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Article

Existence of Ground State Solutions for a Class of Non-Autonomous Fractional Kirchhoff Equations

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Abstract: We take a look at the fractional Kirchhoff problem in this paper. Using a variational approach, we show that there exists a ground state solution for this problem. Furthermore, using the approach developed by Szulkin and Weth, we also find that positive ground state solutions exist for the fractional Kirchhoff equation with $p = 4$.

Keywords: fractional Kirchhoff equation; concentration compactness; variational methods

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1. Introduction

The following non-autonomous fractional Kirchhoff equation will be investigated in this work:

$$\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u+u=Q(x)|u|^{p-2}u,\ x\in\mathbb{R}^3, \tag{1}$$

where $a, b > 0$, $4 < p < 2_s^* := \frac{6}{3-2s}$ and the fractional Laplace operator is defined as

$$(-\Delta)^s u=C_{3,s}P.V.\int_{\mathbb{R}^3}\frac{u(x)-u(y)}{|x-y|^{3+2s}}dy=-\frac{C_{3,s}}{2}\int_{\mathbb{R}^3}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}dy. \tag{2}$$

The constant $C_{3,s}$ is positive and varies with the dimensions 3 and s:

$$C_{3,s}=\left(\int_{\mathbb{R}^3}\frac{(1-cos\xi_1)}{|\xi|^{3+2s}}d\xi\right)^{-1},\ \xi=(\xi_1,\xi_2,\xi_3).$$

It is important to note that the stationary fractional Kirchhoff model with critical growth and homogeneous Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$ was first developed by Fiscella and Valdinoci [1]:

$$\begin{cases} M\left(\int\int_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^2K(x-y)dxdy\right)\mathcal{L}_K u=\lambda f(x,u)+|u|^{2_s^*-2}u & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{3}$$

where the Kirchhoff function M (covering the situation $M(t) = a + bt$) is expressed. The following defines the nonlocal integro-differential operator $\mathcal{L}_K$:

$$\mathcal{L}_K(x):=\frac{1}{2}\int_{\mathbb{R}^N}\Big(u(x+y)+u(x-y)-2u(x)\Big)K(y)dy,\ x\in\mathbb{R}^N,$$

the measurable function $K : \mathbb{R}^N\setminus\{0\} \to (0,+\infty)$ satisfies the following assumptions: there exists $\theta > 0$ and $s \in (0,1)$ such that

$$\theta|x|^{-(N+2s)}\le K(x)\le\theta^{-1}|x|^{-(N+2s)},\forall x\in\mathbb{R}^N\setminus\{0\}.$$

Under appropriate conditions on M and f, they established the existence of non-negative solutions for equation (3). We refer to [1] for further information on (3) on the physical background and its applications. Furthermore, as an extension of the classical d'Alembert wave equations in the fractional setting, the model of (3) can be understood as a generalization of the well-known Kirchhoff model introduced by Kirchhoff [2]. For further information and results about the nonlocal operator $\mathcal{L}_K$, we also refer to [3,4] and their references, where the operator $\mathcal{L}_K(x)$ reduces to $(-\Delta)^s$ if $K(y) = |y|^{-(N+2s)}$.

In [5], Autuori, Fiscella, and Pucci established the existence and asymptotic behavior of non-negative solutions to (3) with a generalized Kirchhoff function. The existence and multiplicity of solutions for a nonhomogeneous fractional p-Laplacian equation of the Schrödinger–Kirchhoff type was subsequently investigated by Pucci, Xiang, and Zhang [6]. For the non-degenerate fractional Kirchhoff equation (i.e., $M(0) = 0$), Caponi and Pucci analyzed the existence and asymptotic behavior of a nontrivial mountain pass solution in [7]. The Pohožaev identity of (1) was established by Zhang et al. [8] using the s-harmonic extension approach developed in [9], and they obtained the existence of ground state solutions for $s \in [\frac{3}{4}, 1)$ as well as the non-existence result for $s \in (0, \frac{3}{4}]$.

Due to the appearance of the term $b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)(-\Delta)^s u$, which is of order four, the fractional Kirchhoff problem (1) is significantly more complex and difficult from the perspective of calculus of variation than the traditional fractional Laplacian equation. Thus, elucidating the implications of this nonlocal term is a basic challenge for research on the problem (1). Yang and Rădulescu [10] recently proved non-degeneracy and uniqueness for positive solutions to Kirchhoff equations with subcritical growth. To be more exact, they showed that the fractional Kirchhoff equation

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)(-\Delta)^s u + u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

where $\frac{N}{4} < s < 1$, $2 < p < 2_s^* = \frac{2N}{N-2s}$, has a unique non-degenerate positive radial solution. Yang [11] proved that for dimensions $N > 4s$, uniqueness breaks down, i.e., there exist two non-degenerate positive solutions that appear to be completely different from the results of the fractional Schrödinger equation or the low dimensional fractional Kirchhoff equation. They derived the existence of solutions to the singularly perturbed problems [12,13] by combining the Lyapunov–Schmidt reduction approach with this non-degeneracy conclusion. We refer to [14–22] for more results of fractional Kirchhoff-type equations employing variational methods.

The fractional Schrödinger equation is as follows, with $a = 1, b = 0$:

$$(-\Delta)^s u + V(x)u = f(x, u),\ x \in \mathbb{R}^3, \tag{4}$$

which is related to the standing wave solution $\psi(t, x) = exp(-ict)u(x)$ of the time-independent fractional Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} + (-\Delta)^s \psi + (V(x) - c)\psi = f(x, \psi),\ x \in \mathbb{R}^3.$$

As a result of extending the Feynman path integral from Brownian-like to Levy-like quantum mechanical paths, the above equation is a fundamental equation of fractional quantum mechanics [23]. Because the fractional Laplace operator $(-\Delta)^s$ is nonlocal, the conventional analytical methods for elliptic PDEs employing classical Laplacian operator $-\Delta$ cannot be directly applied to problem (4). We refer to [24] and the references therein for more information on the distinctions between classical and fractional Laplace. The s-harmonic extension strategy was developed by Cafferelli and Silvestre [9] to convert the nonlocal Equations (1) and (4) into a local problem in the upper half-space. Because of this, a lot of individuals have used this useful method to resolve different fractional Laplacian difficulties. We also refer to [25,26] for more results regarding nonlocal issues using the fractional Laplace operator.

Inspired by the previously listed studies, the focus of this research is on whether a ground state solution to (1) exists under the following conditions:

(Q_1) $\lim_{|x|\to+\infty} Q(x) = Q_\infty > 0$, and $Q(x) \in L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^3)$ with $4 < p < 2_s^*$.

(Q_2) $Q(x) \geq Q_\infty$, $\forall x \in \mathbb{R}^3$ and $Q(x) - Q_\infty > 0$ on a positive measure set.

We also note that Xie and Ma, in [27], obtained the existence of a positive ground state solution (1) with $s = 1$ and $4 < p < 6$ under the aforementioned constraints. The authors in [28] produced a result for the Schrödinger–Poisson system that was comparable. For further results involving the aforementioned comparable conditions, see [29–31], and references therein. To the best of our knowledge, it appears that no studies have been performed regarding the existence of the ground state solution of (1). Assuming $(Q_1) - (Q_2)$, our response in this paper will be in the affirmative.

