lambda_2 A^{\alpha}_{h,q}f(x) + O(h^2),\ f \in \mathfrak{L}^{2+\alpha}(R) \tag{34}$$

where $\lambda_1 = \frac{\alpha-2q}{2(p-q)}, \lambda_2 = \frac{2p-\alpha}{2(p-q)}, p \ne q$.

2.3.2. The Interpolation Approximation of Caputo Derivative

Here, we present two prevalent approximations for the Caputo derivative when $0 < \alpha < 1$.

Let $\tau = \frac{T}{N}$ and $t_k = k\tau (0 \le k \le N)$.

The $L1$ approximation formula is given by [83]

$${}^{C}_{0}D^{\alpha}_t f(t)|_{t=t_n} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[c^{(\alpha)}_0 f(t_n) - \sum_{k=1}^{n-1}(c^{(\alpha)}_{n-k-1} - c^{(\alpha)}_{n-k})f(t_k) - c^{(\alpha)}_{n-1}f(t_0)\right] \tag{35}$$

$$+O(\tau^{2-\alpha}), \tag{36}$$

where $c^{(\alpha)}_l = (l+1)^{1-\alpha} - l^{1-\alpha},\ l \ge 0$.

$L2-1_\sigma$ approximation formula can be formulated as [84]

$$
{}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} c_k^{(n,\alpha)} [f(t_{n-k}) - f(t_{n-k-1})] + O(\tau^{3-\alpha}),\ 1 \le n \le N, \quad (37)
$$

where $c_0^{(1,\alpha)} = \sigma^{1-\alpha}$ and for $n \ge 2$

$$
\begin{cases}
c_0^{(n,\alpha)} = \frac{(1+\sigma)^{2-\alpha}-\sigma^{2-\alpha}}{2-\alpha} - \frac{(1+\sigma)^{1-\alpha}-\sigma^{1-\alpha}}{2}, \\
c_k^{(n,\alpha)} = \frac{1}{2-\alpha}[(k+1+\sigma)^{2-\alpha} - 2(k+\sigma)^{2-\alpha} + (k-1+\sigma)^{2-\alpha}] \\
\qquad -\frac{1}{2}[(k+1+\sigma)^{1-\alpha} - 2(k+\sigma)^{1-\alpha} + (k-1+\sigma)^{1-\alpha}], 1 \le k \le n-2, \\
c_{n-1}^{(n,\alpha)} = \frac{1}{2}[3(n-1+\sigma)^{1-\alpha} - (n-2+\sigma)^{1-\alpha}] \\
\qquad -\frac{1}{2-\alpha}[(n-1+\sigma)^{2-\alpha} - (n-2+\sigma)^{2-\alpha}],
\end{cases}
$$

where $\sigma = 1 - \frac{\alpha}{2}$.

3. Space Fractional Black-Scholes Equations and Solutions

Brownian motion (BM) is an independent incremental continuous random process with normal distribution, which fails to adequately capture the leptokurtic and heavy tail phenomena observed in the dynamic behavior of underlying asset prices. To address this limitation, various modified Lévy processes and fractional BM have been introduced to more accurately represent these features. As a result, a space FBSE was derived.

3.1. Pricing Equations Based on Various Modified Lévy Processes and Their Solutions

Based on FMLS, KoBoL and CGMY processes, Cartea et al. [32] derived three corresponding space fractional B-S (FBS) equations, respectively. These equations have gained significant popularity in financial mathematics and have been extensively utilized for the pricing and risk control of financial derivatives.

3.1.1. KoBoL Pricing Equation

The KoBoL process is a modified Lévy-α-stable process. This modification incorporates a damping effect into the tails of the Lévy stable distribution. This modification ensures the existence of finite moments and improves mathematical tractability. The KoBoL pricing equation based on the KoBoL process can be expressed as [32]

$$
\begin{aligned}
\frac{\partial U}{\partial t} &+ (r - v - \lambda^{\alpha-1}(q-p))\frac{\partial U}{\partial x} \\
&+\frac{\sigma^\alpha}{2}[q(e^{-\lambda x}\,{}_{-\infty}^{RL}D_x^\alpha e^{\lambda x})U + p(e^{\lambda x}\,{}_x^{RL}D_\infty^\alpha e^{-\lambda x})U] = (r + \frac{\sigma^\alpha\lambda^\alpha}{2})U, \quad (38)
\end{aligned}
$$

where λ governs the exponential decay of the tail, $p, q \in [-1, 1]$ with $p + q = 1$ and $v = \frac{1}{2}\sigma^\alpha\{q(\lambda+1)^\alpha + p(\lambda-1)^\alpha - \lambda^\alpha - \alpha\lambda^{\alpha-1}(q-p)\}$. If $\alpha = 2$, the KoBoL pricing equation is restored to the classical B-S equation. It should be mentioned that the FD operators $e^{-\lambda x}\ {}_{-\infty}^{RL}D_x^\alpha e^{\lambda x}$ and $e^{\lambda x}\ {}_x^{RL}D_\infty^\alpha e^{-\lambda x}$ in Equation (38) are called temper fractional operators that can be used to describe the behavior of non-stationary systems.

3.1.2. CGMY Pricing Equation

CGMY process is a damped Lévy process that effectively captures the frequency and magnitude of negative and positive jumps in asset price dynamics. When assuming that log-stock prices follow the CGMY process, the pricing equation is derived as [32]

$$
\begin{aligned}
&\frac{\partial U}{\partial t} + (r - v)\frac{\partial U}{\partial x} + C\Gamma(-Y)(e^{-Gx}\,{}_{-\infty}^{RL}D_x^Y e^{Gx})U \\
+ \quad &C\Gamma(-Y)(e^{Mx}\,{}_x^{RL}D_\infty^Y e^{-Mx})U = (r + C\Gamma(-Y)(M^Y + G^Y))U, \quad (39)
\end{aligned}
$$

where $v = C\Gamma(Y)[(M-1)^Y - M^Y + (G+1)^Y - G^Y]$. The parameter $C > 0$ serves as a quantitative indicator of the overall activity level, while $G \geq 0$ and $M \geq 0$ control the exponential decay rates of the left and right tails, respectively. It is worth mentioning that the distribution exhibits symmetry when $G = M$.

