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A Discrete-Time Fractional-Order Flocking Control Algorithm of Multi-Agent Systems

Haotian Chen, Ming He *, Wei Han, Sicong Liu and Chenyue Wei

Command and Control Engineering College, Army Engineering University of PLA, Nanjing 210007, China; cht0304@126.com (H.C.); mxyacc@nuaa.edu.cn (W.H.); liusc86@sina.com (S.L.); wcy20000425@126.com (C.W.)
* Correspondence: heming@aeu.edu.cn

Abstract: In this paper, a discrete-time fractional flocking control algorithm of multi-agent systems is put forward to address the slow convergence issue of multi-agent systems. Firstly, by introducing Grünwald-Letnikov (G-L) fractional derivatives, the algorithm allows agents to utilize historical information when updating their states. Secondly, based on the Lyapunov stability theory, the convergence of the algorithm is proven. Finally, simulations are conducted to verify the effectiveness of the proposed algorithm. Comparisons are made between the proposed algorithm and other methods. The results show that the proposed algorithm can effectively improve the convergence speed of multi-agent systems.

1. Introduction

A multi-agent system (MAS) consists of multiple autonomous and perceptually-abled agents that interact, communicate, and compete with each other based on specific rules to solve complex problems and accomplish tasks collectively. MASs span various domains, including robotics, network science, sociology, and economics, and have been widely applied in practical scenarios such as consensus control [1], logistics management, and decision support [2].

Flocking control of multi-agent systems has become a research hotspot in recent years. It refers to emergent collective behaviors in a swarm system where agents interact locally and achieve stability, order, and other specific goals [3]. Flocking control originates from the study of natural biological swarms which exhibit intelligent characteristics, such as fish schools, bird flocks, and insect colonies [4–6]. In 1987, Reynolds [7] introduced the Boids model, a discrete-time and space-based multi-agent system used to describe the behavioral interaction in bird-like animal swarms, such as seagulls or flocks of pigeons. The Boids model proposes three basic principles for agent behavior, separation, alignment, and cohesion, which enable collective synergy. On the basis of the Boids model, Vicsek et al. [8] focused only on the alignment principle and proposed the well-known Vicsek model. In this model, each particle adjusts its movement direction by perceiving and responding to the positions and orientations of surrounding particles.

Based on these models, flocking control in multi-agent systems has been extensively studied. Olfati et al. formulated the theoretical framework for swarm control in multi-agent systems based on Reynolds' three principles, providing a theoretical analysis of the flocking phenomenon from the perspective of nonlinear control theory [9]. In 2019, Jia et al. [10] introduced a hierarchical mechanism to the classical Vicsek model, proposing a hierarchical flocking model to explain the collaborative behaviors of specific agents. It is worth noting that the existing research on swarm control in multi-agent systems mainly focuses on integer-order dynamics, such as first-order dynamics [11,12] and second-order

dynamics [13,14], while studies based on fractional-order dynamics are relatively scarce. Fractional calculus has been widely applied in natural sciences and engineering fields, such as linear and nonlinear dynamics, intelligent algorithm optimization, and characterizing the internal structures of complex functions. Fractional calculus is a good approach for modeling complex physical systems [15,16] and improving system convergence rates [17,18]. Due to the "memory" property of fractional calculus, many complex phenomena in nature that integer-order dynamics cannot explain can be naturally described by the cooperative behaviors of agents with fractional-order dynamics, such as the foraging of microorganisms and the collective movement of bacteria [19]. Therefore, it is meaningful to investigate flocking control in multi-agent systems based on fractional-order dynamics. In addition, discrete-time systems are suitable to investigate this, since in continuous systems, in order to execute an implementation, the systems have to be discretized because there are no exact formulas of the solutions.

Inspired by the above results, this paper discusses the flocking control of discrete-time multi-agent systems with fractional dynamics. The main contributions are summarized as follows.

1. A discrete-time multi-agent flocking control algorithm was derived based on Grünwald-Letnikov (G-L) fractional derivatives. Compared with existing flocking methods where only integer-order dynamics are considered, our algorithm allows agents to use historical information, which means that the current states of the agents depend on both recent and historical values. Thus, our method conforms more with the reality that individuals always exhibit a time-dependent memory effect in nature;

2. Compared with existing research [18] where only the leaderless condition is taken into account, this paper investigates the fractional-order flocking control of multi-agent systems under the leader-following strategy. Based on the Lyapunov stability theory, the convergence of this algorithm is proven. Experimental results demonstrate that the proposed algorithm achieves consensus among agents and effectively improves the convergence rate.