For simplicity, we will suppose $a = 1$. Problem (1) then reduces to the following equation:

$$\Big(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx\Big)(-\Delta)^s u + u = Q(x)|u|^{p-2}u,\ x \in \mathbb{R}^3. \tag{5}$$

The critical points of the functional

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q(x)|u|^p dx,$$

are represented by the solutions of (5), where we define on $H^s(\mathbb{R}^3)$ with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx + \int_{\mathbb{R}^3} u^2dx\Big)^{\frac{1}{2}}.$$

It is important to note that we are unable to use the variational methods in the standard sense since the fractional Sobolev space $H^s(\mathbb{R}^3)$ is only continuously embedded into $L^t(\mathbb{R}^3)$, $2 \leq t \leq 2_s^*$. Furthermore, the introduction of the nonlocal term

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx$$

presents another challenge. It makes it difficult to analyze whether Palais–Smale sequences of I are compact because, generally speaking, we do not know

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx$$

from $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. Restricting I to the radial space $H^s_{rad}(\mathbb{R}^3)$, which is compactly embedded into $L^t(\mathbb{R}^3)$, $2 \leq t \leq 2_s^*$, is one technique to recover the compactness. Directly verifying that I satisfies the Palais–Smale condition in $H^s(\mathbb{R}^3)$ is quite challenging. We require certain characteristics of the positive ground state solution of the following equation in order to move beyond these obstacles:

$$(-\Delta)^s u + u = Q(x)|u|^{p-2}u,\ x \in \mathbb{R}^3, \tag{6}$$

which is crucial in determining if the Palais–Smale sequences of I are compact. The solutions of (6) are the critical points of the functional

$$I_Q(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q(x)|u|^p dx.$$

The Nehari manifold associated with I_Q can be defined as follows:

$$\mathcal{N}_Q := \Big\{u \in H^s(\mathbb{R}^3) \setminus \{0\} :< I_Q'(u), u >= 0\Big\}.$$

Regarding the limit problem,

$$(-\Delta)^s u + u = Q_\infty |u|^{p-2}u, \ x \in \mathbb{R}^3. \tag{7}$$

For convenience, we just write $Q_\infty = 1$. For the functional and Nehari manifold, we use the notation I_∞ and $\mathcal{N}_\infty$, respectively; that is,

$$I_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx,$$

and

$$\mathcal{N}_\infty := \left\{ u \in H^s(\mathbb{R}^3) \setminus \{0\} :< I'_\infty(u), u >= 0 \right\}.$$

By (Q_2), we also have that $c_\infty > c_Q$, where

$$c_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u) \text{ and } c_Q := \inf_{u \in \mathcal{N}_Q} I_Q(u).$$

This paper's initial conclusion is as follows.

Theorem 1. *With $(Q_1) - (Q_2)$ as hypotheses, $s \in (\frac{3}{4}, 1)$ where $\mu = \frac{p-4}{p}$, and $0 < b \leq \frac{c_\infty^\mu - c_Q^\mu}{\theta c_Q^{\mu+1}}$, $p \in (4, 2_s^*)$, and $\theta = \frac{2p}{p-2}$. Then there is a ground state solution for problem (1).*

Studying the existence of a positive ground state for the following equation is another goal of this paper:

$$\left(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)(-\Delta)^s u(x) + u = K(x)u^3, \ \text{ in } \mathbb{R}^3. \tag{8}$$

Assume that

(K_1) $\lim_{|x| \to +\infty} K(x) = K_\infty > 0$, and $K(x) \in L^\infty(\mathbb{R}^3)$.

(K_2) $K(x) \geq K_\infty$, $\forall x \in \mathbb{R}^3$ and $K(x) - K_\infty > 0$ on a positive measure set.

This paper's second result says the following.

Theorem 2. *Assume that $s \in (0, 1)$ and $(K_1) - (K_2)$ hold. Then there is a positive ground state solution for problem (8).*

The usual Nehari manifold approach is rendered worthless for problem (8) due to the combination of the nonlocal term $\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)(-\Delta)^s u$ and the nonlinearity $Q(x)u^3$. In order to find a way around this problem, we took a hint from [32], where they discovered that the quasilinear Schrödinger equation with three times growth has infinitely many geometrically distinctive solutions. For a few related papers, see [29,33], along with the citations therein.

2. Preliminary Results

The subsequent vanishing lemma is a version of P.L. Lions' concentration–compactness principle. We have access to [34,35].

Lemma 1. *Let $\{u_n\}$ be a bounded sequence in $H^s(\mathbb{R}^3)$ and it satisfies*

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n(x)|^2 dx = 0,$$

where $R > 0$. Then, $u_n \to 0$ in $L^t(\mathbb{R}^3)$ for every $2 < t < 2_s^*$.

To demonstrate Theorem 1, we first set

$$\mathcal{N} := \Big\{ u \in H^s(\mathbb{R}^3) \setminus \{0\} : \langle I'(u), u \rangle = 0 \Big\}.$$

In the following lemma, we then gather some characteristics of the Nehari manifold $\mathcal{N}$.

Lemma 2.

(i) *$\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the sphere of $H^s(\mathbb{R}^3)$.*
(ii) *I is bounded from below on $\mathcal{N}$.*
(iii) *u is a critical point of I if and only if u is a critical point of I constrained on $\mathcal{N}$.*

Proof. There is a unique $t > 0$ such that $tu \in \mathcal{N}$ for any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ with $\|u\| = 1$. In actuality, $g(t) > 0$ for $t > 0$ small and $g(t) \to -\infty$ as $t \to \infty$, where $g(t) = \langle I'(tu), tu \rangle$, are given by $p \in (4, 2_s^*)$. As a result, g has a positive maximum, and $tu \in \mathcal{N}$ and $g'(t) = 0$ exist for some $t > 0$. Assume that $t_1 > t > 0$ exists and that $t_1 u \in \mathcal{N}$. That implies that

$$\left(\frac{1}{t_1^2} - \frac{1}{t^2}\right)\|u\|^2 = \left(t_1^{p-4} - t^{p-4}\right)\int_{\mathbb{R}^3} Q(x)|u|^p dx,$$

which is not conceivable. Thus, $t'u \in \mathcal{N}$ if and only if $t = t'$.

Note that $u \in \mathcal{N}$, then

$$\begin{aligned} 0 &= \|u\|^2 + b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 - \int_{\mathbb{R}^3} Q(x)|u|^p dx \\ &\geq \|u\|^2 - C_1\|u\|^p, \end{aligned}$$

which means that $\|u\|^2 > C_2$ for some $C_2 > 0$. We can determine that $G(u) := \langle I'(u), u \rangle$ is a C^1 function by using the information that $I \in C^2(H^s(\mathbb{R}^3), \mathbb{R})$. Note that

$$\begin{aligned} \langle G'(u), u \rangle &= \langle I''(u), u \rangle \\ &= 2\|u\|^2 + 4b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 - p\int_{\mathbb{R}^3} Q(x)|u|^p dx \\ &= \Big(2-p\Big)\|u\|^2 + \Big(4-p\Big)b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 \\ &\leq \Big(2-p\Big)\|u\|^2 \leq \Big(2-p\Big)C_2^2 < 0. \end{aligned}$$

(ii) follows immediately from

$$I = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 > C_2 > 0.$$

(iii) The conclusion that $u \in \mathcal{N}$ if $u \neq 0$ is a critical point of I is obvious. Let, however, u represent a critical point of the functional I on $\mathcal{N}$. $G'(u) \neq 0$ and

$$I|'_{\mathcal{N}}(u) = I'(u) - \frac{\langle I'(u),\, u \rangle}{\|G'(u)\|^2} G'(u)$$

lead to the conclusion that $I'(u) = 0$. □

In the following result, we outline some of the known results concerning the positive solutions of (7), which are important to our proof and come from [36,37].

