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Topological Properties of Polymeric Networks Modelled by Generalized Sierpiński Graphs

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Abstract: In this article, we compute the irregularity measures of generalized Sierpiński graphs and obtain some bounds on these irregularities. Moreover, we discuss some bounds on connectivity indices for generalized Sierpiński graphs of any arbitrary graph H along with classification of the extremal graphs used to attain them.

Keywords: generalized Sierpiński graphs; irregularity measure; connectivity indices

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1. Introduction

For many years, the topic of polymer networks has been of great importance in research. The properties of polymer networks rely on the structure of polymer chains and how these chains are cemented to form a network. The primary study of polymer modelling began with linear polymeric structures, but nowadays, researchers are focusing their attention on the complex underlying geometries as well as fractal generalized networks.

In theoretical chemistry, a numerical quantity is used to gain information about the chemical, physical or biological properties of organic substances. This numerical quantity is obtained by applying mathematical definitions to the molecular structures of the substance, and it is called a topological index.

We consider simple, finite, connected and undirected graphs in our present study. Graph theoretic terminologies that are not defined here can be found in [1]. Let $H = H(V, E)$ be a graph, with V as its node set and E its edge set. The cardinalities of V and E are of order n and size m of H, respectively. The total number of nodes adjacent to a node $x \in V$ is the degree of x in H, denoted by $d_H(x)$ or simply $d(x)$. P_n and S_n are path and star graphs, respectively, with n nodes.

In 1998, Bollobás et al. [2] introduced the following definition of the generalized Randić index:

$$R_\gamma(H) = \sum_{xy \in E(H)} \big(d(x)d(y)\big)^\gamma,$$

where γ is any non-zero real number. For $\gamma = \frac{-1}{2}$, we obtain the famous Randić index, while for $\gamma = \frac{1}{2}, 1$ gives the reduced Randić and second Zagreb indices.

The historical and mathematical concepts of the above-discussed topological indices can be found in [3–8].

By replacing product with sum in the above expression, we obtain the general sum-connectivity index. The concept of general sum-connectivity index was put forward by Zhou et. al. in [9] and is defined as

$$\chi_\gamma(H) = \sum_{xy \in E(H)} \big(d(x) + d(y)\big)^\gamma.$$

From the above, the sum-connectivity index can be obtained by replacing γ with $\frac{-1}{2}$. When $\gamma = 1$, the expression is known as the first Zagreb index, while for $\gamma = 2$, we obtain the hyper Zagreb index.

Studies on the general sum-connectivity, sum-connectivity, first Zagreb and hyper Zagreb indices have been conducted by many researchers [5,10–13].

A graph G is regular if the degree of each of its nodes is same; otherwise, it is irregular. In most problems and applications, it is of great interest to know the irregularity of H. The topological characterization of the irregularity of H is useful for investigating the structural properties of random and deterministic networks, as well as systems appearing in social and chemical structures and biology [14].

In 1991, Albertson [15] defined the irregularity of graph H as

$$Irr_{Alb}(H) = \sum_{xy \in E(H)} |d(x) - d(y)|.$$

The above quantity is also named the Albertson index or third Zagreb index.

In [16], the authors proposed a similar kind of quantity (a forgotten topological index):

$$Irr_{Alb2}(H) = \sum_{xy \in E(H)} (d(x) - d(y))^2.$$

Detailed studies on the above irregularities can be found in [17–22].