3.1.3. FMLS Pricing Equation

The FMLS process is a maximally skewed Lévy stable process which can effectively catch the leptokurtic and fat tail feature. If assuming that the log-stock prices follow the FMLS process, then the valuation of a European option can be described by the following FPDE [32]

$$\frac{\partial U}{\partial t} + (r - v)\frac{\partial U}{\partial x} + v\, {}_{-\infty}^{RL}D_x^{\alpha} U = rU, \tag{40}$$

where $v = -\frac{1}{2}\sigma^{\alpha} \sec(\alpha\pi/2)$. This equation is known as the FMLS equation. Moreover, it is evident that the FMLS equation reduces to the classical B-S equation when $\alpha = 2$.

3.1.4. Analytic Solutions of These Three Equations

Chen and her co-workers investigated the option pricing using the FMLS equation [37], the CGMY equation [85], and the KoBoL equation [86]. By utilizing Fourier integral transform, they derived explicit closed-form analytical solutions for each equation based on the corresponding terminal condition. These solutions were expressed in integral form with the Fox function. Subsequently, asymptotic behavior of analytic solutions was discussed. When the log-stock price approaches extreme value, i.e., $x = \pm\infty$, the asymptotic behavior of these analytic solutions aligns with the characteristics of the corresponding option. They also demonstrated the put-call parity for each equation. These findings strongly support the utilization of each equation in option pricing. Furthermore, due to the presence of Fox functions in the integration kernel, the implementation of the analytic solution is not as straightforward as the B-S formula. To overcome this challenge, the authors expressed the Fox function in an infinite series form, thereby significantly facilitating their formulae implementation. Ara et al. [87] utilized the Legendre wavelets optimization method to optimize European options pricing based on the FMLS framework. The novelty of their approach lies in integrating the differential evolution algorithm into the Legendre wavelets method to optimize the approximation of unknown terms.

3.1.5. Numerical Simulation by the Finite Difference Method

Initially, the G-L definition yields the R-L derivative approximations with first-order accuracy [15,88]. However, Meerschaert and Tadjeran [81] highlighted instability issues in numerical methods based on this formula, leading them to propose a shifted G-L formula. As a result, higher-order approximations for the R-L FD have been developed [82,89–92].

The FMLS equation proposed by Cartea et al. [32] was employed to price knock-out barrier options. The numerical solution of the equation was obtained by discretizing the FD using the G-L definition and the Crank-Nicolson (C-N) scheme for t. Furthermore, the comparison between the results obtained using the B-S equation and the FMLS equation revealed that the latter exhibits fat tails and demonstrates superior accuracy in capturing jump characteristics or significant movements for in-the-money options. Subsequently, Marom and Momoniat [93] applied the C-N scheme and the shifted G-L formula to simulate the FMLS, CGMY, and KoBol equations. They also investigated the impact of equation parameters on option prices and explored the convergence conditions for each of these equations. However, no numerical analysis is provided in the aforementioned literature.

Subsequent researchers have developed various difference schemes and presented detailed numerical analyses. Zhang et al. [94,95] investigated the numerical approximation of the CGMY equation, KoBoL equation, and the FMLS equation. They developed implicit difference schemes with second-order accuracy in both temporal and spatial dimensions by employing weighted difference schemes [82,91] for spatial discretization and a C-N scheme for temporal discretization. Furthermore, they employed the bi-conjugate gradient stabilized method [96] in conjunction with the fast Fourier transform to effectively solve the resulting linear systems, leading to a significant reduction in storage requirements and computational cost. Other related studies on the FMLS equation were explored in [97,98].

3.1.6. Numerical Simulation by Finite Difference Coupling with Spectral Method

To improve the accuracy of convergence, the spectral method has been employed for the numerical simulation of fractional equations.

Guo and Ling [99] evaluated the FMLS equation using the Gauss-Jacobi spectral method. By comparing it with the first-order finite difference scheme, they found that the global nature of the Gauss-Jacobi method makes it well suited for solving fractional partial differential equations, allowing for natural consideration of global solution behavior. Building upon this, Xu et al. [100] discussed the numerical approximation of a two-asset option equation based on the FMLS process. They combined an implicit finite difference scheme for the temporal dimension with a collocation method utilizing shifted Chebyshev basis functions of the second kind for the spatial dimension. A similar technique was employed by Aghdam et al. [101] to numerically approximate the CGMY equation, but utilizing shifted Chebyshev polynomials of the fourth kind as basis functions. The advantage of this technique lies in its capability to handle procedures with unconditionally large orders, making it highly suitable for solving corresponding systems.

3.2. FMLS Equation Incorporating Regime Switching Dynamics and Its Solution

Various empirical studies have confirmed the presence of regime switching in financial markets [102], which has sparked extensive discussions on derivative pricing within the framework of regime switching [103,104].

3.2.1. FMLS Equation Incorporating Regime Switching Dynamics

By incorporating the regime-switching mechanism into the FMLS process and allowing the constant volatility to transition between various states following a Markov chain, Zhou et al. obtained a coupled FPDE system that captured the dynamics of option prices [105]

$$\frac{\partial U_i}{\partial t} + (r_i - v_i)\frac{\partial U_i}{\partial x} + v_i\, {}^{RL}_{-\infty}D^{\alpha}_{x} U_i - r_i U_i + \sum_{j=1}^{J} q_{ij} U_j = 0, i \in \mathcal{J} \tag{41}$$

where $1 < \alpha < 2$, q_{ij} is the transition intensity from state i to state j and it satisfies $q_{ij} > 0$ for $i \neq j$. Additionally, $\sum\limits_{j=1}^{J} q_{ij} = 0$ for each $i \in \mathcal{J}$. The interest rate r_i and the volatility σ_i are non-negative constants and $v_i = -\frac{1}{2}\sigma_i^{\alpha} \sec \frac{\alpha\pi}{2}$.

