fractal and fractional

Article

New Version of Fractional Pachpatte-Type Integral Inequalities via Coordinated ℏ-Convexity via Left and Right Order Relation

Tareq Saeed [1], Eze R. Nwaeze [2,*], Muhammad Bilal Khan [3,*] and Khalil Hadi Hakami [4]

[1] Financial Mathematics and Actuarial Science (FMAS)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; tsalmalki@kau.edu.sa
[2] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA
[3] Department of Mathematics and Computer Science, Transilvania University of Brasov, 29 Eroilor Boulevard, 500036 Brasov, Romania
[4] Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia; khakami@jazanu.edu.sa
* Correspondence: enwaeze@alasu.edu (E.R.N.); muhammad.bilal@unitbv.ro (M.B.K.)

Abstract: In particular, the fractional forms of Hermite–Hadamard inequalities for the newly defined class of convex mappings proposed that are known as coordinated left and right ℏ-convexity (*LR*-ℏ-convexity) over interval-valued codomain. We exploit the use of double Riemann–Liouville fractional integral to derive the major results of the research. We also examine the key results' numerical validations that examples are nontrivial. By taking the product of two left and right coordinated ℏ-convexity, some new versions of fractional integral inequalities are also obtained. Moreover, some new and classical exceptional cases are also discussed by taking some restrictions on endpoint functions of interval-valued functions that can be seen as applications of these new outcomes.

Keywords: interval-valued mappings over coordinates; left and right ℏ-convexity; double Riemann–Liouville fractional integral operator; Pachpatte-type inequalities

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1. Introduction

There are many uses for the concepts of convex sets and convex functions in the realms of applied and pure sciences. Furthermore, because of its many applications and tight relationship to the theory of inequalities, convexity has advanced quickly in recent years. When determining exact values for a mathematical problem proves to be challenging, inequalities can be used to approximate the solution. Since many inequalities can be directly derived from convex functions, there is a close relationship between convexity and the theory of inequalities.

The Hermite–Hadamard inequality is one of the most well-known findings in the category of classical convex functions, according to Dragomir and Pearce [1]. This inequality has several applications and a straightforward intrinsic geometric explanation. The result was mainly credited to Hermite (1822–1901), even though Hadamard (1865–1963) was the one who first identified it [2,3]. The following is how this inequality is stated:

Theorem 1. *Assume that the convex mapping* $\mathfrak{J}:[\sigma,\mathrm{i}]\rightarrow\mathfrak{R}$. *Then, the following double inequality holds:*

$$\mathfrak{J}\left(\frac{\sigma+\mathrm{i}}{2}\right)\leq\frac{1}{\mathrm{i}-\sigma}\int_{\sigma}^{\mathrm{i}}\mathfrak{J}(x)dx\leq\frac{\mathfrak{J}(\mathrm{i})+\mathfrak{J}(\sigma)}{2},\tag{1}$$

where $\mathfrak{R}$ *is a set of real numbers. One can check the concavity of the mappings by replacing the symbol "≤" with "≥" in double inequality (1).*

The midpoint and trapezoidal-type inequalities, which are the two sides of the Hermite–Hadamard inequality, are used to estimate error boundaries for specific quadrature rules. The original derivation of these inequalities was in [4,5].

New extended versions of Simpson-type inequalities were derived by Awan et al. [6] using differentiable, strongly (s, m)-convex maps. Simpson's integral inequality has been further expanded upon, refined, and generalized in [7–10].

In the course of research, various variations of the Hermite–Hadamard inequality have been derived by expanding the definition of convex functions. Conversely, the notion of $\mathfrak{s}\}$-convexity [11,12] is divided into two halves, as follows, with the fundamental requirement that $1 \geq s > 0$. The following references [13–17] contain more generalizations and expansions of classical convex functions.

Fractional calculus is the study of integrals and derivatives of any real order. Fractional integrals are used to solve a wide range of problems involving mathematical science's special functions, as well as their generalizations and extensions to one or more variables. Furthermore, compared to traditional derivatives, fractional-order derivatives describe the memory and hereditary characteristics of distinct processes far better. Actually, current applications in fluid mechanics, mathematical biology, electrochemistry, physics, differential and integral equations, signal processing, and fluid mechanics have been the driving forces behind the recent developments in fractional calculus. Without a doubt, fractional calculus can be used to solve a wide range of diverse problems in science, engineering, and mathematics [18–20]. The reference [21] provides a thorough history of fractional calculus.

Creating different kinds of integral inequalities is a modern issue. Utilizing a range of integrals, including the Sugeno integral [22,23], the pseudo integral [24], the Choquet integral [25], and others, a significant amount of important work has been accomplished in recent years. As a notion of generalization of functions and a significant mathematical subject, interval-valued functions [26] have grown in importance as a tool for resolving real-world problems, especially in mathematical economics [27]. Certain classical integral inequalities have been expanded to the domain of interval-valued functions through recent studies. New interval variations of Minkowski and Beckenbach's integral inequalities were introduced by Costa et al. [28]. This generalization established Jensen, Ostrowski, and Hermite–Hadamard-type inequalities [29]. Fractional integrals of Riemann–Liouville with interval values were also used to solve Hermite–Hadamard and Hermite–Hadamard-type inequalities [30]. Zhao and colleagues [31–33] employed the gH -differentiable or h-convex notion to study Jensen and Hermite–Hadamard-type inequalities, Opial-type integral inequalities, and Chebyshev-type inequalities for interval-valued functions. Budaka et al. [34] used the definitions of gH-derivatives to develop new fractional inequalities of the Ostrowski type for interval-valued functions. Log-h-convex fuzzy-interval-valued functions are a new class of convex fuzzy-interval-valued functions that were introduced by Khan et al. [35] using a fuzzy order relation. The Jensen and Hermite–Hadamard inequalities were established in this class. To include the Ostrowski-type inequality in the domain of fuzzy-valued functions, the Hukuhara derivative had to be applied, as Anastassiou [36] showed. Anastassiou's research focused heavily on fuzzy-valued functions, commonly referred to as functions with an interval value. An interesting finding is that Anastassiou's fuzzy Ostrowski-type inequalities could also be used for interval-valued functions. Bede and Gal's [37] and Chalco-Cano et al.'s [38] publications should be studied in order to gain a thorough grasp of the limitations placed on interval-valued functions by the idea of the H-derivative. Significantly, Chalco-Cano et al.'s recent work [39] has produced an Ostrowski-type inequality that is tailored to generalized Hukuhara differentiable interval-valued functions. On the other hand, Lupulescu [40] introduced the concept of left-fractional integral in interval-valued calculus. Then, right fractional integrals are proposed by Budak et al. [41], as well as providing the fractional Hermite–Hadamard-type inequalities for interval-valued mappings over coordinates. Zhao et al. [42] generalized the Riemann integral inequalities for coordinated convex interval-valued mappings and produced the inequalities for the product of coordinated convexity. After that, Khan et al. [43]

provide a new direction in interval-valued calculus by introducing new versions of coordinated integral inequalities via double Riemann integrals and left and right relations. Budak and Sarıkaya [44] and Khan et al. [45] defined Pachpatte's inequalities for the product of coordinated and left and right coordinated convex mappings via fractional integrals, respectively. By using this approach, Khan et al. [46] obtained the left- and right-coordinated interval-valued functions and acquired some integral inequalities in interval fractional calculus for left- and right-coordinated interval-valued functions. Zhang et al. [47] first defined the up and down relations and then discussed some of the properties of these relations. Moreover, he defined the new versions of Jensen-type inequalities using up and down relations.

The structure of this research article is as follows. Some classical notions, definitions, and results are recalled and a new definition of convexity over interval-valued mapping is also introduced, which is known as coordinated LR-$\hbar$-convexity in Section 2. We also report several other findings that follow from this definition of convexity. Using coordinated LR-$\hbar$-convexity defined in Section 2, some new and classical exceptional cases are also obtained in Section 2. In Section 3, involving interval fractional integrals for LR-$\hbar$-convexity, some well-known inequalities have been generalized, as well as nontrivial examples have also been provided to validate the main outcomes of this paper. Section 4 concludes this study and discusses future work.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_I$ containing all bounded and closed intervals within $\mathbb{R}$. $\imath \in \mathbb{R}_I$ should be defined as follows:

$$\imath = [\imath_*,\ \imath^*] = \{\mathrm{y} \in \mathbb{R} | \imath_* \leq \mathrm{y} \leq \imath^*\}, (\imath_*,\ \imath^* \in \mathbb{R}). \tag{2}$$

It is argued that $\imath$ is degenerate if $\imath_* = \imath^*$. The interval $[\imath_*,\ \imath^*]$ is referred to as positive if $\imath_* \geq 0$, $\mathbb{R}_I^+$ represents the set of all positive intervals and is defined as $\mathbb{R}_I^+ = \{[\imath_*,\ \imath^*] : [\imath_*,\ \imath^*] \in \mathbb{R}_I \text{ and } \imath_* \geq 0\}$.

Let $\varrho \in \mathbb{R}$, $\imath, \jmath \in \mathbb{R}_I$ be defined by, with $\jmath = [\jmath_*,\ \jmath^*]$ and $\imath = [\imath_*,\ \imath^*]$, and we may define the interval arithmetic as follows:

- Scaler multiplication:

$$\varrho.\imath = \begin{cases} [\varrho\imath_*,\ \varrho\imath^*] & \text{if } \varrho > 0, \\ \{0\} & \text{if } \varrho = 0, \\ [\varrho\imath^*, \varrho\imath_*] & \text{if } \varrho < 0. \end{cases} \tag{3}$$

- Addition:

$$[\jmath_*,\ \jmath^*] + [\imath_*,\ \imath^*] = [\jmath_* + \imath_*,\ \jmath^* + \imath^*]. \tag{4}$$

- Multiplication:

$$[\jmath_*,\ \jmath^*] \times [\imath_*,\ \imath^*] = [min\{\jmath_*\imath_*,\ \jmath^*\imath_*,\ \jmath_*\imath^*,\ \jmath^*\imath^*\},\ max\{\jmath_*\imath_*,\ \jmath^*\imath_*,\ \jmath_*\imath^*,\ \jmath^*\imath^*\}]. \tag{5}$$

The inclusion " $\supseteq$ " means that $\imath \supseteq \jmath$ if and only if, $[\imath_*, \imath^*] \supseteq [\jmath_*, \jmath^*]$, and if and only if $\imath_* \leq \jmath_*, \jmath^* \leq \imath^*$.

Remark 1 ([47]). *(i) The relation "* $\leq_p$*" is defined on* $\mathbb{R}_I$ *by*

$$[\jmath_*,\ \jmath^*] \leq_p [\imath_*,\ \imath^*] \text{ if and only if } \jmath_* \leq \imath_*,\ \ \jmath^* \leq \imath^*, \tag{6}$$

for all $[\jmath_*,\ \jmath^*],\ [\imath_*,\ \imath^*] \in \mathbb{R}_I$*, and it is a pseudo-order relation. The relation* $[\jmath_*,\ \jmath^*] \leq_p [\imath_*,\ \imath^*]$ *coincident to* $[\jmath_*,\ \jmath^*] \leq [\imath_*,\ \imath^*]$ *on* $\mathbb{R}_I$ *when it is "* $\leq_p$*".*

(ii) It can be easily seen that " $\leq_p$ " looks like "left and right" on the real line $\mathbb{R}$, so we call " $\leq_p$ " "left and right" (or "LR" order, in short).

The Hausdorff–Pompeiu distance between intervals $[\jmath_*, \jmath^*]$, $[\imath_*, \imath^*] \in \mathbb{R}_I$ is given by

$$d([\jmath_*, \jmath^*], [\imath_*, \imath^*]) = max\{|\jmath_* - \imath_*|, |\jmath^* - \imath^*|\}. \tag{7}$$

It is a familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.

Definition 1 ([30,40]). *Let $\mathfrak{J} : [\sigma, \mathfrak{i}] \to \mathbb{R}_I^+$ be an interval-valued mapping (IVM) and $\mathfrak{J} \in \mathcal{IR}_{[\sigma,\mathfrak{i}]}$. Then, interval Riemann–Liouville-type integrals of $\mathfrak{J}$ are defined as*

$$\mathcal{I}^{\alpha}_{\sigma^+}\mathfrak{J}(y) = \frac{1}{\Gamma(\alpha)}\int_{\sigma}^{y}(y - \mathtt{t})^{\alpha-1}\mathfrak{J}(\mathtt{t})d\mathtt{t} \ \ (y > \sigma), \tag{8}$$

$$\mathcal{I}^{\alpha}_{\mathfrak{i}^-}\mathfrak{J}(y) = \frac{1}{\Gamma(\alpha)}\int_{y}^{\mathfrak{i}}(\mathtt{t} - y)^{\alpha-1}\mathfrak{J}(\mathtt{t})d\mathtt{t} \ \ (y < \mathfrak{i}), \tag{9}$$

where $\alpha > 0$ and Γ is the gamma function.

Interval and fuzzy Riemann-type integrals are defined as follows for coordinated VM $\mathfrak{J}(x, y)$.

Theorem 2 ([42]). *Let $\mathfrak{J} : [\sigma, \mathfrak{i}] \subset \mathbb{R} \to \mathbb{R}_I$ be an IVM, given by $\mathfrak{J}(x) = [\mathfrak{J}_*(x), \mathfrak{J}^*(x)]$ for all $x \in [\sigma, \mathfrak{i}]$. Then, $\mathfrak{J}$ is Riemann integrable (IR-integrable) over $[\sigma, \mathfrak{i}]$ if and only if $\mathfrak{J}_*(x)$ and $\mathfrak{J}^*(x)$ both are Riemann integrable (R-integrable) over $[\sigma, \mathfrak{i}]$. Moreover, if $\mathfrak{J}$ is IR-integrable over $[\sigma, \mathfrak{i}]$, then*

$$(IR)\int_{\sigma}^{\mathfrak{i}}\mathfrak{J}(x)dx = \left[(R)\int_{\sigma}^{\mathfrak{i}}\mathfrak{J}_*(x)dx,\ (R)\int_{\sigma}^{\mathfrak{i}}\mathfrak{J}^*(x)dx\right]. \tag{10}$$

The family of all IR-integrable of IVMs over coordinates and R-integrable functions over $[\sigma, \mathfrak{i}]$ are denoted by $IR_{[\sigma,\mathfrak{i}]}$ and $R_{[\sigma,\mathfrak{i}]}$, respectively.

Theorem 3 ([31]). *Let $\mathfrak{J} : \Omega = [\sigma, \mathfrak{i}] \times [\varepsilon, \mathfrak{v}] \subset \mathbb{R}^2 \to \mathbb{R}_I$ be an IVM on coordinates, given by $\mathfrak{J}(x, y) = [\mathfrak{J}_*(x, y), \mathfrak{J}^*(x, y)]$ for all $(x, y) \in \Omega = [\sigma, \mathfrak{i}] \times [\varepsilon, \mathfrak{v}]$. Then, $\mathfrak{J}$ is double integrable (ID-integrable) over Ω if and only if $\mathfrak{J}_*(x, y)$ and $\mathfrak{J}^*(x, y)$ both are D-integrable over Ω. Moreover, if $\mathfrak{J}$ is FD-integrable over Ω, then*

$$(ID)\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}}\mathfrak{J}(x, y)dydx = (IR)\int_{\sigma}^{\mathfrak{i}}(IR)\int_{\varepsilon}^{\mathfrak{v}}\mathfrak{J}(x, y)dydx. \tag{11}$$

The family of all ID-integrable of IVMs over coordinates over coordinates is denoted by $\mathfrak{TD}_{\Omega}$.

