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The Sign-Changing Solution for Fractional (p,q)-Laplacian Problems Involving Supercritical Exponent

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Abstract: In this article, we consider the following fractional (p,q)-Laplacian problem $(-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) + \lambda|u|^{r-2}u$, where $x \in \mathbb{R}^N$, $(-\Delta)^{s_1}_p$ is the fractional p-Laplacian operator ($(-\Delta)^{s_2}_q$ is similar), $0 < s_1 < s_2 < 1 < p < q < \frac{N}{s_2}$, $q^*_{s_2} = \frac{Nq}{N-s_2 q}$, $r \geq q^*_{s_2}$, f is a C^1 real function and V is a coercive function. By using variational methods, we prove that the above problem admits a sign-changing solution if $\lambda > 0$ is small.

Keywords: fractional (p,q)-Laplacian problem; supercritical exponent; sign-changing solution

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1. Introduction

For the fractional (p,q)-Laplacian problem

$$(-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) + \lambda|u|^{r-2}u, \tag{1}$$

we will prove it admits a sign-changing solution, where $x \in \mathbb{R}^N$, $0 < s_1 < s_2 < 1 < p < q < \frac{N}{s_2}$, $q^*_{s_2} = \frac{Nq}{N-s_2 q}$, $r \geq q^*_{s_2}$, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a continuous real function.

1.1. Physical Background

The definition of the fractional p-Laplacian can be seen in [1]. For the physical background, we refer the reader to [2–7]. These papers tell us that the fractional p-Laplacian can describe financial markets, optimization, phase transformation, semi-permeable film, anomalous diffusion and minimal surface problems. Problem (1) models two different materials and it is called the double-phase equation (see e.g., [8]).

1.2. Related Works and Our Main Results

Recently, many authors have been concerned with the fractional (p,q)-Laplacian equations. For the critical and supercritical cases, the existence of multiple solutions is obtained in [8]. In the meantime, for the problem

$$(-\Delta)^{s}_p u + (-\Delta)^{s}_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u),\ x \in \mathbb{R}^N, \tag{2}$$

where $s \in (0,1)$, $1 < p < q < \frac{N}{s}$, $V : \mathbb{R}^N \to \mathbb{R}$, Isernian [9] showed that it admits a positive ground state solution. In 2022, the existence of the least energy sign-changing solution of problem (2) was given by Cheng et al. in [10]. We also quote the papers [11–13] for the p-Laplacian or fractional p-Laplacian in a bounded domain. For other results, please see [1,8,14–20] and the references therein.

The main goal of the present paper is to investigate the problem in (1). We assume that

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 = \inf\limits_{x\in\mathbb{R}^N} V(x)$ and $\lim\limits_{|x|\to+\infty} V(x) = +\infty$, where $V_0 > 0$ is a constant.

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $\lim\limits_{|t|\to 0} \frac{f(t)}{|t|^{p-1}} = 0$.
(f_2) there is $\sigma \in (p, q^*_{s_2})$, $C > 0$ such that $|f(t)| \leq C(|t|^{p-1} + |t|^{\sigma-1})$.
(f_3) there exists $\theta \in (q, q^*_{s_2})$ such that $0 < \theta F(t) \leq f(t)t$ for all $|t| > 0$, where $F(t) = \int_0^t f(s)ds$.
(f_4) the map $t \mapsto \frac{f(t)}{|t|^{q-1}}$ is strictly increasing for all $|t| > 0$.

Theorem 1. *Let $0 < s_1 < s_2 < 1 < p < q < \frac{N}{s_2}$, $q^*_{s_2} = \frac{Nq}{N-s_2 q}$, $r \geq q^*_{s_2}$, (V) and $(f_1) - (f_4)$ hold. Then there is $\lambda_0 > 0$ such that Problem (1) possesses one sign-changing solution w_0 when $\lambda \in (0, \lambda_0)$.*

Remark 1. *$r \geq q^*_{s_2}$ is called critical or supercritical. We do not confirm that w_0 is a least energy sign-changing solution.*

1.3. Our Motivations and Novelties

Like [21], a natural question for us is

For the (p, q)-Laplacian problem (1), does there exist a sign-changing solution?