The regime-switching FMLS equation, which incorporates the phenomenon of regime switching, captures the key characteristics of asset returns and aligns with empirical observations in financial markets. The regime-switching B-S equations can be regarded as a special case of the above coupled FPDE system when $\alpha = 2$. By assigning different values to q_{ij}, the FMLS pricing equation can be obtained.

3.2.2. Numerical Solution and Analytic Solution

Compared to a single fractional partial differential equation, solving the coupled system of equations in the regime switching is more challenging for both numerical and analytical solutions. Zhou et al. [105] developed an iterative Laplace transform method for FPDE system (41) from a numerical perspective. The spatial derivatives were discretized by weighted difference methods, resulting in a system of ordinary differential equations. A Laplace transform was then applied to the temporal direction and numerical contour integral methods were used to compute the inverse Laplace transform. This proposed method exhibits second-order spatial convergence and spectral-order convergence for Laplace transform inversion.

From an analytical perspective, Lin [106] proposed a two-step solution procedure for FPDE system (41) with two states. Firstly, they assumed prior knowledge of future information regarding the Markov chain and obtained the conditional option price by analytically solving a time-dependent FPDE. Secondly, an explicit and exact pricing formula for the unconditioned price was derived using the expansion of the Fourier cosine series. They also illustrated the dynamic behavior of option prices obtained by the FMLS equation with regime switching and the FMLS equation at different expiration dates. The observations indicate that the FMLS equation incorporating regime switching leads to option prices exhibiting significant volatility disparities between different states, which become more pronounced over longer time periods. This provides valuable insights into the behavior of option prices under evolving market conditions.

4. Time and Space Fractional B-S Equations and Solutions

4.1. *Single Parameter Time-Space FBSEs*

The distinguishing characteristic of such equations lies in the interdependence between the order of time FD and the space FD. By employing fractional order Taylor's series and Itô's lemma, a fractional partial differential equation can be derived from the dynamics of stock price that satisfies a fractional stochastic differential equation.

4.1.1. Equation for Fractal Stock Exchange Dynamics and Its Solution

Considering non-random fractional dynamics driven by the usual Brownian motion, Jumarie extended the derivation of the classical B-S equation to fractal processes and subsequently obtained the following dynamical equation [34]

$$\begin{aligned} {}_0^{MR}D_t^{\alpha}V &= t^{1-\alpha}\Big(\frac{r}{\Gamma(2-\alpha)}V - rS^{\alpha}\cdot {}_0^{MR}D_S^{\alpha}V\Big) \\ &\quad -\frac{(\Gamma(1+\alpha))^3[\Gamma(2-\alpha)]^2}{\Gamma(1+2\alpha)}\sigma^2 S^{2\alpha}\cdot {}_0^{MR}D_S^{2\alpha}V,\ 0<\alpha\leq 1. \end{aligned} \tag{42}$$

Jumarie [35] improved this previous work [34] and successfully addressed the solution of (42). By transforming Equation (42) into a fractional heat equation, followed by applying the Fourier transform and inverse Fourier transform, an exact solution was derived in the form of an integral expression involving a M-L function. Yang et al. [46] employed central difference scheme to discretise the spatial derivatives and $L1$ scheme for time FD discretization. By employing parameter $\theta(0\leq\theta\leq 1)$, they obtained a difference scheme, known as the C-N scheme when $\theta=\frac{1}{2}$. This work was further extended by Li et al. [107]. They developed two alternative schemes, namely the explicit-implicit scheme and implicit-explicit scheme, which not only ensure numerical stability but also exhibit favorable parallel characteristics. It has been demonstrated that these schemes can achieve second-order spatial accuracy and temporal accuracy of $2-\alpha$ order.

4.1.2. Equation Driven by the fGBM and Its Solution

It is worth noting that both the classical B-S equation and the aforementioned fractional B-S equations (FBSEs) are based on the hypothesis that the dynamics for the stock price can be described by a SDE, where randomness is modeled by either standard BM or fractional BM. However, these equations fail to fully capture the actual movement of stock prices, including abnormal patterns, long-term dependencies, and uncertainties in volatility [108]. To address this limitation and characterize stock movement more comprehensively, Guo et al. introduced a novel stochastic process known as fractional G-Brownian motion (fGBM), which extends classical BM, fractional BM, and G-Brownian motion. Consequently, they derived a European option pricing equation within a time-space fractional B-S framework as demonstrated below [108]

$$
\begin{aligned}
{}_0^C D_t^\alpha V &= t^{1-\alpha}[\frac{r}{(2-\alpha)}V - rS^\alpha \cdot {}_0^C D_S^\alpha V] \\
&- \frac{\Gamma^2(2-\alpha)\Gamma(1+2H)\Gamma(2-2H)}{\Gamma(1+2\alpha)}\tilde{\sigma}^2\sigma^2 t^{2H-1} S^{2\alpha} \cdot {}_0^C D_S^{2\alpha} V,
\end{aligned} \tag{43}
$$

where the Hurst parameter $H \in (0,1)$ shows the long-term dependence and $\tilde{\sigma}^2$ quantifies the uncertainty in variance for the G-normal distribution. These characteristics further elucidate the volatility uncertainty inherent in the dynamics of stock price $S(t)$. The order of time FD α signifies the persistent memory in both stock and option markets. Moreover, if $H = \frac{1}{2}$ and $\tilde{\sigma}^2 = \Gamma^3(1+\alpha)$, the above equation is reduced to Equation (42). Moreover, the authors employed the Fourier integral transform method to derive explicit option pricing formulas for both European put and call options.

4.2. Bi-Parameter Time-Space FBSEs and Their Solutions

The distinguishing feature of such equations lies in the independent orders of the time FD and the space FD. When formulating these equations, it is customary to consider the dynamic behavior of the underlying asset price as a fractional stochastic process, while regarding the temporal dynamics governing option prices as a transmission mechanism exhibiting fractal properties [109,110].