For a graph H of order $n \geq 2$, let V_l be a set of words with a length of l letters from V, where l is a positive integer. The letters of a word $\mathbf{x} \in V_l$ are indexed by $x_1 x_2 \cdots x_l$. Klavžar [23] introduced the concept of Sierpiński graphs $S(K_n, l)$. Later, Gravier [24] generalized this concept for any graph G and named it the generalized Sierpiński graph $gS(G, l)$. Motived by the study of Gravier [24] and topological index work by Javed et. al. [25], we consider the problem of irregularities and connectivity indices for generalized Sierpiński graphs $gS(G, l)$ and classify the extremal graphs for said invariants. In QSAR/QSPR studies, a TI (topological index) is a numeric entity associated with a chemical graph which can tell the specific physical and chemical properties of the corresponding molecule. In many problems and applications, it is interesting to know how irregular a graph can be. The generalized Sierpiński graph $gS(H, l)$ of H with dimension l is a graph of V_l such that two words, $\mathbf{x}$ and $\mathbf{y}$, form an edge iff (if and only if) there exists $i \in \{1, \cdots, l\}$, satisfying the following:

- $x_j = y_j$, if $j < i$.
- $x_i \neq y_i$ but $\{x_i, y_i\} \in E(H)$.
- $x_j = y_i$ and $y_j = x_i$, whenever $j > i$.

From the above, we note that if $\mathbf{xy} \in E(gS(H, l))$, then there is $uv \in E(H)$ and a word $\mathbf{z}$ satisfying $\mathbf{x} = zuvv \cdots v$ and $\mathbf{y} = zvuu \cdots u$. A node of the representation $xx \cdots x$ is said to be an *extreme* node. For H and an integer $l \geq 2$, $gS(H, l)$ has n extreme nodes. Further, we have $d_H(x) = d_{gS(H,l)}(xx \cdots x)$, $d_H(x) + 1 = d_{gS(G,l)} yxx \cdots x$ and $d_H(y) + 1 = d_{gS(H,l)} xyy \cdots y$. Figures 1 and 2 give $gS(H, l)$ and $gS(C_4, l)$, respectively.

From the construction, we have $\mathbf{x} \in V(gS(H, l))$ and $d_{gS(H,l)}(\mathbf{x}) \in \{d_G(x), d_G(x) + 1\}$, where $d_G(x)$ represents the degree of x in H. We follow the notations and terminologies of [26]. Let $|d_G(x), d_G(y)|_{gS\{H,l\}}$ be the number of copies $\{zxyy \ldots y, zyxx \ldots x\}$ of the edge $\{x, y\}$ whose endpoints have degrees $d_G(x)$ and $d_G(y)$ in $gS(H, l)$. For $x, y \in V(H)$, the number of C_3s in H with nodes x and y is denoted by $\triangleright(x, y)$, and $\triangleright(H)$ is the number of C_3s in H. For an arbitrary $\{x, y\} \in E(H)$, we see that $|N_G(x) \cap N_G(y)| = \triangleright(x, y)$, $|N_G(x) \cup N_G(y)| = d_H(x) + d_H(y) - \triangleright(x, y)$ and $|N_G(x) - N_G(y)| = d_G(x) - \triangleright(x, y)$. For H, we use the identity $1 + n + n^2 + \cdots + n^{l-1} = \frac{n^l - 1}{n - 1}$ and denote it as $\xi_n(l)$.