Our motivation in this paper is to give this question an affirmative answer. It is different from [22,23] since (10), (11) and (15) are new and crucial.

1.4. Methods

We summarize our methods here. We adopt the idea from [22] or [24] to cut off the functional (see (4)). Then we shall prove (6) admits a minimizer w_0. Furthermore, we need to prove that the minimizer w_0 is a critical point of $I_{\lambda,K}$ (see (5)). Finally, we borrow the idea from [22] to make a L^∞-estimation such that $\|w_0\|_{L^\infty(\mathbb{R}^N)} \leq K$, which implies that we do not make any truncations.

1.5. Organization

This paper is organized as follows. Section 2 provides some preliminaries. Section 3 is divided into two parts, which will prove Theorem 1. The last Section is the conclusions and our future direction. Throughout this paper, we use the standard notations.

- C or C_i $(i = 1, 2, ...)$ denote some positive constants (possibly different from line to line) and $C(\cdot)$ denotes some positive constant only dependent on $\cdot$.
- $\|\cdot\|_{L^l}$ $(1 < l < \infty)$ is the standard norm in the usual Lebesgue space $L^l(\mathbb{R}^N)$.
- For a function $u(x)$, $u^+(x) := \max\{u(x), 0\}$, $u^-(x) := \min\{u(x), 0\}$. Clearly, $u = u^+ + u^-$.
- $\mathbb{R}^+ := [0, +\infty)$.

2. Preliminary Results

From now on, we always assume that (V) and $(f_1) - (f_4)$ hold unless a special statement is made. We continue to use the notations and work space $W^{s,p}(\mathbb{R}^N)$ as in [8]. Since the potential V is coercive, we introduce the subspace

$$E = \left\{ u \in W^{s_1,p}(\mathbb{R}^N) \bigcap W^{s_2,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)(|u|^p + |u|^q)dx < \infty \right\}$$

equipped with the following norm $\|u\| = \|u\|_1 + \|u\|_2$, where

$$\|u\|_1 = \left([u]^p_{s_1,p} + \int_{\mathbb{R}^N} V(x)|u|^p dx \right)^{\frac{1}{p}}, \ \|u\|_2 = \left([u]^q_{s_2,q} + \int_{\mathbb{R}^N} V(x)|u|^q dx \right)^{\frac{1}{q}}.$$

For $[\cdot]_{s_1,p}$ and $[\cdot]_{s_2,q}$, see [8]. Formally, the corresponding energy functional of (1) is

$$I(u) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \int_{\mathbb{R}^N} F(u)dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} |u|^r dx, \ u \in E. \tag{3}$$

It is well-known that the functional I is not well-defined on E. In order to overcome this difficulty, similar to [22],

$$h_{\lambda,K}(t) := \begin{cases} f(t) + \lambda|t|^{r-2}t, & \text{if } |t| \le K, \\ f(t) + \lambda K^{r-\sigma}|t|^{\sigma-2}t, & \text{if } |t| > K, \end{cases} \tag{4}$$

where $K > 0, \sigma \in (\theta, q^*_{s_2})$.

Thus, it holds that

(h_1) $\lim\limits_{|t|\to 0} \frac{h_{\lambda,K}(t)}{|t|^{p-1}} = 0$, and for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that

$$|h_{\lambda,K}(t)| \le \varepsilon|t|^{p-1} + (1 + \lambda K^{r-\sigma})C(\varepsilon)|t|^{\sigma-1}, \forall\, t \in \mathbb{R}.$$

(h_2) $0 < \theta H_{\lambda,K}(t) \le h_{\lambda,K}(t)t \ \forall |t| > 0$, and $h_{\lambda,K}(t)t - qH_{\lambda,K}(t)$ is increasing with respect to t, for all $t > 0$ where $H_{\lambda,K}(t) = \int_0^t h_{\lambda,K}(s)ds$.