4.2.1. Equation Derived by Fractional Wiener Process and Its Solution

When the stochastic process is considered as a fractional Wiener process as mentioned in Formula (7) of the study in [36], Liang et al. derived the following equation [36]:

$$
\frac{A_\gamma S^{d_f-1}}{\Gamma(\alpha)} {}_t^{MR} D_T^\gamma V + \frac{\sigma^2 S^{2\alpha}}{\Gamma(2\alpha)} {}_0^{MR} D_S^{2\alpha} V + \frac{rS^\alpha}{\Gamma(\alpha)\Gamma(\alpha-m)} {}_0^{MR} D_S^\alpha V - rV = 0. \tag{44}
$$

It should be noted that researchers often simplify the study of these equations by assuming A_γ and d_f to be 1.

By combining Laplace transform and Fourier transform, along with the expansion theorem for the Laplace transform, explicit option pricing formulas were derived [36]. These formulas were expressed as infinite series containing integrals when $\gamma > 0, 1 \leq \alpha \leq 2$. Edeki et al. [43] investigated the specific form of (44), i.e., the scenario where the spatial derivative D^α reduced to a first-order derivative. They presented approximate analytical solutions expressed as infinite series using a combined approach known as fractional complex transform in conjunction with a modified DTM. Recently, Rezaei and Izadi [111] have proposed a novel approach by combining the fractional calculus theory (FCT) with RPSM to efficiently obtain an analytical solution for the same equation where the fractional operator is considered as a local FD. RPSM is a novel iterative strategy proposed for obtaining Taylor expansion series solutions to systems of linear and nonlinear ODEs and PDEs. This technique was initially introduced by O.A. Arqub [112]. The presented FCT-RPSM technique can be considered as a directly applicable approach, devoid of any form of discretization, linearization, or other additional imposed assumptions.

Meng et al. [113] considered the numerical approximation of the same equation presented in [43] using a combined approach involving the Haar wavelet integration method, the variational iteration method (VIM), and the HPM. The HPM and VIM are widely recognized as efficient tools for solving nonlinear problems. They employed three techniques: (1) Haar wavelet integration method to convert the PDEs into an algebraic equation system, (2) HPM for linearising the problem, and (3) VIM for efficiently solving the algebraic equation system. The proposed algorithm demonstrated high computational efficiency for numerically solving the FBSE.

4.2.2. Equation Derived by FMLS Process and Solution

When the fractional stochastic process is regarded as the FMLS process, Zhang et. al derived the following time-space fractional option pricing equation [114]:

$$ {}^{MR}_{t}D^{\gamma}_{T}U + \left(r + \frac{\sigma^{\alpha}}{2}\sec\left(\frac{\alpha\pi}{2}\right)\right)\frac{\partial U}{\partial x} - \frac{\sigma^{\alpha}}{2}\sec\left(\frac{\alpha\pi}{2}\right)\cdot {}^{RL}_{-\infty}D^{\alpha}_{x}U - rU = 0, \tag{45} $$

where $0 < \gamma < 1, 1 < \alpha \leq 2$.

Later, researchers extended Equation (45) to the two dimensional case as follows [115]:

$$ \begin{aligned} {}^{C}_{0}D^{\gamma}_{t}U \quad &+ \quad \left(r + \frac{\sigma_1^{\alpha_1}}{2}\sec\left(\frac{\alpha_1\pi}{2}\right)\right)\frac{\partial U}{\partial x} + \left(r + \frac{\sigma_2^{\alpha_2}}{2}\sec\left(\frac{\alpha_2\pi}{2}\right)\right)\frac{\partial U}{\partial y} \\ &- \quad \frac{\sigma_1^{\alpha_1}}{2}\sec\left(\frac{\alpha_1\pi}{2}\right)\cdot {}^{RL}_{-\infty}D^{\alpha}_{x}U - \frac{\sigma_2^{\alpha_2}}{2}\sec\left(\frac{\alpha_2\pi}{2}\right)\cdot {}^{RL}_{-\infty}D^{\alpha}_{y}U - rU = 0. \end{aligned} \tag{46} $$

where the log-stock prices $x = \ln S_1$ and $y = \ln S_2$ are assumed to follow two independent FMLS processes.

In [114], the authors constructed a spatial accuracy of second-order and temporal accuracy equivalent to $2 - \gamma$ implicit finite difference scheme to approximate (45). Moreover, the fast bi-conjugate gradient stabilized method was proposed to address the numerical scheme, aiming to reduce storage requirements and computational expenses. Regarding the equation presented in [115], an implicit finite difference scheme was developed by using the backward finite difference formula [15] to approximate the Caputo temporal derivative and employing a shifted G-L type formula to discretize the left R-L spatial derivative. Subsequently, they proposed a parallel all-at-once bi-diagonal block circulant preconditioner for solving the discretized linear system. The resulting discretization exhibits first-order spatial accuracy and $2 - \gamma$ order temporal accuracy.

5. Time Fractional B-S Equations and Their Solutions

In 1997, Carpinterj and Mainardi [116] highlighted the utility of FPDEs in studying fractal geometry and fractal dynamics. With the recognition of fractal structures in financial markets, FPDEs have gradually been integrated into financial theory.

5.1. Simple Time Fractional B-S Equations

By replacing the first-order time derivative with an α-order $(0 < \alpha \leq 1)$ FD, Wyss proposed a time FBSE as follows [31]:

$$ {}^{RL}_{0}D^{\alpha}_{t}V + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{47} $$

Later, assuming that the evolution of option prices over time exhibits a fractal transmission mechanism, and the dynamics of underlying asset prices adheres to geometric Brownian motion, Chen et al. established a specific instance of Liang et al.'s equation (44) in the following form [33]:

$$ {}^{MR}_{t}D^{\alpha}_{T}V + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \tag{48} $$

where D is the dividend yield, and $D = 0$ denotes the absence of dividend payments in the stock price.