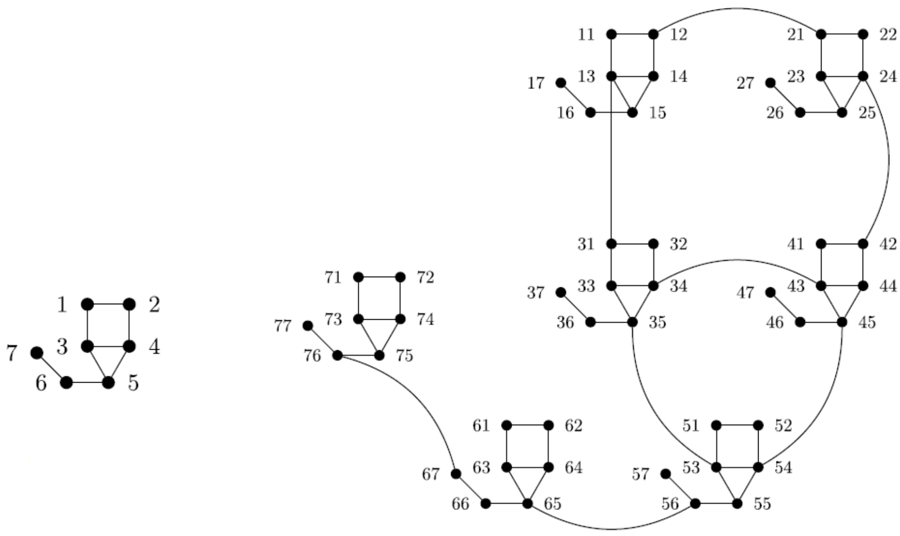

Figure 1. Generalized Sierpiński Graphs $gS(H,1)$ and $gS(H,2)$.

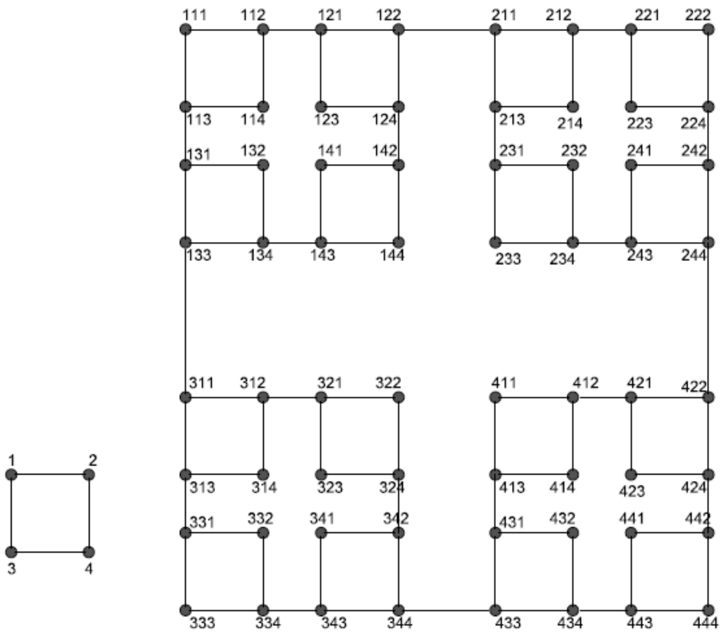

Figure 2. Generalized Sierpiński Graphs $gS(C_4,1)$ and $gS(C_4,3)$.

In [27], the authors used the concept of topological indices to describe the structure of polymers with optimal levels of macroscopic properties. Javed et. al. [25] gave bounds on topological indices for generalized Sierpiński and extended Sierpiński graphs. In this article, we find the irregularity measures of a model of polymer networks based on $gS(H,l)$. Moreover, we present upper/lower bounds on the general sum-connectivity and general Randić indices for $gS(H,l)$ of any graph H. Our results are very general, and the bounds are sharp for large class of graphs. This study answers several questions and fills the gaps in previously published articles.

2. Discussion and Main Results

In this section, we calculate the irregularity of the generalized Sierpiński graph for any arbitrary graph G. Some sharp bounds are also presented as corollaries of the main result.

Moreover, we present the bounds on general connectivity indices, such as the general sum-connectivity and general Randić indices, for the generalized Sierpiński graphs.

First, we mention an important lemma which we use to prove the main results.

Lemma 1 ([26]). *For a graph H of order n and an integer $l \geq 2$, the following holds for an arbitrary edge xy of H.*

1. $|d(x), d(y)|_{gS(H,t)} = n^{l-2}(n - d(x) - d(y) + \triangleright(x,y))$.
2. $|d(x), d(y)+1|_{gS(H,t)} = n^{l-2}(d(y) - \triangleright(x,y)) - \xi_n(l-2)d(x)$.
3. $|d(x)+1, d(y)|_{gS(H,t)} = n^{l-2}(d(x) - \triangleright(x,y)) - \xi_n(l-2)d(y)$.
4. $|d(x)+1, d(y)+1|_{gS(H,t)} = n^{l-2}(\triangleright(x,y)+1) + \xi_n(l-2)(d(x)+d(y)+1)$.