Here is the main definition of fuzzy Riemann–Liouville fractional integral on the coordinates of the function $\mathfrak{J}(x, y)$ by:

Definition 2 ([41]). *Let $\mathfrak{J} : \Omega \to \mathbb{R}_I$ and $\mathfrak{J} \in \mathfrak{TD}_{\Omega}$. The double fuzzy interval Rieman—Liouville-type integrals $\mathcal{I}^{\alpha,\beta}_{\sigma^+,\varepsilon^+}$, $\mathcal{I}^{\alpha,\beta}_{\sigma^+,\mathfrak{v}^-}$, $\mathcal{I}^{\alpha,\beta}_{\mathfrak{i}^-,\varepsilon^+}$, $\mathcal{I}^{\alpha,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-}$ of $\mathfrak{J}$ order $\alpha, \beta > 0$ are defined by:*

$$\mathcal{I}^{\alpha,\beta}_{\sigma^+,\varepsilon^+}\mathfrak{J}(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\sigma}^{x}\int_{\varepsilon}^{y}(x - \mathtt{t})^{\alpha-1}(y - \mathtt{s})^{\beta-1}\mathfrak{J}(\mathtt{t}, \mathtt{s})d\mathtt{s}d\mathtt{t},\ \ (x > \sigma, y > \varepsilon), \tag{12}$$

$$\mathcal{I}^{\alpha,\beta}_{\sigma^+,\mathfrak{v}^-}\mathfrak{J}(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\sigma}^{x}\int_{y}^{\mathfrak{v}}(x - \mathtt{t})^{\alpha-1}(\mathtt{s} - y)^{\beta-1}\mathfrak{J}(\mathtt{t}, \mathtt{s})d\mathtt{s}d\mathtt{t},\ \ (x > \sigma, y < \mathfrak{v}), \tag{13}$$

$$\mathcal{I}^{\alpha,\beta}_{\mathfrak{i}^-,\varepsilon^+}\mathfrak{J}(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\mathfrak{i}}\int_{\varepsilon}^{y}(\mathtt{t} - x)^{\alpha-1}(y - \mathtt{s})^{\beta-1}\mathfrak{J}(\mathtt{t}, \mathtt{s})d\mathtt{s}d\mathtt{t},\ \ (x < \mathfrak{i}, y > \varepsilon), \tag{14}$$

$$\mathcal{J}_{\mathfrak{i}^-,\mathfrak{v}^-}^{\alpha,\,\beta}\mathcal{J}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^{\mathfrak{i}}\int_y^{\mathfrak{v}}(\mathtt{t}-x)^{\alpha-1}(\mathtt{s}-y)^{\beta-1}\mathcal{J}(\mathtt{t},\mathtt{s})d\mathtt{s}d\mathtt{t},\ \ (x<\mathfrak{i},y<\mathfrak{v}). \tag{15}$$

Here is the classical and newly defined concept of coordinated LR-$\hbar$-convexity over fuzzy number space in the codomain via fuzzy relation given by:

Definition 3 ([45]). *The IVM $\mathcal{J}:[\varepsilon,\ \mathfrak{v}]\to\mathbb{R}_I^+$ is referred to be LR-$\hbar$-convex IVM on $[\varepsilon,\ \mathfrak{v}]$ if*

$$\mathcal{J}(\kappa\varepsilon+(1-\kappa)\mathfrak{v})\leq_p\hbar(\kappa)\mathcal{J}(\varepsilon)+\hbar(1-\kappa)\mathcal{J}(\mathfrak{v}), \tag{16}$$

where $\hbar:[0,\ 1]\to\mathbb{R}^+$. If inequality (16) is reversed, then $\mathcal{J}$ is referred to be $\hbar$-concave IVM on $[\varepsilon,\ \mathfrak{v}]$.

Theorem 4 ([45]). *Let $\hbar:[0,\ 1]\to\mathbb{R}^+$ and $\mathcal{J}:[\varepsilon,\ \mathfrak{v}]\to\mathbb{R}_I^+$ be a LR-$\hbar$-convex IVM on $[\varepsilon,\ \mathfrak{v}]$, given by $\mathcal{J}(y)=[\mathcal{J}_*(y),\ \mathcal{J}^*(y)]$ for all $y\in[\varepsilon,\ \mathfrak{v}]$. If $\mathcal{J}\in L([\varepsilon,\ \mathfrak{v}],\mathbb{R}_I^+)$, then*

$$\frac{1}{\alpha\hbar\left(\frac{1}{2}\right)}\mathcal{J}\left(\frac{\varepsilon+\mathfrak{v}}{2}\right)\leq_p\frac{\Gamma(\alpha)}{(\mathfrak{v}-\varepsilon)^{\alpha}}\left[\mathcal{J}_{\varepsilon^+}^{\alpha}\mathcal{J}(\mathfrak{v})+\mathcal{J}_{\mathfrak{v}^-}^{\alpha}\mathcal{J}(\varepsilon)\right]\leq_p[\mathcal{J}(\varepsilon)+\mathcal{J}(\mathfrak{v})]\int_0^1 \upsilon^{\alpha-1}[\hbar(\upsilon)+\hbar(1-\upsilon)]d\upsilon. \tag{17}$$

Definition 4. *The IVM $\mathcal{J}:\Omega\to\mathbb{R}_I^+$ is referred to be coordinated LR-$\hbar$-convex IVM on Ω if*

$$\begin{aligned}&\mathcal{J}(\upsilon\sigma+(1-\upsilon)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\&\leq_p\hbar(\upsilon)\hbar(\kappa)\mathcal{J}(\sigma,\varepsilon)+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{J}(\sigma,\mathfrak{v})+\hbar(1-\upsilon)\hbar(\kappa)\mathcal{J}(\mathfrak{i},\varepsilon)+\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{J}(\mathfrak{i},\mathfrak{v})\end{aligned} \tag{18}$$

for all $(\sigma,\mathfrak{i}),\ (\varepsilon,\mathfrak{v})\in\Omega$, and $\upsilon,\kappa\in[0,\ 1]$,where $\mathcal{J}(x)\geq_p 0$. If inequality (18) is reversed, then $\mathcal{J}$ is referred to be coordinate $\hbar$-concave IVM on Ω. If $\hbar$ is the identity function, we recover the LR-convex IVM on Ω given in [46].

Lemma 1. *Let $\mathcal{J}:\Omega\to\mathbb{R}_I^+$ be a coordinated IVM on Ω. Then, $\mathcal{J}$ is coordinated LR-$\hbar$-convex IVM on Ω if and only if there exist two coordinated LR-$\hbar$-convex IVMs $\mathcal{J}_x:[\varepsilon,\mathfrak{v}]\to\mathbb{R}_I^+$, $\mathcal{J}_x(w)=\mathcal{J}(x,w)$ and $\mathcal{J}_y:[\sigma,\mathfrak{i}]\to\mathbb{R}_I^+$, $\mathcal{J}_y(z)=\mathcal{J}(z,y)$.*

Theorem 5. *Let $\mathcal{J}:\Omega\to\mathbb{R}_I^+$ be a IVM on Ω, given by*

$$\mathcal{J}(x,y)=[\mathcal{J}_*(x,y),\ \mathcal{J}^*(x,y)], \tag{19}$$

for all $(x,y)\in\Omega$. Then, $\mathcal{J}$ is coordinated LR-$\hbar$-convex IVM on Ω, if and only if both $\mathcal{J}_*(x,y)$ and $\mathcal{J}^*(x,y)$ are coordinated LR-$\hbar$-convex.

Proof. Assume that $\mathcal{J}_*(x)$ and $\mathcal{J}^*(x)$ are coordinated LR-$\hbar$-convex and $\hbar$-concave on Ω, respectively. Then, from Equation (19), for all $(\sigma,\mathfrak{i}),\ (\varepsilon,\mathfrak{v})\in\Omega,\ \upsilon$ and $\kappa\in[0,\ 1]$, we have

$$\begin{aligned}&\mathcal{J}_*(\upsilon\sigma+(1-\upsilon)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\&\leq\hbar(\upsilon)\hbar(\kappa)\mathcal{J}_*(\sigma,\varepsilon)+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{J}_*(\sigma,\mathfrak{v})+\hbar(\kappa)\hbar(1-\upsilon)\mathcal{J}_*(\mathfrak{i},\varepsilon)+\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{J}_*(\mathfrak{i},\mathfrak{v})\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\mathcal{J}^*(\upsilon\sigma+(1-\upsilon)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\&\leq\hbar(\upsilon)\hbar(\kappa)\mathcal{J}^*(\sigma,\varepsilon)+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{J}^*(\sigma,\mathfrak{v})+\hbar(\kappa)\hbar(1-\upsilon)\mathcal{J}^*(\mathfrak{i},\varepsilon)+\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{J}^*(\mathfrak{i},\mathfrak{v}).\end{aligned} \tag{21}$$

Then, by Equations (19), (3) and (4), we obtain

$$\begin{aligned}\mathcal{J}(\upsilon\sigma\ &+(1-\upsilon)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\&=[\mathcal{J}_*(\upsilon\sigma+(1-\upsilon)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}),\ \mathcal{J}^*(\upsilon\sigma+(1-\upsilon)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})]\\&\leq_p\hbar(\upsilon)\hbar(\kappa)[\mathcal{J}_*(\sigma,\varepsilon),\ \mathcal{J}^*(\sigma,\varepsilon)]+\hbar(\upsilon)\hbar(1-\kappa)[\mathcal{J}_*(\sigma,\mathfrak{v}),\ \mathcal{J}^*(\sigma,\mathfrak{v})]\\&+\hbar(\kappa)\hbar(1-\upsilon)[\mathcal{J}_*(\mathfrak{i},\varepsilon),\ \mathcal{J}^*(\mathfrak{i},\varepsilon)]+\hbar(1-\upsilon)\hbar(1-\kappa)[\mathcal{J}_*(\mathfrak{i},\mathfrak{v}),\ \mathcal{J}^*(\mathfrak{i},\mathfrak{v})]\end{aligned}$$

From (20) and (21), we have

$$\begin{aligned}
&Ⴇ(v\sigma \quad +(1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\,\mathfrak{v}) \\
&\quad \leq_p \hbar(v)\hbar(\kappa)Ⴇ(\sigma,\varepsilon) + \hbar(v)\hbar(1-\kappa)Ⴇ(\sigma,\mathfrak{v}) + \hbar(1-v)\hbar(1-\kappa)Ⴇ(\mathfrak{i},\varepsilon) + \hbar(1-v)\hbar(1-\kappa)Ⴇ(\mathfrak{i},\mathfrak{v}),
\end{aligned}$$

hence, $Ⴇ$ is coordinated LR-$\hbar$-convex IVM on Ω.

Conversely, let $Ⴇ$ be coordinated LR-$\hbar$-convex IVM on Ω. Then, for all $(\sigma, \mathfrak{i})$, $(\varepsilon, \mathfrak{v}) \in \Omega$, v and $\kappa \in [0,\ 1]$, we have

$$\begin{aligned}
&Ⴇ(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&\leq_p \hbar(v)\hbar(\kappa)Ⴇ(\sigma,\varepsilon) + \hbar(v)\hbar(1-\kappa)Ⴇ(\sigma,\mathfrak{v}) + \hbar(1-v)\hbar(\kappa)Ⴇ(\mathfrak{i},\varepsilon) + \hbar(1-v)\hbar(1-\kappa)Ⴇ(\mathfrak{i},\mathfrak{v}).
\end{aligned}$$

Therefore, again from Equation (20), we have

$$\begin{aligned}
&Ⴇ((v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v})) \\
&= \left[Ⴇ_*(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}),\ Ⴇ^*(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v})\right].
\end{aligned} \tag{22}$$

Again, Equations (3) and (4), we obtain

$$\begin{aligned}
&\hbar(v)\hbar(\kappa)Ⴇ(\sigma,\varepsilon) + \hbar(v)\hbar(1-\kappa)Ⴇ(\sigma,\mathfrak{v}) + \hbar(1-v)\hbar(\kappa)Ⴇ(\mathfrak{i},\varepsilon) + \hbar(1-v)\hbar(1-\kappa)Ⴇ(\mathfrak{i},\mathfrak{v}) \\
&= \hbar(v)\hbar(\kappa)\left[Ⴇ_*(\sigma,\varepsilon),\ Ⴇ^*(\sigma,\varepsilon)\right] + \hbar(v)\hbar(1-\kappa)\left[Ⴇ_*(\sigma,\mathfrak{v}),\ Ⴇ^*(\sigma,\mathfrak{v})\right] \\
&+\hbar(\kappa)\hbar(1-v)\left[Ⴇ_*(\sigma,\varepsilon),\ Ⴇ^*(\sigma,\varepsilon)\right] + \hbar(1-v)\hbar(1-\kappa)\left[Ⴇ_*(\sigma,\mathfrak{v}),\ Ⴇ^*(\sigma,\mathfrak{v})\right],
\end{aligned} \tag{23}$$

for all $x, \omega \in \Omega$ and $v \in [0,\ 1]$. Then, from (22) and (23), we have for all $x, \omega \in \Omega$ and $v \in [0,\ 1]$, such that

$$\begin{aligned}
&Ⴇ_*(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&\leq \hbar(v)\hbar(\kappa)Ⴇ_*(\sigma,\varepsilon) + \hbar(v)\hbar(1-\kappa)Ⴇ_*(\sigma,\mathfrak{v}) + \hbar(1-v)\hbar(\kappa)Ⴇ_*(\mathfrak{i},\varepsilon) + \hbar(1-v)\hbar(1- \\
&\kappa)Ⴇ_*(\mathfrak{i},\mathfrak{v}),
\end{aligned}$$

and

$$\begin{aligned}
&Ⴇ^*(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&\leq \hbar(v)\hbar(\kappa)Ⴇ^*(\sigma,\varepsilon) + \hbar(v)\hbar(1-\kappa)Ⴇ^*(\sigma,\mathfrak{v}) + \hbar(1-v)\hbar(\kappa)Ⴇ^*(\mathfrak{i},\varepsilon) + \\
&\hbar(1-v)\hbar(1-\kappa)Ⴇ^*(\mathfrak{i},\mathfrak{v}),
\end{aligned}$$

hence, the result follows. □

Example 1. *We consider the* IVM $Ⴇ : [0,\ 1] \times [0,\ 1] \to \mathbb{R}_I^+$ *defined by,*

$$Ⴇ(x) = [xy, (6+e^x)(6+e^y)].$$

Endpoint functions $Ⴇ_*(x,y)$, $Ⴇ^*(x,y)$ *are coordinate ħ-concave functions. Hence,* $Ⴇ(x,y)$ *is coordinated LR-ħ-convex IVM.*

From Lemma 1 and Example 1, we can easily note that each LR-$\hbar$-convex IVM is coordinated LR-$\hbar$-convex IVM. But the inverse is not true.