Equation (48) is very similar to Equation (47) in its structure, except that FDs are defined differently. For simplicity, we represent them in a unified equation

$$ D^{\alpha}_{t}V + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{49} $$

Several researchers extended Equation (49) to a more general case, such as incorporating interest rate and volatility as functional variables [50,117,118], and considering a broader range of assets [119].

Remark 11. *The time FD in Equation (49) is commonly considered as either the modified R-L derivative or Caputo derivative. The modified R-L derivative can be transformed into the Caputo derivative through variable substitution for a smooth function [47]. Therefore, if these two derivatives are employed as time FD, further clarification will not be provided in the subsequent literature review.*

5.2. Analytic Solution and Semianalytic Solution

The analytical solution, obtained exclusively through the integral transformation method, is commonly expressed in integral form involving special functions [31], thereby presenting computational challenges. Noting that the homotopy perturbation method (HPM), homotopy analysis method (HAM), RPSM, ADM and Elzaki transform are characterized by their independence from discretization, linearization, restrictive assumptions or transformations, therefore, are widely utilized to solve both linear and nonlinear differential equations. In recent two decades, researchers introduced them together with other techniques to solve Equation (49), aiming to obtain analytical solutions in the form of a computable series.

5.2.1. The HPM

The HPM was originally introduced and applied by J.H.He [120]. Combining the HPM with other transform techniques can effectively obtain (approximate) the analytic solution in a series form that is easy to implement. The LHPM combined HPM with Laplace transform was applied by Kumar et al. [38] to solve (49). Furthermore, the authors also considered the case of non-constant volatility. Subsequently, the same equation as in [38] was considered by Elbeze et al. [121] using a different method, i.e., HPM incorporating the Sumudu transform (SHPM), yielding results that are consistent with those obtained using the LHPM. Later, the LHPM was generalized by Sawangtong et al. [122] and Prathumwan and Trachoo [123] to a two-dimensional case for pricing different types of European options.

The researchers also discuss cases where alternative definitions of the time FD are employed. Yavuz and Özdemir [40] proposed conformable fractional ADM and conformable fractional modified HPM to solve both Equation (49) and a generalized fractional B-S pricing equation with non-constant volatility, when considering the C-F derivative as the time FD. While considering the K-C derivative as time FD, based on the characteristics of the integral kernel in the definition of the K-C derivative, Fall et al. [124] replaced Laplace transform with $\rho-$ Laplace transform [125] and combined HPM to solve the Equation (49) with the usual initial condition. Furthermore, this technique has also been employed by Ampun and Sawangtong [126] to address the same equation subject to a varying initial condition, i.e., the Katugampola integral initial condition. Moreover, Thanompolkrang et al. [127] extended the aforementioned method to a scenario involving two assets.

5.2.2. The HAM

The HAM, originally proposed by Liao et al. [128], offers a distinct advantage over the HPM by overcoming the limitation of the perturbation method dependent on small parameters. Moreover, the HAM offers an efficient approach to control and adjust the convergence of the approximation series by selecting the auxiliary parameter $\hbar$ (or vector $\hbar$) within the series, which is not found in perturbation methods. Yavuz and Özdemir [129] employed a combination of the HAM and Laplace transform to address two identical problems as presented in [40]. However, they considered the C-F derivative with a smooth kernel as the time FD. Furthermore, Fadugba [130] applied HAM to evaluate a European Call Option when taking into account the Caputo derivative.

5.2.3. The RPSM

The RPSM, initially proposed by Arqub et al. [131], is based on the generalized Taylor's formula. This approach offers an approximate analytical solution to the problem by presenting them in a truncated series form. Haq and Hussain [41] utilized the RPSM to derive an exact solution of Equation (49) with both constant and variable coefficients, followed by a numerical approximation using the radial basis function (RBF) mesh-free method. Additionally, they investigated the impact of shape parameter selection on the accuracy of the RBF meshless techniques, and suggested a method to identify an ideal shape parameter for improved accuracy. Subsequently, an efficient approximate iterative mathematical approach, based on the residual power series (RPS) algorithm, was proposed by [132] to obtain an approximate analytical solution for the same example as referenced in [38]. The fundamental superiority of the RPSM over other existing analytical methods stems from its capacity to reduce the complexity and duration of calculations.

5.2.4. Other Coupling Methods

Hariharan et al. [39] employed the Laplace Legendre wavelet method (LLWM) to obtain a rapid and accurate solution, followed by an iterative method to determine the coefficients in the Legendre wavelet expansion expression of the option function. This approach demonstrates superior efficiency in comparison to the conventional Legendre wavelet technique for addressing FPDEs. By integrating the expansion of eigenfunction with the Laplace transform, a comprehensive series solution for a double barrier option was developed in [33]. Kanth and Aruan [133] introduced the fractional DTM and the modified fractional DTM for solving Equation (49) involving non-constant volatility rate and risk-free interest rate. They suggested that fractional DTM and modified fractional DTM were significantly simpler compared to HPM, VIM and ADM. A finite difference method coupled with ADM was applied by Song [42] to obtain an approximate semi-analytical solution. Subsequently, Uddin and Taufiq [134] proposed the Laplace transformed radial kernel method, which utilized the Laplace transform to eliminate the time variable and significantly reduce computational cost. Then, they employed radial kernels to discretising the spatial operator in the local context, leading to sparse differentiation matrices. Finally, the obtained integral form solution was evaluated by a quadrature rule. Recently, Khan et al. [135] employed the Laplace perturbation iteration technique to derive a fractional analytical solution in series representation.

5.3. Numerical Solution Techniques

It is worth mentioning that the limitation of previous analytic solutions is attributed to their reliance on convolution of some special functions or infinite series with integrals. Consequently, numerous researchers have been striving to develop numerical approximations to address the option pricing problem governed by the time fractional B-S equation.