This paper is organized as follows. In Section 2, some notations and basic definitions are given. In Section 3, the model formulation for the discrete-time flocking control algorithm of multi-agent systems is proposed, and the convergence of which is proven. In Section 4, simulations are given to verify our results. Finally, some conclusions are presented in Section 5.

2. Preliminaries

In this section, basin graph theory and definitions of the G-L fractional derivative are given.

2.1. Graph Theory

Let $x_i(t) \in \mathbb{R}^2$, $v_i(t) \in \mathbb{R}^2$ and $u_i(t) \in \mathbb{R}^2$ be the position vector, velocity vector and control input of agent i at time t, respectively. The topology of agents is described as an undirected graph $G = (V, E)$, where $V = \{1, 2, \cdots, N\}$ denotes n agents and $E = \{(i, j) \in V \times V, j \in N_i(t)\}$ indicates that there is a communication link between agent i and agent j. $N_i(t)$ is the set of neighbors of agent i at time t. Let r denote the communication radius, then $N_i(t) = \{j | \|x_i(t) - x_j(t)\| \leq r, j \in V, j \neq i\}$. $A = [a_{ij}]_{N \times N}$ denotes the adjacency matrix with $a_{ij} = 1$ if $j \in N_i$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of graph G is defined as $L = [l_{ij}]_{N \times N}$ with $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

In order to avoid the collision among agents, the agents need to keep a distance between the safety distance r_s and communication radius r; therefore, the expected distance between every two agents is defined as

$$\|x_i(t) - x_j(t)\| = d, (i, j) \in E(t), r_s < d < r. \tag{1}$$

Equation (1) is called the α-lattices system [9]. However, due to the interaction between individuals, the system cannot reach the ideal system and eventually evolves into the α-lattices-like system with error δ, which is shown in Figure 1.

(**a**) α-lattices system

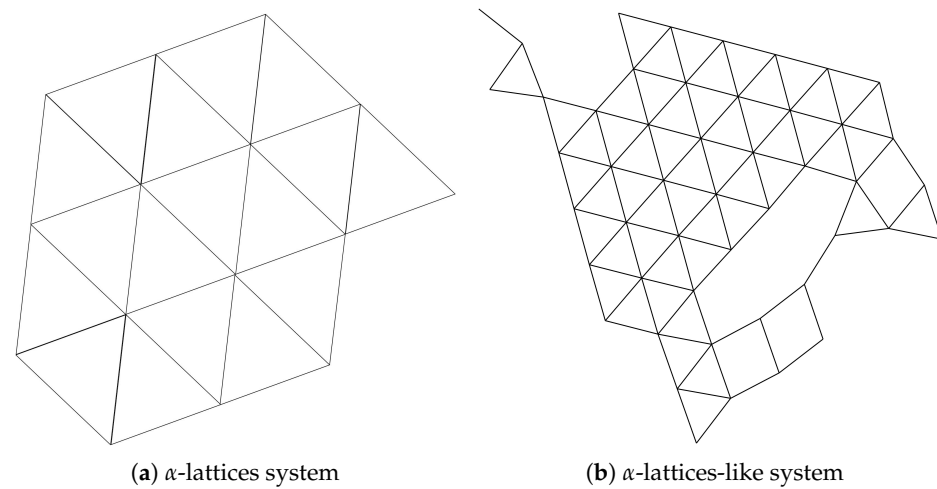(**b**) α-lattices-like system

Figure 1. The α-lattices system and α-lattices-like system.

2.2. Fractional Derivative

Fractions calculus, as a branch of mathematics, has evolved over several centuries, resulting in varying definitions. In this paper, we employ the Grunwald-Letnikov fractional derivative, which is defined as follows [20]:

$$D^{\alpha}(x(t)) = \lim_{h \to 0} \left(\frac{1}{h^{\alpha}} \sum_{k=0}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1) x(t-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \right), \tag{2}$$

where h is a positive real number and $k \geq 0$ is an integer. $\alpha \in (0,1]$ is the fractional order and $\Gamma(\cdot)$ is the Gamma function with $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$. $D^{\alpha}(x(t))$ is called the G-L fractional derivative of $x(t)$ with order α.