Remark 2. *If one assumes that* $Ⴇ_*(x,y) = Ⴇ^*(x,y)$*, then* $Ⴇ$ *is referred to as a classical coordinated LR-ħ-convex function if* $Ⴇ$ *meets the stated inequality here:*

$$\begin{aligned}
&Ⴇ(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&\leq \hbar(v)\hbar(\kappa)Ⴇ(\sigma,\varepsilon) + \hbar(v)\hbar(1-\kappa)Ⴇ(\sigma,\mathfrak{v}) + \hbar(\kappa)\hbar(1-v)Ⴇ(\sigma,\varepsilon) + \hbar(1-v)\hbar(1-\kappa)Ⴇ(\sigma,\mathfrak{v}).
\end{aligned} \tag{24}$$

If one assumes that $\hbar(v) = v$, $\hbar(\kappa) = \kappa$ *and* $Ⴇ_*(x,y) = Ⴇ^*(x,y)$*, then* $Ⴇ$ *is referred to as a classical coordinated convex function if* $Ⴇ$ *meets the stated inequality here:*

$$\begin{aligned}
&Ⴇ(v\sigma + (1-v)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&\leq v\kappa Ⴇ(\sigma,\varepsilon) + v(1-\kappa)Ⴇ(\sigma,\mathfrak{v}) + (1-v)\kappa Ⴇ(\mathfrak{i},\varepsilon) + (1-v)(1-\kappa)Ⴇ(\mathfrak{i},\mathfrak{v}).
\end{aligned} \tag{25}$$

Let one assume that $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and $\mathfrak{J}_*(x,y) \neq \mathfrak{J}^*(x,y)$, and $\mathfrak{J}_*(x,y)$ is an affine function and $\mathfrak{J}^*(x,y)$ is a concave function for the stated inequality here (see [42])

$$\begin{aligned} &\mathfrak{J}(\upsilon\sigma + (1-\upsilon)\mathfrak{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\ &\supseteq \upsilon\kappa\mathfrak{J}(\sigma,\varepsilon) + \upsilon(1-\kappa)\mathfrak{J}(\sigma,\mathfrak{v}) + (1-\upsilon)\kappa\mathfrak{J}(\mathfrak{i},\varepsilon) + (1-\upsilon)(1-\kappa)\mathfrak{J}(\mathfrak{i},\mathfrak{v}), \end{aligned} \tag{26}$$

is true.

Definition 5. *Let $\mathfrak{J} : \Omega \to \mathbb{R}_I^+$ be a IVM on Ω. Then, we have*

$$\mathfrak{J}(x,y) = \left[\mathfrak{J}_*(x,y),\ \mathfrak{J}^*(x,y)\right],$$

for all $(x,y) \in \Omega$. Then, $\mathfrak{J}$ is coordinated left-LR-$\hbar$-convex (concave) IVM on Ω, if and only if, $\mathfrak{J}_*(x,y)$ and $\mathfrak{J}^*(x,y)$ are coordinated LR-$\hbar$-convex (concave) and affine functions on Ω, respectively.

Definition 6. *Let $\mathfrak{J} : \Omega \to \mathbb{R}_I^+$ be a IVM on Ω. Then, we have*

$$\mathfrak{J}(x,y) = \left[\mathfrak{J}_*(x,y),\ \mathfrak{J}^*(x,y)\right],$$

for all $(x,y) \in \Omega$. Then, $\mathfrak{J}$ is coordinated right-LR-$\hbar$-convex (concave) IVM on Ω, if and only if $\mathfrak{J}_*(x,y)$ and $\mathfrak{J}^*(x,y)$ are coordinated $\hbar$-affine and $\hbar$-(concave) functions on Ω, respectively.

Theorem 6. *Let Ω be a coordinated convex set, and let $\mathfrak{J} : \Omega \to \mathbb{R}_I^+$ be a IVM, defined by*

$$\mathfrak{J}(x,y) = \left[\mathfrak{J}_*(x,y),\ \mathfrak{J}^*(x,y)\right],$$

for all $(x,y) \in \Omega$. Then, $\mathfrak{J}$ is coordinated $\hbar$-concave IVM on Ω, if and only if $\mathfrak{J}_*(x,y)$ and $\mathfrak{J}^*(x,y)$ are coordinated $\hbar$-concave and LR-$\hbar$-convex functions, respectively.

Proof. The demonstration of proof of Theorem 6 is similar to the demonstration proof of Theorem 5. □

Example 2. *We consider the $IVMs$ $\mathfrak{J} : [0,\ 1] \times [0,\ 1] \to \mathbb{R}_I^+$ defined by,*

$$\mathfrak{J}(x,y) = [(6 - e^x)(6 - e^y), 40xy].$$

Then, we have endpoint functions $\mathfrak{J}_*(x,y)$, $\mathfrak{J}^*(x,y)$, which are both coordinated $\hbar$-concave functions. Hence, $\mathfrak{J}(x,y)$ is coordinated LR-$\hbar$-concave IVM.

In the next results, to avoid confusion, we will not include the symbols (R), (IR), and (ID) before the integral sign.

3. Main Results

In this section, Hermite–Hadamard and Pachpatte-type inequalities for interval-value functions are given. We first present an inequality of Hermite–Hadamard via coordinated LR-$\hbar$-concave IVMs.

Theorem 7. *Let $\mathfrak{J} : \Omega \to \mathbb{R}_I^+$ be a coordinate LR-$\hbar$-convex IVM on Ω, where $\mathfrak{J}(x,y) = \left[\mathfrak{J}_*(x,y),\ \mathfrak{J}^*(x,y)\right]$ for all $(x,y) \in \Omega$ and let $\hbar : [0,\ 1] \to \mathbb{R}^+$. If $\mathfrak{J} \in \mathfrak{TD}_\Omega$, then the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{\hbar^2\left(\frac{1}{2}\right)} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\
&\leq_p \frac{\Gamma(\alpha+1)}{2\hbar\left(\frac{1}{2}\right)(\mathrm{i}-\sigma)^{\alpha}}\left[\mathcal{J}_{\sigma^+}^{\alpha} Ꟙ\left(\mathrm{i}, \frac{\varepsilon+\mathfrak{v}}{2}\right)+\mathcal{J}_{\mathrm{i}^-}^{\alpha} Ꟙ\left(\sigma, \frac{\varepsilon+\mathfrak{v}}{2}\right)\right]+\frac{\Gamma(\beta+1)}{2\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}_{\varepsilon^+}^{\beta} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \mathfrak{v}\right)+\mathcal{J}_{\mathfrak{v}^-}^{\beta} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \varepsilon\right)\right] \\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}_{\sigma^+,\varepsilon^+}^{\alpha,\beta} Ꟙ(\mathrm{i}, \mathfrak{v})+\mathcal{J}_{\sigma^+,\mathfrak{v}^-}^{\alpha,\beta} Ꟙ(\mathrm{i}, \varepsilon)+\mathcal{J}_{\mathrm{i}^-,\varepsilon^+}^{\alpha,\beta} Ꟙ(\sigma, \mathfrak{v})+\mathcal{J}_{\mathrm{i}^-,\mathfrak{v}^-}^{\alpha,\beta} Ꟙ(\sigma, \varepsilon)\right] \\
&\leq_p \frac{\beta\Gamma(\alpha+1)}{(\mathrm{i}-\sigma)^{\alpha}}\left[\mathcal{J}_{\sigma^+}^{\alpha} Ꟙ(\mathrm{i}, \varepsilon)+\mathcal{J}_{\sigma^+}^{\alpha} Ꟙ(\mathrm{i}, \mathfrak{v})+\mathcal{J}_{\mathrm{i}^-}^{\alpha} Ꟙ(\sigma, \varepsilon)+\mathcal{J}_{\mathrm{i}^-}^{\alpha} Ꟙ(\sigma, \mathfrak{v})\right] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)] d\kappa \\
&+\frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}_{\varepsilon^+}^{\beta} Ꟙ(\sigma, \mathfrak{v})+\mathcal{J}_{\mathfrak{v}^-}^{\beta} Ꟙ(\mathrm{i}, \varepsilon)+\mathcal{J}_{\varepsilon^+}^{\beta} Ꟙ(\mathrm{i}, \mathfrak{v})+\mathcal{J}_{\mathfrak{v}^-}^{\beta} Ꟙ(\mathrm{i}, \varepsilon)\right] \times \int_0^1 v^{\alpha-1} \hbar(v)+\hbar(1-v) dv \\
&\leq_p \alpha\beta[Ꟙ(\sigma, \varepsilon)+Ꟙ(\mathrm{i}, \varepsilon)+Ꟙ(\sigma, \mathfrak{v})+Ꟙ(\mathrm{i}, \mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)] d\kappa \int_0^1 v^{\alpha-1}[\hbar(v)+\hbar(1-v)] dv.
\end{aligned} \tag{27}$$

If $Ꟙ(x,y)$ coordinated $\hbar$-concave IVM, then,

$$\begin{aligned}
&\frac{1}{\hbar^2\left(\frac{1}{2}\right)} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\
&\geq_p \frac{\Gamma(\alpha+1)}{2\hbar\left(\frac{1}{2}\right)(\mathrm{i}-\sigma)^{\alpha}}\left[\mathcal{J}_{\sigma^+}^{\alpha} Ꟙ\left(\mathrm{i}, \frac{\varepsilon+\mathfrak{v}}{2}\right)+\mathcal{J}_{\mathrm{i}^-}^{\alpha} Ꟙ\left(\sigma, \frac{\varepsilon+\mathfrak{v}}{2}\right)\right]+\frac{\Gamma(\beta+1)}{2\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}_{\varepsilon^+}^{\beta} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \mathfrak{v}\right)+\mathcal{J}_{\mathfrak{v}^-}^{\beta} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \varepsilon\right)\right] \\
&\geq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}_{\sigma^+,\varepsilon^+}^{\alpha,\beta} Ꟙ(\mathrm{i}, \mathfrak{v})+\mathcal{J}_{\sigma^+,\mathfrak{v}^-}^{\alpha,\beta} Ꟙ(\mathrm{i}, \varepsilon)+\mathcal{J}_{\mathrm{i}^-,\varepsilon^+}^{\alpha,\beta} Ꟙ(\sigma, \mathfrak{v})+\mathcal{J}_{\mathrm{i}^-,\mathfrak{v}^-}^{\alpha,\beta} Ꟙ(\sigma, \varepsilon)\right] \\
&\geq_p \frac{\beta\Gamma(\alpha+1)}{(\mathrm{i}-\sigma)^{\alpha}}\left[\mathcal{J}_{\sigma^+}^{\alpha} Ꟙ(\mathrm{i}, \varepsilon)+\mathcal{J}_{\sigma^+}^{\alpha} Ꟙ(\mathrm{i}, \mathfrak{v})+\mathcal{J}_{\mathrm{i}^-}^{\alpha} Ꟙ(\sigma, \varepsilon)+\mathcal{J}_{\mathrm{i}^-}^{\alpha} Ꟙ(\sigma, \mathfrak{v})\right] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)] d\kappa \\
&+\frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}_{\varepsilon^+}^{\beta} Ꟙ(\sigma, \mathfrak{v})+\mathcal{J}_{\mathfrak{v}^-}^{\beta} Ꟙ(\mathrm{i}, \varepsilon)+\mathcal{J}_{\varepsilon^+}^{\beta} Ꟙ(\mathrm{i}, \mathfrak{v})+\mathcal{J}_{\mathfrak{v}^-}^{\beta} Ꟙ(\mathrm{i}, \varepsilon)\right] \times \int_0^1 v^{\alpha-1}[\hbar(v)+\hbar(1-v)] dv \\
&\geq_p \alpha\beta[Ꟙ(\sigma, \varepsilon)+Ꟙ(\mathrm{i}, \varepsilon)+Ꟙ(\sigma, \mathfrak{v})+Ꟙ(\mathrm{i}, \mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)] d\kappa \int_0^1 v^{\alpha-1}[\hbar(v)+\hbar(1-v)] dv.
\end{aligned} \tag{28}$$

Proof. Let $Ꟙ : [\sigma, \mathrm{i}] \to \mathbb{R}_I^+$ be a coordinated LR-$\hbar$-convex IVM. Then, by hypothesis, we have

$$\frac{1}{\hbar^2\left(\frac{1}{2}\right)} Ꟙ\left(\frac{\sigma+\mathrm{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \leq_p Ꟙ(v\sigma+(1-v)\mathrm{i}, v\varepsilon+(1-v)\mathfrak{v})+Ꟙ((1-v)\sigma+v\mathrm{i}, (1-v)\varepsilon+v\mathfrak{v})$$

By using Theorem 5, we have

$$\begin{aligned}
&\frac{1}{\hbar^2\left(\frac{1}{2}\right)} Ꟙ_*\left(\frac{\sigma+\mathrm{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\
&\leq Ꟙ_*(v\sigma+(1-v)\mathrm{i}, v\varepsilon+(1-v)\mathfrak{v})+Ꟙ_*((1-v)\sigma+v\mathrm{i}, (1-v)\varepsilon+v\mathfrak{v}), \\
&\frac{1}{\hbar^2\left(\frac{1}{2}\right)} Ꟙ^*\left(\frac{\sigma+\mathrm{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\
&\leq Ꟙ^*(v\sigma+(1-v)\mathrm{i}, v\varepsilon+(1-v)\mathfrak{v})+Ꟙ^*((1-v)\sigma+v\mathrm{i}, (1-v)\varepsilon+v\mathfrak{v}).
\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}
\frac{1}{\hbar\left(\frac{1}{2}\right)} Ꟙ_*\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right) &\leq Ꟙ_*(x, v\varepsilon+(1-v)\mathfrak{v})+Ꟙ_*(x, (1-v)\varepsilon+v\mathfrak{v}), \\
\frac{1}{\hbar\left(\frac{1}{2}\right)} Ꟙ^*\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right) &\leq Ꟙ^*(x, v\varepsilon+(1-v)\mathfrak{v})+Ꟙ^*(x, (1-v)\varepsilon+v\mathfrak{v}),
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\frac{1}{\hbar\left(\frac{1}{2}\right)} Ꟙ_*\left(\frac{\sigma+\mathrm{i}}{2}, y\right) &\leq Ꟙ_*(v\sigma+(1-v)\mathrm{i}, y)+Ꟙ_*((1-v)\sigma+v\mathrm{i}, y), \\
\frac{1}{\hbar\left(\frac{1}{2}\right)} Ꟙ^*\left(\frac{\sigma+\mathrm{i}}{2}, y\right) &\leq Ꟙ^*(v\sigma+(1-v)\mathrm{i}, y)+Ꟙ^*((1-v)\sigma+v\mathrm{i}, y).
\end{aligned} \tag{30}$$