5.3.1. The Finite Difference Method

The time fractional equation typically employs the modified R-L derivative and Caputo derivative to represent time derivative. The former is often transformed into the latter for computational convenience, while common discrete forms of the Caputo derivative include the $L1$ formula, $L2-1\sigma$ formula, and $L1-2$ formula. The spatial and temporal mesh are denoted as h and Δt, respectively, while the order of the time derivative is represented by α. These notations are consistently used throughout this section to maintain simplicity.

Regarding Equation (48), when discretizing the time derivative by $L1$ scheme and central difference for spacial discretization, Zhang et al. [47] obtained a implicit FD scheme with a convergence rate of $O(\Delta t^{2-\alpha}+h^2)$. Similar techniques were utilized in [136,137] to investigate the same problem incorporating different FDs, namely the C-F derivative and Caputo derivative, respectively. In recent study, Rezaei et al. [138] extended Equation (48) by incorporating transaction costs. The new equation considers volatility as a function of spatial derivatives, asset prices and time, with the dividend and interest rate being time dependent. Subsequently, an implicit discrete scheme based on mid-layer, distinct from the previous literature, was developed with $O(\Delta t^{3-\alpha} + h^2)$ accuracy.

To improve the discretization accuracy, the compact scheme in conjunction with higher-order temporal discretization techniques was employed. Staelen and Hendy [139] improved the results in [47] by devising a compact fourth-order spatial difference scheme while maintaining $2 - \alpha$ order temporal accuracy through the $L1$ scheme. Next, Tian et al. [48] further extended the work presented in [139]. They constructed three distinct compact difference schemes, each with their respective order of accuracy. The first scheme has an accuracy of $O(\Delta t^{2-\alpha} + h^4)$, where the time finite difference was discretized using the $L1$ formula. The second scheme achieves an accuracy of $O(\Delta t^2 + h^4)$ by employing the $L2 - 1\sigma$ formula. The third scheme, which utilized the $L1 - 2$ formula, achieves an accuracy of $O(\Delta t^{3-\alpha} + h^4)$. Recently, Abdi et al. [49] constructed two higher-order numerical approaches of order $O(\Delta t^{3-\alpha} + h^6)$ and $O(\Delta t^{3-\alpha} + h^8)$ by combining the $L1 - 2$ scheme with sixth and eighth-order compact schemes. Other similar works can be found in references [140,141].

The non-regularity of a solution resulting from non-smoothness of initial condition may have an impact on the accuracy of the discrete scheme [142–145]. To address this issue, non-uniform meshes have been considered as one of the effective approaches. Cen et al. [146] considered the pricing of the European call option governed by the Formula (49). The authors initially adapted the Equation (48) into an equivalent integral-differential expression. Subsequently, they introduced a first-order accurate integral discretization scheme on a priori graded mesh for temporal discretization, and utilized a central difference scheme on a piecewise uniform mesh to approximate the spatial derivatives. Kazmi [147] later improved Cen's work by utilizing the trapezoidal rule for temporal discretization, resulting in a discrete scheme exhibiting $1 + \alpha$-order temporal accuracy even when considering non-smooth payoff functions. Then a Richardson type extrapolation technique was utilized to improve the accuracy to second order in time. On the other hand, Gu et al. [148] developed a temporal non-uniform mesh combined with the compact difference scheme, which exhibits a convergent order of $O(\Delta t^2 + h^4)$. Soon afterwards, a improved work was presented in [149]. The authors proposed the sixth-order/eighth-order compact difference scheme, incorporating a trapezoidal formulation based on temporally graded meshes for the temporal integral term.

Huang et al. [117] considered a more complex scenario where coefficients of Equation (48) are time dependent, making it challenging to obtain prior information about the exact solution. Consequently, they developed an adaptive moving mesh method to handle potential singularities. Later, Kim et al. [119] extended the discrete technique presented in [117] to the three-dimensional (3D) version of (48), and solved the resulting fully discretized equation using an operator splitting method.

5.3.2. The Meshless Method

The meshless local Petrov-Galerkin (MLPG) and implicit finite difference method were used by Phaochoo et al. [50] to discretise the governing equation, considering the interest rate and volatility as functions. In MLPG method, a moving kriging approximation was utilized to construct the shape function. This method avoids the domain integral in the weak-form.

Another popular meshless method is RBFs, offering several advantages such as spectral convergence, dimension insensitivity, independence from node connectivity requirements, and ease of implementation. The selection of the basis function governs the spectral convergence. The commonly used radial basis functions (RBFs) are presented in Table 1, which provides a range of options for practitioners to choose from based on their specific requirements and problem characteristics [51].

Table 1. Definition of some types of RBFs.

Name of RBF (Abbrevistion)	$\phi(r), r \geq 0$	Smoothness
Gaussian (GA)	e^{-cr^2}	Infinite
Generalised multiquadric (GMQ)	$(c^2 + r^2)^\beta$	Infinite
Inverse multiquadric (IMQ)	$\frac{1}{\sqrt{c^2+r^2}}$	Infinite
Inverse quadric (IQ)	$(c^2 + r^2)^{-1}$	Infinite
Multiquadric MQ)	$\sqrt{c^2 + r^2}$	Infinite
Cubin (CU)	r^3	Piecewise
Linear (LI)	r	Piecewise
Monomial (MN)	r^{2k-1}	Piecewise
Thin plate spline (TPS)	$r^2 log(r)$	Piecewise

A combination of global MQ RBF method and $L1$ scheme was employed by Golbabai et al. [51] to approximate Equation (48). Nikan et al. [150] improved the methodology proposed in [51] by substituting the global RBF method with the a local RBF method. The latter method exclusively utilizes neighboring data points, resulting in a sparse matrix system. Consequently, it effectively addresses the ill-conditioning issues associated with dense system matrices and reduces sensitivity concerns related to the shape parameter used in the overall strategy. The global RBF method was extended by Delpasand and Hosseini [151] to approximate the two-assets case, employing the C-N method based on $L1$ scheme for discretising the time variable. The numerical results demonstrated high accuracy achieved by the proposed method without significant computational costs. Furthermore, different modified $L1$ schemes [152], in conjunction with various RBFs, have been employed to address the diminished convergence order observed for the initial condition with non-smoothness [153–155].