The first result is about the Albertson irregularity of the generalized Sierpiński graph of any graph and presents an explicit formula for it.

Theorem 1. *Let H be a graph. Then, the Albertson irregularity of the generalized Sierpiński graph $gS(H,l)$ with dimension $l \geq 2$ is*

$$Irr_{Alb}(gS(H,l)) = Irr_{Alb}(H)(\xi_n(l) + \xi_n(l-1)) + 2\omega,$$

where

$$\omega = \sum_{\substack{xy \in E(H) \\ d(x)=d(y)}} \Big(n^{l-2}(d(x) - \triangleright(x,y)) - \xi_n(l-2) \cdot d(y) \Big).$$

Proof. Let H be a graph with n nodes. The Albertson irregularity for the generalized Sierpiński graph of H, $gS(H,l)$ is given as

$$Irr_{Alb}(gS(H,l)) = \sum_{xy \in E(H)} \sum_{i,j=0,1} |d(x)+i, d(y)+j|_{gS(H,t)} \big| d(x)+i-(d(y)+j) \big|.$$

From Lemma 1, we have

$$\begin{aligned} Irr_{Alb}(gS(H,t)) = \sum_{xy \in E(H)} \Big[& n^{l-2}(n - d(x) - d(y) + \triangleright(x,y)) \big| d(x) - d(y) \big| \\ & + \Big(n^{l-2}(d(y) - \triangleright(x,y)) - \xi_n(l-2)d(x) \Big) \big| d(x) - (d(y)+1) \big| \\ & + \Big(n^{l-2}(d(x) - \triangleright(x,y)) - \xi_n(l-2)d(y) \Big) \big| (d(x)+1) - d(y) \big| \\ & + \Big(n^{l-2}(\triangleright(x,y)+1) + \xi_n(l-2)(d(x)+d(y)+1) \Big) \\ & \big| (d(x)+1) - (d(y)+1) \big| \Big]. \end{aligned}$$

Now, for each edge xy of H, $|d(x)+i-d(y)-j| = (d(x)+i-d(y)-j)$ for $d(x) > d(y)$, where $0 \leq i,j \leq 1$. For $d(x) = d(y)$, $|d(x)+i-d(y)-j|$ is either 0 or 1. So we have the following:

$$\begin{aligned} &= \sum_{xy \in E(H)} (d(x) - d(y)) \Big[n^{l-1} + n^{l-2} + \xi_n(l-2) + n^{l-2} + \xi_n(l-2) \Big] \\ &+ 2 \sum_{\substack{xy \in E(H) \\ d(x)=d(y)}} \Big[n^{l-2}(d(x) - \triangleright(x,y)) - \xi_n(l-2)d(x) \Big] \\ &= Irr_{Alb}(gS(H,l)) \Big(\xi_n(l) + \xi_n(l-1) \Big) \end{aligned}$$

$$+2\sum_{\substack{xy\in E(H)\\ d(x)=d(y)}}\left[n^{l-2}(d(x)-\triangleright(x,y))-\xi_n(l-2)d(x)\right],$$

which is the required result. □

The importance of the above results lies in the closed formulae for the Albertson irregularity of the generalized Sierpiński graph of an arbitrary graph. The following are its consequences.

Corollary 1. *For a tree T, the generalized Sierpiński graph $gS(T,l)$ satisfies*

$$2(n-3)\left(n^{l-2}\xi_n(l-2)\right)\leq Irr_{Alb}(gS(H,l))\leq(n-1)(n-2)\left(\xi_n(l)+\xi_n(l-1)\right).$$

The lower equality is achieved iff $T=P_n$ and the upper is achieved iff $T=S_n$.

The following result tells us that the Albertson irregularity-2 of the generalized Sierpiński graph of H depends on the Alberston irregularity-2 and the first Zagreb index of H.

