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Properties and Applications of Symmetric Quantum Calculus

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Abstract: Symmetric derivatives and integrals are extensively studied to overcome the limitations of classical derivatives and integral operators. In the current investigation, we explore the quantum symmetric derivatives on finite intervals. We introduced the idea of right quantum symmetric derivatives and integral operators and studied various properties of both operators as well. Using these concepts, we deliver new variants of Young's inequality, Hölder's inequality, Minkowski's inequality, Hermite–Hadamard's inequality, Ostrowski's inequality, and Grüss–Chebyshev inequality. We report the Hermite–Hadamard's inequalities by taking into account the differentiability of convex mappings. These fundamental results are pivotal to studying the various other problems in the field of inequalities. The validation of results is also supported with some visuals.

Keywords: convex; function; Hermite–Hadamard; Holder's; symmetric; quantum; Ostrowski



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1. Introduction

The theory of convex functions has diverse applications in several fields of science but its impact on the growth of inequalities is very significant. Most of the fundamental results related to inequalities are derived through convex mappings.

Several innovative and novel techniques are utilized to derive new counterparts of fundamental results of inequalities. One of them is \hat{q}_1 -calculus because it has advantages over classical concepts. In the perspective of \hat{q}_1 -calculus one can obtain the quantum derivatives of piecewise discontinuous mappings. Meanwhile symmetric calculus is applied to study the non-differentiable mappings for example absolute functions. Quantum symmetric calculus is also a very interesting and intriguing aspect of mathematical analysis. It generalizes the classical symmetric concepts by $\hat{q}_1 \rightarrow 1$.

In Ref. [1] Sudsutad et al. explored the various famous inequalities over finite intervals. In 2018, Alp et al. [2] explored the correct version of Hermite–Hadamard inequality by utilizing the differentiability of convex functions and several other important inequalities. Bin-Mohsin et al. [3] explored the error boundaries for an open method known as the Milne rule implementing the Mercer inequality and quantum calculus. In Ref. [4], Kunt and his fellows gave the idea of right-sided quantum derivatives and integral operators and provided a detailed discussion about these operators. In Ref. [5], Nwaeze and Tameru examined the unified integral inequalities through η -convex functions. Different values of η and other parameters involved in identity produced innovative results. Asawasamrit et al. [6] reported some new integral inequality results concerning Hahn quantum operators. The results produced in the paper can be reduced to integral inequalities established via well-known quantum operators defined on finite intervals. Kunt et al. [7] provided the

quantum Montgomery equality and concluded some of Ostrowski's type bounds. In the sequel, Kalsoom et al. [8] devoted their efforts to come up with Ostrowski estimates by invoking the notion of higher order n -polynomial preinvex functions. In Ref. [9], Ali and his colleagues purported new variants of both midpoint and trapezoidal rule via quantum concepts. In Ref. [10], the authors presented the trapezium-type inequalities in a more general form and deduced some inequalities for comparative study with existing outcomes. In Ref. [11], the authors focused on establishing the post-quantum analogues of trapezium inequality via generalized m -convex mappings. Duo et al. [12] analyzed some quantum estimates through a unified approach to obtain several type inequalities by specifying the values for parameters. In Ref. [13], Khan and his co-authors developed the trapezium type inequalities by taking into account the green functions via quantum calculus and have investigated the post-quantum analogues of Hermite–Hadamard type involving generalized m -convexity. Saleh et al. [14] derived quantum dual Simpson-type error estimates involving convex functions and presented some implications as well. In Ref. [15], the authors utilized the majorization approach and quantum calculus to develop new counterparts of Hermite–Hadamard–Mercer type inequalities. In Ref. [16], Alomari derived the quantum variant of Bernoulli's inequality and its consequences. In Ref. [17], Alp and Sarikaya established the quantum integral inequalities based on newly developed quantum operators known as second sense quantum integral operators. Nosheen et al. [18] studied Ostrowski's type variants through s -convex mappings and quantum symmetric calculus. For further details, see Refs. [19–21].

The principal inspiration of the current study is to investigate the right symmetric derivative and integral operator. To accomplish this study, we divide our complete study into three parts. In the first part of the study, we provide some essential facts and a literature review related to the problem. In successive segments, we introduce the notions of right quantum symmetric operators and explore their essential properties as well. In the third part, we discuss the applications of newly studied concepts in previous sections to integral inequalities. Later on, concluding remarks and future insights are provided. We hope this will create new venues for the investigation.

2. Preliminaries

Let us report the notion of convex mappings:

Definition 1 ([22]). Any mapping $\mathcal{Z} : [C_1, C_2] \rightarrow \mathbb{R}$ is referred to be a convex mapping if,

$$\mathcal{Z}((1 - \tau)x + \tau y) \leq (1 - \tau)\mathcal{Z}(x) + \tau\mathcal{Z}(y), \quad \forall x, y \in [C_1, C_2] \quad (1)$$

where $\tau \in [0, 1]$.

The geometrical interpretation of convex mapping is described as if the chord line joining the point $(C_1, \mathcal{Z}(C_1))$ and $(C_2, \mathcal{Z}(C_2))$ always lies on or above the graph of a mapping. In addition, any mapping is convex if and only if its epigraph is a convex set.

Now we recall a famous consequence of convex mappings, which is known as trapezium inequality proved by Hermite and Hadamard separately and is demonstrated as:

Let $\mathcal{Z} : [C_1, C_2] \rightarrow \mathbb{R}$ be a convex mapping, then

$$\mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) \leq \frac{1}{C_2 - C_1} \int_{C_1}^{C_2} \mathcal{Z}(x) dx \leq \frac{\mathcal{Z}(C_1) + \mathcal{Z}(C_2)}{2}.$$

This inequality serves as criteria for convex mappings and it is widely utilized to determine the error bounds of both mid-point and trapezoidal rules. Moreover, it determines the bounds of the average mean integral. Several studies have been carried out regarding this inequality. For further detail, one may consult the following Refs. [23–25].

Now we will recollect some facts regarding symmetric quantum calculus.

Let $\hat{q}_1 \in (0, 1)$ and let I be arbitrary interval of \mathbb{R} containing 0,

$$I_{\hat{q}_1} = \{\hat{q}_1 x | x \in I\}.$$

Clearly $I_{\hat{q}_1} \subset I$.

Definition 2. ([26]). Let $\mathcal{Z} : I \rightarrow \mathbb{R}$. Then the symmetric quantum derivative operator is described as:

$$D_{\hat{q}_1}^s \mathcal{Z}(\tau) = \frac{\mathcal{Z}(\hat{q}_1^{-1}\tau) - \mathcal{Z}(\hat{q}_1\tau)}{(\hat{q}_1^{-1} - \hat{q}_1)\tau}, \tau \neq 0.$$

Additionally, $D_{\hat{q}_1}^s \mathcal{Z}(0) = \mathcal{Z}'(0)$, $\tau = 0$ provided that \mathcal{Z} is differentiable at $\tau = 0$. If \mathcal{Z} is differentiable mapping at $\tau \in I_{\hat{q}_1}$ then $\lim_{\hat{q}_1 \rightarrow 1} D_{\hat{q}_1}^s \mathcal{Z}(\tau) = \mathcal{Z}'(\tau)$.

Theorem 1. Suppose that \mathcal{Z} and \mathcal{G} are two \hat{q}_1 -symmetric differentiable on I° , then for any $c, m_1, n_1 \in \mathbb{R}$ and $\tau \in I_{\hat{q}_1}$

1. $D_{\hat{q}_1}^s \mathcal{Z}(\tau) = 0 \Leftrightarrow \mathcal{Z} = c$.
2. $D_{\hat{q}_1}^s [m_1 \mathcal{Z}(\tau) + n_1 \mathcal{G}(\tau)] = m_1 D_{\hat{q}_1}^s \mathcal{Z}(\tau) + n_1 D_{\hat{q}_1}^s \mathcal{G}(\tau)$.
3. $D_{\hat{q}_1}^s \mathcal{Z}(\tau) \mathcal{G}(\tau) = \mathcal{G}(\hat{q}_1 \tau) D_{\hat{q}_1}^s \mathcal{Z}(\tau) + \mathcal{Z}(\hat{q}_1^{-1} \tau) D_{\hat{q}_1}^s \mathcal{G}(\tau)$.

Moreover, In 2023, Khan et al. [27] explored the concept of left quantum symmetric derivatives and integrals over finite intervals. Assume that $J = [C_1, C_2] \subset \mathbb{R}$, $0 \in J$ and $0 < \hat{q}_1 < 1$, then the left quantum symmetric derivative is described as:

Definition 3 ([27]). Let $\mathcal{Z} : J \rightarrow \mathbb{R}$ be a continuous mapping, then

$${}_{C_1} D_{\hat{q}_1}^s \mathcal{Z}(\tau) = \frac{\mathcal{Z}(\hat{q}_1^{-1}\tau + (1 - \hat{q}_1^{-1})C_1) - \mathcal{Z}(\hat{q}_1\tau + (1 - \hat{q}_1)C_1)}{(\hat{q}_1^{-1} - \hat{q}_1)(\tau - C_1)}, \tau \neq C_1.$$

And ${}_{C_1} D_{\hat{q}_1}^s \mathcal{Z}(C_1) = \lim_{\hat{q}_1 \rightarrow 1} {}_{C_1} D_{\hat{q}_1}^s \mathcal{Z}(\tau)$, if limit exist. If $C_1 = 0$ then ${}_{C_1} D_{\hat{q}_1}^s \mathcal{Z} = D_{\hat{q}_1}^s \mathcal{Z}$.

Theorem 2. Suppose that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ is quantum symmetric differentiable mapping, then

1. ${}_{C_1} D_{\hat{q}_1}^s [m_1 \mathcal{Z}(\tau) + n_1 \mathcal{G}(\tau)] = m_1 {}_{C_1} D_{\hat{q}_1}^s \mathcal{Z}(\tau) + n_1 {}_{C_1} D_{\hat{q}_1}^s \mathcal{G}(\tau)$.
2. ${}_{C_1} D_{\hat{q}_1}^s \mathcal{Z}(\tau) \mathcal{G}(\tau) = \mathcal{G}(\hat{q}_1 \tau + (1 - \hat{q}_1)C_1) {}_{C_1} D_{\hat{q}_1}^s \mathcal{Z}(\tau) + \mathcal{Z}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})C_1) {}_{C_1} D_{\hat{q}_1}^s \mathcal{G}(\tau)$.