(h_3) the map $t \mapsto \frac{h_{\lambda,K}(t)}{|t|^{q-1}}$ is strictly increasing for all $|t| > 0$.

For the auxiliary functional

$$I_{\lambda,K}(u) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \int_{\mathbb{R}^N} H_{\lambda,K}(u)dx, \tag{5}$$

(h_1) implies that $I_{\lambda,K} \in C^1(E, \mathbb{R})$. We want to prove that

$$m_{\lambda,K} := \inf_{u \in \mathcal{M}_{\lambda,K}} I_{\lambda,K}(u) \tag{6}$$

admits a minimizer, where the corresponding Nehari manifold $\mathcal{M}_{\lambda,K}$ is given by

$$\mathcal{M}_{\lambda,K} = \{u \in E : u^\pm \neq 0, \ \langle I'_{\lambda,K}(u), u^\pm \rangle = 0\}. \tag{7}$$

3. Proof of the Main Results

3.1. Some Lemmas

To begin with, we give several lemmas that will be used in the sequel.

Lemma 1. *For any $u \in E$ with $u^\pm \neq 0$, there is a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_{\lambda,K}$. Moreover,*

$$I_{\lambda,K}(s_u u^+ + t_u u^-) = \max_{(s,t)\in\mathbb{R}^+\times\mathbb{R}^+} I_{\lambda,K}(su^+ + tu^-).$$

Proof. The proof is standard (see e.g., [21]). For $u \in E$ with $u^\pm \neq 0$, we can deduce from (h_1) and (h_2) that there exist $C_1(\lambda, K), C_2(\lambda, K) > 0$ such that

$$H_{\lambda,K}(u) \ge C_1(\lambda, K)|u|^\theta - C_2(\lambda, K)|u|^p. \tag{8}$$

Let $s,t \geq 0$, $\phi_{\lambda,K}(s,t) := I_{\lambda,K}(su^+ + tu^-)$. One has

$$\begin{aligned}
&\phi_{\lambda,K}(s,t)\\
\leq&\frac{1}{p}\|su^+ + tu^-\|_1^p + \frac{1}{q}\|su^+ + tu^-\|_2^q\\
&+ C_2(\lambda,K)s^p\|u^+\|_{L^p} + C_2(\lambda,K)t^p\|u^-\|_{L^p}\\
&- C_1(\lambda,K)s^\theta\|u^+\|_{L^\theta} - C_1(\lambda,K)t^\theta\|u^-\|_{L^\theta}.
\end{aligned} \tag{9}$$

Denote $(\mathbb{R}^N)^+ := \{x \in \mathbb{R}^N : u(x) \geq 0\}$ and $(\mathbb{R}^N)^- := \{x \in \mathbb{R}^N : u(x) < 0\}$. It holds that

$$\begin{aligned}
&\|su^+ + tu^-\|_1^p\\
=&s^p \int_{(\mathbb{R}^N)^+} dy \int_{(\mathbb{R}^N)^+} dx \frac{|u^+(x) - u^+(y)|^p}{|x-y|^{N+ps_1}}\\
&+ \int_{(\mathbb{R}^N)^-} dy \int_{(\mathbb{R}^N)^+} dx \frac{|su^+(x) - tu^-(y)|^p}{|x-y|^{N+ps_1}}\\
&+ \int_{(\mathbb{R}^N)^+} dy \int_{(\mathbb{R}^N)^-} dx \frac{|tu^-(x) - su^+(y)|^p}{|x-y|^{N+ps_1}}\\
&+ t^p \int_{(\mathbb{R}^N)^-} dy \int_{(\mathbb{R}^N)^-} dx \frac{|u^-(x) - u^-(y)|^p}{|x-y|^{N+ps_1}}\\
&+ s^p \int_{\mathbb{R}^N} V(x)|u^+|^p dx + t^p \int_{\mathbb{R}^N} V(x)|u^-|^p dx.
\end{aligned} \tag{10}$$