Proposition 1. *There is only one ground state solution for the equation (7) up to translation $w_\infty \in H^{2s+1}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$, which fulfills the following asymptotic decay conditions and is positive, radially symmetric, and radially declining:*

$$\frac{C_1}{1+|x|^{3+2s}} \leq w_\infty(x) \leq \frac{C_2}{1+|x|^{3+2s}} \quad \text{for all } x \in \mathbb{R}^3, \ \text{where } 0 < C_1 < C_2.$$

Using the concentration–compactness method, which was developed in [38] (also see [35]), the existence of the following result may be obtained.

Lemma 3. *If $(Q_1) - (Q_2)$ holds, then $w_Q \in H^s(\mathbb{R}^3)$ is the positive ground state solution for problem (6).*

As with Lemma 2, it is evident that, given any function $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, there is a unique function $\tau u \in \mathcal{N}_Q$ such that $I_Q(\tau u) = \max_{t>0} I_Q(tu)$.

Lemma 4. *For any $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, assume that tu and τu are its projections on $\mathcal{N}$ and $\mathcal{N}_Q$, respectively. Then, $\tau \leq t$.*

Proof. For $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, it follows from $tu \in \mathcal{N}$ and $\tau u \in \mathcal{N}_Q$ that

$$t^2\|u\|^2 + t^4\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^2 = t^p \int_{\mathbb{R}^3} Q(x)|u|^p dx$$

and

$$\tau^2\|u\|^2 = \tau^p \int_{\mathbb{R}^3} Q(x)|u|^p dx.$$

We deduce from (Q_1) that

$$\begin{aligned}\tau^{p-2} &= \frac{\|u\|^2}{\int_{\mathbb{R}^3} Q(x)|u|^p dx} \\ &= \frac{-t^2\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^2 + t^{p-2}\int_{\mathbb{R}^3} Q(x)|u|^p dx}{\int_{\mathbb{R}^3} Q(x)|u|^p dx} \\ &\leq t^{p-2},\end{aligned}$$

which implies that $\tau \leq t$. □

According to the conditions of Theorem 1, I satisfies the mountain pass structure. Then, we know that

$$d = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in ([0,1],\ H^s(\mathbb{R}^3)) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}$. By using a version of the mountain pass theorem without the (PS) condition (see [35] [Theorem 2.10]), there exists a sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ such that

$$I(u_n) \to d \text{ and } I'(u_n) \to 0 \text{ in } (H^s(\mathbb{R}^3))^{-1}, \quad \text{as } n \to \infty.$$

As in [35], we have that

$$d = \inf_{u\in\mathcal{N}} I(u) = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \max_{t\geq 0} I(tu) > 0.$$

Furthermore, for n large enough,

$$\begin{aligned} d + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx\Big)^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Hence, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^3), \\ &u_n \to u \text{ in } L^q_{loc}(\mathbb{R}^3) \text{ for } q \in [2, 2^*_s), \\ &u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^3. \end{aligned}$$

To finish this section, we give a lemma which will be used later.

Lemma 5. *Let $\{u_n\}$ be a $(PS)_d$ sequence of I. Moreover, there exists $u \in H^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ with $u \neq 0$. Then,*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx.$$

Proof. We claim that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx. \tag{9}$$

In fact, we may assume that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx = A$$

for some $A \geq 0$. Then, we have

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx \leq A.$$

For each $\varphi \in H^s(\mathbb{R}^3)$, the weak convergence of $\{u_n\}$ implies that

$$\int_{\mathbb{R}^3} u_n\varphi dx \to \int_{\mathbb{R}^3} u\varphi, \text{ and } \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi dx \to \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx.$$

Moreover, we deduce that

$$\int_{\mathbb{R}^3} Q(x)|u_n|^{p-2}u_n\varphi dx \to \int_{\mathbb{R}^3} Q(x)|u|^{p-2}u\varphi dx.$$

Using the above facts, we obtain

$$\begin{aligned} o_n(1) &= \langle I'(u_n),\ \varphi\rangle \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi dx + \int_{\mathbb{R}^3} u_n\varphi dx \\ &\quad + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi dx - \int_{\mathbb{R}^3} Q(x)|u_n|^{p-2}u_n\varphi dx + o_n(1) \\ &\geq \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx + \int_{\mathbb{R}^3} u\varphi dx \\ &\quad + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx - \int_{\mathbb{R}^3} Q(x)|u|^{p-2}u\varphi dx. \end{aligned} \tag{10}$$

We only need to consider the case that $\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx < A$. In such a case, $\langle I'(u), \varphi \rangle < 0$. In particular, $\langle I'(u), u \rangle < 0$. Set

$$h_1(t) := \langle I'(tu), tu \rangle \text{ for } t \geq 0.$$

Then, inequality (10) implies that $h_1(1) < 0$. Moreover, we have

$$\begin{aligned} h_1(t) &= t^2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + t^2 \int_{\mathbb{R}^3} u^2 dx + bt^4 \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^2 - t^p \int_{\mathbb{R}^3} Q(x)|u|^p dx \\ &\geq \frac{t^2}{2} \|u\|^2 - t^p \int_{\mathbb{R}^3} Q(x)|u|^p dx \end{aligned}$$

which yields that $h_1(t) > 0$ for $t > 0$ since $4 < p < 2_s^*$. Clearly, $h_1(t)$ is continuous. Thus, there exists a $t_0 \in (0, 1)$ such that $h_1(t_0) = \langle I'(t_0 u), t_0 u \rangle = 0$. That is, $t_0 u \in \mathcal{N}$, and hence, $I(t_0 u) = \max_{t \in [0,1]} I(tu)$. Moreover,

$$\begin{aligned} d \leq I(t_0 u) &= I(t_0 u) - \frac{1}{4} \langle I'(t_0 u),\ t_0 u \rangle \\ &= \frac{t_0^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{t_0^2}{4} \int_{\mathbb{R}^3} u^2 dx + \left(\frac{1}{4} - \frac{1}{p} \right) t_0^p \int_{\mathbb{R}^3} Q(x)|u|^p dx \\ &< \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} u_n^2 dx + \left(\frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} Q(x)|u_n|^p dx \\ &= \liminf_{n \to +\infty} \left\{ I(u_n) - \frac{1}{4} \langle I'(u_n),\ u_n \rangle \right\} \\ &= d + o_n(1), \end{aligned} \tag{11}$$

this contradiction implies that (9) holds true and $u \in \mathcal{N}$. □

3. Proof of Theorem 1

Lemma 6. *Let $\{u_n\}$ be a $(PS)_d$ sequence of I constrained on $\mathcal{N}$, that is, $u_n \in \mathcal{N}$ and*

$$I(u_n) \to d, \ \ I'|_{\mathcal{N}}(u_n) \to 0 \text{ in } (H^s(\mathbb{R}^3))^{-1}.$$

Then, going to a subsequence if necessary, one of the alternatives below holds.