In discrete-time implementation, Equation (2) is given by

$$D^{\alpha}(x(t)) = \frac{1}{T^{\alpha}} \sum_{k=0}^{m} \frac{(-1)^k \Gamma(\alpha+1) x(t-kT)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \tag{3}$$

where $t = eT$, e is a positive integer, T is the sampling period and m is the truncation order.

Remark 1. *Equations* (2) *and* (3) *demonstrate a vital characteristic of fractional differentiation: it comprises an infinite series of terms, unlike integer differentiation, which consists of a finite series. Consequently, integer differentiation is referred to as a local operator, as it solely relies on the function's value at a single point in time and its finite derivative. In contrast, fractional differentiation incorporates historical information throughout the evolutionary process, metaphorically representing the "memory" of all past events. It can be recognized as a non-local operator with memory effects. This particular property is crucial in accurately depicting the dynamic behavior observed in numerous natural, physical, and engineering systems.*

Remark 2. $t = eT$ *where e is a positive integer is applied for all the following equations. Thus all the time t is in the discrete-time domain in the rest of this paper.*

3. The Proposed Flocking Control Method

In the current section, the proposed flocking control of multi-agent systems based on the G-L fractional derivative is given. Furthermore, the convergence of the algorithm is analyzed by using the Lyapunov stability theory.

3.1. Dynamics Model of Multi-Agent Systems

We consider that there are N agents in two-dimensional Euclidean space. In the general flocking control models [21–23], agents always update their states by integer-order dynamics, that is

$$\begin{cases} x_i(t+T) = x_i(t) + v_i(t)T \\ v_i(t+T) = v_i(t) + u_i(t)T \end{cases} \quad i = 1, 2, \cdots, N, \tag{4}$$

where T is the time step. Now, let us see the updating speed of the agents. $v_i(t+T) = v_i(t) + u_i(t)T$ can be written as

$$\frac{v_i(t+T) - v_i(t)}{T} = u_i(t). \tag{5}$$

Equation (5) is equivalent to

$$D^1(v_i(t+T)) = u_i(t). \tag{6}$$

The above equation suggests that in the traditional integer-order flocking control, only current information is used to update the agents' states. To apply the historical information to boost the performance of swarm flocking, we replace the plain derivative with the G-L fractional-order derivative with fractional order $\alpha \in (0, 1]$, as defined in Section 2.2, then we obtain the following relation:

$$D^\alpha(v_i(t+T)) = \frac{1}{T^\alpha} \sum_{k=0}^{m} \frac{(-1)^k \Gamma(\alpha+1) v_i(t+(1-k)T)}{\Gamma(k+1)\Gamma(\alpha-k+1)} = u_i(t), \tag{7}$$

where $D^1(x(t))$ denotes the first-order derivative of $x(t)$. Without loss of generality, the first four terms of the fractional derivative, as given by Equation (3), are considered, then we obtain

$$\begin{cases} x_i(t+T) = x_i(t) + v_i(t)T \\ v_i(t+T) = \alpha v_i(t) + \frac{1}{2}\alpha(1-\alpha)v_i(t-T) + \frac{1}{6}\alpha(1-\alpha)(2-\alpha)v_i(t-2T) \\ \qquad + \frac{1}{24}\alpha(1-\alpha)(2-\alpha)(3-\alpha)v_i(t-3T) + u_i(t)T. \end{cases} \tag{8}$$

Equation (8) reveals that the speed of the agents, based on fractional-order dynamics at time t, is a comprehensive result of historical information from the previous four moments. This contrasts integer-order flocking control, which relies solely on the information of the current state. Importantly, when $\alpha = 0$, the fractional-order flocking control algorithm regresses to the traditional integer-order flocking control.

In order to make all agents move in the desired direction, a virtual leader is added, and the dynamics of which are designed as

$$\begin{cases} x_0(t+T) = x_0(t) + v_0(t) \\ v_0(t+T) = v_0(t) \end{cases}, \tag{9}$$

where $x_0(t)$ and $v_0(t)$ represent the position vector and speed vector of the virtual leader at time t.