Similarly, they investigated the corresponding symmetric quantum integral, which is stated as:

Definition 4. Suppose $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a continuous mapping, then

$$\begin{aligned} \int_{C_1}^{C_2} \mathcal{Z}(\tau) {}_{C_1} d_{\hat{q}_1}^s \tau &= (C_2 - C_1)(\hat{q}_1^{-1} - \hat{q}_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1} C_2 + (1 - \hat{q}_1^{2n+1}) C_1) \\ &= (C_2 - C_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1} C_2 + (1 - \hat{q}_1^{2n+1}) C_1). \end{aligned}$$

If $C_1 = 0$, then it reduces to classical symmetric quantum integrals in Ref. [26]. To get more information about quantum symmetric differences, one may consult Ref. [28].

3. Main Findings

In the following perspective, we introduce the idea of right quantum symmetric derivative and integral operators, which is stated as:

Definition 5. Let $\mathcal{Z} : J \rightarrow \mathbb{R}$ be a continuous mapping, then

$${}_{C_2} D_{\hat{q}_1}^s \mathcal{Z}(\tau) = \frac{\mathcal{Z}(\hat{q}_1 \tau + (1 - \hat{q}_1)C_2) - \mathcal{Z}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})C_2)}{(\hat{q}_1^{-1} - \hat{q}_1)(C_2 - \tau)}, \tau \neq C_2.$$

And ${}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\mathcal{C}_2) = \lim_{\hat{q}_1 \rightarrow 1} {}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\tau)$, if limit exist. If $\mathcal{C}_2 = 0$ then ${}^{C_2}D_{\hat{q}_1}^s \mathcal{Z} = D_{\hat{q}_1}^s \mathcal{Z}$.

Example 1. If $\mathcal{Z}(\tau) = (\mathcal{C}_2 - \tau)^\alpha$, then

$$\begin{aligned} {}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\tau) &= \frac{(\mathcal{C}_2 - \hat{q}_1 \tau - (1 - \hat{q}_1)\mathcal{C}_2)^\alpha - (\mathcal{C}_2 - \hat{q}_1^{-1} \tau - (1 - \hat{q}_1^{-1})\mathcal{C}_2)^\alpha}{(\hat{q}_1^{-1} - \hat{q}_1)(\mathcal{C}_2 - \tau)} \\ &= \frac{\hat{q}_1^\alpha (\mathcal{C}_2 - \tau)^\alpha - \hat{q}_1^{-\alpha} (\mathcal{C}_2 - \tau)^\alpha}{(\hat{q}_1^{-1} - \hat{q}_1)(\mathcal{C}_2 - \tau)} \\ &= \frac{(\hat{q}_1^\alpha - \hat{q}_1^{-\alpha})}{\hat{q}_1^{-1} - \hat{q}_1} (\mathcal{C}_2 - \tau)^{\alpha-1}. \end{aligned}$$

Now, we discuss the algebraic properties of ${}^{C_2}D_{\hat{q}_1}^s$.

Theorem 3. Suppose that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ is a right quantum symmetric differentiable mapping, then

1. ${}^{C_2}D_{\hat{q}_1}^s [m\mathcal{Z}(\tau) + n\mathcal{G}(\tau)] = m {}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\tau) + n {}^{C_2}D_{\hat{q}_1}^s \mathcal{G}(\tau)$.
2. ${}^{C_2}D_{\hat{q}_1}^s [\mathcal{Z}(\tau)\mathcal{G}(\tau)] = \mathcal{G}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2) {}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\tau) + \mathcal{Z}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2) {}^{C_2}D_{\hat{q}_1}^s \mathcal{G}(\tau)$.
or,
 ${}^{C_2}D_{\hat{q}_1}^s [\mathcal{Z}(\tau)\mathcal{G}(\tau)] = \mathcal{Z}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2) {}^{C_2}D_{\hat{q}_1}^s \mathcal{G}(\tau) + \mathcal{G}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2) {}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\tau)$.
3. ${}^{C_2}D_{\hat{q}_1}^s \left[\frac{\mathcal{Z}(\tau)}{\mathcal{G}(\tau)} \right] = \frac{\mathcal{G}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2) {}^{C_2}D_{\hat{q}_1}^s \mathcal{Z}(\tau) - \mathcal{Z}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2) {}^{C_2}D_{\hat{q}_1}^s \mathcal{G}(\tau)}{\mathcal{G}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2)\mathcal{G}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2)}$,
where $\mathcal{G}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2)\mathcal{G}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2) \neq 0$.

Proof. We omit the proof for interested readers. \square

Based on the right symmetric quantum derivative, we construct the quantum symmetric definite integral. For this purpose, we define a new shifting operator

$$E_{\hat{q}_1, s} \mathcal{Z}(\tau) = \mathcal{Z}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2)$$

Additionally,

$$E_{\hat{q}_1^{-1}, s}^s \mathcal{Z}(\tau) = \mathcal{Z}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2)$$

Similarly,

$$\begin{aligned} E_{\hat{q}_1, s}^2 &= E_{\hat{q}_1, s}(E_{\hat{q}_1, s} \mathcal{Z}(\tau)) \\ &= E_{\hat{q}_1, s}(\mathcal{Z}(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2)) \\ &= \mathcal{Z}(\hat{q}_1(\hat{q}_1 \tau + (1 - \hat{q}_1)\mathcal{C}_2) + (1 - \hat{q}_1)\mathcal{C}_2) \\ &= \mathcal{Z}(\hat{q}_1^2 \tau + (1 - \hat{q}_1^2)\mathcal{C}_2). \end{aligned}$$

Applying mathematical induction, we gain

$$E_{\hat{q}_1, s}^n \mathcal{Z}(\tau) = \mathcal{Z}(\hat{q}_1^n \tau + (1 - \hat{q}_1^n)\mathcal{C}_2).$$

Moreover, we note that

$$\begin{aligned} E_{\hat{q}_1, s} E_{\hat{q}_1^{-1}, s} \mathcal{Z}(\tau) &= \mathcal{Z}(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2) \\ &= \mathcal{Z}(\hat{q}_1(\hat{q}_1^{-1} \tau + (1 - \hat{q}_1^{-1})\mathcal{C}_2) + (1 - \hat{q}_1)\mathcal{C}_2) \\ &= \mathcal{Z}(\tau) \end{aligned} \tag{2}$$

From (2), we have

$$E_{\hat{q}_1^{-1}, s} = \frac{1}{E_{\hat{q}_1, s}}.$$

Utilizing this fact, the notion of the right quantum symmetric derivative can be transformed as:

$$\mathcal{G}(\tau) = \frac{(E_{\hat{q}_1^{-1},s} - E_{\hat{q}_1,s})\mathcal{Z}(\tau)}{(\hat{q}_1^{-1} - \hat{q}_1)(\tau - C_2)}$$

Then right quantum symmetric can be defined as:

$$\begin{aligned} \mathcal{Z}(\tau) &= \frac{\mathcal{G}(\tau)(\hat{q}_1^{-1} - \hat{q}_1)(\tau - C_2)}{E_{\hat{q}_1^{-1},s} - E_{\hat{q}_1,s}} \\ &= \frac{\mathcal{G}(\tau)(\hat{q}_1^{-1} - \hat{q}_1)(\tau - C_2)E_{\hat{q}_1,s}}{1 - E_{\hat{q}_1,s}^2} \\ &= (\hat{q}_1^{-1} - \hat{q}_1)E_{\hat{q}_1,s}(1 - E_{\hat{q}_1,s}^2)^{-1}(\tau - C_2)\mathcal{G}(\tau) \\ &= (\hat{q}_1^{-1} - \hat{q}_1)(E_{\hat{q}_1,s} + E_{\hat{q}_1,s}^3 + E_{\hat{q}_1,s}^5 + \dots)(\tau - C_2)\mathcal{G}(\tau) \\ &= (\hat{q}_1^{-1} - \hat{q}_1) \sum_{n=0}^{\infty} E_{\hat{q}_1,s}^{2n+1}(\tau - C_2)\mathcal{G}(\tau) \\ &= (\hat{q}_1^{-1} - \hat{q}_1) \sum_{n=0}^{\infty} (\hat{q}_1^{2n+1}\tau + (1 - \hat{q}_1^{2n+1})C_2 - C_2)\mathcal{G}(\hat{q}_1^{2n+1}\tau + (1 - \hat{q}_1^{2n+1})C_2) \\ &= (\hat{q}_1^{-1} - \hat{q}_1)(\tau - C_2) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1}\mathcal{G}(\hat{q}_1^{2n+1}\tau + (1 - \hat{q}_1^{2n+1})C_2) \\ &= (1 - \hat{q}_1^2)(\tau - C_2) \sum_{n=0}^{\infty} \hat{q}_1^{2n}\mathcal{G}(\hat{q}_1^{2n+1}\tau + (1 - \hat{q}_1^{2n+1})C_2). \end{aligned}$$

Our next definition is the definite right quantum symmetric integral operator.

Definition 6. Assume that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a continuous mapping, then

$$\begin{aligned} \int_{C_1}^{C_2} \mathcal{Z}(\tau) {}_{C_2}d_{\hat{q}_1}^s \tau &= (C_2 - C_1)(\hat{q}_1^{-1} - \hat{q}_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1}C_1 + (1 - \hat{q}_1^{2n+1})C_2) \\ &= (C_2 - C_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1}C_1 + (1 - \hat{q}_1^{2n+1})C_2). \end{aligned}$$

Clearly, a mapping is said to be right quantum symmetric integrable if $\sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1}C_1 + (1 - \hat{q}_1^{2n+1})C_2)$ converges.

Further, we explore some fundamental properties of both right and left quantum symmetric integrals.