From (29) and (30), we have

$$\begin{aligned}
&\frac{1}{\hbar\left(\frac{1}{2}\right)}\left[Ꟙ_*\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right), Ꟙ^*\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right)\right] \\
&\leq_p \left[Ꟙ_*(x, v\varepsilon+(1-v)\mathfrak{v}), Ꟙ^*(x, v\varepsilon+(1-v)\mathfrak{v})\right] \\
&+\left[Ꟙ_*(x, (1-v)\varepsilon+v\mathfrak{v}), Ꟙ^*(x, (1-v)\varepsilon+v\mathfrak{v})\right],
\end{aligned}$$

and

$$\begin{aligned}&\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\left[\text{Ʝ}_*\left(\tfrac{\sigma+\mathfrak{i}}{2},y\right),\text{Ʝ}^*\left(\tfrac{\sigma+\mathfrak{i}}{2},y\right)\right]\\ \leq_p\ &[\text{Ʝ}_*(v\sigma+(1-v)\mathfrak{i},y),\text{Ʝ}^*(v\sigma+(1-v)\mathfrak{i},y)]\\ &+\left[\text{Ʝ}_*(v\sigma+(1-v)\mathfrak{i},y),\text{Ʝ}^*(v\sigma+(1-v)\mathfrak{i},y)\right],\end{aligned}$$

It follows that

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\text{Ʝ}\left(x,\frac{\varepsilon+\mathfrak{v}}{2}\right)\leq_p \text{Ʝ}(x,\ v\varepsilon+(1-v)\mathfrak{v})+\text{Ʝ}(x,\ (1-v)\varepsilon+v\mathfrak{v}), \tag{31}$$

and

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\text{Ʝ}\left(\frac{\sigma+\mathfrak{i}}{2},y\right)\leq_p \text{Ʝ}(v\sigma+(1-v)\mathfrak{i},y)+\text{Ʝ}(v\sigma+(1-v)\mathfrak{i},y). \tag{32}$$

Since $\text{Ʝ}(x,.)$ and $\text{Ʝ}(.,y)$ are both coordinated LR-$\hbar$-convex-IVMs; then, from (17), (31), and (32), we have

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\text{Ʝ}_x\left(\frac{\varepsilon+\mathfrak{v}}{2}\right)\leq_p \frac{\Gamma(\beta)}{(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^\beta_{\varepsilon^+}\text{Ʝ}_x(\mathfrak{v})+\mathcal{J}^\beta_{\mathfrak{v}^-}\ \text{Ʝ}_x(\varepsilon)\right]\leq_p \left[\text{Ʝ}_x(\varepsilon)+\text{Ʝ}_x(\mathfrak{v})\right]\int_0^1\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa \tag{33}$$

and

$$\frac{1}{\alpha\hbar\left(\frac{1}{2}\right)}\text{Ʝ}_y\left(\frac{\sigma+\mathfrak{i}}{2}\right)\leq_p \frac{\Gamma(\alpha)}{(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^\alpha_{\sigma^+}\text{Ʝ}_y(\mathfrak{i})+\mathcal{J}^\alpha_{\mathfrak{i}^-}\text{Ʝ}_y(\sigma)\right]\leq_p \left[\text{Ʝ}_y(\sigma)+\text{Ʝ}_y(\mathfrak{i})\right]\int_0^1 v^{\alpha-1}\hbar(v)+\hbar(1-v)dv \tag{34}$$

Since $\text{Ʝ}_x(w)=\text{Ʝ}(x,w)$, then (34) can be written as

$$\frac{1}{\beta\hbar\left(\frac{1}{2}\right)}\text{Ʝ}\left(x,\frac{\varepsilon+\mathfrak{v}}{2}\right)\leq_p \frac{\Gamma(\beta)}{(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^\alpha_{\varepsilon^+}\text{Ʝ}(x,\mathfrak{v})+\mathcal{J}^\alpha_{\mathfrak{v}^-}\text{Ʝ}(x,\varepsilon)\right]\leq_p [\text{Ʝ}(x,\varepsilon)+\text{Ʝ}(x,\mathfrak{v})]\int_0^1\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa. \tag{35}$$

That is,

$$\begin{aligned}&\tfrac{1}{\beta\hbar\left(\frac{1}{2}\right)}\text{Ʝ}\left(x,\tfrac{\varepsilon+\mathfrak{v}}{2}\right)\leq_p \tfrac{1}{(\mathfrak{v}-\varepsilon)^\beta}\left[\int_\varepsilon^{\mathfrak{v}}(\mathfrak{v}-\kappa)^{\beta-1}\text{Ʝ}(x,\kappa)d\kappa+\int_\varepsilon^{\mathfrak{v}}(\kappa-\varepsilon)^{\beta-1}\text{Ʝ}(x,\kappa)d\kappa\right]\\ &\leq_p [\text{Ʝ}(x,\varepsilon)+\text{Ʝ}(x,\mathfrak{v})]\textstyle\int_0^1\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned} \tag{36}$$

Multiplying double inequality (36) by $\frac{(\mathfrak{i}-x)^{\alpha-1}}{(\mathfrak{i}-\sigma)^\alpha}$ and integrating with respect to x over $[\sigma,\mathfrak{i}]$, we have

$$\begin{aligned}&\tfrac{1}{\beta(\mathfrak{i}-\sigma)^\alpha\hbar\left(\frac{1}{2}\right)}\int_\sigma^{\mathfrak{i}}\text{Ʝ}\left(x,\tfrac{\varepsilon+\mathfrak{v}}{2}\right)(\mathfrak{i}-x)^{\alpha-1}dx\\ &\leq_p \tfrac{1}{(\mathfrak{i}-\sigma)^\alpha(\mathfrak{v}-\varepsilon)^\beta}\int_\sigma^{\mathfrak{i}}\int_\varepsilon^{\mathfrak{v}}(\mathfrak{i}-x)^{\alpha-1}(\mathfrak{v}-\kappa)^{\beta-1}\text{Ʝ}(x,\kappa)d\kappa dx+\int_\sigma^{\mathfrak{i}}\int_\varepsilon^{\mathfrak{v}}(\mathfrak{i}-x)^{\alpha-1}(\kappa-\varepsilon)^{\beta-1}\text{Ʝ}(x,\kappa)d\kappa dx\\ &\leq_p \tfrac{1}{(\mathfrak{i}-\sigma)^\alpha}\left[\int_\sigma^{\mathfrak{i}}(\mathfrak{i}-x)^{\alpha-1}\text{Ʝ}(x,\varepsilon)dx+\int_\sigma^{\mathfrak{i}}(\mathfrak{i}-x)^{\alpha-1}\text{Ʝ}(x,\mathfrak{v})dx\right]\int_0^1\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned} \tag{37}$$

Again, multiplying double inequality (36) by $\frac{(x-\sigma)^{\alpha-1}}{(\mathfrak{i}-\sigma)^\alpha}$ and integrating with respect to x over $[\sigma,\mathfrak{i}]$, we have

$$\begin{aligned}&\tfrac{1}{\beta(\mathfrak{i}-\sigma)^\alpha\hbar\left(\frac{1}{2}\right)}\int_\sigma^{\mathfrak{i}}\text{Ʝ}\left(x,\tfrac{\varepsilon+\mathfrak{v}}{2}\right)(x-\sigma)^{\alpha-1}dx\\ &\leq_p \tfrac{1}{(\mathfrak{i}-\sigma)^\alpha(\mathfrak{v}-\varepsilon)^\beta}\int_\sigma^{\mathfrak{i}}\int_\varepsilon^{\mathfrak{v}}(x-\sigma)^{\alpha-1}(\mathfrak{v}-\kappa)^{\beta-1}\text{Ʝ}(x,\kappa)d\kappa dx\\ &+\tfrac{1}{(\mathfrak{i}-\sigma)^\alpha(\mathfrak{v}-\varepsilon)^\beta}\int_\sigma^{\mathfrak{i}}\int_\varepsilon^{\mathfrak{v}}(x-\sigma)^{\alpha-1}(\kappa-\varepsilon)^{\beta-1}\text{Ʝ}(x,\kappa)d\kappa dx\\ &\leq_p \tfrac{1}{(\mathfrak{i}-\sigma)^\alpha}\left[\int_\sigma^{\mathfrak{i}}(x-\sigma)^{\alpha-1}\text{Ʝ}(x,\varepsilon)dx+\int_\sigma^{\mathfrak{i}}(x-\sigma)^{\alpha-1}\text{Ʝ}(x,\mathfrak{v})dx\right]\int_0^1\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned} \tag{38}$$

From (37), we have

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\hbar\left(\frac{1}{2}\right)\left(\mathfrak{i}-\sigma\right)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\sigma^{+}}\mathfrak{J}\left(\mathfrak{i},\frac{\varepsilon+\mathfrak{v}}{2}\right)\right]\\ &\leq_{p}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left(\mathfrak{i}-\sigma\right)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\mathfrak{J}(\mathfrak{i},\mathfrak{v})+\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\varepsilon^{+}}\mathfrak{J}(\mathfrak{i},\varepsilon)\right]\\ &\leq_{p}\frac{\beta\Gamma(\alpha+1)}{\left(\mathfrak{i}-\sigma\right)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\sigma^{+}}\mathfrak{J}(\mathfrak{i},\varepsilon)+\mathcal{J}^{\alpha}_{\sigma^{+}}\mathfrak{J}(\mathfrak{i},\mathfrak{v})\right]\int_{0}^{1}\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned} \tag{39}$$

From (38), we have

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\hbar\left(\frac{1}{2}\right)\left(\mathfrak{i}-\sigma\right)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\mathfrak{i}^{-}}\mathfrak{J}\left(\sigma,\frac{\varepsilon+\mathfrak{v}}{2}\right)\right]\\ &\leq_{p}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left(\mathfrak{i}-\sigma\right)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\varepsilon^{+}}\mathfrak{J}(\sigma,\mathfrak{v})+\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\mathfrak{v}^{-}}\mathfrak{J}(\sigma,\varepsilon)\right]\\ &\leq_{p}\frac{\beta\Gamma(\alpha+1)}{\left(\mathfrak{i}-\sigma\right)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\mathfrak{i}^{-}}\mathfrak{J}(\sigma,\varepsilon)+\mathcal{J}^{\alpha}_{\mathfrak{i}^{-}}\mathfrak{J}(\sigma,\mathfrak{v})\right]\int_{0}^{1}\kappa^{\beta-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned} \tag{40}$$

Similarly, since $\mathfrak{J}_{y}(z)=\mathfrak{J}(z,y)$, then, from (35), (41), and (42), we have

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\varepsilon^{+}}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\mathfrak{v}\right)\right]\\ &\leq_{p}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left(\mathfrak{i}-\sigma\right)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\mathfrak{J}(\mathfrak{i},\mathfrak{v})+\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\varepsilon^{+}}\mathfrak{J}(\sigma,\mathfrak{v})\right]\\ &\leq_{p}\frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\varepsilon^{+}}\mathfrak{J}(\sigma,\mathfrak{v})+\mathcal{J}^{\beta}_{\varepsilon^{+}}\mathfrak{J}(\mathfrak{i},\mathfrak{v})\right].\end{aligned} \tag{41}$$

and

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^{\alpha}}\left[\mathcal{J}^{\beta}_{\mathfrak{v}^{-}}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\varepsilon\right)\right]\\ &\leq_{p}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left(\mathfrak{i}-\sigma\right)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\mathfrak{v}^{-}}\mathfrak{J}(\mathfrak{i},\varepsilon)+\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\mathfrak{v}^{-}}\mathfrak{J}(\sigma,\varepsilon)\right]\\ &\leq_{p}\frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\mathfrak{v}^{-}}\mathfrak{J}(\sigma,\varepsilon)+\mathcal{J}^{\beta}_{\mathfrak{v}^{-}}\mathfrak{J}(\mathfrak{i},\varepsilon)\right].\end{aligned} \tag{42}$$

The second, third, and fourth inequalities of (27) will be the consequence of adding the inequalities (41) and (42).

Now, we have inequality (17)'s left portion.

$$\frac{1}{\hbar^{2}\left(\frac{1}{2}\right)}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right)\leq_{p}\frac{\Gamma(\beta+1)}{\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\varepsilon^{+}}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\mathfrak{v}\right)+\mathcal{J}^{\beta}_{\mathfrak{v}^{-}}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\varepsilon\right)\right] \tag{43}$$

and

$$\frac{1}{\hbar^{2}\left(\frac{1}{2}\right)}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right)\leq_{p}\frac{\Gamma(\alpha+1)}{\hbar\left(\frac{1}{2}\right)(\mathfrak{i}-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\sigma^{+}}\mathfrak{J}\left(\mathfrak{i},\frac{\varepsilon+\mathfrak{v}}{2}\right)+\mathcal{J}^{\alpha}_{\mathfrak{i}^{-}}\mathfrak{J}\left(\sigma,\frac{\varepsilon+\mathfrak{v}}{2}\right)\right] \tag{44}$$

The following inequality is created by adding the two inequalities (43) and (44):

$$\begin{aligned}\frac{1}{\hbar^{2}\left(\frac{1}{2}\right)}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right)&\leq_{p}\frac{\Gamma(\alpha+1)}{\hbar\left(\frac{1}{2}\right)\left(\mathfrak{i}-\sigma\right)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\sigma^{+}}\mathfrak{J}\left(\mathfrak{i},\frac{\varepsilon+\mathfrak{v}}{2}\right)+\mathcal{J}^{\alpha}_{\mathfrak{i}^{-}}\mathfrak{J}\left(\sigma,\frac{\varepsilon+\mathfrak{v}}{2}\right)\right]\\ &+\frac{\Gamma(\beta+1)}{\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\varepsilon^{+}}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\mathfrak{v}\right)+\mathcal{J}^{\beta}_{\mathfrak{v}^{-}}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\varepsilon\right)\right].\end{aligned}$$

Similarly, since we obtain the set of IVMs $\mathfrak{J}:\Omega\to\mathbb{R}_{I}^{+}$, the inequality can be expressed as follows:

$$\frac{1}{\hbar^2\left(\frac{1}{2}\right)} Ɉ\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\ \leq_p \frac{\Gamma(\alpha+1)}{\hbar\left(\frac{1}{2}\right)(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^{\alpha}_{\sigma^+} Ɉ\left(\mathfrak{i}, \frac{\varepsilon+\mathfrak{v}}{2}\right) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ\left(\sigma, \frac{\varepsilon+\mathfrak{v}}{2}\right)\right] + \frac{\Gamma(\beta+1)}{\hbar\left(\frac{1}{2}\right)(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\beta}_{\varepsilon^+} Ɉ\left(\frac{\sigma+\mathfrak{i}}{2}, \mathfrak{v}\right) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ\left(\frac{\sigma+\mathfrak{i}}{2}, \varepsilon\right)\right]. \tag{45}$$

The first inequality of (27) is this one.