5.3.3. The Spline Interpolation Method

Ghafouri et al. [156] proposed a novel numerical approach that combines the Laplace transform for approximating the time FD and quasi-interpolation using cubic B-spline for spatial discretization. They further employed a C-N method for temporal discretization. Roul [52] employed a spatial discretization based on a quintic B-spline basis function collocation approach, achieving fourth-order accuracy. For temporal discretization, the $L1$ scheme was utilized. Based on the work of [52], Tian et al [157] proposed a $L1-2$ scheme combined with a compact quadratic spline collocation method, which not only ensures fourth-order spatial accuracy but also improves the temporal accuracy to $3-\alpha$ order. $L1$ scheme and a collocation method based upon an extended cubic B-spline with second-order accuracy in spatial direction was constructed by Akram et al. [158]. A similar technique was considered in [159]. Recently, Pan and Zhang [160] proposed a MQ quasi-interpolation technique for approximating space derivative. The advantages of this method lie in its ability to handle a resultant full matrix and eliminate the ill-conditioning issues when employing RBF as a global interpolant function.

5.3.4. The Numerical Technique with Exponential Convergence

He and Zhang [53] implemented $L1$ formula in conjunction with the Fourier-spectral method, which exhibited a convergence order of $O(\Delta t^{2-\alpha} + N^{1-m})$, where m represents the regularity of the option function and N denotes the degree of polynomial. Equation (48) was discussed by She et al. [161] from two perspectives. Firstly, an exact solution was derived through variable separation. The regular analysis of the solution revealed that it blows up as the time is infinitely close to the expiry date. Secondly, to address the challenge posed by the weak regularity of the solution, a modified $L1$ scheme based on a variable transformation was proposed for time discretization, while spatial discretization was achieved using Chebyshev Galerkin methods. X.An et al. [55] employed a space-time spectral method assuming a smooth payoff function. The proposed approach utilizes the Fourier-like basis functions for spatial discretization and Jacobi polynomials for temporal discretization.

Tour et al. [162] employed a Richardson extrapolation approach based on the $L1$ approximation to optimise the reduction in convergence order caused by non-smooth initial values. Additionally, they utilized a spectral element method incorporating Legendre and Laguerre basis functions in the spatial domain. A computational approach combining finite difference with the operational matrix approach (OMA) was developed by Srivastava et al. [163]. The authors used the $L1-2$ scheme to discretise the Caputo derivative. They utilized shifted Legendre polynomials and shifted Chebyshev polynomials within the OMA to simulate the spatial derivatives. Mesgarani et al. [164] proposed the $L1$ scheme and a spatial collocation method based on the third Chebyshev polynomials. Subsequently, they further improved their previous work in [164] by constructing a quadratic interpolation with $3-\alpha$ order accuracy [165] and incorporating a collocation method utilizing Legendre polynomials [166].

5.3.5. Other Solution Techniques with Respect to Other FD

In [167], researchers suggested that a fixed fractional order cannot adequately represent the dynamics of market uncertainties or price fluctuations. To address this issue, they introduced the concept of a random order as a variable fractional order with random values. Motivated by the idea of variable FD in [167], Zhang and Zheng [168] introduced a time-dependent variable FD to a time FBS pricing equation. A fully-discrete finite element scheme was derived by employing a difference scheme for temporal discretization and the finite element technique for spatial discretization.

When considering the distributed order derivative as the time derivative, Kumar and Singh [56] developed a novel approach based on operational matrices derived by Chebyshev and Legendre wavelets. Moreover, by combining the Gauss-Legendre quadrature formula and standard Tau method with the derived operational matrices, they transformed the distributed order time FBSE into a system of linear algebraic equations. On the other hand, Rahimkhani et al. [169] achieved the same goal by utilizing the fractional integral operator of Hahn hybrid functions, the Gauss-Legendre quadrature formula and the collocation method.

Mohammadizadeh et al. [170] investigated a more complex ψ-Hilfer fractional B-S (ψ-HFBS) equation with non-constant volatility. They provided the solvability analysis and implemented a time fractional Chebyshev pseudo-spectral method under appropriate terminal and boundary conditions.

5.4. Complex Time Fractional B-S Equations and Their Solutions

5.4.1. Equations Derived by Fractional Taylor Series and Their Solutions

By utilizing a derivative technique similar to that employed in (42), a time FBSE was derived in the following form [34]

$$ {}^{MR}_{\ \ 0}D^{\alpha}_{t}V = (rV - rS\frac{\partial V}{\partial S})\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}. \tag{50} $$

Similarly, Farhadi et al. derived an alternative time fractional B-S equation [171]

$$ {}^{MR}_{\ \ 0}D^{\alpha}_{t}V = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}(rV - rS\frac{\partial V}{\partial S} - \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2}). \tag{51} $$

As for Equation (50), Jumarie [35] derived an exact solution expressed as an integral form involving the M-L function by using a similar technique as used in (42). The numerical approximation of Equation (50) was addressed by developing an implicit finite difference scheme [45] and a θ finite difference scheme [172,173].

When examining stocks with continuous dividend payments, a pricing equation proposed by Nuugulu et al. [174] closely resembles the one (50). To estimate European put option premiums, an implicit finite difference scheme with $O(\Delta t + h)$ was constructed. Meanwhile, Nuugulu et al. [175] improved upon the results in [174] by introducing a robust numerical method based on extending a C-N finite difference approach with $O(\Delta t^2 + h^2)$. Considering the time-varying dynamics of asset prices in the market, Rezaei et al. [118] proposed a more complicated equation, which incorporates time-varying interest rates and dividend parameters. Subsequently, an $L1$ scheme combined with central difference was employed to obtain a discrete scheme with $O(\Delta t^{2-\alpha} + h^2)$.