Theorem 2. *Let H be a graph and $gS(H,l)$ with $l\geq 2$ be the generalized Sierpiński graph of H. Then, the Albertson-2 irregularity of $gS(H,l)$ is given as follows:*

$$Irr_{Alb2}(gS(H,l))=Irr_{Alb2}(H)\big(\xi_n(l)+2\xi_n(l-1)\big)+\big(\xi_n(l-1)\big)M_1(G)-2n^{l-2}\sum_{xy\in E(H)}\triangleright(x,y).$$

Proof. Let H be a graph with n nodes. The Albertson irregularity-2 for the generalized Sierpiński graph of H, $gS(H,l)$ can be given as follows:

$$Irr_{Alb2}(gS(H,l))=\sum_{xy\in E(H)}\sum_{i,j=0,1}|d(x)+i,d(y)+j|_{gS(H,t)}\big(d(x)+i-(d(y)+j)\big)^2$$

From Lemma 1, we have

$$\begin{aligned}
Irr_{Alb2}(gS(H,t))=&\sum_{xy\in E(H)}\left[n^{l-2}\left(n-d(x)-d(y)+\triangleright(x,y)\right)\big(d(x)-d(y)\big)^2\right.\\
&+\left(n^{l-2}(d(y)-\triangleright(x,y))-\xi_n(l-2)d(x)\right)\big(d(x)-(d(y)+1)\big)^2\\
&+\left(n^{l-2}(d(x)-\triangleright(x,y))-\xi_n(l-2)d(y)\right)\big((d(x)+1)-d(y)\big)^2\\
&\left.+\left(n^{l-2}(\triangleright(x,y)+1)+\xi_n(l-2)(d(x)+d(y)+1)\right)\big((d(x)+1)-(d(y)+1)\big)^2\right]\\
=&\sum_{xy\in E(H)}\left[(d(x)-d(y))^2(n^{l-1}+n^{l-2}+\xi_n(l-2))+\big(n^{l-2}-\xi_n(l-2)\big)(d(x)+d(y))\right.\\
&\left.-2n^{l-2}\triangleright(x,y)+2(d(x)-d(y))^2(n^{l-2}+\xi_n(l-2))\right]\\
=&Irr_{Alb2}(H)\big(\xi_n(l)+\xi_n(l-1)+n^{l-2}\big)+\big(n^{l-2}-\xi_n(l-2)\big)M_1(H)\\
&-2n^{l-2}\sum_{xy\in E(H)}\triangleright(x,y).
\end{aligned}$$

□

Corollary 2. *Let T be any tree. Then, for a given generalized Sierpiński graph $gS(T,l)$, we have*

$$2\xi_n(l) - \xi_n(l-1)(6n-10) + 6(n-1)n^{l-2} \leq Irr_{Alb2}(gS(H,l)) \leq$$
$$(n-1)(n-2)^2\xi_n(l) + \xi_n(l-1)\left[2(n-1)(2n^2-7n+8)\right] + n(n-1)n^{l-2}.$$

The lower bound achieves equality iff $T = P_n$, and the upper inequality achieves equality iff $T = S_n$.

Now, we will present some results related to the general connectivity indices.

Theorem 3. *Let H be a graph and $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, for $\gamma > 0$, we have*

$$\xi_n(l) \cdot \chi_\gamma(H) < \chi_\gamma(gS(H,l)) < \xi_n(l) \cdot \chi_\gamma(H^1),$$

where H^1 is a graph obtained from H by adding one extra weight on each node of H, i.e., for every $x \in V(H)$, we have $d_{H^1}(x) = d_H(x) + 1$.

