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# The Existence Results of Solutions to the Nonlinear Coupled System of Hilfer Fractional Differential Equations and Inclusions

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**Abstract:** This paper is dedicated to studying the existence results of solutions to the nonlinear coupled system of Hilfer fractional differential equations and inclusions, with multi-strip and multi-point mixed boundary conditions. Through tools such as the Leray-Schauder alternative and the nonlinear alternative of Leray-Schauder type, continuous and measurable selection theorems, along with Leray-Schauder degree theory, the main results can be obtained. The Hilfer fractional differential system has practical implications for specific physical phenomena. Examples are provided to clarify the application of our main results.

**Keywords:** the Hilfer fractional derivative; the multi-strip and multi-point mixed boundary conditions; fractional differential system; fractional differential inclusions

**MSC:** primary 34A08; 39A12; secondary 34B16

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## 1. Introduction

Classical calculus began with Newton's invention of forward flow in 1665, while fractional calculus was born in 1695 when Leibniz and L'Hospital discussed the significance of $\frac{1}{2}$ derivatives, see [1]. It can be said that fractional and integer calculus have almost the same research history span. However, fractional calculus has only been revitalized in the last few decades, thanks to the development of various fields that promote research in fractional calculus. The new study of fractional calculus can also offer novel ideas for solving challenging problems in various fields, see [2–7].

Although Riemann-Liouville and Caputo fractional derivatives are considered valuable tools for modeling many real-world problems, R. Hilfer found that the traditional fractional derivatives of Riemann-Liouville and Caputo could not meet the requirements for solving new problems during the study of fractional time evolution [8]. Therefore, in order to separate fractional integrals, a generalized definition of fractional derivatives is proposed based on the Riemann-Liouville integral by R. Hilfer, which is ${}^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^{n}I^{(1-\beta)(n-\alpha)}u(t)$, where $n-1 < \alpha < n, 0 \leq \beta \leq 1, D = \frac{d}{dt}$. Many authors later called this definition as the Hilfer fractional derivative. The reader is referred to references [9] for the distinction between the same order $\alpha$ but with different values of $\beta$.

Initial value and boundary value problems involving the Hilfer fractional derivative have attracted a lot of research. In [10], K.M. Furati discussed the existence of solutions to a Hilfer fractional differential equation for the following initial value problem. Moreover, the stability of the solution to a weighted Cauchy-type problem is also analyzed.

$$\begin{cases} {}^{H}D_{a^+}^{\alpha,\beta}y(x) = f(x,y),\ a < x, 0 < \alpha < 1, 0 \leq \beta \leq 1, \\ I_{a^+}^{1-\gamma}y(a^+) = y_a, \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha$, and type $\beta$.

In [11], K. Dhawan investigated the coupled Hilfer fractional differential equations with nonlocal conditions. By applying the Leray-alternative Schauder's and the Contraction principle, the author proved the existence and uniqueness of the solution. Furthermore, the Ulam stability of the solution was discussed for the defined problem.

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}x(t) = f(t,x(t),y(t)), \alpha_1 < v_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1, t \in [0,T], \\ {}^{H}D^{\alpha_2,\beta_2}y(t) = f(t,x(t),y(t)), \alpha_2 < v_2 = \alpha_2 + \beta_2 - \alpha_2\beta_2, t \in [0,T], \\ I_{0^+}^{1-v_1}x(0) = \sum_{i=1}^{n} b_{1i}x(\xi_i), \xi_i \in [0,T], \\ I_{0^+}^{1-v_2}y(0) = \sum_{i=1}^{n} b_{2i}y(\eta_i), \eta_i \in [0,T], \end{cases} \tag{2}$$

where ${}^{H}D^{\alpha_i,\beta_i}$ represents the Hilfer fractional derivative of order $\alpha_i$ and type $\beta_i$, $i = 1,2$, $\alpha_i \in (0,1)$, $\beta_i \in [0,1]$. $I_{0^+}^{1-v_i}$ is the left-side Riemann-Liouville integral of order $1 - v_i$, $i = 1,2$. Also, $f_1, f_2 : [0,T] \times R \times R \to R$ are given continuous nonlinear functions and $T > 0$, $b_{11}, b_{12}$ are real numbers, $\xi_i, \eta_i (i = 1,2,\ldots,m)$ are prefixed points satisfying $0 < \xi_1 < \xi_2 < \ldots < T$ and $0 < \eta_1 < \eta_2 < \ldots < T$, respectively.

In [12], B. Ahmad studied the nonlinear generalized coupled fractional differential equations accompanied by nonlocal coupled multipoint Riemann-Stieltjes and generalized fractional integral boundary conditions using the Leray-Schauder alternative and Banach contraction mapping principle.

$$\begin{cases} {}^{\rho}D_{0^+}^{\alpha}u(t) = f(t,u(t),v(t),{}^{\rho}D_{0^+}^{\gamma_1}v(t)), t \in [0,T], \\ {}^{\rho}D_{0^+}^{\beta}v(t) = f(t,u(t),{}^{\rho}D_{0^+}^{\gamma_2}u(t),v(t)), t \in [0,T], \\ u(0) = v(0) = 0, \\ \int_0^T u(s)dH_1(s) = \lambda_1^{\rho} I_{0^+}^{\delta_1}v(\xi_1) + \sum_{p=1}^{m} a_p v(\eta_p), \xi_1 a \in (0,T), \\ \int_0^T v(s)dH_2(s) = \lambda_2^{\rho} I_{0^+}^{\delta_2}v(\xi_2) + \sum_{p=1}^{m} b_p u(\eta_p), \xi_2 \in (0,T), \end{cases} \tag{3}$$

where ${}^{\rho}D_{0^+}^{\alpha}$ and ${}^{\rho}D_{0^+}^{\beta}$ are the generalized fractional derivative operators of order $1 < \alpha, \beta < 2$, respectively, $0 < \gamma_1, \gamma_2 < 1$, ${}^{\rho}I_{0^+}^{\delta_1}$ and ${}^{\rho}I_{0^+}^{\delta_2}$ are the generalized fractional integral operators of order $\delta_1, \delta_2 > 0$, respectively, $\int_0^T u(s)dH_i(s) (i = 1,2)$ are the Riemann-Stieltjes integrals with respect to the functions $H_i : [0,T] \to \mathbb{R}$, $f, g \in C([0,T] \times \mathbb{R}^3, \mathbb{R})$, $\lambda_1, \lambda_2, a_p, b_p \in \mathbb{R}$ and $\eta_p \in (0,T), p = 1,2,\ldots,m$.