Obviously,

$$\begin{aligned}
&\int_{(\mathbb{R}^N)^-} dy \int_{(\mathbb{R}^N)^+} dx \frac{|su^+(x) - tu^-(y)|^p}{|x-y|^{N+ps_1}}\\
\leq&\max\{|s|^p, |t|^p\} \int_{(\mathbb{R}^N)^-} dy \int_{(\mathbb{R}^N)^+} dx \frac{|u^+(x) - u^-(y)|^p}{|x-y|^{N+ps_1}}.
\end{aligned} \tag{11}$$

The result also holds for the third integral in (10). We can estimate the term $\|su^+ + tu^-\|_2^p$ similarly. Thus, we obtain

$$\phi_{\lambda,K}(s,t) \to -\infty \text{ as } |(s,t)| \to +\infty. \tag{12}$$

Note that $\phi_{\lambda,K}(0,0) = 0$. The continuity of $\phi_{\lambda,K}$ shows that it admits a global maximum point $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Next, similar to ([21], Lemma 2.3), we can prove that the maximum point cannot be achieved on the boundary of $(0,+\infty) \times (0,+\infty)$.

The remaining part sets out to prove the uniqueness. We divide it into two cases. **Case 1:** For $u \in \mathcal{M}_{\lambda,K}$. If there exists a pair $\bar{s} > 0, \bar{t} > 0$ such that $\bar{s}u^+ + \bar{t}u^- \in \mathcal{M}_{\lambda,K}$. We shall discuss the case $0 < \bar{t} \leq \bar{s}$. Clearly,

$$\begin{aligned}
&\int_{\mathbb{R}^N} h_{\lambda,K}(\bar{s}u^+)\bar{s}u^+ dx\\
\leq&\bar{s}^p\|u^+\|_1^p + \bar{s}^q\|u^+\|_2^q
\end{aligned}$$

$$
\begin{aligned}
&+\bar{s}^p\int_{(\mathbb{R}^N)^-}\int_{(\mathbb{R}^N)^+}\frac{|u^+(x)-u^-(y)|^{p-1}u^+(x)-|u^+(x)|^p}{|x-y|^{N+s_1p}}dxdy\\
&+\bar{s}^p\int_{(\mathbb{R}^N)^+}\int_{(\mathbb{R}^N)^-}\frac{|u^-(x)-u^+(y)|^{p-1}u^+(y)-|u^+(y)|^p}{|x-y|^{N+s_1p}}dxdy\\
&+\bar{s}^q\int_{(\mathbb{R}^N)^-}\int_{(\mathbb{R}^N)^+}\frac{|u^+(x)-u^-(y)|^{q-1}u^+(x)-|u^+(x)|^q}{|x-y|^{N+s_2q}}dxdy\\
&+\bar{s}^q\int_{(\mathbb{R}^N)^+}\int_{(\mathbb{R}^N)^-}\frac{|u^-(x)-u^+(y)|^{q-1}u^+(y)-|u^+(y)|^q}{|x-y|^{N+s_2q}}dxdy\\
&:=\bar{s}^p\|u^+\|_1^p+\bar{s}^q\|u^+\|_2^q+\bar{s}^pA_1+\bar{s}^pA_2+\bar{s}^qA_3+\bar{s}^qA_4.
\end{aligned}
$$

$u\in\mathcal{M}_{\lambda,K}$ gives

$$
\int_{\mathbb{R}^N}h_{\lambda,K}(u^+)u^+dx=\|u^+\|_1^p+\|u^+\|_2^q+A_1+A_2+A_3+A_4. \tag{13}
$$

If $\bar{s}>1$, in view of (h_3), we obtain

$$
\begin{aligned}
0<&\int_{\mathbb{R}^N}\left[\frac{h_{\lambda,K}(\bar{s}u^+)}{(\bar{s}u^+)^{q-1}}-\frac{h_{\lambda,K}(u^+)}{u^{+q-1}}\right](u^+)^qdx\\
\le&(\bar{s}^{p-q}-1)\|u^+\|_1^p+(\bar{s}^{p-q}-1)A_1+(\bar{s}^{p-q}-1)A_2\\
<&0.
\end{aligned} \tag{14}
$$