Proof. Let $gS(H,l)$ be a generalized Sierpiński graph of a graph H with order n. The general sum-connectivity index for $gS(H,l)$ is given as

$$\chi_\gamma(gS(H,l)) = \sum_{xy \in E(H)} \sum_{i,j=0,1} |d(x)+i, d(y)+j|_{gS(H,t)} \big(d(x)+i+d(y)+j\big)^\gamma.$$

From Lemma 1, we have

$$\begin{aligned}
\chi_\gamma(gS(H,l)) &= \sum_{xy \in E(H)} \Big[n^{l-2}\big(n - d(x) - d(y) + \triangleright(x,y)\big)\big(d(x)+d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(y) - \triangleright(x,y)\big) - \xi_n(l-2)d(x)\Big)\big(d(x) + (d(y)+1)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(x) - \triangleright(x,y)\big) - \xi_n(l-2)d(y)\Big)\big((d(x)+1) + d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(\triangleright(x,y)+1\big) + \xi_n(l-2)(d(x)+d(y)+1)\Big)\big((d(x)+1) + (d(y)) + 1\big)^\gamma\Big] \\
&< \sum_{xy \in E(H)} (d(x)+d(y)+2)^\gamma \Big[n^{l-1} + n^{l-2} + \xi_n(l-2)\Big] \\
&= \xi_n(l) \cdot \chi_\gamma(H^1).
\end{aligned}$$

Now, for the lower bound, we have

$$\begin{aligned}
\chi_\gamma(gS(H,l)) &> \sum_{xy \in E(H)} \Big[n^{l-2}\big(n - d(x) - d(y) + \triangleright(x,y)\big)\big((d(x)+i) + (d(y)+j)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(y) - \triangleright(x,y)\big) - \xi_n(l-2)d(x)\Big)\big(d(x)+d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(x) - \triangleright(x,y)\big) - \xi_n(l-2)d(y)\Big)\big(d(x)+d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(\triangleright(x,y)+1\big) + \xi_n(l-2)(d(x)+d(y)+1)\Big)\big(d(x)+d(y))\big)^\gamma\Big] \\
&= \sum_{xy \in E(H)} (d(x)+d(y))^\gamma \Big[n^{l-1} + n^{l-2} + \xi_n(l-2)\Big] \\
&= \xi_n(l) \cdot \chi_\gamma(H),
\end{aligned}$$

which is the required result. □

For $\gamma < 0$, the inequalities in the above result become reverse. For $\gamma = 1, 2$, we have the following corollaries.

Corollary 3. *Let H be a graph and $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, for the first Zagreb index, we have*

$$\xi_n(l) \cdot M_1(H) < M_1(gS(H,l)) < \xi_n(l) \cdot M_1(H^1).$$

With the bound for the first Zagreb index for the trees, unicyclic graphs and bicyclic graphs for Zagreb indices of [6], we have the following consequence from Theorem 3.

Corollary 4. *Let T, U and B be a tree, a unicyclic and a bicyclic graph. Then, for $l \geq 2$, we have*

$$\xi_n(l) \cdot 2(n-3) < M_1(gS(T,l)) < \xi_n(l) \cdot (n^2+4n-4),$$

$$\xi_n(l) \cdot 4n < M_1(gS(U,l)) < \xi_n(l) \cdot (n^2+4n+6),$$

$$\xi_n(l) \cdot (4n+10) < M_1(gS(B,l)) < \xi_n(l) \cdot (n^2+4n+18).$$

Corollary 5. *Let H be a graph and $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, for the first hyper Zagreb index, we have*

$$\xi_n(l) \cdot HM_1(H) < HM_1(gS(H,l)) < \xi_n(l) \cdot HM_1(H^1).$$

With the help of the results from [11] (see Theorems 6, 7 and 8), and Theorem 3, we have the following result.

Corollary 6. *Let T, U and B be a tree, a unicyclic and a bicyclic graph, respectively, with n and $l \geq 2$. Then,*

$$\xi_n(l) \cdot 2(8n-15) < HM_1(gS(T,l)) < \xi_n(l) \cdot (n-1)(n+2)^2,$$

$$\xi_n(l) \cdot 16n < HM_1(gS(U,l)) < \xi_n(l) \cdot (3n^2+16n+58),$$

$$\xi_n(l) \cdot 2(8n+35) < HM_1(gS(B,l)) < \xi_n(l) \cdot (n^3+3n^2+8n+116).$$

For $\gamma = \frac{-1}{2}$, we have the following results.