Now, we have inequality (17)'s right portion:

$$\frac{\Gamma(\beta)}{(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\beta}_{\varepsilon^+} Ɉ(\sigma, \mathfrak{v}) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ(\sigma, \varepsilon)\right] \leq_p [Ɉ(\sigma, \varepsilon) + Ɉ(\sigma, \mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa) + \hbar(1-\kappa)]d\kappa \tag{46}$$

$$\frac{\Gamma(\beta)}{(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\beta}_{\varepsilon^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ(\mathfrak{i}, \varepsilon)\right] \leq_p [Ɉ(\mathfrak{i}, \varepsilon) + Ɉ(\mathfrak{i}, \mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa) + \hbar(1-\kappa)]d\kappa \tag{47}$$

$$\frac{\Gamma(\alpha)}{(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^{\alpha}_{\sigma^+} Ɉ(\mathfrak{i}, \varepsilon) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ(\sigma, \varepsilon)\right] \leq_p [Ɉ(\sigma, \varepsilon) + Ɉ(\mathfrak{i}, \varepsilon)] \times \int_0^1 v^{\alpha-1}\hbar(v) + \hbar(1-v)dv \tag{48}$$

$$\frac{\Gamma(\alpha)}{(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^{\alpha}_{\sigma^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ(\sigma, \mathfrak{v})\right] \leq_p [Ɉ(\sigma, \mathfrak{v}) + Ɉ(\mathfrak{i}, \mathfrak{v})] \times \int_0^1 v^{\alpha-1}\hbar(v) + \hbar(1-v)dv \tag{49}$$

Summing inequalities (46), (47), (48), and (49), and then taking the multiplication of the resultant with $\alpha\beta$, we have

$$\frac{\beta\Gamma(\alpha+1)}{(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^{\alpha}_{\sigma^+} Ɉ(\mathfrak{i}, \varepsilon) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ(\sigma, \varepsilon) + \mathcal{J}^{\alpha}_{\sigma^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ(\sigma, \mathfrak{v})\right] \\ + \frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\beta}_{\varepsilon^+} Ɉ(\sigma, \mathfrak{v}) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ(\sigma, \varepsilon) + \mathcal{J}^{\beta}_{\varepsilon^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ(\mathfrak{i}, \varepsilon)\right] \\ \leq_p [Ɉ(\sigma, \varepsilon) + Ɉ(\sigma, \mathfrak{v}) + Ɉ(\mathfrak{i}, \varepsilon) + Ɉ(\mathfrak{i}, \mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\hbar(\kappa) + \hbar(1-\kappa)]d\kappa \int_0^1 v^{\alpha-1}\hbar(v) + \hbar(1-v)dv. \tag{50}$$

This is the final inequality of (27) and the conclusion has been established. □

Example 3. *We assume the IVMs* $Ɉ : [0,\ 2] \times [0,\ 2] \to \mathbb{R}_I^+$ *defined by*

$$Ɉ(x, y) = \left[\left(2 - \sqrt{x}\right)\left(2 - \sqrt{y}\right), 2\left(2 - \sqrt{x}\right)\left(2 - \sqrt{y}\right)\right], \tag{51}$$

then, for each $\gamma \in [0,\ 1]$, *we have. Endpoint functions* $Ɉ_*(x, y)$, $Ɉ^*(x, y)$ *are coordinate LR-ħ-convex and ħ-concave functions. Hence,* $\widetilde{Ɉ}(x, y)$ *is ħ-coordinated LR-ħ-convex IVM.*

$$Ɉ\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) = [1, 2],$$
$$\frac{\Gamma(\alpha+1)}{4(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^{\alpha}_{\sigma^+} Ɉ\left(\mathfrak{i}, \frac{\varepsilon+\mathfrak{v}}{2}\right) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ\left(\sigma, \frac{\varepsilon+\mathfrak{v}}{2}\right)\right] + \frac{\Gamma(\beta+1)}{4(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\beta}_{\varepsilon^+} Ɉ\left(\frac{\sigma+\mathfrak{i}}{2}, \mathfrak{v}\right) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ\left(\frac{\sigma+\mathfrak{i}}{2}, \varepsilon\right)\right]$$
$$= 2 - \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{8}\pi \cdot [1, 2]$$
$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^\alpha(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\alpha,\ \beta}_{\sigma^+, \varepsilon^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\alpha,\ \beta}_{\sigma^+, \mathfrak{v}^-} Ɉ(\mathfrak{i}, \varepsilon) + \mathcal{J}^{\alpha,\ \beta}_{\mathfrak{i}^-, \varepsilon^+} Ɉ(\sigma, \mathfrak{v}) + \mathcal{J}^{\alpha,\ \beta}_{\mathfrak{i}^-, \mathfrak{v}^-} Ɉ(\sigma, \varepsilon)\right]$$
$$= \frac{33}{8} - \sqrt{2} - \frac{\sqrt{2}}{2}\pi + \frac{\pi}{8} + \frac{\pi^2}{32} \cdot [1, 2]$$
$$\frac{\Gamma(\alpha+1)}{8(\mathfrak{i}-\sigma)^\alpha}\left[\mathcal{J}^{\alpha}_{\sigma^+} Ɉ(\mathfrak{i}, \varepsilon) + \mathcal{J}^{\alpha}_{\sigma^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ(\sigma, \varepsilon) + \mathcal{J}^{\alpha}_{\mathfrak{i}^-} Ɉ(\sigma, \mathfrak{v})\right]$$
$$+ \frac{\Gamma(\beta+1)}{8(\mathfrak{v}-\varepsilon)^\beta}\left[\mathcal{J}^{\beta}_{\varepsilon^+} Ɉ(\sigma, \mathfrak{v}) + \mathcal{J}^{\beta}_{\varepsilon^+} Ɉ(\mathfrak{i}, \mathfrak{v}) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ(\sigma, \varepsilon) + \mathcal{J}^{\beta}_{\mathfrak{v}^-} Ɉ(\mathfrak{i}, \varepsilon)\right]$$
$$= \frac{34\sqrt{2} + \left(\sqrt{2} - 4\right)\pi - 24}{8\sqrt{2}} \cdot [1, 2]$$
$$\frac{Ɉ(\varepsilon, \mathfrak{i}) + Ɉ(\sigma, \mathfrak{i}) + Ɉ(\varepsilon, \mathfrak{v}) + Ɉ(\sigma, \mathfrak{v})}{4} = \left(\frac{9}{2} - 2\sqrt{2}\right) \cdot [1, 2].$$

That is

$$[1, 2] \leq_p 2 - \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{8}\pi \cdot [1, 2] \leq_p \frac{33}{8} - \sqrt{2} - \frac{\sqrt{2}}{2}\pi + \frac{\pi}{8} + \frac{\pi^2}{32} \cdot [1, 2] \leq_p \frac{34\sqrt{2} + \left(\sqrt{2} - 4\right)\pi - 24}{8\sqrt{2}} \cdot [1, 2] \leq_p \left(\frac{9}{2} - 2\sqrt{2}\right) \\ \cdot [1, 2]$$

Hence, Theorem 6 has been verified.

Remark 3. *If one assumes that $\alpha = 1$ and $\beta = 1$, and $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$, then from (27), as a result, there will be inequality (see [43]):*

$$\begin{aligned}
&\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\\
&\leq_p \frac{1}{2}\left[\frac{1}{\mathfrak{i}-\sigma}\int_{\sigma}^{\mathfrak{i}} \mathcal{J}\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right)dx + \frac{1}{\mathfrak{v}-\varepsilon}\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2}, y\right)dy\right] \leq_p \frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)}\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(x,y)dydx\\
&\leq_p \frac{1}{4(\mathfrak{i}-\sigma)}\left[\int_{\sigma}^{\mathfrak{i}} \mathcal{J}(x,\varepsilon)dx + \int_{\sigma}^{\mathfrak{i}} \mathcal{J}(x,\mathfrak{v})dx\right] + \frac{1}{4(\mathfrak{v}-\varepsilon)}\left[\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(\sigma,y)dy + \int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(\mathfrak{i},y)dy\right]\\
&\leq_p \frac{\mathcal{J}(\sigma,\varepsilon)+\mathcal{J}(\mathfrak{i},\varepsilon)+\mathcal{J}(\sigma,\mathfrak{v})+\mathcal{J}(\mathfrak{i},\mathfrak{v})}{4}.
\end{aligned} \tag{52}$$

If one assumes that $\alpha = 1$ and $\beta = 1$, $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and $\mathcal{J}$ is coordinated left-LR-$\hbar$-convex, then from (27), as a result, there will be inequality (see [42]):

$$\begin{aligned}
&\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\\
&\supseteq \frac{1}{2}\left[\frac{1}{\mathfrak{i}-\sigma}\int_{\sigma}^{\mathfrak{i}} \mathcal{J}\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right)dx + \frac{1}{\mathfrak{v}-\varepsilon}\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2}, y\right)dy\right] \supseteq \frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)}\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(x,y)dydx\\
&\supseteq \frac{1}{4(\mathfrak{i}-\sigma)}\left[\int_{\sigma}^{\mathfrak{i}} \mathcal{J}(x,\varepsilon)dx + \int_{\sigma}^{\mathfrak{i}} \mathcal{J}(x,\mathfrak{v})dx\right] + \frac{1}{4(\mathfrak{v}-\varepsilon)}\left[\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(\sigma,y)dy + \int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(\mathfrak{i},y)dy\right]\\
&\supseteq \frac{\mathcal{J}(\sigma,\varepsilon)+\mathcal{J}(\mathfrak{i},\varepsilon)+\mathcal{J}(\sigma,\mathfrak{v})+\mathcal{J}(\mathfrak{i},\mathfrak{v})}{4}.
\end{aligned} \tag{53}$$

If $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and $\mathcal{J}_*(x,y) \neq \mathcal{J}^*(x,y)$, then from (27), we succeed in bringing about the upcoming inequality (see [46]):

$$\begin{aligned}
&\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\\
&\leq_p \frac{\Gamma(\alpha+1)}{4(\mathfrak{i}-\sigma)^{\alpha}}\left[\mathscr{I}_{\sigma^+}^{\alpha}\mathcal{J}\left(\mathfrak{i}, \frac{\varepsilon+\mathfrak{v}}{2}\right) + \mathscr{I}_{\mathfrak{i}^-}^{\alpha}\mathcal{J}\left(\sigma, \frac{\varepsilon+\mathfrak{v}}{2}\right)\right] + \frac{\Gamma(\beta+1)}{4(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathscr{I}_{\varepsilon^+}^{\beta}\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \mathfrak{v}\right) + \mathscr{I}_{\mathfrak{v}^-}^{\beta}\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \varepsilon\right)\right]\\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathscr{I}_{\sigma^+,\varepsilon^+}^{\alpha,\,\beta}\mathcal{J}(\mathfrak{i},\mathfrak{v}) + \mathscr{I}_{\sigma^+,\mathfrak{v}^-}^{\alpha,\,\beta}\mathcal{J}(\mathfrak{i},\varepsilon) + \mathscr{I}_{\mathfrak{i}^-,\varepsilon^+}^{\alpha,\,\beta}\mathcal{J}(\sigma,\mathfrak{v}) + \mathscr{I}_{\mathfrak{i}^-,\mathfrak{v}^-}^{\alpha,\,\beta}\mathcal{J}(\sigma,\varepsilon)\right]\\
&\leq_p \frac{\Gamma(\alpha+1)}{8(\mathfrak{i}-\sigma)^{\alpha}}\left[\mathscr{I}_{\sigma^+}^{\alpha}\mathcal{J}(\mathfrak{i},\varepsilon) + \mathscr{I}_{\sigma^+}^{\alpha}\mathcal{J}(\mathfrak{i},\mathfrak{v}) + \mathscr{I}_{\mathfrak{i}^-}^{\alpha}\mathcal{J}(\sigma,\varepsilon) + \mathscr{I}_{\mathfrak{i}^-}^{\alpha}\mathcal{J}(\sigma,\mathfrak{v})\right].\\
&\quad + \frac{\Gamma(\beta+1)}{8(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathscr{I}_{\varepsilon^+}^{\beta}\mathcal{J}(\sigma,\mathfrak{v}) + \mathscr{I}_{\mathfrak{v}^-}^{\beta}\mathcal{J}(\sigma,\varepsilon) + \mathscr{I}_{\varepsilon^+}^{\beta}\mathcal{J}(\mathfrak{i},\mathfrak{v}) + \mathscr{I}_{\mathfrak{v}^-}^{\beta}\mathcal{J}(\mathfrak{i},\varepsilon)\right]\\
&\leq_p \frac{\mathcal{J}(\sigma,\varepsilon)+\mathcal{J}(\mathfrak{i},\varepsilon)+\mathcal{J}(\sigma,\mathfrak{v})+\mathcal{J}(\mathfrak{i},\mathfrak{v})}{4}.
\end{aligned} \tag{54}$$

If $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and $\mathcal{J}_*(x,y) \neq \mathcal{J}^*(x,y)$, then by (27), we succeed in bringing about the upcoming inequality (see [43]):

$$\begin{aligned}
&\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\\
&\leq_p \frac{1}{2}\left[\frac{1}{\mathfrak{i}-\sigma}\int_{\sigma}^{\mathfrak{i}} \mathcal{J}\left(x, \frac{\varepsilon+\mathfrak{v}}{2}\right)dx + \frac{1}{\mathfrak{v}-\varepsilon}\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2}, y\right)dy\right] \leq_p \frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)}\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(x,y)dydx\\
&\leq_p \frac{1}{4(\mathfrak{i}-\sigma)}\left[\int_{\sigma}^{\mathfrak{i}} \mathcal{J}(x,\varepsilon)dx + \int_{\sigma}^{\mathfrak{i}} \mathcal{J}(x,\mathfrak{v})dx\right] + \frac{1}{4(\mathfrak{v}-\varepsilon)}\left[\int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(\sigma,y)dy + \int_{\varepsilon}^{\mathfrak{v}} \mathcal{J}(\mathfrak{i},y)dy\right]\\
&\leq_p \frac{\mathcal{J}(\sigma,\varepsilon)+\mathcal{J}(\mathfrak{i},\varepsilon)+\mathcal{J}(\sigma,\mathfrak{v})+\mathcal{J}(\mathfrak{i},\mathfrak{v})}{4}.
\end{aligned} \tag{55}$$