Theorem 4. Assume that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a continuous mapping and $m_1, n_1 \in \mathbb{R}$, then

1. $\int_{C_1}^{C_2} [m_1\mathcal{Z}(\tau) + n_1\mathcal{G}(\tau)] {}_{C_1}d_{\hat{q}_1}^s \tau = m_1 \int_{C_1}^{C_2} \mathcal{Z}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau + n_1 \int_{C_1}^{C_2} \mathcal{G}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau.$
2. $\int_0^1 \mathcal{Z}(\tau C_2 + (1 - \tau)C_1) {}_{C_1}d_{\hat{q}_1}^s \tau = \frac{1}{C_2 - C_1} \int_{C_1}^{C_2} \mathcal{Z}(u) {}_{C_1}d_{\hat{q}_1}^s u.$
3. $\int_{C_1}^t {}_{C_1}D_{\hat{q}_1}^s \mathcal{Z}(u) {}_{C_1}d_{\hat{q}_1}^s u = \mathcal{Z}(t)$
4. ${}_{C_1}D_{\hat{q}_1}^s \int_{C_1}^t \mathcal{Z}(u) {}_{C_1}d_{\hat{q}_1}^s u = \mathcal{Z}(t)$
5. $\int_e^{C_2} {}_{C_1}D_{\hat{q}_1}^s \mathcal{Z}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau = \mathcal{Z}(C_2) - \mathcal{Z}(e), e \in (C_1, C_2).$

Proof. From the definition of the right quantum symmetric integral, we have

$$\begin{aligned}
 & \int_{\mathcal{C}_1}^{\mathcal{C}_2} [m_1 \mathcal{Z}(\tau) + n_1 \mathcal{G}(\tau)]_{\mathcal{C}_1} d_{\hat{q}_1}^s \tau \\
 &= (\mathcal{C}_2 - \mathcal{C}_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} [m_1 \mathcal{Z}((\hat{q}_1^{2n+1} \tau + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1)) + n_1 \mathcal{G}((\hat{q}_1^{2n+1} \tau + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1))] \\
 &= m_1 (\mathcal{C}_2 - \mathcal{C}_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}((\hat{q}_1^{2n+1} \tau + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1)) \\
 &+ n_1 (\mathcal{C}_2 - \mathcal{C}_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{G}((\hat{q}_1^{2n+1} \tau + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1)) \\
 &= m_1 \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(\tau)_{\mathcal{C}_1} d_{\hat{q}_1}^s \tau + n_1 \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{G}(\tau)_{\mathcal{C}_1} d_{\hat{q}_1}^s \tau.
 \end{aligned}$$

For the second property, we again consider the definition of the right symmetric quantum integral,

$$\begin{aligned}
 & \int_0^1 \mathcal{Z}(\tau \mathcal{C}_2 + (1 - \tau) \mathcal{C}_1) d_{\hat{q}_1}^s \tau \\
 &= (1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1} \mathcal{C}_2 + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1) \\
 &= \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(u)_{\mathcal{C}_1} d_{\hat{q}_1}^s u.
 \end{aligned}$$

For the third property, we consider Definitions 3 and 4, then

$$\begin{aligned}
 & \int_{\mathcal{C}_1}^t {}_{\mathcal{C}_1} D_{\hat{q}_1}^s \mathcal{Z}(u)_{\mathcal{C}_1} d_{\hat{q}_1}^s u \\
 &= \int_{\mathcal{C}_1}^t \left[\frac{\mathcal{Z}(\hat{q}_1^{-1} u + (1 - \hat{q}_1^{-1}) \mathcal{C}_1) - \mathcal{Z}(\hat{q}_1 u + (1 - \hat{q}_1) \mathcal{C}_1)}{(\hat{q}_1^{-1} - \hat{q}_1)(u - \mathcal{C}_1)} \right]_{\mathcal{C}_1} d_{\hat{q}_1}^s u \\
 &= \sum_{n=0}^{\infty} \frac{(t - \mathcal{C}_1) \hat{q}_1^{2n+1}}{(\hat{q}_1^{2n+1} t + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1) - \mathcal{C}_1} \left[\mathcal{Z}(\hat{q}_1^{-1} (\hat{q}_1^{2n+1} t + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1) + (1 - \hat{q}_1^{-1}) \mathcal{C}_1) \right. \\
 &\quad \left. - \mathcal{Z}(\hat{q}_1 (\hat{q}_1^{2n+1} t + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1) + (1 - \hat{q}_1) \mathcal{C}_1) \right] \\
 &= \sum_{n=0}^{\infty} \mathcal{Z}(\hat{q}_1^{2n} t + (1 - \hat{q}_1^{2n}) \mathcal{C}_1) - \sum_{n=0}^{\infty} \mathcal{Z}(\hat{q}_1^{2n+2} t + (1 - \hat{q}_1^{2n+2}) \mathcal{C}_1) \\
 &= \sum_{n=0}^{\infty} \mathcal{Z}(\hat{q}_1^{2n} t + (1 - \hat{q}_1^{2n}) \mathcal{C}_1) - \sum_{n=1}^{\infty} \mathcal{Z}(\hat{q}_1^{2n} t + (1 - \hat{q}_1^{2n}) \mathcal{C}_1) \\
 &= \mathcal{Z}(t).
 \end{aligned}$$

Again, we consider the Definitions 3 and 4, then

$$\begin{aligned}
 & {}_{\mathcal{C}_1} D_{\hat{q}_1}^s \int_{\mathcal{C}_1}^t \mathcal{Z}(u)_{\mathcal{C}_2} d_{\hat{q}_1}^s u \\
 &= {}_{\mathcal{C}_1} D_{\hat{q}_1}^s \left[(\hat{q}_1^{-1} - \hat{q}_1)(t - \mathcal{C}_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1} t + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1) \right] \\
 &= \frac{(\hat{q}_1^{-1} t + (1 - \hat{q}_1^{-1}) \mathcal{C}_1 - \mathcal{C}_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1} (\hat{q}_1^{-1} t + (1 - \hat{q}_1^{-1}) \mathcal{C}_1) + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1)}{(t - \mathcal{C}_1)} \\
 &\quad - \frac{(\hat{q}_1 t + (1 - \hat{q}_1) \mathcal{C}_1 - \mathcal{C}_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1} (\hat{q}_1 t + (1 - \hat{q}_1) \mathcal{C}_1) + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_1)}{(t - \mathcal{C}_1)} \\
 &= \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n} t + (1 - \hat{q}_1^{2n}) \mathcal{C}_1) - \sum_{n=0}^{\infty} \hat{q}_1^{2n+2} \mathcal{Z}(\hat{q}_1^{2n+2} t + (1 - \hat{q}_1^{2n+2}) \mathcal{C}_1) \\
 &= \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n} t + (1 - \hat{q}_1^{2n}) \mathcal{C}_1) - \sum_{n=1}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n} t + (1 - \hat{q}_1^{2n}) \mathcal{C}_1) \\
 &= \mathcal{Z}(t).
 \end{aligned}$$

Now we prove our last property by using simple facts,

$$\begin{aligned} \int_e^{C_2} c_1 D_{\hat{q}_1}^s \mathcal{Z}(\tau) c_1 d_{\hat{q}_1}^s \tau &= \int_0^{C_2} c_1 D_{\hat{q}_1}^s \mathcal{Z}(\tau) {}_0 d_{\hat{q}_1}^s \tau - \int_0^e c_1 D_{\hat{q}_1}^s \mathcal{Z}(\tau) {}_0 d_{\hat{q}_1}^s \tau \\ &= \mathcal{Z}(C_2) - \mathcal{Z}(e). \end{aligned}$$

Hence the result is achieved. \square

Theorem 5. Assume that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ be a continuous mapping and $m, n \in \mathbb{R}$, then

1. $\int_{C_1}^{C_2} [m\mathcal{Z}(\tau) + n\mathcal{G}(\tau)] c_2 d_{\hat{q}_1}^s \tau = m \int_{C_1}^{C_2} \mathcal{Z}(\tau) c_2 d_{\hat{q}_1}^s \tau + n \int_{C_1}^{C_2} \mathcal{G}(\tau) c_2 d_{\hat{q}_1}^s \tau.$
2. $\int_0^1 \mathcal{Z}(\tau C_2 + (1-\tau)C_1) d_{\hat{q}_1}^s \tau = \frac{1}{C_2 - C_1} \int_{C_1}^{C_2} \mathcal{Z}(u) c_2 d_{\hat{q}_1}^s u.$
3. $c_2 D_{\hat{q}_1}^s \int_t^{C_2} \mathcal{Z}(u) c_2 d_{\hat{q}_1}^s u = -\mathcal{Z}(t)$
4. $\int_t^{C_2} c_2 D_{\hat{q}_1}^s \mathcal{Z}(u) c_1 d_{\hat{q}_1}^s u = -\mathcal{Z}(t)$
5. $\int_e^{C_2} c_2 D_{\hat{q}_1}^s \mathcal{Z}(\tau) c_1 d_{\hat{q}_1}^s \tau = \mathcal{Z}(C_2) - \mathcal{Z}(e).$
6. $\int_{C_1}^c f(\hat{q}_1^{-1}\tau + (1-\hat{q}_1^{-1})C_2) c_2 D_{\hat{q}_1}^s \mathcal{G}(\tau) c_2 d_{\hat{q}_1}^s \tau$
 $= \mathcal{Z}(c)\mathcal{G}(c) - \mathcal{Z}(C_1)\mathcal{G}(C_1) - \int_{C_1}^c g(\hat{q}_1\tau + (1-\hat{q}_1)\tau) c_2 D_{\hat{q}_1}^s \mathcal{Z}(\tau) c_2 d_{\hat{q}_1}^s \tau.$

Proof. The proofs of Properties 1 and 2 are straightforward from Definition 6.

Now we prove the third property,

$$\begin{aligned} &\int_0^1 \mathcal{Z}(\tau C_2 + (1-\tau)C_1) d_{\hat{q}_1}^s \tau \\ &= (\hat{q}_1^{-1} - \hat{q}_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}((1-\hat{q}_1^{2n+1})C_2 + \hat{q}_1^{2n+1}C_1) \\ &= \frac{1}{C_2 - C_1} \int_{C_1}^{C_2} \mathcal{Z}(u) c_2 d_{\hat{q}_1}^s u. \end{aligned}$$

For the fourth property, we consider Definitions 5 and 6, then

$$\begin{aligned} &\int_t^{C_2} c_2 D_{\hat{q}_1}^s \mathcal{Z}(u) c_2 d_{\hat{q}_1}^s u \\ &= \int_t^{C_2} \left[\frac{\mathcal{Z}(\hat{q}_1^{-1}u + (1-\hat{q}_1^{-1})C_2) - \mathcal{Z}(\hat{q}_1 u + (1-\hat{q}_1)C_2)}{(\hat{q}_1^{-1} - \hat{q}_1)(u - C_2)} \right] c_2 d_{\hat{q}_1}^s u \\ &= \sum_{n=0}^{\infty} \frac{(C_2 - t)\hat{q}_1^{2n+1}}{(\hat{q}_1^{2n+1}t + (1-\hat{q}_1^{2n+1})C_2) - C_2} \left[\mathcal{Z}(\hat{q}_1^{-1}(\hat{q}_1^{2n+1}t + (1-\hat{q}_1^{2n+1})C_2) + (1-\hat{q}_1^{-1})C_2) \right. \\ &\quad \left. - \mathcal{Z}(\hat{q}_1(\hat{q}_1^{2n+1}t + (1-\hat{q}_1^{2n+1})C_2) + (1-\hat{q}_1)C_2) \right] \\ &= \sum_{n=0}^{\infty} \mathcal{Z}(\hat{q}_1^{2n+2}t + (1-\hat{q}_1^{2n+2})C_2) - \sum_{n=0}^{\infty} \mathcal{Z}(\hat{q}_1^{2n}t + (1-\hat{q}_1^{2n})C_2) \\ &= \sum_{n=1}^{\infty} \mathcal{Z}(\hat{q}_1^{2n}t + (1-\hat{q}_1^{2n})C_2) - \sum_{n=0}^{\infty} \mathcal{Z}(\hat{q}_1^{2n}t + (1-\hat{q}_1^{2n})C_2) \\ &= -\mathcal{Z}(t). \end{aligned}$$