Corollary 7. *Let H be a graph and $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, for the sum-connectivity index, we have*

$$\xi_n(l) \cdot \chi_{-\frac{1}{2}}(H) > \chi_{-\frac{1}{2}} gS(H,l) > \xi_n(l) \cdot \chi_{-\frac{1}{2}}(H^1).$$

Using the extremal values for the general sum-connectivity and harmonic indices of the unicylic graph and bicyclic graphs from [10,28], the following result follows from the above corollary.

Corollary 8. *Let T be a tree, U be a unicyclic and B be a bicyclic graph of order n. Then,*

$$\xi_n(l) \cdot (\frac{2}{\sqrt{3}} + \frac{(n-3)}{\sqrt{4}}) < \chi_{-\frac{1}{2}} gS(T,l) < \xi_n(l) \cdot \frac{n-1}{\sqrt{n+2}},$$

$$\xi_n(l) \cdot \left(\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{n+3}} + \frac{n-3}{\sqrt{n+2}}\right) < \chi_{-\frac{1}{2}} gS(U,l),$$

$$\chi_{-\frac{1}{2}} gS(B,l) < \xi_n(l) \cdot \left(\frac{n-4}{\sqrt{6}} + \frac{4}{\sqrt{7}} + \frac{1}{\sqrt{8}}\right).$$

In the following result, we present the lower/upper bounds on the general Randić index for $gS(H,l)$.

Theorem 4. *Let H be a graph and $gS(H,l)$ be the generalized Sierpiński graph of H for $l \geq 2$. Then, for $\gamma > 0$, the general Randić index has the following bounds:*

$$\xi_n(l) \cdot R_\gamma(H) < R_\gamma(gS(H,l)) < \xi_n(l) \cdot R_\gamma(H^1)$$

where H^1 is a graph which is obtained from H by adding one extra weight on each node of H, i.e., for every $x \in V(H)$, we have $d_{H^1}(x) = d_H(x) + 1$.

Proof. Let $gS(H,l)$ be the generalized Sierpiński graph of a graph H with order n. The general Randić index of $gS(H,l)$ can be written as

$$R_\gamma(gS(H,l)) = \sum_{xy \in E(H)} \sum_{i,j=0,1} |d(x)+i, d(y)+j|_{gS(H,t)} \big((d(x)+i) \cdot (d(y)+j)\big)^\gamma.$$

From Lemma 1, we have

$$\begin{aligned}
R_\gamma(gS(H,l)) &= \sum_{xy \in E(H)} \Big[n^{l-2}\big(n - d(x) - d(y) + \triangleright(x,y)\big)\big(d(x) \cdot d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(y) - \triangleright(x,y)\big) - \xi_n(l-2)d(x)\Big)\big(d(x) \cdot (d(y)+1)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(x) - \triangleright(x,y)\big) - \xi_n(l-2)d(y)\Big)\big((d(x)+1) \cdot d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(\triangleright(x,y)+1\big) + \xi_n(l-2)(d(x)+d(y)+1)\Big)\big((d(x)+1) \cdot (d(y)) + 1\big)^\gamma \Big] \\
&< \sum_{xy \in E(H)} ((d(x)+1) \cdot (d(y)+1))^\gamma \Big[n^{l-1} + n^{l-2} + \xi_n(l-2) \Big] \\
&= \xi_n(l) \cdot R_\gamma(H^1).
\end{aligned}$$

Now, for the lower bound,

$$R_\gamma(gS(H,l)) = \sum_{xy \in E(H)} \sum_{i,j=0,1} |d(x)+i, d(y)+j|_{gS(H,t)} \big((d(x)+i) \cdot (d(y)+j)\big)^\gamma.$$