If $\mathcal{J}$ is coordinated LR-$\hbar$-convex with $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and $\mathcal{J}_*(x,y) = \mathcal{J}^*(x,y)$, then from (28), we succeed in bringing about the upcoming classical inequality:

$$
\begin{aligned}
&\mathcal{J}\left(\frac{\sigma+\mathrm{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\\
&\leq \frac{\Gamma(\alpha+1)}{4(\mathrm{i}-\sigma)^{\alpha}}\left[\mathcal{I}^{\alpha}_{\sigma^{+}}\ \mathcal{J}\left(\mathrm{i},\frac{\varepsilon+\mathfrak{v}}{2}\right)+\mathcal{I}^{\alpha}_{\mathrm{i}^{-}}\mathcal{J}\left(\sigma,\frac{\varepsilon+\mathfrak{v}}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\beta}_{\varepsilon^{+}}\ \mathcal{J}\left(\frac{\sigma+\mathrm{i}}{2},\mathfrak{v}\right)+\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\ \mathcal{J}\left(\frac{\sigma+\mathrm{i}}{2},\varepsilon\right)\right]\\
&\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\ \mathcal{J}(\mathrm{i},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\sigma^{+},\mathfrak{v}^{-}}\ \mathcal{J}(\mathrm{i},\varepsilon)+\mathcal{I}^{\alpha,\,\beta}_{\mathrm{i}^{-},\varepsilon^{+}}\ \mathcal{J}(\sigma,\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathrm{i}^{-},\mathfrak{v}^{-}}\ \mathcal{J}(\sigma,\varepsilon)\right]\\
&\leq \frac{\Gamma(\alpha+1)}{8(\mathrm{i}-\sigma)^{\alpha}}\left[\mathcal{I}^{\alpha}_{\sigma^{+}}\ \mathcal{J}(\mathrm{i},\varepsilon)\mathcal{J}\mathcal{I}^{\alpha}_{\sigma^{+}}\ \mathcal{J}(\mathrm{i},\mathfrak{v})+\mathcal{I}^{\alpha}_{\mathrm{i}^{-}}\ \mathcal{J}(\sigma,\varepsilon)+\mathcal{I}^{\alpha}_{\mathrm{i}^{-}}\ \mathcal{J}(\sigma,\mathfrak{v})\right].\\
&+\frac{\Gamma(\beta+1)}{8(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\beta}_{\varepsilon^{+}}\ \mathcal{J}(\sigma,\mathfrak{v})\ \tilde{+}\ \mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\ \mathcal{J}(\sigma,\varepsilon)+\mathcal{I}^{\beta}_{\varepsilon^{+}}\ \mathcal{J}(\mathrm{i},\mathfrak{v})+\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\ \mathcal{J}(\mathrm{i},\varepsilon)\right]\\
&\leq \frac{\mathcal{J}(\sigma,\varepsilon)+\mathcal{J}(\mathrm{i},\varepsilon)+\mathcal{J}(\sigma,\mathfrak{v})+\mathcal{J}(\mathrm{i},\mathfrak{v})}{4}.
\end{aligned}
\tag{56}
$$

In the next outcomes, we are going to find very interesting outcomes that will be obtained over the product of two coordinated LR-ℏ-convex IVMs. These inequalities are known as Pachpatte's inequalities.

Theorem 7. *Let* $\mathcal{J},\ \mathcal{g}:\Omega\to\mathbb{R}^{+}_{I}$ *be two coordinated LR-ℏ-convex IVMs on* Ω*, given by* $\mathcal{J}(x,y)=\left[\mathcal{J}_{*}(x,y),\ \mathcal{J}^{*}(x,y)\right]$ *and* $\mathcal{g}(x,y)=\left[\mathcal{g}_{*}(x,y),\ \mathcal{g}^{*}(x,y)\right]$ *for all* $(x,y)\in\Omega$ *and let* $\hbar_1,\hbar_2:[0,\ 1]\to\mathbb{R}^{+}$*. If* $\mathcal{J}\times\mathcal{g}\in\mathfrak{TD}_{\Omega}$*, then the following inequalities hold:*

$$
\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\mathcal{J}(\mathrm{i},\mathfrak{v})\times\mathcal{g}(\mathrm{i},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\sigma^{+},\mathfrak{v}^{-}}\mathcal{J}(\mathrm{i},\varepsilon)\times\mathcal{g}(\mathrm{i},\varepsilon)\right]\\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathrm{i}^{-},\varepsilon^{+}}\mathcal{J}(\sigma,\mathfrak{v})\times\mathcal{g}(\sigma,\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathrm{i}^{-},\mathfrak{v}^{-}}\mathcal{J}(\sigma,\varepsilon)\times\mathcal{g}(\sigma,\varepsilon)\right]\\
&\leq_{p}\ \mathcal{M}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]d\upsilon d\kappa\\
&+P(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(1-\\
&\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(1-\\
&\upsilon)\,\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]d\upsilon d\kappa\\
&+\mathcal{N}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]d\upsilon d\kappa\\
&+Q(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)+\\
&\hbar_1(1-\upsilon)\,\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]d\upsilon d\kappa.
\end{aligned}
\tag{57}
$$

If $\mathcal{J}$ *and* $\mathcal{g}$ *are both coordinated ℏ-concave IVMs on* Ω*, then the inequality above can be expressed as follows:*

$$
\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\mathcal{J}(\mathrm{i},\mathfrak{v})\times\mathcal{g}(\mathrm{i},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\sigma^{+},\mathfrak{v}^{-}}\mathcal{J}(\mathrm{i},\varepsilon)\times\mathcal{g}(\mathrm{i},\varepsilon)\right]\\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathrm{i}^{-},\varepsilon^{+}}\mathcal{J}(\sigma,\mathfrak{v})\times\mathcal{g}(\sigma,\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathrm{i}^{-},\mathfrak{v}^{-}}\mathcal{J}(\sigma,\varepsilon)\times\mathcal{g}(\sigma,\varepsilon)\right]\\
&\geq_{p}\ \mathcal{M}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]d\upsilon d\kappa\\
&+P(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(1-\\
&\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(1-\\
&\upsilon)\,\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]d\upsilon d\kappa\\
&+\mathcal{N}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&+\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]d\upsilon d\kappa\\
&+Q(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})\int_0^1 \upsilon^{\alpha-1}\kappa^{\beta-1}[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)+\\
&\hbar_1(1-\upsilon)\,\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]d\upsilon d\kappa
\end{aligned}
\tag{58}
$$

where

$$\begin{aligned}
\mathcal{M}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) &= \mathbb{J}(\sigma,\varepsilon)\times\jmath(\sigma,\varepsilon)+\mathbb{J}(\mathrm{i},\varepsilon)\times\jmath(\mathrm{i},\varepsilon)+\mathbb{J}(\sigma,\mathfrak{v})\times\jmath(\sigma,\mathfrak{v})+\mathbb{J}(\mathrm{i},\mathfrak{v})\times\jmath(\mathrm{i},\mathfrak{v}),\\
P(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) &= \mathbb{J}(\sigma,\varepsilon)\times\jmath(\mathrm{i},\varepsilon)+\mathbb{J}(\mathrm{i},\varepsilon)\times\jmath(\sigma,\varepsilon)+\mathbb{J}(\sigma,\mathfrak{v})\times\jmath(\mathrm{i},\mathfrak{v})+\mathbb{J}(\mathrm{i},\mathfrak{v})\times\jmath(\sigma,\mathfrak{v}),\\
\mathcal{N}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) &= \mathbb{J}(\sigma,\varepsilon)\times\jmath(\sigma,\mathfrak{v})+\mathbb{J}(\mathrm{i},\varepsilon)\times\jmath(\mathrm{i},\mathfrak{v})+\mathbb{J}(\sigma,\mathfrak{v})\times\jmath(\sigma,\varepsilon)+\mathbb{J}(\mathrm{i},\mathfrak{v})\times\jmath(\mathrm{i},\varepsilon),\\
Q(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) &= \mathbb{J}(\sigma,\varepsilon)\times\jmath(\mathrm{i},\mathfrak{v})+\mathbb{J}(\mathrm{i},\varepsilon)\times\jmath(\sigma,\mathfrak{v})+\mathbb{J}(\sigma,\mathfrak{v})\times\jmath(\mathrm{i},\varepsilon)+\mathbb{J}(\mathrm{i},\mathfrak{v})\times\jmath(\sigma,\varepsilon)\ ,
\end{aligned}$$

and $\mathcal{M}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})$, $P(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})$, $\mathcal{N}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})$ *and* $Q(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})$ *are defined as follows:*

$$\mathcal{M}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) = [\mathcal{M}_*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}),\ \mathcal{M}^*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})],$$

$$P(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) = [P_*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}),\ P^*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})],$$

$$\mathcal{N}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) = [\mathcal{N}_*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}),\ \mathcal{N}^*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})],$$

$$Q(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) = [Q_*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}),\ Q^*(\sigma,\mathrm{i},\varepsilon,\mathfrak{v})].$$

Proof. Let $\mathbb{J}$ and $\jmath$ be two coordinated LR-$\hbar_1$ and LR-$\hbar_2$-convex IVMs on $[\sigma,\mathrm{i}]\times[\varepsilon,\ \mathfrak{v}]$, respectively. Then,

$$\begin{gathered}
\mathbb{J}(\upsilon\sigma+(1-\upsilon)\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\
\leq_p \hbar_1(\upsilon)\hbar_1(\kappa)\mathbb{J}(\sigma,\varepsilon)+\hbar_1(\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\sigma,\mathfrak{v})+\hbar_1(1-\upsilon)\hbar_1(\kappa)\mathbb{J}(\mathrm{i},\varepsilon)\\
+\hbar_1(1-\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\mathrm{i},\mathfrak{v}),\\
\mathbb{J}(\upsilon\sigma+(1-\upsilon)\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\\
\leq_p \hbar_1(\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\sigma,\varepsilon)+\hbar_1(\upsilon)\hbar_1(\kappa)\mathbb{J}(\sigma,\mathfrak{v})+\hbar_1(1-\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\mathrm{i},\varepsilon)+\hbar_1(1-\upsilon)\hbar_1(\kappa)\mathbb{J}(\mathrm{i},\mathfrak{v}),
\end{gathered}$$

$$\begin{gathered}
\mathbb{J}((1-\upsilon)\sigma+\upsilon\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\
\leq_p \hbar_1(1-\upsilon)\hbar_1(\kappa)\mathbb{J}(\sigma,\varepsilon)+\hbar_1(1-\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\sigma,\mathfrak{v})+\hbar_1(\upsilon)\hbar_1(\kappa)\mathbb{J}(\mathrm{i},\varepsilon)\\
+\hbar_1(\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\mathrm{i},\mathfrak{v}),\\
\mathbb{J}((1-\upsilon)\sigma+\upsilon\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\\
\leq_p \hbar_1(1-\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\sigma,\varepsilon)+\hbar_1(1-\upsilon)\hbar_1(\kappa)\mathbb{J}(\sigma,\mathfrak{v})+\hbar_1(\upsilon)\hbar_1(1-\kappa)\mathbb{J}(\mathrm{i},\varepsilon)\\
+\hbar_1(\upsilon)\hbar_1(\kappa)\mathbb{J}(\mathrm{i},\mathfrak{v}),
\end{gathered}$$

and

$$\begin{gathered}
\jmath(\upsilon\sigma+(1-\upsilon)\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\
\leq_p \hbar_2(\upsilon)\hbar_2(\kappa)\jmath(\sigma,\varepsilon)+\hbar_2(\upsilon)\hbar_2(1-\kappa)\jmath(\sigma,\mathfrak{v})+\hbar_2(1-\upsilon)\hbar_2(\kappa)\jmath(\mathrm{i},\varepsilon)\\
+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)\jmath(\mathrm{i},\mathfrak{v}),\\
\jmath(\upsilon\sigma+(1-\upsilon)\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\\
\leq_p \hbar_2(\upsilon)\hbar_2(1-\kappa)\jmath(\sigma,\varepsilon)+\hbar_2(\upsilon)\hbar_2(\kappa)\jmath(\sigma,\mathfrak{v})+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)\jmath(\mathrm{i},\varepsilon)+\hbar_2(1-\upsilon)\hbar_2(\kappa)\jmath(\mathrm{i},\mathfrak{v}),
\end{gathered}$$

$$\begin{gathered}
((1-\upsilon)\sigma+\upsilon\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\\
\leq_p \hbar_2(1-\upsilon)\hbar_2(\kappa)\jmath(\sigma,\varepsilon)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)\jmath(\sigma,\mathfrak{v})+\hbar_2(\upsilon)\hbar_2(\kappa)\jmath(\mathrm{i},\varepsilon)\\
+\hbar_2(\upsilon)\hbar_2(1-\kappa)\jmath(\mathrm{i},\mathfrak{v}),\\
\jmath((1-\upsilon)\sigma+\upsilon\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\\
\leq_p \hbar_2(1-\upsilon)\hbar_2(1-\kappa)\jmath(\sigma,\varepsilon)+\hbar_2(1-\upsilon)\hbar_2(\kappa)\jmath(\sigma,\mathfrak{v})+\hbar_2(\upsilon)\hbar_2(1-\kappa)\jmath(\mathrm{i},\varepsilon)\\
+\hbar_2(\upsilon)\hbar_2(\kappa)\jmath(\mathrm{i},\mathfrak{v}),
\end{gathered}$$

Since $\mathcal{J}$ and $\jmath$ both are coordinated LR-$\hbar_1$ and LR-$\hbar_2$-convex IVMs on $[\sigma, \mathrm{i}] \times [\varepsilon,\ \mathfrak{v}]$, respectively, we have

$$\begin{aligned}
&\mathcal{J}(v\sigma + (1-v)\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \times \jmath(v\sigma + (1-v)\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&\quad + \mathcal{J}(v\sigma + (1-v)\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \times \jmath(v\sigma + (1-v)\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \\
&\quad + \mathcal{J}((1-v)\sigma + v\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \times \jmath((1-v)\sigma + v\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \\
&+ \mathcal{J}((1-v)\sigma + v\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \times \jmath((1-v)\sigma + v\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \\
&\leq_p \mathcal{M}(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa) + \hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)] \\
&\quad + P(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})[\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) + \hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)] \\
&\quad + \mathcal{N}(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(\kappa) \\
&\qquad + \hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)] \\
&+ Q(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})[\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)].
\end{aligned}$$

Taking the multiplication of the above fuzzy inclusion with $v^{\alpha-1}\kappa^{\beta-1}$ and then taking the double integration of the resultant over $[0,\ 1] \times [0,\ 1]$ with respect to $(v,\ \kappa)$, such that

$$\begin{aligned}
&\int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}(v\sigma + (1-v)\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \times \jmath(v\sigma + (1-v)\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v})dvd\kappa \\
&\quad + \int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}(v\sigma + (1-v)\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \times \jmath(v\sigma + (1-v)\mathrm{i}, (1-\kappa)\varepsilon + \\
&\quad \kappa\mathfrak{v})dvd\kappa \\
&\quad + \int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}((1-v)\sigma + v\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \times \jmath((|1-v)\sigma + v\mathrm{i}, \kappa\varepsilon + (1- \\
&\quad \kappa)\mathfrak{v})dvd\kappa \\
&+ \int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}((1-v)\sigma + v\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \times \jmath((1-v)\sigma + v\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v})dvd\kappa \\
&\leq_p \mathcal{M}(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})\int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa) + \hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa \\
&\quad + P(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})\int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa) + \hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa \\
&\quad + \mathcal{N}(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})\int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(\kappa) \\
&\qquad + \hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa \\
&+ Q(\sigma, \mathrm{i}, \varepsilon, \mathfrak{v})\int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa) \\
&\qquad + \hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa
\end{aligned} \tag{59}$$