3.2. Control Protocol of the Agents

Now, we consider the control protocol of the agents. Assume that the control input of agent i at time t is

$$u_i(t) = f_i^g(t) + f_i^v(t) + f_i^\gamma(t), \tag{10}$$

where $f_i^g(t)$ is the relative distance control term, which is used to achieve aggression and separation. The definition of $f_i^g(t)$ is

$$f_i^g(t) = -\nabla_{x_i(t)} \sum_{j \in N_i(t)} \phi_a(\|x_j(t) - x_i(t)\|_\sigma), \tag{11}$$

where $\|z\|_\sigma = \frac{1}{\varepsilon}[\sqrt{1 + \|z\|^2} - 1]$ with parameter $\varepsilon > 0$, and

$$\phi_a(\|z\|_\sigma) = \frac{1}{2}\left[(a+b)\frac{\|z\|_\sigma + c}{\sqrt{1 + (\|z\|_\sigma + c)^2}} + (a-b)\right], \tag{12}$$

where the parameters a, b and c satisfy $0 < a < b$ and $c = \|a - b\| / \sqrt{4ab}$. The gradient of σ-norm is $\nabla\|z\|_\sigma = z / \sqrt{1 + (\|z\|_\sigma + c)^2}$. The term $f_i^v(t)$ is used to control the agents to achieve speed alignment, which is expressed as

$$f_i^v(t) = \sum_{j \in N_i(t)} (v_j(t) - v_i(t)). \tag{13}$$

$f_i^\gamma(t)$ is used for the agents to follow the virtual leader, which is defined as

$$f_i^\gamma(t) = -c_1(x_i(t) - x_0(t)) - c_2(v_i(t) - v_0(t)), \tag{14}$$

where c_1 and c_2 are positive real numbers, representing the feedback parameters.

Assumption 1. *The initial states of the multi-agent systems are connected, which means that the undirected graph $G(0)$ is connected.*

Assumption 2. *There are no collisions between agents in the initial state of the system.*

Assumption 3. *All the agents can simultaneously receive instructions from the virtual leader.*

Theorem 1. *Consider that Assumptions 1–3 are valid. For the multi-agent system described in Equations (8) and (9) with the control input in (10), if the sampling period is small enough such that $T \to 0$, then we can obtain*

1. *The system will be asymptotic stable, and the agents' positions will eventually tend to lattices;*
2. *The speed of all agents will tend towards the virtual leader;*
3. *There will be no collisions among the agents.*

Proof of Theorem 1. Here, we use the same technique as described in the stability analysis of flocking in [9,24] to prove the stability of our proposed model. Let $\tilde{x}_i(t) = x_i(t) - x_0(t)$, $\tilde{v}_i(t) = v_i(t) - v_0(t)$; hence, it is obviously that $\tilde{x}_{ij}(t) = x_{ij}(t)$, where $x_{ij}(t) = x_j(t) - x_i(t)$. The total energy of a multi-agent system is composed of the total potential energy and the relative kinetic energy. According to Equations (10)–(14), the total energy of the system at time t is

$$Q(t) = \frac{1}{2} \sum_{i=1}^{N} (U_i(t) + \tilde{v}_i(t)^T \tilde{v}_i(t)), \tag{15}$$

where $U_i(t)$ is the total potential energy at time t [24]

$$U_i(\tilde{x}_i(t)) = \sum_{j=1, j \neq i}^{N} \phi_a(\|\tilde{x}_{ij}(t)\|_\sigma) + c_1 \tilde{x}_i(t)^T \tilde{x}_i(t). \tag{16}$$

where $\tilde{x}_{ij}(t) = \tilde{x}_j(t) - \tilde{x}_i(t)$. For simplicity, let $W_i(\tilde{x}_{ij}) = \sum_{j=1, j \neq i}^{N} \phi_a(\|\tilde{x}_{ij}(t)\|_\sigma)$, then we have

$$\lim_{T \to 0} \frac{1}{2} \frac{U_i(t+T) - U_i(t)}{T} = \frac{1}{2} \dot{U}_i(t) = \tilde{v}_i(t)^T \nabla_{\tilde{x}_i(t)} W_i(\tilde{x}_{ij}(t)) + c_1 \tilde{v}_i(t)^T \tilde{x}_i(t). \quad (17)$$

The energy function can be written as

$$\begin{aligned}
Q(t) &= \frac{1}{2} \sum_{i=1}^{N} (U_i(t) + \tilde{v}_i(t)^T \tilde{v}_i(t)) \\
&= \frac{1}{2} \sum_{i=1}^{N} (W_i(\tilde{x}_{ij}(t)) + c_1 \tilde{x}_i(t)^T \tilde{x}_i(t) + \tilde{v}_i(t)^T \tilde{v}_i(t)).
\end{aligned} \quad (18)$$