This is a contradiction. Similarly, we can have $\bar{t}\ge1$. Therefore, $\bar{s}=\bar{t}=1$. **Case 2:** For $u\notin\mathcal{M}_{\lambda,K}$. Using the method in ([25], page 90), the desired conclusion is obtained. □

Lemma 2. *There exists $\beta>0$ such that $\|u^\pm\|\ge\beta$ for all $u\in\mathcal{M}_{\lambda,K}$.*

Proof. For $u\in\mathcal{M}_{\lambda,K}$, we only prove the result for u^+. It is easy to check that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|^{N+ps_1}}dxdy\\
\ge&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^+(x)-u^+(y)|^p}{|x-y|^{N+ps_1}}dxdy.
\end{aligned} \tag{15}
$$

Combining with (h_1), it is shown that

$$
\|u^+\|_1^p+\|u^+\|_2^q\le\varepsilon\int_{\mathbb{R}^N}|u^+|^pdx+(1+\lambda K^{r-\sigma})C(\varepsilon)\int_{\mathbb{R}^N}|u^+|^\sigma dx. \tag{16}
$$

Choosing $\varepsilon=\frac{V_0}{2}$, we obtain

$$
\frac{1}{2}\|u^+\|_1^p+\|u^+\|_2^q\le C\|u^+\|^\sigma, \tag{17}
$$

which implies the desired conclusion. □

Lemma 3. $m_{\lambda,K}>0$.

Proof. For $u \in \mathcal{M}_{\lambda,K}$, with (h_2), (10), (15) in hand, we have

$$\begin{aligned} I_{\lambda,K}(u) =& I_{\lambda,K}(u) - \frac{1}{\theta}\langle I'_{\lambda,K}(u), u^+\rangle - \frac{1}{\theta}\langle I'_{\lambda,K}(u), u^-\rangle \\ \geq& \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u^+\|_1^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u^+\|_2^q \\ &+ \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u^-\|_1^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u^-\|_2^q. \end{aligned} \tag{18}$$

Jointly with Lemma 2, we derive $m_{\lambda,K} > 0$. □

Lemma 4. *$m_{\lambda,K}$ is achieved.*

Proof. Let $\{u_n\}$ be a minimization sequence. Since $\theta > q > p$, similar to (18), we obtain

$$\begin{aligned} m_{\lambda,K} + o_n(1) \geq& \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n^+\|_1^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n^+\|_2^q \\ &+ \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n^-\|_1^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n^-\|_2^q. \end{aligned} \tag{19}$$

Thus, $\{u_n^\pm\}$ is bounded in E. Using ([26], Lemma 2.5), up to a subsequence, there exists a $u_0 \in E$ satisfying $u_0^\pm \neq 0$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_{\lambda,K}(u_n^\pm)u_n^\pm dx = \int_{\mathbb{R}^N} h_{\lambda,K}(u_0^\pm)u_0^\pm ddx. \tag{20}$$

Here, we only prove $u_0^\pm \neq 0$. If not, in consideration of (10), we deduce that

$$0 = \langle I'_{\lambda,K}(u_n), u_n^\pm\rangle \geq \frac{1}{p}\|u_n^\pm\|_1^p + \frac{1}{q}\|u_n^\pm\|_2^q + o_n(1). \tag{21}$$

This is in contradiction with Lemma 2. Based on Lemma 1, there exist $s_0, t_0 > 0$ such that

$$\langle I'_{\lambda,K}(s_0u_0{}^+ + t_0u_0{}^-), s_0u_0{}^+\rangle = \langle I'_{\lambda,K}(s_0u_0{}^+ + t_0u_0{}^-), t_0u_0{}^-\rangle = 0. \tag{22}$$