From the right-hand side of (59), we have

$$\begin{aligned}
&\int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}(v\sigma + (1-v)\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \times \jmath(v\sigma + (1-v)\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v})dvd\kappa \\
&\quad + \int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}(v\sigma + (1-v)\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \times \jmath(v\sigma + (1-v)\mathrm{i}, (1-\kappa)\varepsilon + \\
&\quad \kappa\mathfrak{v})dvd\kappa \\
&\quad + \int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}((1-v)\sigma + v\mathrm{i}, \kappa\varepsilon + (1-\kappa)\mathfrak{v}) \times \jmath((1-v)\sigma + v\mathrm{i}, \kappa\varepsilon + (1- \\
&\quad \kappa)\mathfrak{v})dvd\kappa \\
&+ \int_0^1 \int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}((1-v)\sigma + v\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v}) \times \jmath((1-v)\sigma + v\mathrm{i}, (1-\kappa)\varepsilon + \kappa\mathfrak{v})dvd\kappa \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathrm{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\sigma^+,\varepsilon^+}\mathcal{J}(\mathrm{i}, \mathfrak{v}) \times \jmath(\mathrm{i}, \mathfrak{v}) + \mathcal{I}^{\alpha,\ \beta}_{\sigma^+,\mathfrak{v}^-}\mathcal{J}(\mathrm{i}, \varepsilon) \times \jmath(\mathrm{i}, \varepsilon)\right]
\end{aligned} \tag{60}$$

Combining (59) and (60), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\mathfrak{J}(\mathfrak{i},\mathfrak{v})\times \jmath(\mathfrak{i},\mathfrak{v})+\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\mathfrak{v}^{-}}\mathfrak{J}(\mathfrak{i},\varepsilon)\times \jmath(\mathfrak{i},\varepsilon)\right]\\
&\leq_p \mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\
&+P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)\\
&+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\
&+\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&+\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa\\
&+Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa.
\end{aligned}$$

Hence, the required result. □

Remark 4. *If one assumes that $\mathfrak{J}$ is coordinated left-LR-$\hbar$-convex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and $\alpha=1$ and $\beta=1$, then from (59), as a result, there will be inequality (see [42]):*

$$\begin{aligned}
&\frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)}\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}}\mathfrak{J}(x,y)\times \jmath(x,y)dydx\\
&\supseteq \tfrac{1}{9}\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\tfrac{1}{18}[P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})]+\tfrac{1}{36}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}).
\end{aligned} \tag{61}$$

If $\mathfrak{J}$ is coordinated LR-$\hbar$-convex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and one assumes that $\alpha=1$ and $\beta=1$, then from (59), as a result, there will be inequality (see [43]):

$$\begin{aligned}
&\frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)}\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}}\mathfrak{J}(x,y)\times \jmath(x,y)dydx\\
&\leq_p \tfrac{1}{9}\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\tfrac{1}{18}[P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})]+\tfrac{1}{36}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}).
\end{aligned} \tag{62}$$

If $\mathfrak{J}_*(x,y)\neq\mathfrak{J}^*(x,y)$ and $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ then, by (57), we succeed in bringing about the upcoming inequality (see [46]):

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\varepsilon^{+}}\mathfrak{J}(\mathfrak{i},\mathfrak{v})\times J(\mathfrak{i},\mathfrak{v})+\mathcal{J}^{\alpha,\,\beta}_{\sigma^{+},\mathfrak{v}^{-}}\mathfrak{J}(\mathfrak{i},\varepsilon)\times J(\mathfrak{i},\varepsilon)\right]\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\varepsilon^{+}}\mathfrak{J}(\sigma,\mathfrak{v})\times J(\sigma,\mathfrak{v})+\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^{-},\mathfrak{v}^{-}}\mathfrak{J}(\sigma,\varepsilon)\times J(\sigma,\varepsilon)\right]\\
&\leq_p \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\\
&+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\\
&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}).
\end{aligned} \tag{63}$$

If $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and $\mathfrak{J}_*(x,y)\neq\mathfrak{J}^*(x,y)$, then by (57), we succeed in bringing about the upcoming inequality (see [43]):

$$\begin{aligned}
&\frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)}\int_{\sigma}^{\mathfrak{i}}\int_{\varepsilon}^{\mathfrak{v}}\mathfrak{J}(x,y)\times \jmath(x,y)dydx\\
&\leq_p \tfrac{1}{9}\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\tfrac{1}{18}[P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})]+\tfrac{1}{36}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}).
\end{aligned} \tag{64}$$

If $\mathfrak{J}_*(x,y) = \mathfrak{J}^*(x,y)$ and $\jmath_*(x,y) = \jmath^*(x,y)$ and $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, then from (57), we succeed in bringing about the upcoming classical inequality:

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\jmath^{\alpha,\,\beta}_{\sigma^+,\varepsilon^+}\mathfrak{J}(\mathfrak{i},\mathfrak{v})\times\jmath(\mathfrak{i},\mathfrak{v})+\jmath^{\alpha,\,\beta}_{\sigma^+,\mathfrak{v}^-}\mathfrak{J}(\mathfrak{i},\varepsilon)\times\jmath(\mathfrak{i},\varepsilon)\right]\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[+\jmath^{\alpha,\,\beta}_{\mathfrak{i}^-,\varepsilon^+}\mathfrak{J}(\sigma,\mathfrak{v})\times\jmath(\sigma,\mathfrak{v})+\jmath^{\alpha,\,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-}\mathfrak{J}(\sigma,\varepsilon)\times\jmath(\sigma,\varepsilon)\right]\\
&\le\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\right.\\
&\left.\frac{\beta}{(\beta+1)(\beta+2)}\right)P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\\
&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}).
\end{aligned} \tag{65}$$

Theorem 8. *Let $\mathfrak{J}$, $\jmath : \Omega \to \mathbb{R}^+_I$ be two coordinated LR-$\hbar$-convex IVMs on Ω, given by $\mathfrak{J}(x,y) = [\mathfrak{J}_*(x,y),\ \mathfrak{J}^*(x,y)]$ and $\jmath(x,y) = [\jmath_*(x,y),\ \jmath^*(x,y)]$ for all $(x,y) \in \Omega$ and let $\hbar : [0,\ 1] \to \mathbb{R}^+$. If $\mathfrak{J} \times \jmath \in \mathfrak{TO}_{\Omega}$, then the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{2\alpha\beta\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\times\jmath\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\\
&\le_p\frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\jmath^{\alpha,\,\beta}_{\sigma^+,\varepsilon^+}\mathfrak{J}(\mathfrak{i},\mathfrak{v})\times\jmath(\mathfrak{i},\mathfrak{v})+\jmath^{\alpha,\,\beta}_{\sigma^+,\mathfrak{v}^-}\mathfrak{J}(\mathfrak{i},\varepsilon)\times\jmath(\mathfrak{i},\varepsilon)\right]\\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\jmath^{\alpha,\,\beta}_{\mathfrak{i}^-,\varepsilon^+}\mathfrak{J}(\sigma,\mathfrak{v})\times\jmath(\sigma,\mathfrak{v})+\jmath^{\alpha,\,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-}\mathfrak{J}(\sigma,\varepsilon)\times\jmath(\sigma,\varepsilon)\right]\\
&+\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)\\
&\quad+\hbar_2(1-v)\hbar_2(1-\kappa)]\\
&\quad+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)\\
&\quad+\hbar_2(1-v)\hbar_2(\kappa)]]dvd\kappa\\
&+P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\\
&\hbar_2(v)\hbar_2(1-\kappa)]+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\\
&\hbar_2(v)\hbar_2(\kappa)]]dvd\kappa\\
&+\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)\\
&\quad+\hbar_2(1-v)\hbar_2(\kappa)]\\
&\quad+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)\\
&\quad+\hbar_2(1-v)\hbar_2(1-\kappa)]]dvd\kappa\\
&+Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]+\\
&\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]]dvd\kappa.
\end{aligned} \tag{66}$$

If $\mathfrak{J}$ and $\jmath$ both are coordinate $\hbar$-concave IVMs on Ω, then the inequality above can be expressed as follows:

$$\begin{aligned}
&\frac{1}{2\alpha\beta\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)} Л\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right) \\
&\geq_p \frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\sigma^+,\varepsilon^+} Л(\mathfrak{i},\mathfrak{v}) \times \jmath(\mathfrak{i},\mathfrak{v}) + \mathcal{J}^{\alpha,\,\beta}_{\sigma^+,\mathfrak{v}^-} Л(\mathfrak{i},\varepsilon) \times \jmath(\mathfrak{i},\varepsilon)\right] \\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^-,\varepsilon^+} Л(\sigma,\mathfrak{v}) \times \jmath(\sigma,\mathfrak{v}) + \mathcal{J}^{\alpha,\,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-} Л(\sigma,\varepsilon) \times \jmath(\sigma,\varepsilon)\right] \\
&+\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa) \\
&\quad+\hbar_2(1-v)\hbar_2(1-\kappa)] \\
&\quad+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa) \\
&\quad+\hbar_2(1-v)\hbar_2(\kappa)]]dvd\kappa \\
&+P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+ \\
&\hbar_2(v)\hbar_2(1-\kappa)]+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+ \\
&\hbar_2(v)\hbar_2(\kappa)]]dvd\kappa \\
&+\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa) \\
&\quad+\hbar_2(1-v)\hbar_2(\kappa)] \\
&\quad+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa) \\
&\quad+\hbar_2(1-v)\hbar_2(1-\kappa)]]dvd\kappa \\
&+Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]+ \\
&\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]]dvd\kappa.
\end{aligned} \tag{67}$$

where $\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$, $P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$, $\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$, *and* $Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$ *are given in Theorem 7.*

Proof. Since $Л, \jmath : \Omega \to \mathbb{R}_I^+$ is two LR-$\hbar$-convex IVMs, then from inequality (17), we have

$$Л\left(\frac{\sigma+\mathfrak{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right)$$

$$
\begin{aligned}
&= \mathcal{J}\left(\tfrac{v\sigma+(1-v)\mathrm{i}}{2}+\tfrac{(1-v)\sigma+v\mathrm{i}}{2}, \tfrac{\kappa\varepsilon+(1-\kappa)\mathfrak{v}}{2}+\tfrac{\varepsilon+\mathfrak{v}}{2}\right) \\
&\quad\times \jmath\left(\tfrac{v\sigma+(1-v)\mathrm{i}}{2}+\tfrac{(1-v)\sigma+v\mathrm{i}}{2}, \tfrac{\kappa\varepsilon+(1-\kappa)\mathfrak{v}}{2}+\tfrac{(1-\kappa)\varepsilon+\kappa\mathfrak{v}}{2}\right) \\
&\leq_p \hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right) \times \begin{bmatrix} \mathcal{J}(v\sigma+(1-v)\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})+\mathcal{J}((1-v)\sigma+v\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}) \\ +\mathcal{J}(v\sigma+(1-v)\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})+\mathcal{J}((1-v)\sigma+v\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v}) \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \jmath(v\sigma+(1-v)\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})+\jmath((1-v)\sigma+v\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}) \\ +\jmath(v\sigma+(1-v)\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})+\jmath((1-v)\sigma+v\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v}) \end{bmatrix} \\
&\leq_p \hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right) \times \begin{bmatrix} \mathcal{J}(v\sigma+(1-v)\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\times\jmath(v\sigma+(1-v)\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}) \\ +\mathcal{J}((1-v)\sigma+v\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\times\jmath((1-v)\sigma+v\mathrm{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}) \\ +\mathcal{J}(v\sigma+(1-v)\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\times\jmath(v\sigma+(1-v)\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v}) \\ +\mathcal{J}((1-v)\sigma+v\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\times\jmath((1-v)\sigma+v\mathrm{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v}) \end{bmatrix} \\
&\quad +\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right) \\
&\quad \times \begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \end{bmatrix} \mathcal{M}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) \\
&\quad +\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right) \\
&\quad \times \begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \end{bmatrix} P(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) \\
&\quad +\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right) \\
&\quad \times \begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)] \\ +\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \end{bmatrix} \mathcal{N}(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}) \\
&\qquad +\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right) \\
&\quad \times \begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \end{bmatrix} Q(\sigma,\mathrm{i},\varepsilon,\mathfrak{v}).
\end{aligned}
$$

Taking the multiplication of the above fuzzy inclusion with $v^{\alpha-1}\kappa^{\beta-1}$ and then taking the double integration of the resultant over $[0,\ 1]\times[0,\ 1]$ with respect to $(v,\ \kappa)$, we have

$$
\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}\mathcal{J}\left(\frac{\sigma+\mathrm{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right)\times\jmath\left(\frac{\sigma+\mathrm{i}}{2},\frac{\varepsilon+\mathfrak{v}}{2}\right)dv\,d\kappa
$$

$$\leq_p \hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)$$
$$\times\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \mathcal{J}(v\sigma+(1-v)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\times g(v\sigma+(1-v)\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}) \\ +\mathcal{J}((1-v)\sigma+v\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v})\times g((1-v)\sigma+v\mathfrak{i},\kappa\varepsilon+(1-\kappa)\mathfrak{v}) \\ +\mathcal{J}(v\sigma+(1-v)\mathfrak{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\times g(v\sigma+(1-v)\mathfrak{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v}) \\ +\mathcal{J}((1-v)\sigma+v\mathfrak{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v})\times g((1-v)\sigma+v\mathfrak{i},(1-\kappa)\varepsilon+\kappa\mathfrak{v}) \end{bmatrix} dv d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$$
$$\times\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \end{bmatrix} dv d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$$
$$\times\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \end{bmatrix} dv d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$$
$$\times\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)] \\ +\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \end{bmatrix} dv d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})$$
$$\times\int_0^1\int_0^1 v^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)] \\ +\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \\ +\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)] \end{bmatrix} dv d\kappa,$$

which implies that

$$\frac{1}{\alpha\beta}\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)\times g\left(\frac{\sigma+\mathfrak{i}}{2},\ \frac{\varepsilon+\mathfrak{v}}{2}\right)$$
$$\leq_p \frac{\Gamma(\alpha)\Gamma(\beta)\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}{(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\sigma^+,\varepsilon^+}\mathcal{J}(\mathfrak{i},\mathfrak{v})\times g(\mathfrak{i},\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\sigma^+,\mathfrak{v}^-}\mathcal{J}(\mathfrak{i},\varepsilon)\times g(\mathfrak{i},\varepsilon)\right]$$
$$+\frac{\Gamma(\alpha)\Gamma(\beta)\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}{(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{i}^-,\varepsilon^+}\mathcal{J}(\sigma,\mathfrak{v})\times g(\sigma,\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{i}^-,\mathfrak{v}^-}\mathcal{J}(\sigma,\varepsilon)\times g(\sigma,\varepsilon)\right]$$
$$+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)$$
$$+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]$$
$$+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)$$
$$+\hbar_2(1-v)\hbar_2(\kappa)]]dv d\kappa$$
$$+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)$$
$$+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]$$
$$+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)$$
$$+\hbar_2(v)\hbar_2(\kappa)]]dv d\kappa$$
$$+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)$$
$$+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]$$
$$+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)$$
$$+\hbar_2(1-v)\hbar_2(1-\kappa)]]dv d\kappa$$
$$+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})\int_0^1 v^{\alpha-1}\kappa^{\beta-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]]dv d\kappa,$$

hence, the required result. □

Remark 5. *If one assumes that $\mathfrak{J}$ is coordinated left-LR-$\hbar$-convex with $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and $\alpha = 1$ and $\beta = 1$, then from (66), as a result, there will be inequality (see [42]):*

$$\begin{gathered} 4\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\ \supseteq \frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)} \int_{\sigma}^{\mathfrak{i}} \int_{\varepsilon}^{\mathfrak{v}} \mathfrak{J}(x,y) \times \jmath(x,y)dydx + \frac{5}{36}\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\frac{7}{36}[P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) + \mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})] + \frac{2}{9}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}). \end{gathered} \tag{68}$$