It is worth noting that J. Pradeesh considered the existence of a mild solution for the following Hilfer fractional stochastic differential system in [13].

$$\begin{cases} {}^{H}D_{0^+}^{v,\mu}z(t) \in Az(t) + f(t,z(t)) + G(t,z(t))\dfrac{dW(t)}{dt}, t \in (0,C], \\ (I_{0^+}^{2-\gamma}z)(0) = z_0, \ (I_{0^+}^{2-\gamma}z)'(0) = z_1, \end{cases} \tag{4}$$

where ${}^{H}D_{0^+}^{v,\mu}$ denotes the Hilfer fractional derivative of order $\mu$ and type $v$ such that $1 < \mu < 2$ and $0 \le v \le 1$, while $I_{0^+}^{2-\gamma}$ is the Riemann–Liouville integral operator with order $(2-\gamma)$, where $\gamma = \mu + v(2-\mu)$. $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family $C(t)_{t\ge 0}$ of uniformly bounded linear operators and $z(\cdot)$ takes values in a separable Hilbert space $H$ with inner product $\langle \cdot,\cdot \rangle$ and norm $\| \cdot \|$. The authors employed fractional calculus, multivalued analysis, sine and cosine operators, and Bohnenblust–Karlin's fixed point theorem to investigate the existence of a mild solution for the Hilfer fractional stochastic differential system.

However, there are very few articles that have studied Hilfer fractional differential equations and inclusions simultaneously. Inspired by the aforementioned works, we are initiating a study on the existence of solutions to a coupled system of Hilfer fractional differential equations

$$\begin{cases} {}^H D^{\alpha_1,\beta} u(t) = f_1(t, u(t), v(t), {}^H D^{\gamma_1,\beta} u(t), {}^H D^{\gamma_2,\beta} v(t)), & t \in [0,1], \\ {}^H D^{\alpha_2,\beta} v(t) = f_2(t, u(t), v(t), {}^H D^{\gamma_1,\beta} u(t), {}^H D^{\gamma_2,\beta} v(t)), & t \in [0,1], \end{cases} \tag{5}$$

and inclusions

$$\begin{cases} {}^H D^{\alpha_1,\beta} u(t) \in F(t, u(t), v(t), {}^H D^{\gamma_1,\beta} u(t), {}^H D^{\gamma_2,\beta} v(t)), & t \in [0,1], \\ {}^H D^{\alpha_2,\beta} v(t) \in G(t, u(t), v(t), {}^H D^{\gamma_1,\beta} u(t), {}^H D^{\gamma_2,\beta} v(t)), & t \in [0,1], \end{cases} \tag{6}$$

subject to the coupled fractional integral and discrete mixed boundary conditions

$$\begin{cases} u(0) = 0, \ \ v(0) = 0, \\ u(1) = \sum_{i=1}^{m} \lambda_{1i} I^{l_{1i}} v(\xi_i) + \sum_{j=1}^{n} b_{1j} v(\eta_j), \\ v(1) = \sum_{i=1}^{m} \lambda_{2i} I^{l_{2i}} u(\xi_i) + \sum_{j=1}^{n} b_{2j} u(\eta_j), \end{cases} \tag{7}$$

where ${}^H D^{\alpha_k,\beta}$, ${}^H D^{\gamma_k,\beta}$ denote the Hilfer fractional derivative of order $\alpha_k$, $\gamma_k$ respectively and parameter $\beta$, $I^{l_{ki}}$ is the Riemann-Liouville fractional integral of order $l_{ki}$, $1 < \alpha_k \leq 2$, $0 \leq \beta \leq 1$, $0 < \gamma_k < \alpha_k - 1$, $\lambda_{ki} \geq 0$, $l_{ki} \geq 0$, $0 \leq \xi_i \leq 1$, $b_{kj} \geq 0$, $0 \leq \eta_j \leq 1$, for $k = 1,2$; $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

In the research of fractional time evolution, a significant challenge arises when generalizing traditional equations of motion, as it involves deciding whether to utilize the Riemann-Liouville fractional derivative or the Caputo fractional derivative. R. Hilfer introduced the Hilfer fractional derivative during the study. When $\beta = 0$, the Hilfer system can be simplified to the Riemann-Liouville system studied by authors such as X. Zhao, A. Guezane-Lakoud, T. Jankowski, and others. For more details, refer to [14–16]. When $\beta = 1$, the Hilfer system can be reduced to the Caputo system discussed by investigators like Y. Zi, A. Alsaedi, J. Xie, B. Ahmad, and others. References can be found in [17–20]. We learned that models of fractional diffusion equations with Hilfer fractional derivatives are used in the context of glass relaxation and aquifer problems [21]. Also, fractional reaction-diffusion equations and space-time fractional diffusion equations involving the Hilfer fractional derivative are studied in [22,23]. And more importantly, differential inclusion, which can handle uncertainty problems, has been applied to dynamical systems and stochastic processes, such as control problems and sweeping processes [24,25]. Therefore, our studies on the nonlinear coupled Hilfer fractional differential inclusions have great practical applications in physical phenomena.