Again, we consider Definitions 5 and 6, then

$$\begin{aligned}
 & {}_{C_2}D_{\hat{q}_1}^s \int_t^{C_2} \mathcal{Z}(u) {}_{C_2}d_{\hat{q}_1}^s u \\
 &= {}_{C_2}D_{\hat{q}_1}^s \left[(\hat{q}_1^{-1} - \hat{q}_1)(C_2 - t) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1}t + (1 - \hat{q}_1^{2n+1})C_2) \right] \\
 &= \frac{(C_2 - \hat{q}_1^{-1}t - (1 - \hat{q}_1^{-1})C_2) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1}(\hat{q}_1^{-1}t + (1 - \hat{q}_1^{-1})C_2) + (1 - \hat{q}_1^{2n+1})C_2)}{(t - C_2)} \\
 &\quad - \frac{(C_2 - \hat{q}_1 t - (1 - \hat{q}_1)C_1) \sum_{n=0}^{\infty} \hat{q}_1^{2n+1} \mathcal{Z}(\hat{q}_1^{2n+1}(\hat{q}_1 t + (1 - \hat{q}_1)C_2) + (1 - \hat{q}_1^{2n+1})C_2)}{(t - C_2)} \\
 &= \sum_{n=0}^{\infty} \hat{q}_1^{2n+2} \mathcal{Z}(\hat{q}_1^{2n+2}t + (1 - \hat{q}_1^{2n+2})C_2) - \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n}t + (1 - \hat{q}_1^{2n})C_2) \\
 &= \sum_{n=1}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n}t + (1 - \hat{q}_1^{2n})C_2) - \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n}t + (1 - \hat{q}_1^{2n})C_2) \\
 &= -\mathcal{Z}(t).
 \end{aligned}$$

Now lastly, we prove the integration by parts formula. For this purpose, we consider the product rule property, which is proved in Theorem 4.

$${}_{C_1}D_{\hat{q}_1}^s [\mathcal{Z}(\tau)\mathcal{G}(\tau)] = \mathcal{G}(\hat{q}_1\tau + (1 - \hat{q}_1)C_2) {}_{C_1}D_{\hat{q}_1}^s \mathcal{Z}(\tau) + \mathcal{Z}(\hat{q}_1^{-1}\tau + (1 - \hat{q}_1^{-1})C_2) {}_{C_1}D_{\hat{q}_1}^s \mathcal{G}(\tau)$$

Applying the right quantum symmetric integral operator on the above expression with respect to τ over $[C_1, c] \subset [C_1, C_2]$, then

$$\begin{aligned}
 & \int_{C_1}^{C_2} D_{\hat{q}_1}^s [\mathcal{Z}(\tau)\mathcal{G}(\tau)] {}_{C_1}d_{\hat{q}_1}^s \tau \\
 &= \int_{C_1}^c g(\hat{q}_1\tau + (1 - \hat{q}_1)C_2) {}_{C_1}D_{\hat{q}_1}^s \mathcal{Z}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau + \int_{C_1}^c f(\hat{q}_1^{-1}\tau + (1 - \hat{q}_1^{-1})C_2) {}_{C_1}D_{\hat{q}_1}^s \mathcal{G}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 & \int_{C_1}^c f(\hat{q}_1^{-1}\tau + (1 - \hat{q}_1^{-1})C_2) {}_{C_1}D_{\hat{q}_1}^s \mathcal{G}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau \\
 &= \mathcal{Z}(c)\mathcal{G}(c) - \mathcal{Z}(C_1)\mathcal{G}(C_1) - \int_{C_1}^c g(\hat{q}_1\tau + (1 - \hat{q}_1)C_2) {}_{C_1}D_{\hat{q}_1}^s \mathcal{Z}(\tau) {}_{C_1}d_{\hat{q}_1}^s \tau.
 \end{aligned}$$

Hence, the result is acquired. \square

Lemma 1. For $\alpha \in \mathbb{R} - \{-1\}$, then

$$\int_{C_1}^{\tau} (u - C_1)^\alpha {}_{C_1}d_{\hat{q}_1}^s u = \left(\frac{\hat{q}_1^{-1} - \hat{q}_1}{\hat{q}_1^{-\alpha} - \hat{q}_1^\alpha} \right) (\tau - C_1)^{\alpha+1}.$$

Proof. Let $\mathcal{Z}(u) = (u - C_1)^{\alpha+1}$, then

$${}_{C_1}D_{\hat{q}_1}^s \mathcal{Z}(u) = \left(\frac{\hat{q}_1^{-(\alpha+1)} - \hat{q}_1^{\alpha+1}}{\hat{q}_1^{-1} - \hat{q}_1} \right) (\tau - C_1)^\alpha$$

Now applying the left symmetric integral operator with respect to u over $[C_1, \tau]$, then we acquired our desired outcome. \square

Lemma 2. For $\alpha \in \mathbb{R} - \{-1\}$, then

$$\int_{\tau}^{\mathcal{C}_2} (\mathcal{C}_2 - u)^{\alpha} {}_{\mathcal{C}_2} d_{\hat{q}_1}^s u = \left(\frac{\hat{q}_1 - \hat{q}_1^{-1}}{\hat{q}_1^{-\alpha} - \hat{q}_1^{\alpha}} \right) (\mathcal{C}_2 - \tau)^{\alpha+1}.$$

Proof. Let $\mathcal{Z}(u) = (\mathcal{C}_2 - u)^{\alpha+1}$, then

$${}_{\mathcal{C}_2} D_{\hat{q}_1}^s \mathcal{Z}(u) = \left(\frac{\hat{q}_1^{-(\alpha+1)} - \hat{q}_1^{\alpha+1}}{\hat{q}_1^{-1} - \hat{q}_1} \right) (\mathcal{C}_2 - u)^{\alpha}$$

Now, by applying the right quantum symmetric integral operator with respect to u over $[\tau, \mathcal{C}_2]$, we have acquired our desired outcome. \square

Now we give the quantum symmetric analogue of Young's inequality.

Theorem 6. For $\mathcal{C}_1, \mathcal{C}_2 > 0$ and $\frac{1}{p} + \frac{1}{r} = 1$ with $p, \hat{q}_1 > 1$, then

$$\mathcal{C}_1 \mathcal{C}_2 \leq \frac{\mathcal{C}_1^p}{[p]_{\hat{q}_1, s}} + \frac{\mathcal{C}_2^r}{[r]_{\hat{q}_1, s}}.$$

Proof. Consider $y = x^{p-1}$ and $x = y^{\frac{1}{p-1}}$ for $p > 1$ and $\frac{1}{p} + \frac{1}{r} = 1$, let us draw the graph of $y = x^{p-1}$ as shown in Figure 1,

$$s_1 = \int_0^{\mathcal{C}_1} x^{p-1} {}_0 d_{\hat{q}_1}^s x = \frac{\mathcal{C}_1^p}{[p]_{\hat{q}_1, s}}.$$

And

$$s_2 = \int_0^{\mathcal{C}_2} y^{\frac{1}{p-1}} {}_0 d_{\hat{q}_1}^s y = \frac{\mathcal{C}_2^r}{[r]_{\hat{q}_1, s}}.$$

According to the graph, we have

$$\mathcal{C}_1 \mathcal{C}_2 \leq s_1 + s_2 = \frac{\mathcal{C}_1^p}{[p]_{\hat{q}_1, s}} + \frac{\mathcal{C}_2^r}{[r]_{\hat{q}_1, s}}.$$

Hence, the proof is completed.

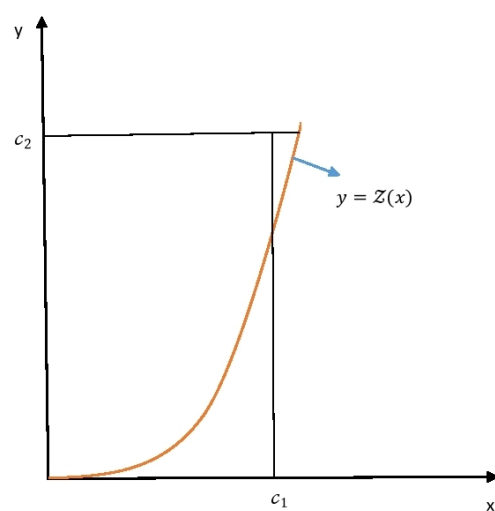


Figure 1. This figure shows the graph of $\mathcal{Z}(x) = x^{p-1}$.

\square

Theorem 7. Let $x \in [\mathcal{C}_1, \mathcal{C}_2]$, $0 < \hat{q}_1 < 1$ and $p, r > 1$ such that $\frac{1}{p} + \frac{1}{r} = 1$, then

$$\int_x^{\mathcal{C}_2} |\mathcal{Z}(\tau)\mathcal{G}(\tau)|^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \leq \left(\int_x^{\mathcal{C}_2} |\mathcal{Z}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \left(\int_x^{\mathcal{C}_2} |\mathcal{G}(\tau)|^r d_{\hat{q}_1}^s \tau \right)^{\frac{1}{r}}.$$

Proof. Considering the Definition of the right quantum symmetric integral operator, we have

$$\begin{aligned} & \int_x^{\mathcal{C}_2} |\mathcal{Z}(\tau)\mathcal{G}(\tau)|^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \\ &= (1 - \hat{q}_1^2)(\mathcal{C}_2 - x) \sum_{n=0}^{\infty} \hat{q}_1^{2n} |\mathcal{Z}(\hat{q}_1^{2n+1}x + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2)\mathcal{G}(\hat{q}_1^{2n+1}x + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2)| \\ &\leq (1 - \hat{q}_1^2)^{\frac{1}{p} + \frac{1}{r}} (\mathcal{C}_2 - x)^{\frac{1}{p} + \frac{1}{r}} \sum_{n=0}^{\infty} (\hat{q}_1^{2n})^{\frac{1}{p} + \frac{1}{r}} |\mathcal{Z}(\hat{q}_1^{2n+1}x + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2)| \\ & \quad |\mathcal{G}(\hat{q}_1^{2n+1}x + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2)| \\ &\leq \left((1 - \hat{q}_1^2)(\mathcal{C}_2 - x) \sum_{n=0}^{\infty} (\hat{q}_1^{2n}) |\mathcal{Z}(\hat{q}_1^{2n+1}x + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2)|^p \right)^{\frac{1}{p}} \\ & \quad \times \left((1 - \hat{q}_1^2)(\mathcal{C}_2 - x) \sum_{n=0}^{\infty} (\hat{q}_1^{2n}) |\mathcal{G}(\hat{q}_1^{2n+1}x + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2)|^r \right)^{\frac{1}{r}} \\ &= \left(\int_x^{\mathcal{C}_2} |\mathcal{Z}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \left(\int_x^{\mathcal{C}_2} |\mathcal{G}(\tau)|^r d_{\hat{q}_1}^s \tau \right)^{\frac{1}{r}}. \end{aligned}$$

This completes the proof. \square

Theorem 8. Assume that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ are continuous mappings, then

$$\left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \leq \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{G}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{r} = 1$.