(i) There exists a solution $\tilde{u} \neq 0$ of problem (5), a number $l \in \mathbb{N}$, and $\{y_n^k\} \subset \mathbb{R}^3$ with $|y_n^k| \to +\infty$ for each $1 \leq k \leq l$ and $|y_n^{k_1} - y_n^{k_2}| \to +\infty$ for $k_1 \neq k_2$(as $n \to +\infty$), nontrivial solutions $w^1, \cdots, w^l$ of the problem (7), such that

$$\begin{aligned} &\left\| u_n - u - \sum_{k=1}^{l} w^k(\cdot - y_n^k) \right\| \to 0, \\ &I(u_n) \to I(\tilde{u}) + \sum_{k=1}^{l} I_\infty(w^k) \end{aligned} \tag{12}$$

(ii) There exists a solution $\tilde{w}^0 \in H^s(\mathbb{R}^3)$, $\bar{y}_n \in \mathbb{R}^3$ with $|\bar{y}_n| \to +\infty$, a number $l \in \mathbb{N}$, and $\{y_n^k\} \subset \mathbb{R}^3$ with $|y_n^k| \to +\infty$ for each $1 \leq k \leq l$, $|\bar{y}_n - y_n^{k_2}| \to +\infty$ and $|y_n^{k_1} - y_n^{k_2}| \to +\infty$ for $k_1 \neq k_2$(as $n \to +\infty$), nontrivial solutions $w^1, \cdots, w^l$ of the problem (7), such that

$$\begin{aligned} &\left\| u_n - \tilde{w}^0(\cdot - \bar{y}_n) - \sum_{k=1}^{l} w^k(\cdot - y_n^k) \right\| \to 0, \\ &I(u_n) \to J(\tilde{w}^0) + \sum_{k=1}^{l} I_\infty(w^k) \end{aligned} \tag{13}$$

where $\tilde{w}^0$ is a nontrivial weak solution of the following problem:

$$\Big(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx\Big)(-\Delta)^s u + u = |u|^{p-2}u \tag{14}$$

and

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx. \tag{15}$$

Proof. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, we can prove that $I'(u_n) \to 0$. In fact, we have

$$o_n(1) = I'|_{\mathcal{N}}(u_n) = I'(u_n) - \lambda_n G'(u_n),$$

for some $\lambda_n \in \mathbb{R}$. Thus, we obtain

$$o_n(1) = \langle I'|_{\mathcal{N}}(u_n), u_n\rangle = \langle I'(u_n), u_n\rangle - \lambda_n\langle G'(u_n),\ u_n\rangle.$$

By $u_n \in \mathcal{N}$ and $\langle G'(u_n),\ u_n\rangle < 0$, we have $\lambda_n \to 0$ for $n \to +\infty$. Thus, $I'(u_n) \to 0$. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, then, up to a subsequence, $u_n \rightharpoonup \tilde{u}$ in $H^s(\mathbb{R}^3)$, $u_n \to \tilde{u}$ in $L^q_{loc}(\mathbb{R}^3)$ for $q \in [2, 2^*_s)$, $u_n(x) \to \tilde{u}(x)$ a.e. in $\mathbb{R}^3$.

We consider separately the two cases $\tilde{u} \neq 0$ and $\tilde{u} = 0$.

Case (**i**): $\tilde{u} \neq 0$. If $u_n \to \tilde{u}$ in $H^s(\mathbb{R}^3)$, then the proof is complete. It follows from Lemma 5 that $I'(\tilde{u}) = 0$; hence, $\tilde{u}$ is a weak solution of Equation (5). Thus, we assume that $u_n \nrightarrow \tilde{u}$ in $H^s(\mathbb{R}^3)$. Setting $u_n^1 = u_n - \tilde{u}$, we have $u_n^1 \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. We claim that

$$I(u_n) = I(\tilde{u}) + I_\infty(u_n^1) + o_n(1),\ \text{ and }\ I_\infty(u_n^1) = o_n(1)\ \text{in}(H^s(\mathbb{R}^3))^{-1}$$

Indeed, it follows from the Brezis–Lieb lemma that
(a.1) $\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n^1|^2dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx + o_n(1)$;
(a.2) $|u_n^1|_2^2 = |u_n|_2^2 - |\tilde{u}|_2^2 + o_n(1)$;
(a.3) $\int_{\mathbb{R}^3} Q(x)|u_n^1|^p dx = \int_{\mathbb{R}^3} Q(x)|u_n|^p dx - \int_{\mathbb{R}^3} Q(x)|\tilde{u}|^p dx + o_n(1)$.

Moreover, by [35], we have

$$\int_{\mathbb{R}^3}(Q(x)-1)|u_n^1|^p dx = o_n(1),\ \int_{\mathbb{R}^3}(Q(x)-1)|u_n^1|^{p-2}u_n^1\varphi dx = o_n(1),\ \forall\varphi \in H^s(\mathbb{R}^3),$$

and

$$\int_{\mathbb{R}^3} Q(x)|u_n|^{p-2}u_n\varphi dx = \int_{\mathbb{R}^3} Q(x)|u_n^1|^{p-2}u_n^1\varphi dx + \int_{\mathbb{R}^3} Q(x)|\tilde{u}|^{p-2}\tilde{u}\varphi dx + o_n(1),\ \forall\varphi \in H^s(\mathbb{R}^3).$$

Therefore,

$$\begin{aligned}
I(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q(x)|u_n|^p dx \\
&= \frac{1}{2}\|u_n^1\|^2 + \frac{1}{2}\|\tilde{u}\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q(x)|\tilde{u}|^p dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} Q(x)|u_n^1|^p dx \\
&= \frac{1}{2}\|u_n^1\|^2 + \frac{1}{2}\|\tilde{u}\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q(x)|\tilde{u}|^p dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} |u_n^1|^p dx \\
&= I(\tilde{u}) + I_\infty(u_n^1) + o_n(1).
\end{aligned}$$

And, for all $\varphi \in H^s(\mathbb{R}^3)$, we obtain

$$\begin{aligned} o_n(1) &= \langle I'(u_n), \varphi \rangle \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} u_n \varphi dx \\ &\quad + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx - \int_{\mathbb{R}^3} Q(x)|u_n|^{p-2} u_n \varphi dx + o_n(1) \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \tilde{u} (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} \tilde{u} \varphi dx + b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \tilde{u} (-\Delta)^{\frac{s}{2}} \varphi dx \\ &\quad - \int_{\mathbb{R}^3} Q(x)|\tilde{u}|^{p-2} \tilde{u} \varphi dx - \int_{\mathbb{R}^3} |u_n^1|^{p-2} u_n^1 \varphi dx + o_n(1) \\ &= \langle I'(\tilde{u}), \varphi \rangle + \langle I'_\infty(u_n^1), \varphi \rangle + o_n(1). \end{aligned}$$

Thus, we obtain that $I_\infty(u_n^1) = o_n(1)$ in$(H^s(\mathbb{R}^3))^{-1}$. So, the claim is proved.

Let us define

$$\delta := \limsup_{n \to +\infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_n^1|^2 dx.$$

If vanishing occurs, i.e., $\delta = 0$, then $u_n^1 \rightharpoonup 0$, and $u_n^1 \to 0$ in $L^t(\mathbb{R}^3)$ for $t \in (2, 2_s^*)$ is evident from Lemma 1. The proof is then finished when $\|u_n^1\| \to 0$ and consequently $u_n^1 \to 0$ in $H^s(\mathbb{R}^3)$.