If $\mathfrak{J}$ is coordinated LR-$\hbar$-convex with $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$ and one assumes that $\alpha = 1$ and $\beta = 1$, then from (66), as a result, there will be inequality (see [43]):

$$\begin{gathered} 4\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\ \leq_p \frac{1}{(\mathfrak{i}-\sigma)(\mathfrak{v}-\varepsilon)} \int_{\sigma}^{\mathfrak{i}} \int_{\varepsilon}^{\mathfrak{v}} \mathfrak{J}(x,y) \times \jmath(x,y)dydx + \frac{5}{36}\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\frac{7}{36}[P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) + \mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v})] + \frac{2}{9}Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}). \end{gathered} \tag{69}$$

If $\mathfrak{J}$ is coordinated left-LR-$\hbar$-convex and $\mathfrak{J}_*(x,y) \neq \mathfrak{J}^*(x,y)$ with $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$, then from (66), we succeed in bringing about the upcoming inequality (see [41]):

$$\begin{gathered} 4\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\ \supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}} \left[\begin{array}{c} \mathcal{J}^{\alpha,\beta}_{\sigma^+,\varepsilon^+}\mathfrak{J}(\mathfrak{i},\mathfrak{v}) \times \jmath(\mathfrak{i},\mathfrak{v}) + \mathcal{J}^{\alpha,\beta}_{\sigma^+,\mathfrak{v}^-}\mathfrak{J}(\mathfrak{i},\varepsilon) \times \jmath(\mathfrak{i},\varepsilon) \\ +\mathcal{J}^{\alpha,\beta}_{\mathfrak{i}^-,\varepsilon^+}\mathfrak{J}(\sigma,\mathfrak{v}) \times \jmath(\sigma,\mathfrak{v}) + \mathcal{J}^{\alpha,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-}\mathfrak{J}(\sigma,\varepsilon) \times \jmath(\sigma,\varepsilon) \end{array}\right] \\ +\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} + \frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\left[\frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\left[\frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\left[\frac{1}{4} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}). \end{gathered} \tag{70}$$

If $\mathfrak{J}_*(x,y) \neq \mathfrak{J}^*(x,y)$ and $\hbar(\upsilon) = \upsilon$, $\hbar(\kappa) = \kappa$, then from (66), we succeed in bringing about the upcoming inequality (see [46]):

$$\begin{gathered} 4\mathfrak{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\ \leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}} \left[\begin{array}{c} \mathcal{J}^{\alpha,\beta}_{\sigma^+,\varepsilon^+}\mathfrak{J}(\mathfrak{i},\mathfrak{v}) \times \jmath(\mathfrak{i},\mathfrak{v}) + \mathcal{J}^{\alpha,\beta}_{\sigma^+,\mathfrak{v}^-}\mathfrak{J}(\mathfrak{i},\varepsilon) \times \jmath(\mathfrak{i},\varepsilon) \\ +\mathcal{J}^{\alpha,\beta}_{\mathfrak{i}^-,\varepsilon^+}\mathfrak{J}(\sigma,\mathfrak{v}) \times \jmath(\sigma,\mathfrak{v}) + \mathcal{J}^{\alpha,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-}\mathfrak{J}(\sigma,\varepsilon) \times \jmath(\sigma,\varepsilon) \end{array}\right] \\ +\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} + \frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\mathcal{M}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\left[\frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]P(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\left[\frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}) \\ +\left[\frac{1}{4} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]Q(\sigma,\mathfrak{i},\varepsilon,\mathfrak{v}). \end{gathered} \tag{71}$$

If $\mathcal{J}_*(x, y) = \mathcal{J}^*(x, y)$ *and* $\jmath_*(x, y) = \jmath^*(x, y)$ *and* $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, *then from (66), we succeed in bringing about the upcoming classical inequality.*

$$\begin{aligned} & 4\mathcal{J}\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \times \jmath\left(\frac{\sigma+\mathfrak{i}}{2}, \frac{\varepsilon+\mathfrak{v}}{2}\right) \\ \leq & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{i}-\sigma)^{\alpha}(\mathfrak{v}-\varepsilon)^{\beta}} \left[\begin{array}{c} \mathcal{I}^{\alpha,\beta}_{\sigma^+,\varepsilon^+} \mathcal{J}(\mathfrak{i}, \mathfrak{v}) \times \jmath(\mathfrak{i}, \mathfrak{v}) + \mathcal{I}^{\alpha,\beta}_{\sigma^+,\mathfrak{v}^-} \mathcal{J}(\mathfrak{i}, \varepsilon) \times \jmath(\mathfrak{i}, \varepsilon) \\ + \mathcal{I}^{\alpha,\beta}_{\mathfrak{i}^-,\varepsilon^+} \mathcal{J}(\sigma, \mathfrak{v}) \times \jmath(\sigma, \mathfrak{v}) + \mathcal{I}^{\alpha,\beta}_{\mathfrak{i}^-,\mathfrak{v}^-} \mathcal{J}(\sigma, \varepsilon) \times \jmath(\sigma, \varepsilon) \end{array} \right] \\ & + \left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} + \frac{\beta}{(\beta+1)(\beta+2)} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \right] \mathcal{M}(\sigma, \mathfrak{i}, \varepsilon, \mathfrak{v}) \\ & + \left[\frac{1}{2} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) + \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} \right] P(\sigma, \mathfrak{i}, \varepsilon, \mathfrak{v}) \\ & + \left[\frac{1}{2} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) + \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} \right] \mathcal{N}(\sigma, \mathfrak{i}, \varepsilon, \mathfrak{v}) \\ & + \left[\frac{1}{4} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} \right] Q(\sigma, \mathfrak{i}, \varepsilon, \mathfrak{v}). \end{aligned} \tag{72}$$

4. Conclusions and Future Plans

To sum up, this study offers a new extension of interval-valued convexity. Through the use of fractional integral operators, various inequalities for *LR*-$\hbar$-convexity are produced by applying interval-valued mapping. Many exceptional cases are discussed and new and classical versions of integral inequalities are also acquired that can be considered as applications of this article's outcomes. Some very interesting examples are also given to discuss the validation of the main results. The results of this research paper could potentially have applications in various areas of mathematics, physics, and engineering. The extension of the proposed iterative method to systems of equations could be an interesting future research problem. In the future, we will try to explore these concepts in quantum calculus.

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References

1. Dragomir, S.S.; Pearce, C. *Selected Topics on Hermite-Hadamard Inequality and Applications*; Victoria University: Melbourne, Australia, 2000.
2. Hadamard, J. Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *9*, 171–216.
3. Hermite, C. Sur deux limites d'une intégrale définie. *Mathesis* **1883**, *3*, 1–82.
4. Dragomir, S.S.; Agarwal, R. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
5. Kirmaci, U.S. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comput.* **2004**, *147*, 137–146. [CrossRef]
6. Awan, M.U.; Javed, M.Z.; Rassias, M.T.; Noor, M.A.; Noor, K.I. Simpson type inequalities and applications. *J. Anal.* **2021**, *29*, 1403–1419. [CrossRef]
7. Alomari, M.; Darus, M. On some inequalities of Simpson-type via quasi-convex functions and applications. *Transylv. J. Math.* **2010**, *2*, 15–24.
8. İşcan, İ. Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions. *J. Math.* **2014**, *2014*, 346305. [CrossRef]

9. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson's type for convex functions. *Res. Group Math. Inequalities Appl.* **2010**, *60*, 2191–2199. [CrossRef]
10. Set, E.; Akdemir, A.O.; Özdemir, E.M. Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. *Filomat* **2017**, *31*, 4415–4420. [CrossRef]
11. Breckner, W.W. Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. *Publ. Inst. Math.* **1978**, *23*, 13–20.
12. Hudzik, H.; Maligranda, L. Some remarks on s-convex functions. *Aequationes Math.* **1994**, *48*, 100–111. [CrossRef]
13. Godunova, E.K.; Levin, V.I. Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. *Numer. Math. Math. Phys.* **1985**, *138*, 166.
14. Noor, M.A.; Noor, K.I.; Awan, M.U.; Li, J. On Hermite-Hadamard Inequalities for h-Preinvex Functions. *Filomat* **2014**, *28*, 1463–1474. [CrossRef]
15. Awan, M.U.; Talib, S.; Noor, M.A.; Noor, K.I. On γ-preinvex functions. *Filomat* **2020**, *34*, 4137–4159. [CrossRef]
16. Tunç, M.; Göv, E.; Sanal, Ü. On *tgs*-convex function and their inequalities. *Facta Univ. Ser. Math. Inform.* **2015**, *30*, 679–691.
17. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
18. Rentería-Baltiérrez, F.Y.; Reyes-Melo, M.E.; Puente-Córdova, J.G.; López-Walle, B. Application of fractional calculus in the mechanical and dielectric correlation model of hybrid polymer films with different average molecular weight matrices. *Polym. Bull.* **2023**, *80*, 6327–6347. [CrossRef]
19. Sun, H.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 213–231. [CrossRef]
20. Viera-Martin, E.; Gómez-Aguilar, J.F.; Solís-Pérez, J.E.; Hernández-Pérez, J.A.; Escobar-Jiménez, R.F. Artificial neural networks: A practical review of applications involving fractional calculus. *Eur. Phys. J. Spec. Top.* **2022**, *231*, 2059–2095. [CrossRef]
21. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
22. Abbaszadeh, S.; Gordji, M.E.; Pap, E.; Szakál, A. Jensen-type inequalities for Sugeno integral. *Inf. Sci.* **2017**, *376*, 148–157. [CrossRef]
23. Agahi, H.; Mesiar, R.; Ouyang, Y. General Minkowski type inequalities for Sugeno integrals. *Fuzzy Sets Syst.* **2010**, *161*, 708–715. [CrossRef]
24. Pap, E.; Štrboja, M. Generalization of the Jensen inequality for pseudo-integral. *Inf. Sci.* **2010**, *180*, 543–548. [CrossRef]
25. Wang, R.S. Some inequalities and convergence theorems for Choquet integral. *J. Appl. Math. Comput.* **2011**, *35*, 305–321. [CrossRef]
26. Aumann, R.J. Integrals of set-valued functions. *J. Math. Anal. Appl.* **1965**, *12*, 1–12. [CrossRef]
27. Klein, E.; Thompson, A.C. *Theory of Correspondences*; A Wiley-Interscience Publication: New York, NY, USA, 1984.
28. Costa, T.M.; Román-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
29. Khan, M.B.; Zaini, H.G.; Treanţă, S.; Soliman, M.S.; Nonlaopon, K. Riemann–liouville fractional integral inequalities for generalized pre-invex functions of interval-valued settings based upon pseudo order relation. *Mathematics* **2022**, *10*, 204. [CrossRef]
30. Budak, H.; Tun, T.; Sarikaya, M.Z. Fractional hermite-hadamard-type inequalities for interval-valued functions. *Proc. Am. Math. Soc.* **2019**, *148*, 705–718. [CrossRef]
31. Zhao, D.; An, T.; Ye, G.; Liu, W. Chebyshev type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2020**, *396*, 82–101. [CrossRef]
32. Zhao, D.; An, T.; Ye, G.; Liu, W. Some generalizations of opial type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2021**, *436*, 128–151. [CrossRef]
33. Zhao, D.; Zhao, G.; Ye, G.; Liu, W.; Dragomir, S.S. On hermite–hadamard-type inequalities for coordinated h-convex interval-valued functions. *Mathematics* **2021**, *9*, 2352. [CrossRef]
34. Budak, H.; Kashuri, A.; Butt, S. Fractional Ostrowski type inequalities for interval valued functions. *Mathematics* **2022**, *36*, 2531–2540. [CrossRef]
35. Khan, M.B.; Abdullah, L.; Noor, M.A.; Noor, K.I. New hermite–hadamard and jensen inequalities for log-h-convex fuzzy interval valued functions. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 155. [CrossRef]
36. Anastassiou, G.A. Fuzzy ostrowski type inequalities. *Comput. Appl. Math.* **2003**, *22*, 279–292. [CrossRef]
37. Bede, B.; Gal, S.G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets Syst.* **2005**, *151*, 581–599. [CrossRef]
38. Chalco-Cano, Y.; Román-Flores, H.; Jiménez-Gamero, M.D. Generalized derivative and π-derivative for set-valued functions. *Inf. Sci.* **2011**, *181*, 2177–2188. [CrossRef]
39. Chalco-Cano, Y.; Román-Flores, H.; Jiménez-Gamero, M.D. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* **2011**, *181*, 2177–2188.
40. Lupulescu, V. Fractional calculus for interval-valued functions. *Fuzzy Set. Syst.* **2015**, *265*, 63–85. [CrossRef]
41. Budak, H.; Kara, H.; Ali, M.A.; Khan, S.; Chu, Y. Fractional Hermite-Hadamard-type inequalities for interval-valued co-ordinated convex functions. *Open Math.* **2021**, *19*, 1081–1097. [CrossRef]
42. Zhao, D.F.; Ali, M.A.; Murtaza, G. On the Hermite-Hadamard inequalities for interval-valued coordinated convex functions. *Adv. Differ. Equ.* **2020**, *2020*, 570. [CrossRef]

43. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Nonlaopon, K.; Hamed, Y.S. Some new estimates on coordinates of left and right convex interval-valued functions based on pseudo order relation. *Symmetry* **2022**, *14*, 473. [CrossRef]
44. Budak, H.; Sarıkaya, M.Z. Hermite-Hadamard type inequalities for products of two coordinated convex mappings via fractional integrals. *Int. J. Appl. Math. Stat.* **2019**, *58*, 11–30.
45. Khan, M.B.; Treanță, T.; Alrweili, H.; Saeed, T.; Soliman, M.S. Some new Riemann-Liouville fractional integral inequalities for interval-valued mappings. *AIMS Math.* **2022**, *7*, 15659–15679. [CrossRef]
46. Khan, M.B.; Zaini, H.G.; Macías-Díaz, J.E.; Treanţă, S.; Soliman, M.S. Some integral inequalities in interval fractional calculus for left and right coordinated interval-valued functions. *AIMS Math.* **2022**, *7*, 10454–10482. [CrossRef]
47. Zhang, D.; Guo, C.; Chen, D.; Wang, G. Jensen's inequalities for set-valued and fuzzy set-valued functions. *Fuzzy Sets Syst.* **2020**, *404*, 178–204. [CrossRef]