Proof. From the following expression, we have

$$\begin{aligned} & \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^p d_{\hat{q}_1}^s \tau \\ &= \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{p-1} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)| d_{\hat{q}_1}^s \tau \\ &\leq \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{p-1} |\mathcal{Z}(\tau)| d_{\hat{q}_1}^s \tau + \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{p-1} |\mathcal{G}(\tau)| d_{\hat{q}_1}^s \tau. \end{aligned}$$

Implementing the classical Hölder's inequality on the above relation

$$\begin{aligned} & \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{p-1} |\mathcal{Z}(\tau)| d_{\hat{q}_1}^s \tau + \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{p-1} |\mathcal{G}(\tau)| d_{\hat{q}_1}^s \tau \\ &\leq \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{r(p-1)} d_{\hat{q}_1}^s \tau \right)^{\frac{1}{r}} \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \\ & \quad + \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{r(p-1)} d_{\hat{q}_1}^s \tau \right)^{\frac{1}{r}} \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{G}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau) + \mathcal{G}(\tau)|^{r(p-1)} d_{\hat{q}_1}^s \tau \right)^{\frac{1}{r}} \left[\left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} + \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{G}(\tau)|^p d_{\hat{q}_1}^s \tau \right)^{\frac{1}{p}} \right]. \end{aligned}$$

After simple computations, we obtain our required result. \square

Now, we give the Hermite–Hadamard’s inequalities involving new quantum symmetric calculus. To prove the Hermite–Hadamard’s inequalities, we draw the following Figure 2.

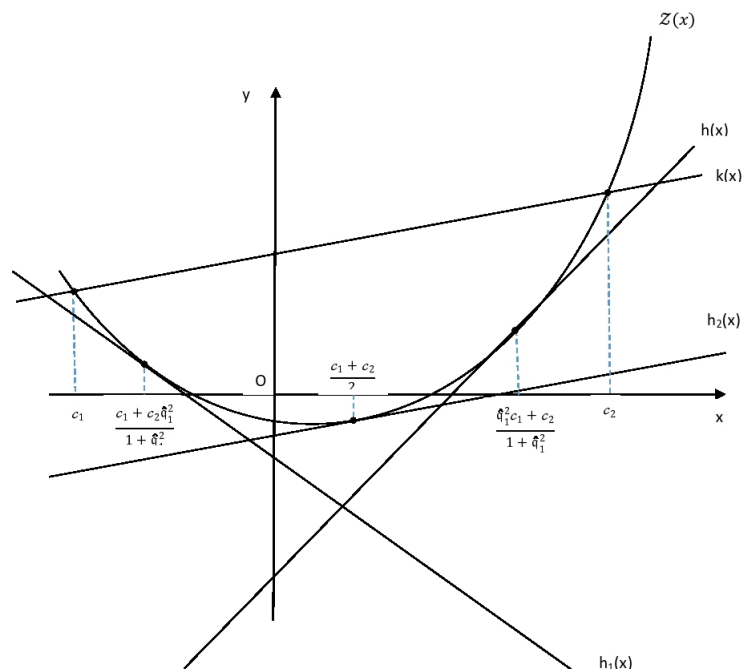


Figure 2. This figure presents the secant and tangent lines of convex mapping.

Theorem 9. Assume that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a differentiable convex mapping on (C_1, C_2) and $\hat{q}_1 \in (0, 1)$, then

$$\begin{aligned} & \mathcal{Z}\left(\frac{C_1 \hat{q}_1^2 + C_2}{1 + \hat{q}_1^2}\right) - \frac{\hat{q}_1(C_2 - C_1)}{1 + \hat{q}_1} \mathcal{Z}'\left(\frac{C_1 \hat{q}_1^2 + C_2}{1 + \hat{q}_1^2}\right) \leq \frac{1}{C_2 - C_1} \int_{C_1}^{C_2} \mathcal{Z}(x) c_2 d_{\hat{q}_1^s} x \\ & \leq \frac{\hat{q}_1 \mathcal{Z}(C_1) + (1 + \hat{q}_1^2 - \hat{q}_1) \mathcal{Z}(C_2)}{1 + \hat{q}_1^2}. \end{aligned}$$

Proof. From the given assumption, \mathcal{Z} is a differentiable convex mapping on (C_1, C_2) , so there exists a tangent line for point $\frac{C_1 \hat{q}_1^2 + C_2}{1 + \hat{q}_1^2} \in (C_1, C_2)$ and the equation of the tangent line is given as:

$$h(u) = \mathcal{Z}\left(\frac{C_1 \hat{q}_1^2 + C_2}{1 + \hat{q}_1^2}\right) + \mathcal{Z}'\left(\frac{C_1 \hat{q}_1^2 + C_2}{1 + \hat{q}_1^2}\right) \left(u - \frac{C_1 \hat{q}_1^2 + C_2}{1 + \hat{q}_1^2}\right).$$

As \mathcal{Z} is convex mapping, then $h(u) \leq \mathcal{Z}(u)$. Now taking the right quantum symmetric integral on both sides of the preceding inequality, we have

$$\begin{aligned}
& \int_{\mathcal{C}_1}^{\mathcal{C}_2} h(u)^{\mathcal{C}_2} d_{\hat{q}_1^s} u \\
&= \int_{\mathcal{C}_1}^{\mathcal{C}_2} \left[\mathcal{Z} \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) + \mathcal{Z}' \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \left(u - \frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \right] \mathcal{C}_2 d_{\hat{q}_1^s} u \\
&= (\mathcal{C}_2 - \mathcal{C}_1) \mathcal{Z} \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) + \mathcal{Z}' \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \int_{\mathcal{C}_1}^{\mathcal{C}_2} \left(u - \frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \mathcal{C}_2 d_{\hat{q}_1^s} u \\
&= (\mathcal{C}_2 - \mathcal{C}_1) \mathcal{Z} \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) + \mathcal{Z}' \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} u \mathcal{C}_2 d_{\hat{q}_1^s} u - (\mathcal{C}_2 - \mathcal{C}_1) \frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \\
&= (\mathcal{C}_2 - \mathcal{C}_1) \mathcal{Z} \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) + (\mathcal{C}_2 - \mathcal{C}_1) \mathcal{Z}' \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \left(\frac{(1 + \hat{q}_1^2) \mathcal{C}_2 - \hat{q}_1 (\mathcal{C}_2 - \mathcal{C}_1)}{1 + \hat{q}_1^2} - \frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \\
&= (\mathcal{C}_2 - \mathcal{C}_1) \left[\mathcal{Z} \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) - \mathcal{Z}' \left(\frac{\mathcal{C}_1 \hat{q}_1^2 + \mathcal{C}_2}{1 + \hat{q}_1^2} \right) \frac{\hat{q}_1 (\mathcal{C}_2 - \mathcal{C}_1)}{1 + \hat{q}_1^2} \right] \\
&\leq \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(u)^{\mathcal{C}_2} d_{\hat{q}_1^s} u.
\end{aligned}$$

In addition, due to convexity of \mathcal{Z} then secant lines $k(u)$ joining the points $(\mathcal{C}_1, \mathcal{Z}(\mathcal{C}_1))$ and $(\mathcal{C}_2, \mathcal{Z}(\mathcal{C}_2))$ always lies on or above the graph of \mathcal{Z} , so $\mathcal{Z}(u) \leq k(u)$, where $k(u)$ is given as:

$$k(u) = \mathcal{Z}(\mathcal{C}_1) + \frac{\mathcal{Z}(\mathcal{C}_2) - \mathcal{Z}(\mathcal{C}_1)}{\mathcal{C}_2 - \mathcal{C}_1} (u - \mathcal{C}_1), \forall u \in [\mathcal{C}_1, \mathcal{C}_2].$$

Now, implementing the right quantum symmetric integral operator, we have

$$\begin{aligned}
& \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(u)^{\mathcal{C}_2} d_{\hat{q}_1^s} u \\
&\leq \int_{\mathcal{C}_1}^{\mathcal{C}_2} k(u)^{\mathcal{C}_2} d_{\hat{q}_1^s} u \\
&= \int_{\mathcal{C}_1}^{\mathcal{C}_2} \left[\mathcal{Z}(\mathcal{C}_1) + \frac{\mathcal{Z}(\mathcal{C}_2) - \mathcal{Z}(\mathcal{C}_1)}{\mathcal{C}_2 - \mathcal{C}_1} (u - \mathcal{C}_1) \right] \mathcal{C}_2 d_{\hat{q}_1^s} u \tag{3} \\
&= (\mathcal{C}_2 - \mathcal{C}_1) \mathcal{Z}(\mathcal{C}_1) + (\mathcal{Z}(\mathcal{C}_2) - \mathcal{Z}(\mathcal{C}_1)) \left(\frac{(1 + \hat{q}_1^2) \mathcal{C}_2 - \hat{q}_1 (\mathcal{C}_2 - \mathcal{C}_1)}{1 + \hat{q}_1^2} - \mathcal{C}_1 \right) \\
&= (\mathcal{C}_2 - \mathcal{C}_1) \left[\frac{\hat{q}_1 \mathcal{Z}(\mathcal{C}_1) + (1 + \hat{q}_1^2 - \hat{q}_1) \mathcal{Z}(\mathcal{C}_2)}{1 + \hat{q}_1^2} \right].
\end{aligned}$$

In the following manner, we reach our desired result. \square

Example 2. We consider $\mathcal{Z} : [0, 2] \rightarrow \mathbb{R}$ such that $\mathcal{Z}(x) = x^2$ is a differentiable convex functions, then applying Theorem 9 for $\hat{q}_1 \in (0, 1)$, we have

$$\frac{4}{(1 + \hat{q}_1^2)^2} - \frac{8}{(1 + \hat{q}_1)(1 + \hat{q}_1^2)} \leq 4 + \frac{4\hat{q}_1^2}{1 + \hat{q}_1^2 + \hat{q}_1^4} - \frac{8\hat{q}_1}{1 + \hat{q}_1^2} \leq \frac{4(1 + \hat{q}_1^2 - \hat{q}_1)}{1 + \hat{q}_1^2}.$$

For $\hat{q}_1 = 0.6$, we obtain

$$-1.70667 < 1.5619 < 2.4.$$

For graphic visualization, we use $\hat{q}_1 \in (0, 1)$ as the variable. Figure 3 gives a validation of Theorem 9.