Then we obtain

$$\begin{aligned}
\lim_{T \to 0} \frac{Q(t+T) - Q(t)}{T} &= \dot{Q}(t) \\
&= \frac{1}{2} \sum_{i=1}^{N} (\dot{U}(t) + \tilde{v}_i(t)^T u_i(t)).
\end{aligned} \quad (19)$$

From the definition of $u_i(t)$, invoking Equations (10)–(14),

$$\begin{aligned}
\dot{Q}(t) &= \sum_{i=1}^{N} (\tilde{v}_i(t)^T \nabla_{\tilde{x}_i(t)} W_i(\tilde{x}_{ij}(t)) + c_1 \tilde{v}_i(t)^T x_i(t) + \tilde{v}_i(t)^T u_i(t)) \\
&= -\tilde{v}(t)^T (L(t) \otimes I_n) \tilde{v}(t) - c_2 \sum_{i=1}^{N} \tilde{v}_i(t)^T \tilde{v}_i(t) \\
&= -\tilde{v}(t)[(L(t) + c_2 I_N) \otimes I_n] \tilde{v}(t),
\end{aligned} \quad (20)$$

where $\tilde{v}(t) = [\tilde{v}_1(t) \ \tilde{v}_2(t) \ \cdots \ \tilde{v}_N(t)]^T$, $L(t)$ is the Laplacian matrix of the system at time t, and $\otimes$ denotes the Kronecker product. Since $L(t)$ is a positive semi-definite matrix, $L(t) + c_2 I_N$ is a positive matrix. Then we obtain $Q(t+T) - Q(t) < 0$, which means $Q(t)$ is decreasing. Therefore, $Q(t) < Q(0)$ for any $t > 0$, and the system will be asymptotically stable.

Assume that agents i and j collide during $t_c \in [t_m, t_n]$. Then $U_i(t)$ and $Q_i(t)$ will increase, which contradicts that $Q(t)$ is decreasing. Therefore, the agents will not collide. Using the same method, the agents will not collide at each period $[t_k, t_{k+1}]$. This completes the proof. □

3.3. Flow Chart of Our Proposed Algorithm

Figure 2 illustrates the procedure of the discrete-time fractional-order flocking control algorithm of multi-agent systems. First, the agent's speed and position are set up. Next, the interaction force on the agent is calculated using Equations (10)–(14). After this, the agent updates its fractional speed based on the interaction force and Equation (8). Finally, the agent's position is updated using Equation (8).
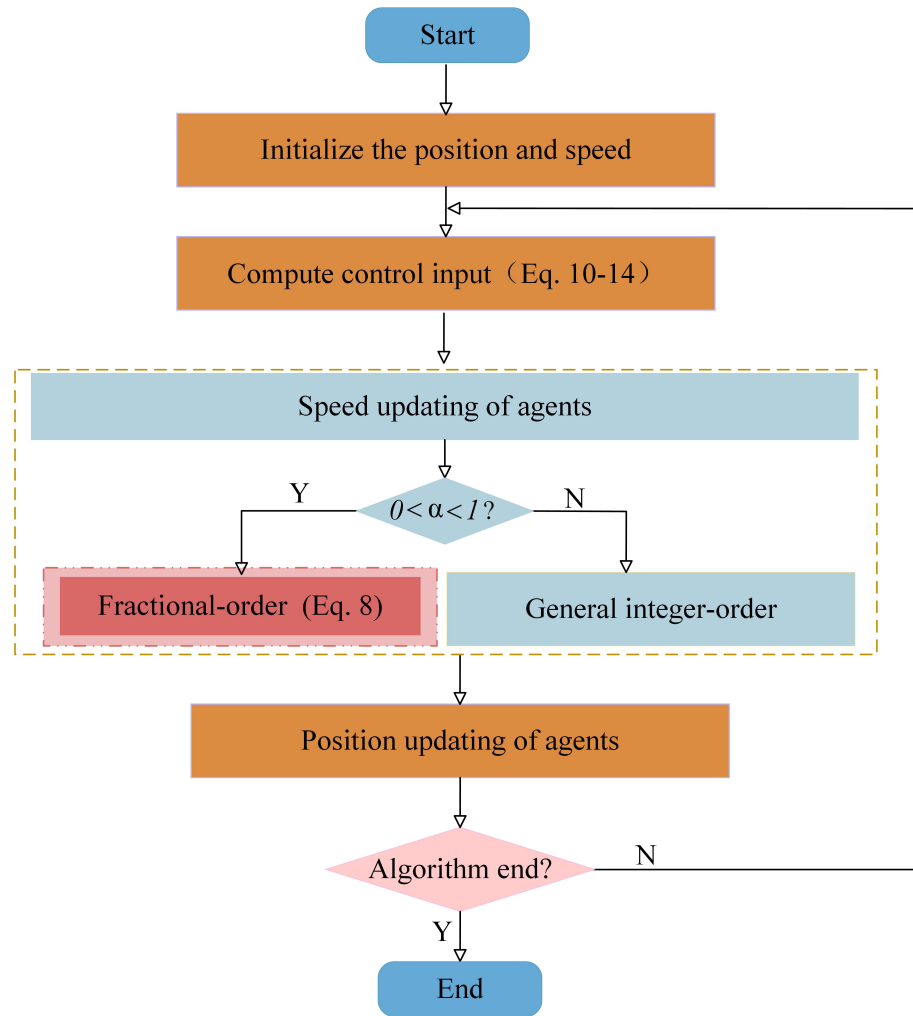
Figure 2. Flow chart of our proposed flocking algorithm.