In the coupled Hilfer fractional differential system, there are $u(t)$, $v(t)$ that mutually influence each other. The nonlinear terms, $f_1$ and $f_2$, consist of $u(t)$, $v(t)$, ${}^H D^{\gamma_1,\beta} u(t)$, ${}^H D^{\gamma_2,\beta} v(t)$, where $\gamma_1, \gamma_2$ are less than $\alpha_1, \alpha_2$. The inclusion of ${}^H D^{\gamma_1,\beta} u(t)$, ${}^H D^{\gamma_2,\beta} v(t)$ in $f_1, f_2$ enhances the model's capability to address real-world problems The coupled Hilfer fractional differential inclusions have the same structure. Moreover, the nonlocal boundary conditions consist of the Riemann-Liouville fractional integral and numerous discrete points. The value of the unknown function $u(t)$ at the right endpoint $t = 1$ is equal to the sum of the values of the Riemann–Liouville fractional integral of the unknown function $v(t)$ on the subinterval $[0, \xi_i] (i = 1,2,\ldots,m)$ and the discrete values of the unknown function $v(t)$ at $\eta_j (j = 1,2,\ldots,n)$.

In addition, there are now many well-established methods for studying fractional differential equations and inclusions, such as Guo-Krasnoselskii's fixed-point theorem, the Banach contraction mapping principle, and monotone iteration techniques. We employed the Leray-Schauder alternative and the nonlinear alternative of Leray-Schauder type, continuous and measurable selection theorems, along with Leray-Schauder degree theory, to explore the existence of solutions for the Hilfer fractional differential equations and inclusions, respectively.

In fact, fractional derivatives have been greatly developed and applied, leading to the emergence of several mature definitions such as the Riemann-Liouville fractional derivative, the Hadamard fractional derivative, the Caputo-Katugampola fractional derivative, the Katugampola fractional derivative, and others. In contrast, the derivative under Hilfer's definition requires more research efforts to promote its development. Whether transferring mature research techniques to Hilfer or developing new technical methods, the work is meaningful. Differential inclusions can be regarded as a collection of differential equations and inequalities. Moreover, this paper examines the coupled system of Hilfer differential equations and inclusions, which is of great significance for practical applications.

The rest of the paper is structured as follows: In Section 2, some fundamental concepts of fractional calculus and lemmas are presented. Section 3 is dedicated to presenting the main results, which are illustrated through examples. Section 4 contains a summary of previous work and future prospects.

## 2. Preliminaries

In this section, we present some basic definitions, lemmas, and auxiliary results for the proof that will be utilized in the next section.

Let $C(J)$, $AC(J)$ and $C^n(J)$ denote the spaces of continuous, absolutely continuous and $n$ times continuously differentiable functions on $J := [0,1]$, respectively. We denote by $L^p(0,1)$, $p \geq 1$, the spaces of Lebesgue integrable functions on $(0,1)$.

**Definition 1** ([10,26])**.** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $u \in L^1(0,1)$ is defined by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds,\ n-1 < \alpha < n,$$

*where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of real number $\alpha$.*

**Definition 2** ([10,26])**.** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u \in AC(0,1)$ is defined by*

$$D^{\alpha}u(t) := D^n I^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}u(s)ds,\ n-1 < \alpha < n,$$

*where $n = [\alpha]+1$.*

**Definition 3** ([27])**.** *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u \in AC(0,1)$ is defined by*

$${}^{C}D^{\alpha}u(t) := I^{n-\alpha}D^n u(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n u(s)ds,\ n-1 < \alpha < n,$$

*where $n = [\alpha]+1$.*

**Definition 4** ([10])**.** *The Hilfer fractional derivative of order $\alpha$ and parameter $\beta$ of a function $u \in AC(0,1)$ is defined by*

$${}^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^n I^{(1-\beta)(n-\alpha)}u(t),$$

*where $n-1 < \alpha < n, 0 \leq \beta \leq 1, t > a, D = \frac{d}{dt}$.*

**Remark 1.** *From Definitions 2 and 3, when $\beta = 0$, the Hilfer fractional derivative corresponds to the Riemann-Liouville fractional derivative*

$$^{H}D^{\alpha,0}u(t) = D^{n}I^{n-\alpha}u(t),$$

*while when $\beta = 1$, the Hilfer fractional derivative corresponds to the Caputo fractional derivative*

$$^{H}D^{\alpha,1}u(t) = I^{n-\alpha}D^{n}u(t).$$

For convenience, let $\theta = \alpha + 2\beta - \alpha\beta$, $1 < \alpha \leq 2$, we can know $1 < \theta \leq 2$ and $0 \leq 2-\theta < 1$. We have the weighted spaces of continuous functions

$$C_{2-\theta}(J) = \{u : (0,1] \to R : t^{2-\theta}u(t) \in C(J)\}, \tag{8}$$

with the norm

$$\|u\|_{C_{2-\theta}} = \sup_{t\in J} |t^{2-\theta}u(t)|, \tag{9}$$

and

$$C^{\alpha,\beta}_{2-\theta}(J) = \{u \in C_{2-\theta}(J), ^{H}D^{\alpha,\beta}u \in C_{2-\theta}(J)\}, \tag{10}$$

$$C^{\theta}_{2-\theta}(J) = \{u \in C_{2-\theta}(J), D^{\theta}u \in C_{2-\theta}(J)\}. \tag{11}$$

**Lemma 1** ([10])**.** *For $\alpha > 0$, $I^{\alpha}$ maps $C(J)$ into $C(J)$.*

**Lemma 2** ([10])**.** *Let $\alpha > 0$, and $0 \leq 2-\theta < 1$. Then $I^{\alpha}$ is bounded from $C_{2-\theta}(J)$ into $C_{2-\theta}(J)$.*

**Lemma 3** ([10])**.** *Let $\alpha > 0$, and $0 \leq 2-\theta < 1$. If $2-\theta \leq \alpha$, then $I^{\alpha}$ is bounded from $C_{2-\theta}(J)$ into $C(J)$.*

**Lemma 4** ([27])**.** *If $\alpha > 0, b > 0$, then*

$$I^{\alpha}t^{b} = \frac{\Gamma(b+1)}{\Gamma(b+1+\alpha)}t^{b+\alpha},$$

$$D^{\alpha}t^{b} = \frac{\Gamma(b+1)}{\Gamma(b+1-\alpha)}t^{b-\alpha},$$

$$^{H}D^{\alpha,\beta}t^{b} = \frac{\Gamma(b+1)}{\Gamma(b+1-\alpha)}t^{b-\alpha}.$$