Obviously, similar to (10), $u_n \in \mathcal{M}_{\lambda,K}$ shows that $\langle I'_{\lambda,K}(u_0), u_0^+\rangle \leq 0$. Similarly, $\langle I'_{\lambda,K}(u_0), u_0^-\rangle \leq 0$. In the spirit of ([27], Lemma 3.2), we have $0 < s_0, t_0 \leq 1$. Using (10) again, we find that

$$\|s_0u_0{}^+ + t_0u_0{}^-\|_1^p \leq \|u_0{}^+ + u_0{}^-\|_1^p. \tag{23}$$

Taking into account (h_2), we have

$$\begin{aligned} &m_{\lambda,K} \\ \leq& I_{\lambda,K}(s_0u_0{}^+ + t_0u_0{}^-) - \frac{1}{q}\langle I'(s_0u_0{}^+ + t_0u_0{}^-), (s_0u_0{}^+ + t_0u_0{}^-)\rangle \\ \leq& \liminf_{n\to\infty}\left(I_{\lambda,K}(u_n) - \frac{1}{q}\langle I'_{\lambda,K}(u_n), u_n\rangle\right) \\ =& \lim_{n\to\infty} I_{\lambda,K}(u_n) = m_{\lambda,K}. \end{aligned} \tag{24}$$

It indicates that $s_0u_0{}^+ + t_0u_0{}^-$ is the minimizer. □

3.2. *Proof of Theorem 1*

We are devoted to proving Theorem 1 in this section. From Lemmas 1–4, we find that (6) possesses a minimizer $w_0 := s_0u_0{}^+ + t_0u_0{}^-$. There are two methods to ensure that the minimizer w_0 is a critical point of $I_{\lambda,K}$. One method can be seen in ([21], Section 3). The other method can be used as ([28], lemma 3.6). Using a standard Moser iteration (see e.g., [22] or [23]), we can draw the conclusion that $\lambda_0 > 0$ such that $\|w_0\|_{L^\infty(\mathbb{R}^N)} \leq K$ when $\lambda \in (0, \lambda_0)$. Thus, w_0 is a sign-changing solution of the initial problem (1), which means that in (4), we do not make any truncations.

4. Conclusions and Future Studies

With the above analysis made, the following conclusions can be drawn. Under the assumptions of Theorem 1, problem (1) admits a sign-changing solution. As mentioned in Remark 1, the sign-changing solution w_0 does not necessarily mean that it is a least energy sign-changing solution. Our future work will study the ground state or least energy sign-changing solution to (1). Maybe it is an open problem since it appears as the supercritical term $|u|^{r-2}u$.

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References

1. Ambrosio, V.; Rădulescu, V. Fractional double-phase patterns: Concentration and multiplicity of solutions. *J. Math. Pure Appl.* **2020**, *142*, 101–145. [CrossRef]
2. Aris, R. *Mathematical Modelling Techniques*; Pitman: Boston, MA, USA, 1979.
3. Fife, P. *Mathematical Aspects of Reacting and Diffusing Systems*; Springer: Berlin/Heidelberg, Germany, 1979.
4. Hilfer, R. *Applications of Fractional Falculus in Physics*; World Scientific: Singaopore, 2000.
5. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 17–77. [CrossRef]
6. Sabatier, J.; Agrawal, O.; José, A. Advances in fractional calculus: Theoretical developments and applications in physics and engineering. *Biochem. J.* **2007**, *361*, 97–103.
7. Wilhelmsson, H. Explosive instabilities of reaction-diffusion equations. *Phys. Rev. A* **1987**, *36*, 965–966. [CrossRef] [PubMed]
8. Ambrosio, V. Fractional (p, q)-Laplacian problems in $\mathbb{R}^N$ with critical growth. *Z. Anal. Anwend.* **2020**, *39*, 289–314. [CrossRef]
9. Isernia, T. Fractional (p, q)-Laplacian problems with potentials vanishing at infinity. *Opuscula Math.* **2020**, *40*, 93–110. [CrossRef]
10. Cheng, K.; Wang, L. Existence of least energy sign-changing solution for a class of fractional (p, q)-Laplacian problems with potentials vanishing at infinity. *Complex Var. Elliptic Equ.* **2022**, *69*, 425–448. [CrossRef]
11. Gabriella, B.; Drábek, P. The p-Laplacian equation with superlinear and supercritical growth, multiplicity of radial solutions. *Nonlinear Anal.* **2005**, *60*, 719–728.
12. Sunra, M.; Kanishka, P.; Marco, S.; Yang, Y. The Brezis-Nirenberg problem for the fractional p-Laplacian. *Calc. Var. Partial Differ. Equ.* **2016**, *55*, 1–25.
13. Vladimir, B.; Tanaka, M. On subhomogeneous indefinite p-Laplace equations in the supercritical spectral interval. *Calc. Var. Partial Differ. Equ.* **2023**, *62*, 22.
14. Ambrosio, V.; Isernia, T. Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional p-Laplacian. *Discrete Contin. Dyn. Syst.* **2018**, *38*, 5835–5881. [CrossRef]
15. Bhakta, M.; Mukherjee, D. Multiplicity results for (p, q) fractional elliptic equations involving critical nonlinearities. *Adv. Differ. Equ.* **2019**, *24*, 185–228. [CrossRef]
16. Chang, X.; Nie, Z.; Wang, Z. Sign-changing solutions of fractional p-Laplacian problems. *Adv. Nonlinear Stud.* **2019**, *19*, 29–53. [CrossRef]
17. Cherfils, L.; Ilyasov, Y. On the stationary solutions of generalized reaction diffusion equations with (p, q)-Laplacian. *Commun. Pure Appl. Anal.* **2005**, *4*, 9–22. [CrossRef]