4. Simulation Results

In this section, some numerical simulations based on MATLAB are provided to illustrate the effectiveness of our proposed fractional-order flocking control algorithm.

4.1. Tests of the Fractional-Order Flocking Control Algorithm

The current simulation investigates the speed and position changes of agents based on G-L fractional-order dynamics. The multi-agent system is composed of 100 agents, and the initial location and direction angle are randomly generated in a $[0, 100] \times [0, 100]$ area and $[-\pi, \pi]$, respectively, which is shown in Figure 3a. The red arrow represents the direction of the agents. Consider the communication radius $r = 6$, the subsequent expected distance between every two agents is $d = 5$, and besides $c_1 = 0.2$ and $c_2 = 0.5$, the sampling period $T = 0.02$. The initial position of the virtual leader is $(25, 25)$, and the speed of it is constant $(0.5, 0.5)$. The fractional order is $\alpha = 0.8$.

Figure 3 illustrates the status of the multi-agent system based on our method at different time intervals. Figure 3a shows the initial states of the system, which are complete disorder. Figure 3b,c present the agents' states after 150 and 300 iterations, respectively. Observably, with increasing iterations, the system transitions from disorder to order, and the speed of the agents gradually approach that of the virtual leader. The system's final state is shown in Figure 3d, where all agents converge to a uniform state, exhibiting a lattice-like structure (verifying the first condition in Theorem 1).
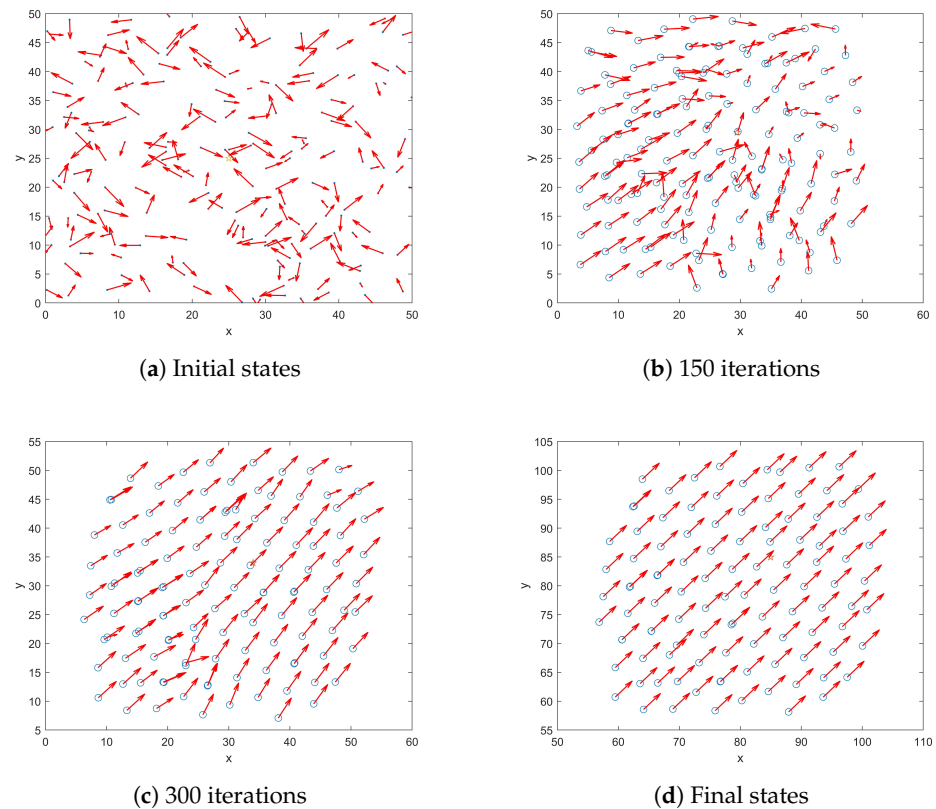
(**a**) Initial states