**Lemma 5** ([10])**.** *Let $0 \leq 2-\theta < 1$ and $u \in C_{2-\theta}[0,1]$. Then*

$$I^{\alpha}u(0) := \lim_{t\to 0^{+}} I^{\alpha}u(t) = 0,\ 0 \leq 2-\theta < \alpha.$$

**Lemma 6** ([10])**.** *Let $\alpha > 0$, $0 \leq 2-\theta < 1$, and $u \in C_{2-\theta}(J)$. Then*

$$D^{\alpha}I^{\alpha}u(t) = u(t), for\ \ all\ t \in (0,1].$$

**Lemma 7** ([10])**.** *For $t > 0$, we have*

$$D^{\alpha}_{0^{+}}t^{\alpha-1} = 0,\ 1 < \alpha \leq 2.$$

**Lemma 8** ([27])**.** *Let $1 < \alpha \leq 2$, $0 \leq 2-\theta < 1$. If $u \in C_{2-\theta}(J)$, and $I^{2-\alpha}u \in C^{2}_{2-\theta}(J)$, then*

$$I^{\alpha}D^{\alpha}u(t) = u(t) - \frac{(I^{1-\alpha}u)(t)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{(I^{2-\alpha}u)(t)}{\Gamma(\alpha-1)}t^{\alpha-2}.$$

*for all $t \in (0,1]$.*

**Lemma 9** ([27]). *Let $1 < \alpha \le 2$, and $0 \le \beta \le 1$. If $u \in C_{2-\theta}(J)$, and $I^{1-\beta(2-\alpha)}u \in C^1_{2-\theta}(J)$, then ${}^H D^{\alpha,\beta} I^\alpha u$ exists in $(0,1]$ and*

$$ {}^H D^{\alpha,\beta} I^\alpha u(t) = u(t),\ t \in (0,1]. $$

**Lemma 10** ([10]). *Let $1 < \alpha \le 2$, and $0 \le \beta \le 1$. If $f \in C^\theta_{2-\theta}(J)$, then*

$$ I^\theta D^\theta u = I^\alpha\, {}^H D^{\alpha,\beta} u,\ D^\theta I^\alpha u = D^{\beta(2-\alpha)} u. $$

**Lemma 11** (Leray–Schauder alternative [28]). *Let $X$ be a Banach space, $T : X \to X$ be a completely continuous operator. Let*

$$ \xi(T) = \{x \in X : X = \lambda T(x), 0 < \lambda < 1\}. $$

*Then, either the set $\xi(T)$ is unbounded, or $T$ has at least one fixed point.*

**Lemma 12** (Nonlinear alternative of Leray-Schauder type [29]). *Let $X$ be a Banach space, $\Omega$ be a closed convex subset of $X$, and $U$ be an open subset of $\Omega$ with $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(\Omega)$ is an upper semicontinuous compact map. Then either (1) $F$ has a fixed point in $\overline{U}$, or (2) there is $\partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda F(x)$.*

For each $z \in C(J)$, we defined the set of selections of $F$ by

$$ S_{F\circ z} = \{w \in L^1(J) : w(t) \in F(t, z(t)), a.e.\ t \in J\}. $$

For convenience, we denote

$$ \begin{cases} l_1 = \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\theta_2 - 1} ds + \sum_{j=1}^{n} b_{1j} \eta_j^{\theta_2 - 1}, \\ l_2 = \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\theta_1 - 1} ds + \sum_{j=1}^{n} b_{2j} \eta_j^{\theta_1 - 1}. \end{cases} \tag{12} $$

In the forthcoming analysis, we always need to make the following assumptions:

($F_1$) $1 < \alpha_k \le 2$, $0 < \gamma < \alpha_k - 1$, $l_{ki} > 0, 1 < \theta_k \le 2$, where $\theta_k = \alpha_k + 2\beta - \alpha_k\beta$, for $k = 1,2$ and $i = 1,2,\ldots,m$;
($F_2$) $0 \le \xi_i, \eta_j \le 1$, $\lambda_{1i}, \lambda_{2i} \ge 0$, $b_{1j}, b_{2j} \ge 0$, for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$;
($F_3$) $1 - l_1 l_2 > 0$, where $l_1, l_2$ are defined by (12).

Subject to BVP (5) and (7), we consider a corresponding linear differential system as follows and establish the expression of the corresponding Green's functions.

**Lemma 13.** *Assume that ($F_1$)–($F_3$) hold. For $h_1 \in C_{2-\theta_1}(J), h_2 \in C_{2-\theta_2}(J)$, the fractional differential system*

$$ \begin{cases} {}^H D^{\alpha_1,\beta} u(t) = h_1(t), \\ {}^H D^{\alpha_2,\beta} v(t) = h_2(t), \end{cases} \tag{13} $$

*with boundary conditions (5) has an integral representation*

$$ \begin{aligned} u(t) = {} & I^{\alpha_1} h_1(t) + \frac{t^{\theta_1 - 1}}{(1 - l_1 l_2)} \\ & \Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} h_1(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} h_1(s) ds - I^{\alpha_2} h_2(1) \Bigg) \\ & + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} h_2(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} h_2(s) ds - I^{\alpha_1} h_1(1) \Bigg], \end{aligned} \tag{14} $$

$$v(t) = I^{\alpha_2}h_2(t) + \frac{t^{\theta_2-1}}{(1-l_1l_2)}$$
$$\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds - I^{\alpha_1}h_1(1)\right)\right. \tag{15}$$
$$\left.+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds - I^{\alpha_2}h_2(1)\right].$$

**Proof.** From Definition 1 and Lemma 10, using $I^{\alpha_1}$, $I^{\alpha_2}$ to (13), we have:

$$\begin{cases} I^{\theta_1}D^{\theta_1}u(t) = I^{\alpha_1}h_1(t),\ \theta_1 = \alpha_1 + \beta(2-\alpha_1), \\ I^{\theta_2}D^{\theta_2}v(t) = I^{\alpha_2}h_2(t),\ \theta_2 = \alpha_2 + \beta(2-\alpha_2). \end{cases} \tag{16}$$