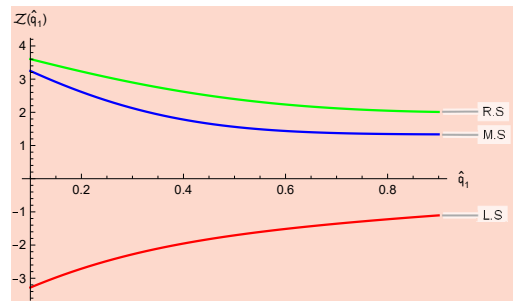


Figure 3. This figure validates the accuracy of Theorem 9.

Theorem 10. Assume that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a differentiable convex mapping on (C_1, C_2) and $\hat{q}_1 \in (0, 1)$, then

$$\mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + \frac{(1 - \hat{q}_1)(C_2 - C_1)}{1 + \hat{q}_1^2} \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \leq \frac{1}{C_2 - C_1} \int_{C_1}^{C_2} \mathcal{Z}(x) {}^{C_2}d_{\hat{q}_1^s} x \leq \frac{\hat{q}_1 \mathcal{Z}(C_1) + (1 + \hat{q}_1^2 - \hat{q}_1) \mathcal{Z}(C_2)}{1 + \hat{q}_1^2}.$$

Proof. From the given assumption, \mathcal{Z} is a differentiable convex mapping on (C_1, C_2) , so there exists a tangent line for point $\frac{C_1 + bq^2}{1 + \hat{q}_1^2} \in (C_1, C_2)$ and the equation of the tangent line is given as:

$$h_1(u) = \mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \left(u - \frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right).$$

As \mathcal{Z} is convex mapping, then $h_1(u) \leq \mathcal{Z}(u)$, Now, taking the right quantum symmetric integral on both sides of the preceding inequality, we have

$$\begin{aligned} & \int_{C_1}^{C_2} h_1(u) {}^{C_2}d_{\hat{q}_1^s} u \\ &= \int_{C_1}^{C_2} \left[\mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \left(u - \frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \right] {}^{C_2}d_{\hat{q}_1^s} u \\ &= (C_2 - C_1) \mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \int_{C_1}^{C_2} \left(u - \frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) {}^{C_2}d_{\hat{q}_1^s} u \\ &= (C_2 - C_1) \mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \left(\int_{C_1}^{C_2} u {}^{C_2}d_{\hat{q}_1^s} u - (C_2 - C_1) \frac{C_1 + bq^2}{1 + \hat{q}_1^2} \right) \tag{4} \\ &= (C_2 - C_1) \mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + (C_2 - C_1) \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \left(\frac{(1 + \hat{q}_1^2)C_2 - \hat{q}_1(C_2 - C_1)}{1 + \hat{q}_1^2} - \frac{C_1 + bq^2}{1 + \hat{q}_1^2} \right) \\ &= (C_2 - C_1) \left[\mathcal{Z}\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) + \mathcal{Z}'\left(\frac{C_1 + bq^2}{1 + \hat{q}_1^2}\right) \frac{(1 - \hat{q}_1)(C_2 - C_1)}{1 + \hat{q}_1^2} \right] \\ &\leq \int_{C_1}^{C_2} \mathcal{Z}(u) {}^{C_2}d_{\hat{q}_1^s} u. \end{aligned}$$

Now, comparing (3) and (4), we achieve our desired result. \square

Example 3. We consider $\mathcal{Z} : [0, 2] \rightarrow \mathbb{R}$ such that $\mathcal{Z}(x) = x^2$ is a differentiable convex functions, then applying on Theorem 10 for $\hat{q}_1 \in (0, 1)$, we have

$$\frac{4\hat{q}_1^4}{(1 + \hat{q}_1^2)^2} + \frac{8\hat{q}_1^2(1 - \hat{q}_1)}{(1 + \hat{q}_1^2)^2} \leq 4 + \frac{4\hat{q}_1^2}{1 + \hat{q}_1^2 + \hat{q}_1^4} - \frac{8\hat{q}_1}{1 + \hat{q}_1^2} \leq \frac{4(1 + \hat{q}_1^2 - \hat{q}_1)}{1 + \hat{q}_1^2}.$$

For $\hat{q}_1 = 0.6$, we obtain

$$0.903114 < 1.43729 < 2.23529.$$

For graphic visualization, we take $\hat{q}_1 \in (0, 1)$ as the variable. Figure 4 gives a validation of Theorem 10.

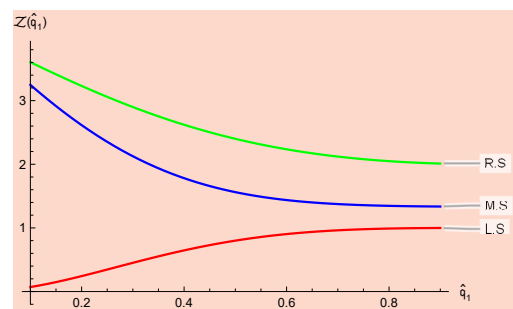


Figure 4. This figure validates the accuracy of Theorem 10.

Theorem 11. Assume that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a differentiable convex mapping on (C_1, C_2) and $\hat{q}_1 \in (0, 1)$, then

$$\begin{aligned} \mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + \frac{(1 + \hat{q}_1^2 - 2\hat{q}_1)(C_2 - C_1)}{2(1 + \hat{q}_1^2)} \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) &\leq \int_{C_1}^{C_2} \mathcal{Z}(x)^{C_2} d_{\hat{q}_1}^s x \\ &\leq \frac{\hat{q}_1 \mathcal{Z}(C_1) + (1 + \hat{q}_1^2 - \hat{q}_1) \mathcal{Z}(C_2)}{1 + \hat{q}_1^2}. \end{aligned}$$

Proof. From the given assumption, \mathcal{Z} is a differentiable convex mapping on (C_1, C_2) , so there exists a tangent line for point $\frac{C_1 + C_2}{2} \in (C_1, C_2)$ and the equation of tangent line is given as:

$$h_2(u) = \mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) \left(u - \frac{C_1 + C_2}{2}\right).$$

As \mathcal{Z} is convex mapping, then $\mathcal{T}_1(u) \leq \mathcal{Z}(u)$. Now, taking the right quantum symmetric integral on both sides of the preceding inequality, we have

$$\begin{aligned} &\int_{C_1}^{C_2} h_2(u)^{C_2} d_{\hat{q}_1}^s u \\ &= \int_{C_1}^{C_2} \left[\mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) \left(u - \frac{C_1 + C_2}{2}\right) \right]^{C_2} d_{\hat{q}_1}^s u \\ &= (C_2 - C_1) \mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) \int_{C_1}^{C_2} \left(u - \frac{C_1 + C_2}{2}\right)^{C_2} d_{\hat{q}_1}^s u \\ &= (C_2 - C_1) \mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) \left(\int_{C_1}^{C_2} u^{C_2} d_{\hat{q}_1}^s u - (C_2 - C_1) \frac{C_1 + C_2}{2} \right) \\ &= (C_2 - C_1) \mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + (C_2 - C_1) \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) \left(\frac{(1 + \hat{q}_1^2)C_2 - \hat{q}_1(C_2 - C_1)}{1 + \hat{q}_1^2} - \frac{C_1 + C_2}{2} \right) \\ &= (C_2 - C_1) \left[\mathcal{Z}\left(\frac{C_1 + C_2}{2}\right) + \mathcal{Z}'\left(\frac{C_1 + C_2}{2}\right) \frac{(1 + \hat{q}_1^2 - 2\hat{q}_1)(C_2 - C_1)}{(1 + \hat{q}_1^2)2} \right] \\ &\leq \int_{C_1}^{C_2} \mathcal{Z}(u)^{C_2} d_{\hat{q}_1}^s u. \end{aligned} \tag{5}$$

Now, comparing (3) and (5), we achieve our desired result. \square

Example 4. We consider $\mathcal{Z} : [0, 2] \rightarrow \mathbb{R}$ such that $\mathcal{Z}(x) = x^2$ is a differentiable convex function, then applying Theorem 11 for $\hat{q}_1 \in (0, 1)$, we have

$$1 + \frac{2(1 + \hat{q}_1^2 - 2\hat{q}_1)}{(1 + \hat{q}_1^2)} \leq 4 + \frac{4\hat{q}_1^2}{1 + \hat{q}_1^2 + \hat{q}_1^4} - \frac{8\hat{q}_1}{1 + \hat{q}_1^2} \leq \frac{4(1 + \hat{q}_1^2 - \hat{q}_1)}{1 + \hat{q}_1^2}.$$

For $\hat{q}_1 = 0.6$, we obtain

$$1.23529 < 1.43729 < 2.23529.$$

For graphical visualization, we take $\hat{q}_1 \in (0, 1)$ as the variable. Figure 5 gives a validation of Theorem 11.

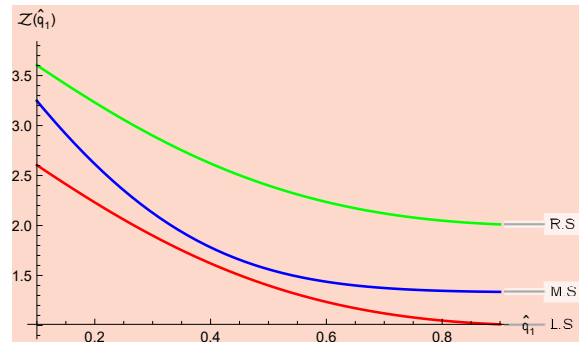


Figure 5. This figure validates the accuracy of Theorem 11.

Now we give an analytical proof of Hermite–Hadamard’s inequality.