18. De, F.; Palatucci, G. Hölder regularity for nonlocal double phase equations. *J. Differ. Equ.* **2019**, *267*, 547–586.
19. Pucci, P.; Xiang, M.; Zhang, B. Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. *Calc. Var. Partial Differ. Equ.* **2015**, *54*, 2785–2806. [CrossRef]
20. Xiang, M.; Zhang, B.; Rădulescu, V. Existence of solutions for perturbed fractional p-Laplacian equations. *J. Differ. Equ.* **2016**, *260*, 1392–1413. [CrossRef]
21. Shuai, W.; Wang, Q. Existence and asymptotic behavior of sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$. *Z. Angew. Math. Phys.* **2015**, *66*, 3267–3282. [CrossRef]
22. Li, Q.; Nie, J.; Wang, W. Nontrivial solutions for fractional Schrödinger equations with electromagnetic fields and critical or supercritical growth. *Qual. Theory Dyn. Syst.* **2024**, *23*, 21. [CrossRef]
23. Wang, W.; Zhou, J.; Li, Y.; Li, Q. Existence of positive solutions for fractional Schrödinger-Poisson system with critical or supercritical growth. *Acta Math. Sin. (Chin. Ser.)* **2021**, *64*, 269–280.
24. Li, Q.; Teng, K.; Wu, X.; Wang, W. Existence of nontrivial solutions for fractional Schrödinger equations with critical or supercritical growth. *Math. Meth. Appl. Sci.* **2019**, *42*, 1480–1487. [CrossRef]
25. Feng, S.; Wang, L.; Huang, L. Least energy sign-changing solutions for fractional Kirchhoff-Schrödinger-Poisson system with critical and logarithmic nonlinearity. *Complex Var. Elliptic* **2021**, *2021*, 1–26. [CrossRef]
26. Lv, H.; Zheng, S.; Feng, Z. Existence results for nonlinear Schrödinger equations involving the fractional (p, q)-Laplacian and critical nonlinearities. *Electron. J. Differ. Equ.* **2021**, *2021*, 1–24. [CrossRef]
27. Wang, D.; Ma, Y.; Guan, W. Least energy sign-changing solutions for the fractional Schrödinger-Poisson systems in $\mathbb{R}^3$. *Bound. Value Probl.* **2019**, *25*, 18.
28. Wang, W.; Li, Q. Existence of signed and sign-changing solutions for quasilinear Schrödinger-Poisson system. *Acta Math. Sin. (Chin. Ser.)* **2018**, *61*, 685–694.