From Lemma 8, we obtain

$$\begin{cases} I^{\theta_1}D^{\theta_1}u(t) = u(t) - \frac{(I^{1-\theta_1}u)(t)}{\Gamma(\theta_1)}t^{\theta_1-1} - \frac{(I^{2-\theta_1}u)(t)}{\Gamma(\theta_1-1)}t^{\theta_1-2}, \\ I^{\theta_2}D^{\theta_2}v(t) = v(t) - \frac{(I^{1-\theta_2}u)(t)}{\Gamma(\theta_2)}t^{\theta_2-1} - \frac{(I^{2-\theta_2}v)(t)}{\Gamma(\theta_2-1)}t^{\theta_2-2}. \end{cases} \tag{17}$$

Combining (16) and (17), we can simplify (13) to the following equivalent integral equations

$$\begin{cases} u(t) = I^{\alpha_1}h_1(t) + \frac{c_{11}}{\Gamma(\theta_1)}t^{\theta_1-1} + \frac{c_{12}}{\Gamma(\theta_1-1)}t^{\theta_1-2}, \\ v(t) = I^{\alpha_2}h_2(t) + \frac{c_{21}}{\Gamma(\theta_2)}t^{\theta_2-1} + \frac{c_{22}}{\Gamma(\theta_2-1)}t^{\theta_2-2}, \end{cases} \tag{18}$$

where $c_{11}$, $c_{12}$, $c_{21}$, $c_{22}$ are constants.

From $u(0) = v(0) = 0$, we obtain $c_{12} = c_{22} = 0$, and we get

$$\begin{cases} u(t) = I^{\alpha_1}h_1(t) + \frac{c_{11}}{\Gamma(\theta_1)}t^{\theta_1-1}, \\ v(t) = I^{\alpha_2}h_2(t) + \frac{c_{21}}{\Gamma(\theta_2)}t^{\theta_2-1}. \end{cases} \tag{19}$$

From the remaining conditions in (5), it can be inferred that

$$\begin{cases} u(1) = I^{\alpha_1}h_1(1) + \frac{c_{11}}{\Gamma(\theta_1)} = \sum_{i=1}^{m}\lambda_{1i}I^{l_{1i}}v(\xi_i) + \sum_{j=1}^{n}b_{1j}v(\eta_j), \\ v(1) = I^{\alpha_2}h_2(1) + \frac{c_{21}}{\Gamma(\theta_2)} = \sum_{i=1}^{m}\lambda_{2i}I^{l_{2i}}u(\xi_i) + \sum_{j=1}^{n}b_{2j}u(\eta_j). \end{cases} \tag{20}$$

$$\begin{cases} c_{11} = \Gamma(\theta_1)\left(\sum_{i=1}^{m}\lambda_{1i}I^{l_{1i}}v(\xi_i) + \sum_{j=1}^{n}b_{1j}v(\eta_j) - I^{\alpha_1}h_1(1)\right), \\ c_{21} = \Gamma(\theta_2)\left(\sum_{i=1}^{m}\lambda_{2i}I^{l_{2i}}u(\xi_i) + \sum_{j=1}^{n}b_{2j}u(\eta_j) - I^{\alpha_2}h_2(1)\right). \end{cases} \tag{21}$$

Further, we can reduce (18) to

$$\begin{cases} u(t) = I^{\alpha_1}h_1(t) + t^{\theta_1-1}\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}v(s)ds + \sum_{j=1}^{n}b_{1j}v(\eta_j) - I^{\alpha_1}h_1(1)\right), \\ v(t) = I^{\alpha_2}h_2(t) + t^{\theta_2-1}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}u(s)ds + \sum_{j=1}^{n}b_{2j}u(\eta_j) - I^{\alpha_2}h_2(1)\right). \end{cases} \tag{22}$$

Then we can get

$$\begin{aligned}&\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}v(s)ds+\sum_{j=1}^{n}b_{1j}v(\eta_j)-I^{\alpha_1}h_1(1)\\&=\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\theta_2-1}ds+\sum_{j=1}^{n}b_{1j}\eta_j^{\theta_2-1}\right)\\&\quad\cdot\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}u(s)ds+\sum_{j=1}^{n}b_{2j}u(\eta_j)-I^{\alpha_2}h_2(1)\right)\\&\quad+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^{s}(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds-I^{\alpha_1}h_1(1),\end{aligned}\tag{23}$$

$$\begin{aligned}&\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}u(s)ds+\sum_{j=1}^{n}b_{2j}u(\eta_j)-I^{\alpha_2}h_2(1)\\&=\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\theta_1-1}ds+\sum_{j=1}^{n}b_{2j}\eta_j^{\theta_1-1}\right)\\&\quad\cdot\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}v(s)ds+\sum_{j=1}^{n}b_{1j}v(\eta_j)-I^{\alpha_1}h_1(1)\right)\\&\quad+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^{s}(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds-I^{\alpha_2}h_2(1).\end{aligned}\tag{24}$$

Combining (23) and (24), it can be seen that

$$\begin{aligned}&\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}v(s)ds+\sum_{j=1}^{n}b_{1j}v(\eta_j)-I^{\alpha_1}h_1(1)\\&=\frac{1}{1-l_1l_2}\Bigg[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^{s}(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds-I^{\alpha_2}h_2(1)\right)\\&+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^{s}(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds-I^{\alpha_1}h_1(1)\Bigg],\\&\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}u(s)ds+\sum_{j=1}^{n}b_{1j}u(\eta_j)-I^{\alpha_2}h_2(1)\\&=\frac{1}{1-l_1l_2}\Bigg[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^{s}(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds-I^{\alpha_1}h_1(1)\right)\\&+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^{s}(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds-I^{\alpha_2}h_2(1)\Bigg],\end{aligned}\tag{25}$$

where $l_k$ $(k=1,2)$ is defined by (12). From (22) and (25), we have

$$\begin{aligned}&u(t)=I^{\alpha_1}h_1(t)+\frac{t^{\theta_1-1}}{(1-l_1l_2)}\\&\Bigg[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^{s}(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds-I^{\alpha_2}h_2(1)\right)\\&+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^{s}(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds+\sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds-I^{\alpha_1}h_1(1)\Bigg],\end{aligned}\tag{26}$$

$$v(t) = I^{\alpha_2}h_2(t) + \frac{t^{\theta_2-1}}{(1-l_1l_2)}$$
$$\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds - I^{\alpha_1}h_1(1)\right)\right. \tag{27}$$
$$\left. + \sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds - I^{\alpha_2}h_2(1)\right].$$