If nonvanishing happen, that is, $\delta > 0$, then we can presume that there is a $\{y_n^1\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |u_n^1|^2 dx > \frac{\delta}{2}.$$

Then, for some $w^1 \in H^s(\mathbb{R}^3)$, we obtain, going to a subsequence if needed,

$$|y_n^1| \to +\infty, \quad w_n^1 := u_n^1(\cdot + y_n^1) \rightharpoonup w^1.$$

Since

$$\int_{B_1(0)} |w_n^1|^2 dx > \frac{\delta}{2},$$

then

$$\int_{B_1(0)} |w^1|^2 dx > \frac{\delta}{2},$$

and $w^1 \neq 0$. However, $\{y_n^1\}$ must be unbounded since $u_n^1 \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$ implies it. It is assumed that $|y_n^1| \to +\infty$, up to a subsequence. Furthermore, we derive $I'_\infty(w^1) = 0$ from $I_\infty(u_n^1) = o_n(1)$.

Let us define $u_n^2 = u_n - u - w^1(\cdot - y_n^1)$ next. In $H^s(\mathbb{R}^3)$, we have $u_n^2 \rightharpoonup 0$. We conclude the result from the Brezis–Lieb lemma:

(b.1) $\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n^2|^2 dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w^1|^2 dx + o_n(1)$,

(b.2) $|u_n^2|_2^2 = |u_n|_2^2 - |u|_2^2 - |w^1|_2^2 + o_n(1)$,

(b.3)

$$\begin{aligned} \int_{\mathbb{R}^3} Q(x)|u_n^2|^p dx &= \int_{\mathbb{R}^3} Q(x)|u_n^1|^p dx - \int_{\mathbb{R}^3} Q(x)|w^1(x - y_n^1)|^p dx + o_n(1) \\ &= \int_{\mathbb{R}^3} Q(x)|u_n|^p dx - \int_{\mathbb{R}^3} Q(x)|\tilde{u}|^p dx - \int_{\mathbb{R}^3} Q(x)|w^1(x - y_n^1)|^p dx + o_n(1). \end{aligned}$$

Thus,

$$I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(w^1) + o_n(1)$$

and

$$I(u_n) = I(\tilde{u}) + I_\infty(u_n^1) + o_n(1) = I(\tilde{u}) + I_\infty(w^1) + I_\infty(u_n^2) + o_n(1).$$

Similarly, we have that

$$I'_\infty(u_n^2) = o_n(1) \text{ in } (H^s(\mathbb{R}^3))^{-1}.$$

Similar to the above arguments, define

$$\sigma := \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n^2|^2 dx.$$

If vanishing takes place, the proof is finished and $\|u_n^2\| \to 0$. A sequence $\{y_n^2\} \subset \mathbb{R}^3$ and a nontrivial $w^2 \in H^s(\mathbb{R}^3)$ such that $w_n^2 := u_n^2(\cdot + y_n^2) \rightharpoonup w^2 \neq 0$ in $H^s(\mathbb{R}^3)$ exist if nonvanishing happens. Thus, $I'_\infty(w^2) \to 0$ remains. Furthermore, $y_n^2| \to +\infty$ and $|y_n^2 - y_n^1| \to +\infty$ are implied by $u_n^2 \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. The next step is iteration: we obtain sequences of points $\{y_n^k\} \subset \mathbb{R}^3$ such that, for $k_1 \neq k_2$ and $u_n^k = u_n^{k-1} - w^{k-1}(\cdot - y_n^{k-1})$, $|y_n^k| \to +\infty$, such that $k \geq 2$

$$u_n^k \rightharpoonup 0 \ \text{ in } \ H^s(\mathbb{R}^3) \ \text{ and } \ I'_\infty(w^k) = 0$$

and

$$\begin{cases} \left\| u_n - u - \sum\limits_{k=1}^{l} w^k(\cdot - y_n^k) \right\| \to 0 \\ I(u_n) \to I(\tilde{u}) + \sum\limits_{k=1}^{l} I_\infty(w^1). \end{cases} \tag{16}$$

Recall that if w^k is a nontrivial solution of I_∞, then $I_\infty(w^k) \geq c_\infty > 0$, where c_∞ is the least energy associated with the functional I_∞. Equation (16) implies that the iteration stops at some finite index $l+1$. This is because $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Therefore, $u_n^{l+1} \to 0$ in H_ε. The proof is now complete.

Case $(\mathbf{ii})$ $\tilde{u} = 0$, that is, $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. Similar to *Case (i)*, we obtain that

$$I(u_n) = J(u_n) + o_n(1), \text{and } J'(u_n) = o_n(1) \text{ in } (H^s(\mathbb{R}^3))^{-1}. \tag{17}$$

Letting

$$\delta := \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx.$$

If $\delta = 0$, it follows from Lemma 1 that $u_n \to 0$ in $L^t(\mathbb{R}^3)$ for $t \in (2, 2_s^*)$. Then, $\|u_n\| \to 0$ and $I(u_n) \to 0$, which is a contradiction.

Our arguments are then extended to the following case: $\delta > 0$. Since $\{\tilde{y}_n^0\} \subset \mathbb{R}^3$ exists, we may assume that the following holds true:

$$\int_{B_1(\tilde{y}_n^0)} |u_n|^2 dx > \frac{\delta}{2}.$$

Following that, if a subsequence needs to be extracted, we obtain, for some $\tilde{w}^1 \in H^s(\mathbb{R}^3)$,

$$|\tilde{y}_n^0| \to +\infty, \quad w_n^0 := u_n(\cdot + \tilde{y}_n^0) \rightharpoonup \tilde{w}^0.$$

Since

$$\int_{B_1(0)} |w_n^0|^2 dx > \frac{\delta}{2},$$

then

$$\int_{B_1(0)} |\tilde{w}^0|^2 dx > \frac{\delta}{2},$$

and $w^1 \neq 0$. $\{\tilde{y}_n^0\}$ must be unbounded, as $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$ implies. We assume that $|\tilde{y}_n^0| \to +\infty$, up to a subsequence.

Furthermore, we derive

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n^0|^2 dx \to \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{w}^0|^2 dx \text{ and } J'(\tilde{w}^0) = 0 \tag{18}$$

from Lemma 5 and (17). Similarly, let us define $w_n^1 = w_n^0 - \tilde{w}^0(\cdot - \tilde{y}_n^0)$, we have $w_n^1 \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. From the Brezis–Lieb lemma, we obtain that

(c.1) $\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n^1|^2 dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n^0|^2 dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{w}^0|^2 dx$;
(c.2) $|w_n^1|_2^2 = |w_n^0|_2^2 - |\tilde{w}^0|_2^2 + o_n(1)$;
(c.3) $\int_{\mathbb{R}^3} |w_n^1|^p dx = \int_{\mathbb{R}^3} |w_n^0|^p dx - \int_{\mathbb{R}^3} |\tilde{w}^0|^p dx + o_n(1)$.