From Lemma 1, we have

$$\begin{aligned}
R_\gamma(gS(H,l)) &= \sum_{xy \in E(H)} \Big[n^{l-2}\big(n - d(x) - d(y) + \triangleright(x,y)\big)\big(d(x) \cdot d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(y) - \triangleright(x,y)\big) - \xi_n(l-2)d(x)\Big)\big(d(x) \cdot (d(y)+1)\big)^\gamma \\
&+ \Big(n^{l-2}\big(d(x) - \triangleright(x,y)\big) - \xi_n(l-2)d(y)\Big)\big((d(x)+1) \cdot d(y)\big)^\gamma \\
&+ \Big(n^{l-2}\big(\triangleright(x,y)+1\big) + \xi_n(l-2)(d(x)+d(y)+1)\Big)\big((d(x)+1) \cdot (d(y)) + 1\big)^\gamma \Big] \\
&> \sum_{xy \in E(H)} (d(x) \cdot d(y))^\gamma \Big[n^{l-1} + n^{l-2} + \xi_n(l-2) \Big] \\
&= \xi_n(l) \cdot R_\gamma(H),
\end{aligned}$$

which is the required result. □

The inequalities in the above result flip for $\gamma < 0$.
The following results hold for $\gamma = \frac{1}{2}, 1$.

Corollary 9. *Let $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, the bounds of the reciprocal Randić index are as follows:*

$$\xi_n(l) \cdot RR(H) < RR(gS(H,l)) < \xi_n(l) \cdot RR(H^1).$$

Corollary 10. *Let $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, the second Zagreb index has the following bounds:*

$$\xi_n(l) \cdot M_2(H) < M_2(gS(H,l)) < \xi_n(l) \cdot M_2(H^1).$$

For $\gamma = \frac{-1}{2}$, we have the following:

Corollary 11. *Let $gS(H,l)$ be the generalized Sierpiński graph of H with $l \geq 2$. Then, the Randić index has the following bounds:*

$$\xi_n(l) \cdot R(H) > R(gS(H,l)) > \xi_n(l) \cdot R(H^1)$$

With values for the smallest general Randić index for trees from [29], the above corollary implies the following consequence.

Corollary 12. *Let $gS(T,l)$ be the generalized Sierpiński graph of T with $l \geq 2$. The Randić index has the bounds*

$$\xi_n(l) \cdot \left(\frac{n-1}{\sqrt{2n}}\right) < R(gS(T,l)) < \xi_n(l) \cdot \left(\frac{n-3}{2} + \sqrt{2}\right),$$

$$\xi_n(l) \cdot \left(2\sqrt{3} + 2(n-3)\right) < RR(gS(T,l)) < \xi_n(l) \cdot \left((n-1)\sqrt{n+2}\right),$$

$$\xi_n(l) \cdot \left(4(n-2)\right) < M_2(gS(T,l)) < \xi_n(l) \cdot \left(4n^2(n-1))\right).$$

Finally for the general Randić index and M_2 from [30], we have the following immediate consequence of the above result.

Corollary 13. *Let U be a unicyclic graph and $gS(U,l)$ be the generalized Sierpiński graph of U for $l \geq 2$. Then, we have*

$$\xi_n(l) \cdot 4n < M_2(gS(U,l)) < \xi_n(l) \cdot \left(2n^2 + 9\right),$$

$$\xi_n(l) \cdot 16n < RR(gS(U,l)) < \xi_n(l) \cdot \left((n-3)\sqrt{2n} + 2\sqrt{3n} + 3\right),$$

$$\xi_n(l) \cdot \left(\frac{1}{3} + \frac{2}{\sqrt{3n}} + \frac{n-3}{\sqrt{2n}}\right) < R(gS(U,l)) < \xi_n(l) \cdot \left(\frac{n}{2}\right).$$

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