This completes the proof of the lemma. □

Moreover, according to Lemma 4, and fractional order derivative of solution (11) can be expressed as

$${}^{H}D^{\gamma_1,\beta}u(t) = I^{\alpha_1-\gamma_1}h_1(t) + \frac{\Gamma(\theta_1)t^{\theta_1-\gamma_1-1}}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)}$$
$$\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds - I^{\alpha_2}h_2(1)\right)\right. \tag{28}$$
$$\left. + \sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds - I^{\alpha_1}h_1(1)\right],$$

$${}^{H}D^{\gamma_2,\beta}v(t) = I^{\alpha_2-\gamma_2}h_2(t) + \frac{\Gamma(\theta_2)t^{\theta_2-\gamma_2-1}}{\Gamma(\theta_2-\gamma_2)(1-l_1l_2)}$$
$$\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}h_2(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}h_2(s)ds - I^{\alpha_1}h_1(1)\right)\right. \tag{29}$$
$$\left. + \sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}h_1(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}h_1(s)ds - I^{\alpha_2}h_2(1)\right].$$

**Lemma 14.** *Let $f_k : J \times R^4 \to R$ be a function such that $f_k \in C_{2-\theta_k}(J), k = 1,2$. If $u \in C^{\theta_1}_{2-\theta_1}(J)$, $v \in C^{\theta_2}_{2-\theta_2}(J)$, then $u, v$ satisfie (5) and (7) if and only if $u, v$ satisfies (30) and (31).*

$$u(t) = I^{\alpha_1}f_1(u,v)(t) + \frac{t^{\theta_1-1}}{(1-l_1l_2)}$$
$$\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}f_1(u,v)(s)ds - I^{\alpha_2}f_2(u,v)(1)\right)\right. \tag{30}$$
$$\left. + \sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}f_2(u,v)(s)ds - I^{\alpha_1}f_1(u,v)(1)\right],$$

$$v(t) = I^{\alpha_2}f_2(u,v)(t) + \frac{t^{\theta_2-1}}{(1-l_1l_2)}$$
$$\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}f_2(u,v)(s)ds - I^{\alpha_1}f_1(u,v)(1)\right)\right. \tag{31}$$
$$\left. + \sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}f_1(u,v)(s)ds - I^{\alpha_2}f_2(u,v)(1)\right],$$

*where*

$$\begin{cases} f_1(u,v)(s) \triangleq f_1(s,u(s),v(s),{}^{H}D^{\gamma_1,\beta}u(s),{}^{H}D^{\gamma_2,\beta}v(s)), \\ f_2(u,v)(s) \triangleq f_2(s,u(s),v(s),{}^{H}D^{\gamma_1,\beta}u(s),{}^{H}D^{\gamma_2,\beta}v(s)). \end{cases} \tag{32}$$

**Proof.** First we prove the necessity. Let $u \in C^{\theta_1}_{2-\theta_1}(J)$, $v \in C^{\theta_2}_{2-\theta_2}(J)$ be a solution of (5) and (7). We want to prove that $u, v$ are the solutions of the integral Equations (30) and (31). By the definition of $C^{\theta}_{2-\theta}(J)$, Lemma 3, we have

$$I^{2-\theta_1}u \in C(J),\ I^{2-\theta_2}v \in C(J), \tag{33}$$

and

$$D^{\theta_1}u = D^2(I^{2-\theta_1}u) \in C_{2-\theta_1}(J), D^{\theta_2}v = D^2(I^{2-\theta_2}v) \in C_{2-\theta_2}(J). \tag{34}$$

Thus by (11), we have

$$I^{2-\theta_1}u \in C^2_{2-\theta_1}(J), I^{2-\theta_2}v \in C^2_{2-\theta_2}(J). \tag{35}$$

We apply Lemma 8 to obtain

$$\begin{aligned} I^{\theta_1}D^{\theta_1}u(t) &= u(t) - \frac{(I^{1-\theta_1}u)(t)}{\Gamma(\theta_1)}t^{\theta_1-1} - \frac{(I^{2-\theta_2}u)(t)}{\Gamma(\theta_1-1)}t^{\theta_1-2}, \\ I^{\theta_2}D^{\theta_2}v(t) &= v(t) - \frac{(I^{1-\theta_2}v)(t)}{\Gamma(\theta_2)}t^{\theta_2-1} - \frac{(I^{2-\theta_2}v)(t)}{\Gamma(\theta_2-1)}t^{\theta_2-2}. \end{aligned} \tag{36}$$

Since by $D^{\theta_1}u \in C_{2-\theta_1}$, $D^{\theta_2}v \in C_{2-\theta_2}$, Lemma 10, we have

$$I^{\theta_1}D^{\theta_1}u = I^{\alpha_1}\ {}^{H}D^{\alpha_1,\beta}u = I^{\alpha_1}f_1,\ I^{\theta_2}D^{\theta_2}v = I^{\alpha_2}\ {}^{H}D^{\alpha_1,\beta}v = I^{\alpha_2}f_2. \tag{37}$$