(**b**) 150 iterations

(**c**) 300 iterations

(**d**) Final states

Figure 3. The status of the agents.

The motion trajectories of the agents are shown in Figure 4. It can be seen that, at first, the whole system is disordered. With an increase in iterations, the degree of order of the system is improved through interactions between agents; in the end, the system reaches convergence, and all the agents move towards the upper-right direction.
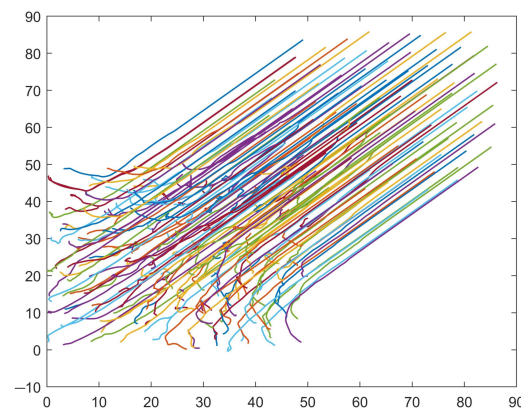

Figure 4. The trajectories of the agents.

We define the speed error between agents and the virtual leader as

$$\begin{cases} e_i^x(t) = v_i^x(t) - v_0^x(t) \\ e_i^y(t) = v_i^y(t) - v_0^y(t) \end{cases} \quad i = 1, 2, \cdots, N, \tag{21}$$

where $e_i^x(t)$ and $e_i^y(t)$ are the speed error in the direction of X-axis and Y-axis at time t, respectively. $v_i^x(t)$, $v_0^x(t)$ and $v_i^y(t)$, $v_0^y(t)$ are the speed components in the direction of

X-axis and Y-axis at time t, respectively. Then, the global error between the agents and the virtual leader can be expressed as $e_i(t) = \sqrt{(e_i^x(t))^2 + (e_i^y(t))^2}$. The speed error between the agents and the virtual leader based on fractional-order flocking control is shown in Figure 5. Figure 5a is the speed error in the direction of the X-axis, and Figure 5b is the speed error in the direction of the Y-axis. Figure 5c shows the global error between the agents and the virtual leader. We can see that in the initial time, the speed error between the agents and the virtual leader is significant because the initial speed of the agents is random. Under the control of the fractional-order flocking algorithm, the agent's speed gradually approaches the leader's speed, and finally the speed error tends to 0. At this time, the speed of the agents and the virtual leader is consistent (verifying the second condition in Theorem 1).
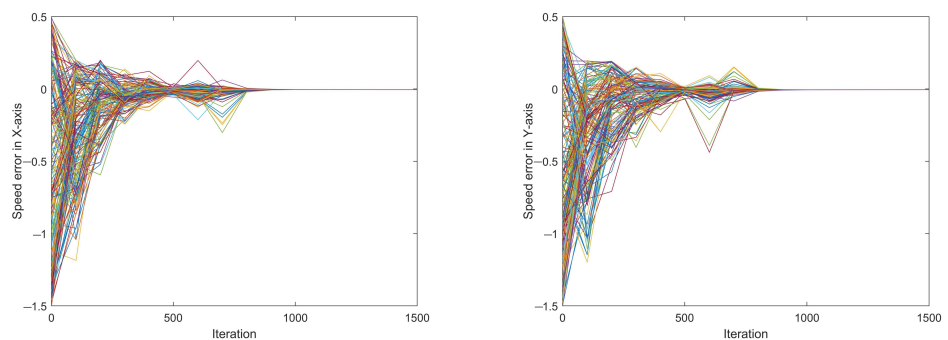
(**a**) Speed error in the direction of the X-axis   (**b**) Speed error in the direction of the Y-axis
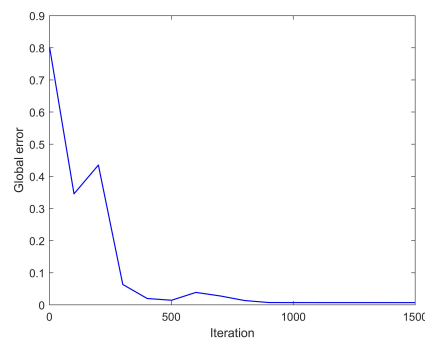
(**c**) Global error

Figure 5. The speed error between the agents and the virtual leader.