In [171], Equation (51) was solved using a reconstructed variational iteration method, which yielded a series solution. Later, Rezaei et al. [176] employed a similar technique presented in [118] to evaluate double barrier options in the European market with dynamically adjusting barriers, which are governed by an extension of Equation (51) incorporating time-dependent interest rates, volatility, and dividend parameters.

5.4.2. Equation with Markov-Switching Properties and Its Solution

Modifying the classical B-S equation for the dynamics of the price of the underlying asset to incorporate Markov-switching properties, a B-S equation with regime-switching can be derived. Building upon this concept, Laura proposed a time FBSE with regime-switching [78]:

$$\begin{aligned} {}_0^C D_t^{\alpha_i} V_i - \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} - (r_i - \delta_i) S \frac{\partial V_i}{\partial S} + r_i V_i - q_{ii} V_i - \sum_{i \neq j} q_{ij} V_j = 0,& \\ 0 < \alpha_i < 1, i \in \mathcal{J},& \end{aligned} \tag{52}$$

by using Itô's lemma within the framework of the no-arbitrage principle and assuming that the evolution of option price over time is a fractal transmission system. Here $\mathcal{J} = \{1, 2, \cdots, J\}$ is a finite state space. r_i, δ_i and σ_i are the risk-free interest rate, the dividend rate and the asset price volatility in the state i, respectively. q_{ij} is the same as that in FPDE system (41).

In [78], researchers proposed the $L1$ scheme combined with the Richardson extrapolation approach to improve the temporal accuracy. For spatial discretization, a local RBF-generated finite difference method was constructed, where the finite difference weights were determined using Gaussian RBFs. This approach achieved a fourth-order convergence rate. Then they further explored the spectral element method. For the European option with a non-smooth payoff at the strike price, the entire domain was divided into two sub-domains. The Legendre polynomials and stable Gauss-Radau-Laguerre polynomials were employed in their respective sub-domains to derive a fully discrete scheme with spectral accuracy. Finally, the author applied these numerical techniques to address the pricing of some special options such as the butterfly call option and double barrier call option.

5.4.3. Equation with Jump-Diffusion and Its Solution

A time FBSE under jump-diffusion can be described as follows [177]:

$$\begin{aligned} {}_0^C D_t^{\alpha} V + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k) S \frac{\partial V}{\partial S} &- (r + \lambda) V \\ &+ \lambda \int_0^\infty V(S\xi, t)\varphi(\xi) d\xi = 0, \end{aligned} \tag{53}$$

where $\lambda > 0$ denotes the intensity of the independent Poisson process and k is the expected relative jump size. $\varphi(\xi)$ represents the probability density function of the jump with amplitude ξ, which satisfies that $\forall \xi, \varphi(\xi) \geq 0$ with $\int_0^\infty \varphi(\xi) d\xi = 1$.

In [177], the authors considered both numerical and analytical solutions for Equation (53). The equation was initially transformed into a fractional integro-differential equation with the Fredholm integral operator. By utilizing the $L1$ scheme on a graded mesh for time discretization, combined with a second-order central difference scheme for spatial derivatives and the composite trapezoidal approximation for the integral component, a fully discrete framework was derived. Furthermore, an analytical approximate solution was obtained using the ADM. Then they compared this result with numerical solution.

6. Inverse Problem of Fractional B-S Equations

To the best of our knowledge, research on the inverse problem of FBSEs is still limited.

The coefficients of the FMLS Equation (40), including the tail index α and implied volatility σ, were estimated by Jiang [178] using three well-established statistical inversion schemes, namely Markov Chain Monte Carlo, slice sampling algorithm, and Hamiltonian/hybrid Monte Carlo algorithm. Jiang and Xu [179] investigated a time FBSE (48) in which the implicit volatility is thought to be related to the underlying asset price. Firstly, they constructed a robust $L1$-central difference implicit approximation scheme to effectively solve Equation (40). Subsequently, a linearization method was employed to transform an inverse problem into a Fredholm integral equation. Then the implied volatility was reconstructed via additional data. To tackle this ill-posed problem, the Fredholm integral equation was solved by the Tikhonov regularization method. Recently, X.An et al. [180] aimed to estimate the parameters of Equation (49) with Caputo derivative by using the real option prices of the S&P 500 index options. They initially developed a high order difference scheme using the $L1-2$ formula and central-difference technique, achieving an accuracy of $O(\Delta t^{3-\alpha}+h^2)$. Then they employed a modified hybrid Nelder-Mead simplex search and particle swarm optimization (MH-NMSS-PSO) to determine the values of the implied volatility and the order of Caputo derivative. Based on empirical results, it was observed that the order of FD can serve as a market fluctuation indicator for classifying financial environments, exhibiting exceptional adaptability in fitting real data compared to the traditional B-S equation.

7. Conclusions

The real financial market exhibit inherent nonlinearity and memory effects that can be more accurately captured using fractional derivatives and integrals, leading to more precise modeling methods. In traditional integer-order calculus, market fluctuations are typically measured using volatility as an essential risk indicator in finance. However, due to the presence of long-range memory and heavy-tailed along with leptokurtic characteristics in financial markets, traditional methods may not fully reflect the actual situation. The introduction of the fractional index parameter in FBSEs allows for adjusting the behavior of long tails and non-normal features in stochastic processes. This enables better characterization of long-term correlations and provides more accurate and reliable risk measurement methods. The solution techniques of FBSEs offer efficient and stable ways to analyse and predict market behaviors.

With the increasing popularity of computational methods and ongoing advancements in financial mathematics, it is worth expecting further improvements and innovations in the field of FBSEs. These may include utilizing machine learning and neural network techniques for approximating solutions, as well as integrating hybrid equations and multi-factor equations into the fractional-order B-S framework. We believe that as these equations continue to evolve, they will undoubtedly play a crucial role in informed decision-making for financial professionals and policymakers.

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