Thus, by (18) we have

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u_n|^p dx \\ &= \frac{1}{2}\|w_n^1\|^2 + \frac{1}{2}\|\tilde{w}^0\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{w}^0|^2 dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^3} |w_n^1|^p dx - \frac{1}{p}\int_{\mathbb{R}^3} |\tilde{w}^0|^p dx \\ &= J(\tilde{w}^0) + I_\infty(w_n^1) + o_n(1). \end{aligned}$$

For all $\varphi \in H^s(\mathbb{R}^3)$, we obtain

$$\begin{aligned} o_n(1) &= \langle J'(u_n), \varphi(\cdot - \tilde{y}_n^0)\rangle \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \tilde{w}_n^0 (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} \tilde{w}_n^0 \varphi dx \\ &+ b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{w}_n^0|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \tilde{w}_n^0 (-\Delta)^{\frac{s}{2}} \varphi dx - \int_{\mathbb{R}^3} |\tilde{w}_n^0|^{p-2} \tilde{w}_n^0 \varphi dx + o_n(1) \\ &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \tilde{w}^0 (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} \tilde{w}^0 \varphi dx + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{w}^0|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \tilde{w}^0 (-\Delta)^{\frac{s}{2}} \varphi dx \\ &- \int_{\mathbb{R}^3} |\tilde{w}^0|^{p-2} \tilde{w}^0 \varphi dx - \int_{\mathbb{R}^3} |w_n^1|^{p-2} w_n^1 \varphi dx + o_n(1) \\ &= \langle J'(\tilde{w}^0), \varphi\rangle + \langle I'_\infty(w_n^1), \varphi\rangle + o_n(1). \end{aligned}$$

Using the above facts, we obtain

$$I_\infty(w_n^1) = o_n(1) \text{ in} (H^s(\mathbb{R}^3))^{-1}.$$

If $w_n^1 \to 0$ in $H^s(\mathbb{R}^3)$, then this completes the proof. Otherwise, $w_n^1 \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. We also repeat the procedure carried out in the proof of *Case (i)*, there exist sequences of points $\{\tilde{y}_n^i\} \in \mathbb{R}^3$ with

$$|\tilde{y}_n^k| \to +\infty \text{ and } |\tilde{y}_n^{k_1} - \tilde{y}_n^{k_2}| \to +\infty \text{ for } k_1 \neq k_2.$$

Letting

$$w_n^k = w_n^{k-1} - w^{k-1}(\cdot - \tilde{y}_n^{k-1}),$$

from which, for $k \geq 2$, we have that $w_n^k \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$, $I'_\infty(\tilde{w}^k) = 0$ and

$$I(u_n) \to J(\tilde{w}^0) + \sum_{k=1}^{l} I_\infty(w^k).$$

Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and $I_\infty(w^k) \geq c_\infty > 0$, then the above iteration stops at some finite index $l+1$. This completes the proof. □

Remark 1. *It is easy to see that case* (i) *of Lemma 6 occurs if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *with* $u \neq 0$. *On the other hand, if* $u = 0$, *then case* (ii) *of Lemma 6 holds true, and the min–max energy level must be* $d \leq d_\infty$, *where* d_∞ *is the ground state energy of* J.

Lemma 7. *Let* $\{u_n\}$ *be a* $(PS)_d$ *sequence. Then,* $\{u_n\}$ *is relatively compact for all* $d \in (0, c_\infty]$.

Proof. Let us consider a $(PS)_d$ sequence $\{u_n\}$ and use Lemma 6, since $I_\infty(w^k) > c_\infty$ for all k, when $I(u_n) \to d \leq c_\infty$, it is easy to see that $l = 0$ holds true in case (i) of Lemma 6, and case (ii) in Lemma 6 does not occur for $d \leq c_\infty < d_\infty$. □

Proof of Theorem 1. In order to prove the existence of a ground state solution of (5), by Lemma 6 and Remark 1, it is sufficient to check that

$$d \le c_\infty. \tag{19}$$

Now, remark again that w_Q is a positive ground state solution of (6), then it follows from Lemma 4 that there exists $t > 0$ such that $tw_Q \in \mathcal{N}$ with $t \ge 1$. Thus,

$$\begin{aligned} d \le I(tw_Q) &= I(tw_Q) - \frac{1}{4}\langle I'(tw_Q),\, tw_Q\rangle \\ &= \frac{t^2}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_Q|^2 dx + \frac{t^2}{4}\int_{\mathbb{R}^3} w_Q^2 dx + \left(\frac{1}{4} - \frac{1}{p}\right) t^p \int_{\mathbb{R}^3} Q(x)|w_Q|^p dx \\ &< t^p\left[\frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_Q|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} w_Q^2 dx + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} Q(x)|w_Q|^p dx\right] \\ &= t^p\left(\frac{1}{2} - \frac{1}{p}\right)\|w_Q\|^2 = t^p c_Q. \end{aligned}$$

Moreover,

$$\begin{aligned} 0 &= t^2\|w_Q\|^2 + t^4 b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_Q|^2 dx\right)^2 - t^p \int_{\mathbb{R}^3} Q(x)|w_Q|^p dx \\ &\le t^4\|w_Q\|^2 + t^4 b\|w_Q\|^4 - t^p\|w_Q\|^2, \end{aligned}$$

which satisfies the following properties,

$$t \le \left(1 + b\|w_Q\|^2\right)^{\frac{1}{p-4}} = \left(1 + \frac{2p}{p-2} b c_Q\right)^{\frac{1}{p-4}}. \tag{20}$$

Recalling $\theta = \frac{2(p+1)}{p-1}$, we deduce that

$$d \le I(tw_Q) \le t^p c_Q \le \left(1 + \theta b c_Q\right)^{\frac{p}{p-4}} c_Q \le c_\infty.$$

The proof of Theorem 1 is thus concluded. □

4. Proof of Theorem 2

This section demonstrates the existence of the following problem's ground state solution.

$$\left(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx\right)(-\Delta)^s u(x) + u = K(x)u^3, \text{ in } \mathbb{R}^3. \tag{21}$$

The weak solutions of (21) correspond to critical points of the following functional:

$$\mathcal{I}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)u^4 dx.$$

Let $\mathcal{M}$ be the Nehari manifold associated to $\mathcal{I}(u)$:

$$\mathcal{M} = \left\{u \in H^s(\mathbb{R}^3) \setminus \{0\} : \langle \mathcal{I}'(u), u\rangle = 0\right\}.$$

The competing effect of the nonlocal term $\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)$ with the nonlinearity $K(x)u^3$ makes it impossible to find ground states on $\mathcal{M}$. We need the following set to construct the variational framework:

$$\Theta = \left\{u \in H^s(\mathbb{R}^3) : b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 < \int_{\mathbb{R}^3} K(x)|u|^4 dx\right\}$$

It is easy to see that $\Theta \neq \emptyset$ since $K(x) > K_\infty$.

Lemma 8. *Let* $(K_1) - (K_2)$ *hold, then we have that*

(i) *For all* $u \in \Theta$*, there is a unique* $t_u > 0$ *such that* $g'(t) > 0$ *for* $0 < t < t_u$ *and* $g'(t) < 0$ *for* $t > t_u$*, where* $g(t) = \mathcal{I}(tu)$*. Moreover,* $t_u u \in \mathcal{M}$ *and* $\mathcal{I}(t_u u) = \max_{t>0} \mathcal{I}(tu)$*.*

(ii) *There is* $\rho > 0$ *such that* $d := \inf_{u\in\mathcal{M}} \mathcal{I}(u) \geq \inf_{u\in S_\rho} \mathcal{I}(u) > 0$*, where* $S_\rho := \{u \in H^s(\mathbb{R}^3) : \|u\| = \rho\}$*.*

Proof. (i) For each $u \in \Theta$, it is easy to check that $g(t) > 0$ for $t > 0$ small enough, and $g(t) < 0$ for $t > 0$ large enough. Then, $g(t)$ has a positive maximum point in $(0, +\infty)$. Moreover, the maximum point t satisfies that $g'(t) = 0$, that is,

$$\|u\|^2 = t^2\left(\int_{\mathbb{R}^3} K(x)|u|^4 dx - b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2\right).$$

Thus, there exists a unique $t_u > 0$ such that $g'(t_u) = 0$.