Combining (36) and (37), we have

$$\begin{cases} u(t) = I^{\alpha_1}f_1(u,v)(t) + \dfrac{d_{11}}{\Gamma(\theta_1)}t^{\theta_1-1} + \dfrac{d_{12}}{\Gamma(\theta_1-1)}t^{\theta_1-2}, \\ v(t) = I^{\alpha_2}f_2(u,v)(t) + \dfrac{d_{21}}{\Gamma(\theta_2)}t^{\theta_2-1} + \dfrac{d_{22}}{\Gamma(\theta_2-1)}t^{\theta_2-2}, \end{cases} \tag{38}$$

where $d_{11}$, $d_{12}$, $d_{21}$, $d_{22}$ are constants. According to boundary conditions (5) and calculation steps in Lemma 13, we obtain

$$\begin{aligned} u(t) &= I^{\alpha_1}f_1(u,v)(s) + \frac{t^{\theta_1-1}}{(1-l_1l_2)} \\ &\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}f_1(u,v)(s)ds - I^{\alpha_2}f_2(u,v)(1)\right)\right. \\ &\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}f_2(u,v)(s)ds - I^{\alpha_1}f_1(u,v)(1)\right], \end{aligned} \tag{39}$$

$$\begin{aligned} v(t) &= I^{\alpha_2}f_2(u,v)(s) + \frac{t^{\theta_2-1}}{(1-l_1l_2)} \\ &\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}f_2(u,v)(s)ds - I^{\alpha_1}f_1(u,v)(1)\right)\right. \\ &\left.+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}f_1(u,v)(s)ds - I^{\alpha_2}f_2(u,v)(1)\right], \end{aligned} \tag{40}$$

which is the integral Equations (30) and (31).

Now we prove the sufficiency. Let $u \in C^{\theta_1}_{2-\theta_1}(J)$, $v \in C^{\theta_2}_{2-\theta_2}(J)$ satisfy (30) and (31). Since ${}^{H}D^{\alpha_1,\beta}u = I^{\beta(2-\alpha_1)}D^{\theta_1}u$, ${}^{H}D^{\alpha_2,\beta}v = I^{\beta(2-\alpha_2)}D^{\theta}v$, it follows from Lemma 2 that $C^{\theta_1}_{2-\theta_1}(J) \subset$

$C^{\alpha_1,\beta}_{2-\theta_1}(J)$ and $C^{\theta_1}_{2-\theta_1}(J) \subset C^{\alpha_1,\beta}_{2-\theta_1}(J)$. We can get ${}^{H}D^{\alpha_1,\beta}u$ and ${}^{H}D^{\alpha_2,\beta}v$ exist. Applying the operator $D^{\theta_1}$, $D^{\theta_2}$ to (30) and (31), it follows from Lemma 7, Lemma 10 that

$$D^{\theta_1}u = D^{\beta(2-\alpha_1)}f_1,\ D^{\theta_2}v = D^{\beta(2-\alpha_2)}f_2. \tag{41}$$

From (41) and $D^{\theta_1}u \in C_{2-\theta_1}(J)$, $D^{\theta_2}v \in C_{2-\theta_2}(J)$, we have

$$DI^{1-\beta(2-\alpha_1)}f_1 = D^{\beta(2-\alpha_1)}f_1 \in C_{2-\theta_1}(J), \tag{42}$$

$$DI^{1-\beta(2-\alpha_2)}f_2 = D^{\beta(2-\alpha_2)}f_2 \in C_{2-\theta_2}(J). \tag{43}$$

Since $f_k \in C_{2-\theta_k}$, $k = 1,2$, by Lemma 2, we have

$$I^{1-\beta(2-\alpha_1)}f_1 \in C_{2-\theta_1}(J),\ I^{1-\beta(2-\alpha_2)}f_2 \in C_{2-\theta_2}(J). \tag{44}$$

It follows from (11), (42)–(44) that

$$I^{1-\beta(2-\alpha_1)}f_1 \in C^1_{2-\theta_1}(J),\ I^{1-\beta(2-\alpha_2)}f_2 \in C^1_{2-\theta_2}(J). \tag{45}$$

Now by applying $I^{\beta(2-\alpha_1)}$, $I^{\beta(2-\alpha_2)}$ to (41), using Lemma 8, we have

$${}^{H}D^{\alpha_1,\beta}u(t) = f_1 + \frac{(I^{1-\beta(2-\alpha_1)}f_1)(0)}{\Gamma(\beta(2-\alpha_1))}t^{\beta(2-\alpha_1)-1}, \tag{46}$$

$${}^{H}D^{\alpha_2,\beta}v(t) = f_2 + \frac{(I^{1-\beta(2-\alpha_2)}f_2)(0)}{\Gamma(\beta(2-\alpha_2))}t^{\beta(2-\alpha_2)-1}. \tag{47}$$

According to $\theta_k = \alpha_k + 2\beta - \alpha_k\beta$, $1 < \alpha_k \le 2$ and $0 \le \beta \le 1$, we have $2 - \theta_k < 1 - \beta(2-\alpha_k)$ for $k = 1,2$. Lemma 5 implies that

$$(I^{1-\beta(2-\alpha_1)}f_1)(0) = 0,\ (I^{1-\beta(2-\alpha_2)}f_2)(0) = 0. \tag{48}$$

Hence the (46) and (47) reduce to

$${}^{H}D^{\alpha_1,\beta}u(t) = f_1,\ {}^{H}D^{\alpha_2,\beta}v(t) = f_2. \tag{49}$$