Above all, the multi-agent systems based on the fractional-order flocking control algorithm proposed in this paper can achieve flocking effectively. The speed of all agents gradually tends to that of the virtual leader over time. In addition, the distance between agents is always greater than 0, and thus Theorem 1 is verified.

4.2. Performance Test

In order to further illustrate the advantages of the proposed fractional-order flocking control method in this paper, comparisons were made among this method and two other commonly used flocking control algorithm, which are flocking control based on integer-order dynamics [9] and betweenness centrality with the influence degree (BCID) [25]. The selected system consists of 200 agents in a $[0, 50] \times [0, 50]$ area. To avoid randomness and to obtain general results, 50 simulations are conducted, each with a maximum of 300 iterations.

Other parameters were the same as those in Section 4.1. The states of the agents can be quantitatively expressed by the velocity direction order parameter Φ, which is defined as

$$\Phi = \frac{1}{N} \left\| \sum_{i=1}^{N} \frac{v_i(t)}{\|v_i(t)\|} \right\|. \tag{22}$$

Φ describes the degree of order of the agents' movements. When $\Phi = 0$, all agents in the system move in random directions and the system is completely disordered. When $\Phi = 1$, all agents move in the same direction, and the system is ultimately ordered. It was found that the system will show apparent order when $\Phi = 0.9$ [26].

The results are shown in Figure 6. The initial state of the system is strongly disordered; therefore, the degree of order for all three methods is poor in the beginning. With the increase in iterations, the speed of the agents tend towards the virtual leader's speed, and consequently the system gradually becomes ordered. Note that when there are fewer than 120 iterations, the convergence rate of our proposed fractional-order flocking control algorithm is no better than the other two algorithms. This is because the agents with fractional-order dynamics utilize the historical information to update their states; however, the order of the system at each moment in this period (less than 120 iterations) is relatively poor, and the use of such poor information will reduce the order of the system. However, as the order of the system gradually increases, "high-quality" historical information (the degree of order is relatively high) will accelerate the convergence rate of the system. Therefore, when the number of iterations reaches 120, the convergence rate of the multi-agent system based on the fractional-order flocking control algorithm is significantly higher than that of the integer-order method and BCID method, and the consistency of the system is eventually realized much faster.
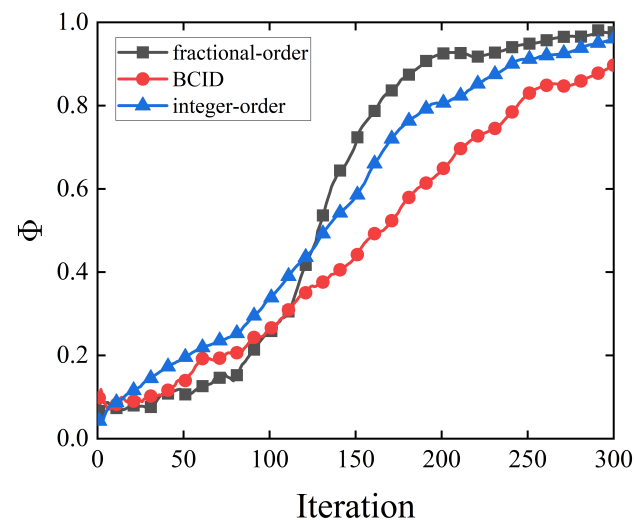

Figure 6. The convergence rate comparison of the algorithms.

5. Conclusions

A discrete-time flocking control algorithm for multi-agent systems was proposed by introducing a G-L fractional derivative to the speed updating process of agents, which enables agents to utilize historical information. The convergence of this algorithm is proven. The simulation results demonstrate that multi-agent systems can achieve flocking based on this algorithm. Furthermore, our proposed algorithm can effectively improve the convergence rate of multi-agent systems. In future work, we will focus on applying this algorithm to multi-vehicle and multi-robot systems.

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