(ii) By the Sobolev embedding theorem, we have

$$\mathcal{I}(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^4.$$

Then, there is a small ρ such that $\inf_{u\in S_\rho} \mathcal{I}(u) > 0$. For any $u \in \mathcal{M}$, there exists $\theta u > 0$ such that $\theta u \in S_\rho$; thus, we have that $\mathcal{I}(u) = \mathcal{I}(t_u u) \geq \mathcal{I}(\theta u)$ and (ii) follows. □

Lemma 9.

(i) *For each compact subset* W *of* $\Theta \cap S_1$*, there exists* $T_W > 0$ *such that* $t_w \leq T_W$ *for all* $w \in W$*.*

(ii) *If* $u \notin \Theta$*, then* $tu \notin \mathcal{M}$ *for any* $t > 0$*.*

Proof. (i) Suppose by contradiction that there exist a compact subset W of $\Theta \cap S_1$ and a sequence $w_n \in W$ such that $t_{w_n} \to +\infty$ and $w_n \to w$ in $H^s(\mathbb{R}^3)$, then

$$b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} K(x)|w_n|^4 dx \to b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w|^2 dx\right)^2 - \int_{\mathbb{R}^3} K(x)|w|^4 dx < 0.$$

Hence,

$$\frac{\mathcal{I}(t_{w_n} w_n)}{t_{w_n}^2} = \frac{1}{2} + \frac{t_{w_n}^2}{4}\left(b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} K(x)|w_n|^4 dx\right) \to -\infty,$$

which yields a contradiction by the fact that $\mathcal{I}(t_{w_n} w_n) > 0$.

(ii) If there exists a $t > 0$ such that $tu \in \mathcal{M}$, we have that

$$\|u\|^2 = t^2\left(\int_{\mathbb{R}^3} K(x)|u|^4 dx - b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2\right) > 0,$$

which implies that $u \in \Theta$. This completes the proof. □

We define the mapping $\tilde{m} : \Theta \to \mathcal{M}$ by $\tilde{m}(u) = t_u u$. Let $U := \Theta \cap W$ and define $m := \tilde{m}|_U$. Then, $\tilde{m}$ is a bijection from U to $\mathcal{M}$. Moreover, by Lemmas 8 and 9, as in the proof of [39], we have that the mapping m is a homeomorphism between U and $\mathcal{M}$, and the inverse of m is given by $m^{-1}(u) = \frac{u}{\|u\|}$. Define the functional $\Phi(u) : U \to \mathbb{R}$ by

$$\Phi(u) = \mathcal{I}(m(u)).$$

The following properties play important roles in the proof.

Lemma 10.

(i) $\{m(w_n)\}$ *is a* (PS) *sequence of* I *if* $\{w_n\}$ *is a* (PS) *sequence of* Φ*;* $\{m^{-1}(u_n)\}$ *is a* (PS) *sequence of* Φ *if* $\{u_n\} \subset \mathcal{M}$ *is a bounded* (PS) *sequence of* $\mathcal{I}$*.*

(ii) w *is a critical point of* Φ *if and only if* $m(w)$ *is a nontrivial point of* $\mathcal{I}$*. Moreover, the corresponding values of* Φ *and* $\mathcal{I}$ *coincide and* $\inf\limits_{U}\Phi = \inf\limits_{\mathcal{M}}\mathcal{I}$*.*

(iii) A minimizer of $\mathcal{I}$ *on* $\mathcal{M}$ *is a ground state of Equation (1).*

Proof of Theorem 2. Given a minimizing sequence $\{w_n\} \subset U$ such that $\Phi(w_n) \to \inf\limits_{U}\Phi$, this allows us to use the Ekeland variational principle to suppose $\Phi'(w_n) \to 0$. Then, using Lemma 10 we can deduce that for $u_n = m(w_n) \in \mathcal{M}$, $\mathcal{I}'(u_n) \to 0$ and $\mathcal{I}(u_n) = \Phi(w_n) \to c$. Be aware that in $H^s(\mathbb{R}^3)$, u_n is bounded. Then, up to a subsequence, $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^3)$ for $q \in [2, 2_s^*)$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$.

We then carry on with our reasoning by making a distinction between $u = 0$ and $u \neq 0$.

In the first case, we claim that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx = \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx. \tag{22}$$

Indeed, we may suppose that

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx \le \lim_{n\to+\infty}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx = \tilde{A} > 0.$$

By the fact that $\mathcal{I}'(u_n) \to 0$, u is a solution of the following equation:

$$(1+b\tilde{A})(-\Delta)^s u(x) + u = K(x)u^3, \ \text{ in } \mathbb{R}^3.$$

Then,

$$\begin{aligned}
&\Big(1+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \int_{\mathbb{R}^3}u^2dx \\
&\le (1+b\tilde{A})\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \int_{\mathbb{R}^3}u^2dx \\
&= \int_{\mathbb{R}^3}K(x)u^4dx.
\end{aligned} \tag{23}$$

Set

$$h_2(t) := \langle \mathcal{I}'(tu), tu\rangle, \text{ for } t \ge 0.$$

If $\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx = \tilde{A}$, the (22) holds. If $\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx < A$, by (22) we have that $h_2(1) < 0$. Similar to Lemma 8, by (23) we obtain that there exists $t_1 > 0$ such that $t_1u \in \mathcal{M}$. It follows from $\langle \mathcal{I}'t_1u, t_1u\rangle = 0$ and $t_1 > 1$ that

$$\begin{aligned}
&\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \int_{\mathbb{R}^3}u^2dx + b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\Big)^2 \\
&> \frac{1}{t_1^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \frac{1}{t_1^2}\int_{\mathbb{R}^3}u^2dx + b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\Big)^2 \\
&= \int_{\mathbb{R}^3}K(x)u^4dx.
\end{aligned}$$

This contradicts with (23) and then $t_1 \le 1$. Thus,

$$
\begin{aligned}
d \leq \mathcal{I}(t_1 u) &= \mathcal{I}(t_1 u) - \frac{1}{4}\langle \mathcal{I}'(t_1 u),\ t_1 u\rangle = \frac{t_1^2}{4}\|u\|^2 \\
&\leq \frac{1}{4}\|u\|^2 \leq \frac{1}{4}\|u_n\|^2 + o_n(1) \\
&= \mathcal{I}(u_n) - \frac{1}{4}\langle \mathcal{I}'(u_n),\ u_n\rangle + o_n(1) \\
&= d + o_n(1).
\end{aligned} \tag{24}
$$

So, $t_1 = 1$ and (22) hold. Thus, $\mathcal{I}'(u) = 0$. Again by (24), we obtain $\mathcal{I}(u) = d$.

We now consider the second case, that is, is $\{u_n\}$ vanishing or nonvanishing. If $\{u_n\}$ is vanishing, i.e.,

$$
\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n(x)|^2 dx = 0,
$$

it follows from Lemma 1 that $u_n \to 0$, in $L^t(\mathbb{R}^3), t \in (2, 2_s^*)$, then $\|u_n\| \to 0$ and $\mathcal{I}(u_n) \to 0$, which is a contradiction of $\mathcal{I}(u_n) \to d$. Therefore, $\{u_n\}$ is nonvanishing. Then, there exist $\{x_n\} \subset \mathbb{R}^3$ and $\delta_0 > 0$ such that

$$
\int_{B_1(x_n)} |u_n(x)|^2 dx \geq \delta_0. \tag{25}
$$

Denote $\tilde{u}_n(\cdot) := u_n(\cdot + x_n)$, up to a subsequence, $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^s(\mathbb{R}^3)$, $\tilde{u}_n \to \tilde{u}$ in $L^q_{loc}(\mathbb{R}^3)$ for $q \in [2, 2_s^*)$ and $\tilde{u}_n(x) \to \tilde{u}(x)$ a.e. in $\mathbb{R}^3$. By using (25) we have $\tilde{u} \neq 0$.