Theorem 12. Suppose that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a convex mapping, then

$$\mathcal{Z}\left(\frac{\mathcal{C}_1 + \mathcal{C}_2}{2}\right) \leq \frac{1}{2(\mathcal{C}_2 - \mathcal{C}_1)} \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}_{\mathcal{C}_1} d_{\hat{q}_1}^s x + \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}_{\mathcal{C}_2} d_{\hat{q}_1}^s x \right] \leq \frac{\mathcal{Z}(\mathcal{C}_1) + \mathcal{Z}(\mathcal{C}_2)}{2}.$$

Proof. Since \mathcal{Z} is a convex mapping and for $x, y \in J$ such that $x = \tau\mathcal{C}_1 + (1 - \tau)\mathcal{C}_2$ and $y = (1 - \tau)\mathcal{C}_1 + \tau\mathcal{C}_2$, we have

$$\mathcal{Z}\left(\frac{\mathcal{C}_1 + \mathcal{C}_2}{2}\right) \leq \frac{1}{2} [\mathcal{Z}((1 - \tau)\mathcal{C}_1 + \tau\mathcal{C}_2) + \mathcal{Z}(\tau\mathcal{C}_1 + (1 - \tau)\mathcal{C}_2)].$$

Now, by applying the quantum symmetric integration with respect to τ over $[0, 1]$, then

$$\int_0^1 \mathcal{Z}\left(\frac{\mathcal{C}_1 + \mathcal{C}_2}{2}\right) {}_0 d_{\hat{q}_1}^s x \leq \frac{1}{2} \int_0^1 [\mathcal{Z}((1 - \tau)\mathcal{C}_1 + \tau\mathcal{C}_2) + \mathcal{Z}(\tau\mathcal{C}_1 + (1 - \tau)\mathcal{C}_2)] {}_0 d_{\hat{q}_1}^s x. \quad (6)$$

In addition, note that

$$\begin{aligned} & \int_0^1 \mathcal{Z}((1 - \tau)\mathcal{C}_1 + \tau\mathcal{C}_2) {}_0 d_{\hat{q}_1}^s \tau \\ &= (1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}\left((1 - \hat{q}_1^{2n+1})\mathcal{C}_1 + \hat{q}_1^{2n+1}\mathcal{C}_2\right) \\ &= \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}_{\mathcal{C}_1} d_{\hat{q}_1}^s x. \end{aligned} \quad (7)$$

Additionally,

$$\int_0^1 \mathcal{Z}(\tau\mathcal{C}_1 + (1 - \tau)\mathcal{C}_2) {}_0 d_{\hat{q}_1}^s \tau = \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}_{\mathcal{C}_2} d_{\hat{q}_1}^s x. \quad (8)$$

Inserting the values of (7) and (8) in (4), we obtain the left inequality.

To prove the right inequality, we utilize the convexity of \mathcal{Z}

$$\frac{\mathcal{Z}((1 - \tau)\mathcal{C}_1 + \tau\mathcal{C}_2) + \mathcal{Z}(\tau\mathcal{C}_1 + (1 - \tau)\mathcal{C}_2)}{2} \leq \frac{\mathcal{Z}(\mathcal{C}_1) + \mathcal{Z}(\mathcal{C}_2)}{2}.$$

By implementing the quantum symmetric integration with respect to τ over $[0, 1]$, we get the right side inequality. \square

Example 5. We consider $\mathcal{Z} : [0, 2] \rightarrow \mathbb{R}$ such that $\mathcal{Z}(x) = x^2$ is a differentiable convex functions, then applying on Theorem 12 for $\hat{q}_1 \in (0, 1)$, we have

$$1 \leq 2 + \frac{4\hat{q}_1^2}{1+\hat{q}_1^2+\hat{q}_1^4} - \frac{4\hat{q}_1}{1+\hat{q}_1^2} \leq 2.$$

For $\hat{q}_1 = 0.6$, we obtain

$$1 < 1.202 < 2.$$

For graphical visualization, we take $\hat{q}_1 \in (0, 1)$ as the variable. Figure 6 gives a validation of Theorem 12.

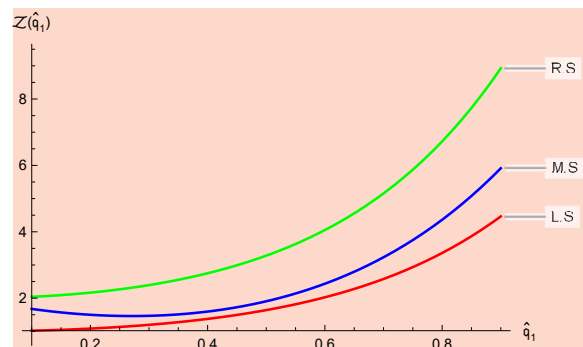


Figure 6. This figure validates the accuracy of Theorem 12.

Next, we compute another quantum symmetric analogue of Ostrowski's inequality.

Theorem 13. Assume that $\mathcal{Z} : J \rightarrow \mathbb{R}$ is a continuous and symmetric quantum differentiable mappings, then

$$\left| \mathcal{Z}(x) - \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(\tau) {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \right| \leq \frac{\|{}^{\mathcal{C}_2} D_{\hat{q}_1}^s\|}{(\mathcal{C}_2 - \mathcal{C}_1)} \left[\frac{\hat{q}_1^2 (x - \mathcal{C}_1)^2 + (1 + \hat{q}_1^2 - \hat{q}_1)(\mathcal{C}_2 - x)^2}{1 + \hat{q}_1^2} \right].$$

Proof. We start with the following expression and make use of the Lagrange mean value theorem,

$$\begin{aligned} & \left| \mathcal{Z}(x) - \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(\tau) {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \right| = \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \left| \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(\tau) - \mathcal{Z}(x)) {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \right| \\ & \leq \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} |\mathcal{Z}(x) - \mathcal{Z}(\tau)| {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \\ & \leq \frac{\|{}^{\mathcal{C}_2} D_{\hat{q}_1}^s\|}{(\mathcal{C}_2 - \mathcal{C}_1)} \int_{\mathcal{C}_1}^{\mathcal{C}_2} |x - \tau| {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \\ & \leq \frac{\|{}^{\mathcal{C}_2} D_{\hat{q}_1}^s\|}{(\mathcal{C}_2 - \mathcal{C}_1)} \left[\int_{\mathcal{C}_1}^x (x - \tau) {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau + \int_x^{\mathcal{C}_2} (\mathcal{C}_2 - x) {}^{\mathcal{C}_2} d_{\hat{q}_1}^s \tau \right] \\ & \leq \frac{\|{}^{\mathcal{C}_2} D_{\hat{q}_1}^s\|}{(\mathcal{C}_2 - \mathcal{C}_1)} \left[\frac{\hat{q}_1 (x - \mathcal{C}_1)^2}{1 + \hat{q}_1^2} + \frac{(1 + \hat{q}_1^2 - \hat{q}_1)(\mathcal{C}_2 - x)^2}{1 + \hat{q}_1^2} \right]. \end{aligned}$$

Hence, we have achieved our desired result. \square

Here, we present Korkine's identity, which is beneficial in determining the Guss–Chebyshev inequality.

Lemma 3. Assume that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ is a continuous mapping on J , then

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{G}(x) - \mathcal{G}(y)) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \\ &= (\mathcal{C}_2 - \mathcal{C}_1) \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x)\mathcal{G}(x) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x - \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x \right) \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{G}(x) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x \right). \end{aligned}$$

Proof. Utilizing the notion of the right quantum symmetric integral operator,

$$\begin{aligned} & \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{G}(x) - \mathcal{G}(y)) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \\ &= \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(x)\mathcal{G}(x) - \mathcal{Z}(x)\mathcal{G}(y) - \mathcal{Z}(y)\mathcal{G}(x) + \mathcal{Z}(y)\mathcal{G}(y)) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \\ &= (\mathcal{C}_2 - \mathcal{C}_1)^2 (1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \mathcal{G}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \\ &+ (\mathcal{C}_2 - \mathcal{C}_1)^2 (1 - \hat{q}_1^2)^2 \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{G}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \\ &+ (\mathcal{C}_2 - \mathcal{C}_1)^2 (1 - \hat{q}_1^2)^2 \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{G}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \\ &+ (\mathcal{C}_2 - \mathcal{C}_1)^2 (1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \mathcal{Z}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2) \mathcal{G}(\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2). \end{aligned}$$

We obtain

$$\begin{aligned} & \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{G}(x) - \mathcal{G}(y)) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \\ &= 2(\mathcal{C}_2 - \mathcal{C}_1) \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x)\mathcal{G}(x) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x - 2 \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x \right) \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{G}(x) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x \right). \end{aligned}$$

Hence, the proof is achieved. \square

Here, we prove the Cauchy–Bunyakovsky–Schwarz integral inequality for double integrals.

Theorem 14. Assume that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ is continuous mapping on J , then

$$\begin{aligned} & \left| \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(x,y)\mathcal{G}(x,y)) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \right| \\ & \leq \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}^2(x,y) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \right]^{\frac{1}{2}} \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{G}^2(x,y) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \right]^{\frac{1}{2}}. \end{aligned}$$

Proof. To prove our result, we consider the following double symmetric quantum integral,

$$\begin{aligned} & \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (\mathcal{Z}(x,y)\mathcal{G}(x,y)) {}^{\mathcal{C}_2}d_{\hat{q}_1}^s x {}^{\mathcal{C}_2}d_{\hat{q}_1}^s y \right]^2 \\ &= \left[(1 - \hat{q}_1)^2 (\mathcal{C}_2 - \mathcal{C}_1)^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{q}_1^{2n+2m} \mathcal{Z} \left((\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2), \right. \right. \\ & \quad \left. \left. \mathcal{Z}(\hat{q}_1^{2m+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2m+1})\mathcal{C}_2) \right) \right. \\ & \quad \left. \times \mathcal{G} \left((\hat{q}_1^{2n+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2n+1})\mathcal{C}_2), \mathcal{Z}(\hat{q}_1^{2m+1}\mathcal{C}_1 + (1 - \hat{q}_1^{2m+1})\mathcal{C}_2) \right) \right]^2. \end{aligned}$$