Without loss of generality, we may assume that $K_\infty = 1$. Since $u_n \rightharpoonup 0$, for all $\varphi \in H^s(\mathbb{R}^3)$, we obtain

$$
\begin{aligned}
o_n(1) &= \langle \mathcal{I}'(u_n), \varphi\rangle \\
&= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} u_n \varphi dx \\
&\quad + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx - \int_{\mathbb{R}^3} K(x) u_n^3 \varphi dx + o_n(1) \\
&= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} u_n \varphi dx \\
&\quad + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx - \int_{\mathbb{R}^3} u_n^3 \varphi dx + o_n(1) \\
&:= \langle \mathcal{J}'(u_n), \varphi\rangle,
\end{aligned}
$$

where $\mathcal{J}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\Big)^2 - \frac{1}{4}\int_{\mathbb{R}^3} u^4 dx$ is the energy functional associated with the following equation:

$$
\Big(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx\Big)(-\Delta)^s u(x) + u = u^3, \ \text{ in } \mathbb{R}^3.
$$

By the arbitrary nature of φ, $\mathcal{J}' \rightharpoonup 0$ in $(H^s(\mathbb{R}^3))^{-1}$. Since $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^s(\mathbb{R}^3)$ we may suppose that

$$
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \leq \lim_{n\to+\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \bar{A} > 0.
$$

By the fact that $\mathcal{J}'(\tilde{u}_n) \to 0$, $\tilde{u}$ is a solution of the following equation:

$$
\Big(1 + b\bar{A}\Big)(-\Delta)^s u(x) + u = u^3, \ \text{ in } \mathbb{R}^3.
$$

Then,

$$
\begin{aligned}
&\left(1+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx\right)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx+\int_{\mathbb{R}^3}\tilde{u}^2dx\\
&\le (1+b\bar{A})\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx+\int_{\mathbb{R}^3}\tilde{u}^2dx\\
&=\int_{\mathbb{R}^3}\tilde{u}^4dx,
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\mathcal{J}(\tilde{u})-\frac{1}{4}\langle\mathcal{J}'(\tilde{u}),\,\tilde{u}\rangle &=\frac{1}{4}\|\tilde{u}\|^2\\
&\le \frac{1}{4}\|\tilde{u}_n\|^2+o_n(1)=\frac{1}{4}\|u_n\|^2+o_n(1)\\
&=\mathcal{I}(u_n)-\frac{1}{4}\langle\mathcal{I}'(u_n),\,u_n\rangle+o_n(1)\\
&=d+o_n(1).
\end{aligned}
\tag{27}
$$

Letting

$$\mathcal{M}_1=\Big\{u\in H^s(\mathbb{R}^3)\setminus\{0\}:\langle\mathcal{J}'(u),u\rangle=0\Big\}.$$

$$\Theta_1=\Big\{u\in H^s(\mathbb{R}^3):b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\Big)^2<\int_{\mathbb{R}^3}|u|^4dx\Big\},\ \text{ and }\ d_1:=\inf_{u\in\mathcal{M}_1}\mathcal{J}(u).$$

It follows from (26) that $\tilde{u}\in\Theta_1$. Similar to Lemma 8, there exists $t_2>0$ such that $t_2\tilde{u}\in\mathcal{M}_1$, and $\mathcal{J}(t_2\tilde{u})=\max_{t>0}\mathcal{J}(tu)$. Moreover, we claim that $t_2\le 1$. Otherwise, $t_2>1$. We deduce from $\langle\mathcal{J}(t_2\tilde{u}),t_2\tilde{u}\rangle=0$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx+\int_{\mathbb{R}^3}\tilde{u}^2dx+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx\Big)^2\\
&>\frac{1}{t_2^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx+\frac{1}{t_2^2}\int_{\mathbb{R}^3}\tilde{u}^2dx+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2dx\Big)^2\\
&=\int_{\mathbb{R}^3}\tilde{u}^4dx,
\end{aligned}
$$

which contradicts (26). Then, $t_2\le 1$. Thus,

$$
\begin{aligned}
d_1\le\mathcal{J}(t_2\tilde{u})=\mathcal{J}(t_2\tilde{u})-\frac{1}{4}\langle\mathcal{J}'(t_2\tilde{u}),\,t_2\tilde{u}\rangle &=\frac{t_2^2}{4}\|\tilde{u}\|^2\\
&\le\frac{1}{4}\|\tilde{u}\|^2\\
&=\mathcal{J}(\tilde{u})-\frac{1}{4}\langle\mathcal{J}'(\tilde{u}),\,\tilde{u}\rangle\le d+o_n(1).
\end{aligned}
\tag{28}
$$

We deduce from the fact that $\mathcal{I}(u)\le\mathcal{J}(u)$ that

$$d=\inf_{u\in U}\max_{t>0}\mathcal{I}(tu)\le\inf_{u\in\Theta_1\cap S_1}\max_{t>0}\mathcal{J}(tu)=d_1.$$

Then, it follows from (28) that $t_2=1$ and $d_1=\mathcal{J}(\tilde{u})=d$. By (26) and $K(x)\ge 1$, we have $\tilde{u}\in\Theta$. From Lemma 8, there exists $\tilde{t}>0$ such that $\tilde{t}\tilde{u}\in\mathcal{M}$. Then, by $d_1=\mathcal{J}(\tilde{u})=d$, we have

$$d\le\mathcal{I}(\tilde{t}\tilde{u})\le\mathcal{J}(\tilde{t}\tilde{u})\le\mathcal{J}(\tilde{u})=d.$$

Therefore, $\mathcal{I}(\tilde{t}\tilde{u})=d$.

By using the arguments of the two cases above, we can conclude that d has been reached. The ground state of (21) is the corresponding minimizer. Next, we shall demonstrate that (21)'s ground state solution is positive. It is possible to perform the entire analysis above word for word by substituting $\mathcal{I}(u)$ with $\mathcal{I}^+(u)$, with the functional $\mathcal{I}^+(u)$ defined by

$$I^+(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)|u^+|^4 dx.$$

In this way, we obtain a ground state solution u for the following equation:

$$\Big(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)(-\Delta)^s u + u = K(x)|u^+|^3, \text{ in } \mathbb{R}^3. \tag{29}$$

By using u^- as a test function in (29) we obtain

$$\begin{aligned} 0 =& \Big(1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}u^- dx + \int_{\mathbb{R}^3} |u^-|^2\, dx \\ \geq& \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u^-|^2 dx + \int_{\mathbb{R}^3} |u^-|^2 dx \geq 0, \end{aligned}$$

which yields that $u^- = 0$, and hence, $u \geq 0$. Furthermore, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$, then $(-\Delta)^s u(x_0) = 0$. It follows from (2) and $u(x_0) = 0$ that

$$(-\Delta)^s u(x_0) = -\frac{C_{3,s}}{2}\int_{\mathbb{R}^3} \frac{u(x_0 + y) + u(x_0 - y)}{|y|^{3+2s}} dy = 0,$$

which implies $u \equiv 0$. Therefore, u is positive. □

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