Now, employing the discrete Cauchy–Schwarz inequality, we have

$$\begin{aligned}
& \left[(1 - \hat{q}_1)^2 (\mathcal{C}_2 - \mathcal{C}_1)^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{q}_1^{2n+2m} \mathcal{Z}((\hat{q}_1^{2n+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_2), \mathcal{Z}(\hat{q}_1^{2m+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2m+1}) \mathcal{C}_2)) \right. \\
& \times \mathcal{G}((\hat{q}_1^{2n+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_2), \mathcal{Z}(\hat{q}_1^{2m+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2m+1}) \mathcal{C}_2)) \left. \right]^2 \\
& \leq \left[(1 - \hat{q}_1)^2 (\mathcal{C}_2 - \mathcal{C}_1)^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{q}_1^{2n+2m} \mathcal{Z}^2((\hat{q}_1^{2n+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_2), \right. \\
& \mathcal{Z}(\hat{q}_1^{2m+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2m+1}) \mathcal{C}_2)) \\
& \times (1 - \hat{q}_1)^2 (\mathcal{C}_2 - \mathcal{C}_1)^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{q}_1^{2n+2m} \mathcal{G}^2((\hat{q}_1^{2n+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2n+1}) \mathcal{C}_2), \\
& \left. \mathcal{Z}(\hat{q}_1^{2m+1} \mathcal{C}_1 + (1 - \hat{q}_1^{2m+1}) \mathcal{C}_2)) \right]^2 \\
& = \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}^2(x, y) {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x {}^{\mathcal{C}_2} d_{\hat{q}_1^s} y \right]^{\frac{1}{2}} \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{G}^2(x, y) {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x {}^{\mathcal{C}_2} d_{\hat{q}_1^s} y \right]^{\frac{1}{2}}.
\end{aligned}$$

Hence, the proof is obtained. \square

Theorem 15. Assume that $\mathcal{Z}, \mathcal{G} : J \rightarrow \mathbb{R}$ is two Lipschitzian continuous mappings on J , then

$$\begin{aligned}
& \left| \frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) \mathcal{G}(x) {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x - \left(\frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{Z}(x) {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x \right) \left(\frac{1}{\mathcal{C}_2 - \mathcal{C}_1} \int_{\mathcal{C}_1}^{\mathcal{C}_2} \mathcal{G}(x) {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x \right) \right| \\
& \leq \frac{L_1 L_2 \hat{q}_1^4 (\mathcal{C}_2 - \mathcal{C}_1)^2}{(1 + \hat{q}_1^2 + \hat{q}_1^4)(1 + \hat{q}_1^2)^2}.
\end{aligned}$$

Proof. Since both \mathcal{Z}, \mathcal{G} are Lipschitzian continuous mappings, then for any $L_1, L_2 \in \mathbb{R}$, we have

$$\begin{aligned}
|\mathcal{Z}(x) - \mathcal{Z}(y)| & \leq L_1 |x - y| \\
|\mathcal{G}(x) - \mathcal{G}(y)| & \leq L_2 |x - y|.
\end{aligned}$$

Then

$$|(\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{Z}(x) - \mathcal{Z}(y))| \leq L_1 L_2 (x - y)^2 \quad (9)$$

Now, applying the double right quantum symmetric integral operator on both sides of (9), then

$$\begin{aligned}
& \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} |(\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{Z}(x) - \mathcal{Z}(y))| {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x {}^{\mathcal{C}_2} d_{\hat{q}_1^s} y \\
& \leq L_1 L_2 \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (x - y)^2 {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x {}^{\mathcal{C}_2} d_{\hat{q}_1^s} y \\
& = L_1 L_2 \int_{\mathcal{C}_1}^{\mathcal{C}_2} \int_{\mathcal{C}_1}^{\mathcal{C}_2} (x^2 - 2xy + y^2) {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x {}^{\mathcal{C}_2} d_{\hat{q}_1^s} y \\
& = L_1 L_2 2 \left[\int_{\mathcal{C}_1}^{\mathcal{C}_2} x^2 {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x - \left(\int_{\mathcal{C}_1}^{\mathcal{C}_2} x {}^{\mathcal{C}_2} d_{\hat{q}_1^s} x \right)^2 \right].
\end{aligned} \quad (10)$$

Moreover, by taking into account the definition of the right quantum symmetric integral operator, we have

$$\begin{aligned}
& \int_{C_1}^{C_2} x^2 {}^{C_2}d_{\hat{q}_1^s} x \\
&= (C_2 - C_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \left(C_2 - \hat{q}_1^{2n+1}(C_2 - C_1) \right)^2 \\
&= (C_2 - C_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \left(C_2^2 + \hat{q}_1^{4n+2}(C_2 - C_1)^2 - \hat{q}_1^{2n+1} C_2(C_2 - C_1) \right) \quad (11) \\
&= (C_2 - C_1)(1 - \hat{q}_1^2) \left[\frac{C_2^2}{1 - \hat{q}_1^2} + \frac{\hat{q}_1^2(C_2 - C_1)^2}{1 - \hat{q}_1^6} - \frac{2\hat{q}_1 C_2(C_2 - C_1)}{1 - \hat{q}_1^4} \right] \\
&= C_2^2(C_2 - C_1) + \frac{\hat{q}_1^2(C_2 - C_1)^3}{1 + \hat{q}_1^2 + \hat{q}_1^4} - \frac{2\hat{q}_1 C_2(C_2 - C_1)^2}{1 + \hat{q}_1^2}.
\end{aligned}$$

Additionally,

$$\begin{aligned}
& \int_{C_1}^{C_2} x {}^{C_2}d_{\hat{q}_1^s} x \\
&= (C_2 - C_1)(1 - \hat{q}_1^2) \sum_{n=0}^{\infty} \hat{q}_1^{2n} \left(C_2 - \hat{q}_1^{2n+1}(C_2 - C_1) \right) \quad (12) \\
&= C_2(C_2 - C_1) - \frac{\hat{q}_1(C_2 - C_1)^2}{1 + \hat{q}_1^2}.
\end{aligned}$$

Introducing the values of (11) and (12) in (10) results the following inequality,

$$\begin{aligned}
& \int_{C_1}^{C_2} \int_{C_1}^{C_2} |(\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{G}(x) - \mathcal{G}(y))| {}^{C_2}d_{\hat{q}_1^s} x {}^{C_2}d_{\hat{q}_1^s} y \\
&\leq \frac{2L_1 L_2 \hat{q}_1^4 (C_2 - C_1)^4}{(1 + \hat{q}_1^2 + \hat{q}_1^4)(1 + \hat{q}_1^2)^2}. \quad (13)
\end{aligned}$$

Now from Korkine equality and inequality (13), we have

$$\begin{aligned}
& \left| (C_2 - C_1) \int_{C_1}^{C_2} \mathcal{Z}(x) \mathcal{G}(x) {}^{C_2}d_{\hat{q}_1^s} x - \left(\int_{C_1}^{C_2} \mathcal{Z}(x) {}^{C_2}d_{\hat{q}_1^s} x \right) \left(\int_{C_1}^{C_2} \mathcal{G}(x) {}^{C_2}d_{\hat{q}_1^s} x \right) \right| \\
&= \left| \frac{1}{2} \int_{C_1}^{C_2} \int_{C_1}^{C_2} (\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{G}(x) - \mathcal{G}(y)) {}^{C_2}d_{\hat{q}_1^s} x {}^{C_2}d_{\hat{q}_1^s} y \right| \\
&\leq \frac{1}{2} \int_{C_1}^{C_2} \int_{C_1}^{C_2} |(\mathcal{Z}(x) - \mathcal{Z}(y))(\mathcal{G}(x) - \mathcal{G}(y))| {}^{C_2}d_{\hat{q}_1^s} x {}^{C_2}d_{\hat{q}_1^s} y \\
&= \frac{2L_1 L_2 \hat{q}_1^4 (C_2 - C_1)^4}{(1 + \hat{q}_1^2 + \hat{q}_1^4)(1 + \hat{q}_1^2)^2}. \quad (14)
\end{aligned}$$

By dividing both sides of (14) by $(C_2 - C_1)^2$, we attain our desired result. \square

Example 6. We consider $\mathcal{Z}, \mathcal{G} : [0, 2] \rightarrow \mathbb{R}$ such that $\mathcal{Z}(x) = \frac{x^2}{2}$ and $\mathcal{G}(x) = \frac{x}{3}$ are two continuous Lipschitzian mappings, then applying on Theorem 15 for $\hat{q}_1 \in (0, 1)$, we have

$$\begin{aligned}
& \left| \frac{4}{3} \left[\frac{1 + \hat{q}_1^2 - 3\hat{q}_1}{1 + \hat{q}_1^2} + \frac{3\hat{q}_1^2}{1 + \hat{q}_1^2 + \hat{q}_1^4} - \frac{\hat{q}_1^3}{(1 + \hat{q}_1^4)(1 + \hat{q}_1^2)} \right] - \frac{(2 + 2\hat{q}_1^2 - 2\hat{q}_1)}{(3(1 + \hat{q}_1^2))} \left(\frac{2 + 4\hat{q}_1^2 + 2\hat{q}_1^4}{1 + \hat{q}_1^2 + \hat{q}_1^4} - \frac{4\hat{q}_1}{1 + \hat{q}_1^2} \right) \right| \\
&\leq \frac{8\hat{q}_1^4}{3(1 + \hat{q}_1^2 + \hat{q}_1^4)(1 + \hat{q}_1^2)^2}.
\end{aligned}$$

For $\hat{q}_1 = 0.6$, we obtain

$$0.0801305 < 0.125437.$$

For graphical visualization, we take $\hat{q}_1 \in (0, 1)$ as the variable. Figure 7 gives a validation of Theorem 15.

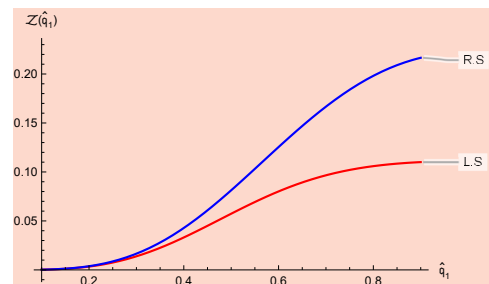


Figure 7. This figure validates the accuracy of Theorem 15.

4. Conclusions

In this study, we introduced the novel concepts of right analogues of symmetric operators and examined some key properties. Additionally, we presented the quantum symmetric analogues of several well-known inequalities, such as Hermite–Hadamard’s, Young’s, Ostrowski’s, and Hölder’s inequalities. Moreover, we have proposed the geometrical and analytical proof of Hermite–Hadamard’s inequality. Furthermore, the correctness of the results is verified through numerical examples and visuals. These operators will play a significant role in the study of non-differentiable mappings. In the future, we will try to develop new symmetric analogues of Hermite–Hadamard–Mercer inequality, Simpson’s inequality, and Newton’s inequality associated with various generalizations of convex functions. In addition, based on these derivative operators, new integral operators can be established. Furthermore, we will extend these results for set-valued mappings and will also establish necessary and sufficient conditions for the differentiability of interval-valued mapping. We hope that these results will play a significant contribution to the development of inequalities and optimization theory.

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