Now we show that the boundary conditions (5) also hold. From Lemma 5, (30) and (31), we have

$$u(0) = I^{\alpha_1}f_1(0) = 0,\ v(0) = I^{\alpha_2}f_2(0) = 0. \tag{50}$$

From (25), (30) and (31), we obtain

$$\begin{aligned}
u(1) &= I^{\alpha_1}f_1(u,v)(1) + \frac{1}{(1-l_1l_2)} \\
&\Bigg[l_1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}f_1(u,v)(s)ds - I^{\alpha_2}f_2(u,v)(1)\Bigg) \\
&+ \sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}f_2(u,v)(s)ds - I^{\alpha_1}f_1(u,v)(1)\Bigg] \\
&= I^{\alpha_1}f_1(u,v)(1) + \Bigg(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}v(s)ds + \sum_{j=1}^{n}b_{1j}v(\eta_j) - I^{\alpha_1}f_1(u,v)(1)\Bigg) \\
&= \sum_{i=1}^{m}\lambda_{1i}I^{l_{1i}}v(\xi_i) + \sum_{j=1}^{n}b_{1j}v(\eta_j),
\end{aligned} \tag{51}$$

$$
\begin{aligned}
&v(1) = I^{\alpha_2} f_2(u,v)(1) + \frac{1}{(1-l_1 l_2)} \\
&\Bigg[ l_2 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^{s} (s-\tau)^{\alpha_2 - 1} f_2(u,v)(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} f_2(u,v)(s) ds - I^{\alpha_1} f_1(u,v)(1) \Bigg) \\
&+ \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^{s} (s-\tau)^{\alpha_1 - 1} f_1(u,v)(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_1(u,v)(s) ds - I^{\alpha_2} f_2(u,v)(1) \Bigg] \\
&= I^{\alpha_2} f_2(u,v)(1) + \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} u(s) ds + \sum_{j=1}^{n} b_{1j} u(\eta_j) - I^{\alpha_2} f_2(u,v)(1) \Bigg) \\
&= \sum_{i=1}^{m} \lambda_{2i} I^{l_{2i}} u(\xi_i) + \sum_{j=1}^{n} b_{2j} u(\eta_j).
\end{aligned} \tag{52}
$$

This completes the proof. □

## 3. Main Results

For computational convenience, we introduce the notations:

$$M_1 = \frac{2 - l_1 l_2}{(1-l_1 l_2)\Gamma(\alpha_1+1)} + \frac{l_1}{(1-l_1 l_2)} \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})\Gamma(\alpha_1+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\alpha_1} ds + \sum_{j=1}^{n} \frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)} \Bigg), \tag{53}$$

$$M_2 = \frac{1}{(1-l_1 l_2)} \Bigg( \frac{l_1}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})\Gamma(\alpha_2+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\alpha_2} ds + \sum_{j=1}^{n} \frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg), \tag{54}$$

$$M_3 = \frac{1}{\Gamma(\alpha_1-\gamma_1+1)} + \frac{\Gamma(\theta_1)}{\Gamma(\theta_1-\gamma_1)(1-l_1 l_2)} \Bigg( \sum_{i=1}^{m} \frac{l_1\lambda_{2i}}{\Gamma(l_{2i})\Gamma(\alpha_1+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\alpha_1} ds + \sum_{j=1}^{n} \frac{l_1 b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(\alpha_1+1)} \Bigg), \tag{55}$$

$$M_4 = \frac{\Gamma(\theta_1)}{\Gamma(\theta_1-\gamma_1)(1-l_1 l_2)} \Bigg( \frac{l_1}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})\Gamma(\alpha_2+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\alpha_2} ds + \sum_{j=1}^{n} \frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg), \tag{56}$$

$$M_5 = \frac{1}{(1-l_1 l_2)} \Bigg( \frac{l_2}{\Gamma(\alpha_1+1)} + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})\Gamma(\alpha_1+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\alpha_1} ds + \sum_{j=1}^{n} \frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)} \Bigg), \tag{57}$$

$$M_6 = \frac{2 - l_1 l_2}{(1-l_1 l_2)\Gamma(\alpha_2+1)} + \frac{l_2}{(1-l_1 l_2)} \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})\Gamma(\alpha_2+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\alpha_2} ds + \sum_{j=1}^{n} \frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg), \tag{58}$$

$$M_7 = \frac{\Gamma(\theta_2)}{\Gamma(\theta_2-\gamma_2)(1-l_1 l_2)} \Bigg( \frac{l_2}{\Gamma(\alpha_1+1)} + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})\Gamma(\alpha_1+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\alpha_1} ds + \sum_{j=1}^{n} \frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)} \Bigg), \tag{59}$$

$$M_8 = \frac{1}{\Gamma(\alpha_2-\gamma_2+1)} + \frac{\Gamma(\theta_2)}{\Gamma(\theta_2-\gamma_2)(1-l_1 l_2)} \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i} l_2}{\Gamma(l_{1i})\Gamma(\alpha_2+1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\alpha_2} ds + \sum_{j=1}^{n} \frac{l_2 b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(\alpha_2+1)} \Bigg). \tag{60}$$

Also, Let $C_1 = \{u | u \in C_{2-\theta_1}, {}^H D^{\gamma_1,\beta} u(t) \in C_{2-\theta_1}\}$ be a Banach space endowed with the norm

$$\|u\|_{C_1} = \|u\|_{C_{2-\theta_1}} + \|{}^H D^{\gamma_1,\beta} u\|_{C_{2-\theta_1}}, \tag{61}$$

which defined by (9) and $C_2 = \left\{v | v \in C_{2-\theta_2}, {}^H D^{\gamma_2,\beta} v(t) \in C_{2-\theta_2}\right\}$ be a Banach space endowed with the norm

$$\|v\|_{C_2} = \|v\|_{C_{2-\theta_2}} + \|{}^H D^{\gamma_2,\beta} v\|_{C_{2-\theta_2}}, \tag{62}$$

which defined by (9). We can have the poiduct space $\mathcal{C} := C_1 \times C_2$ with the norm

$$\|(u,v)\|_{\mathcal{C}} = \|u\|_{C_1} + \|v\|_{C_2}. \tag{63}$$

*3.1. The Existence Results of the Coupled Hilfer Fractional Differential System*

**Theorem 1.** *Let $f_k : J \times R^4 \to R$ be a function such that $f_k \in C_{2-\theta_k}(J), k = 1,2$. Assume that there exist real constants $a_i, b_i \geq 0$ for $i = 1,2,3,4$ and $a_i, b_i > 0$ such that, for any $x_i \in R(i = 1,2,3,4)$, we have*

$$\begin{aligned} |f_1(t,x_1,x_2,x_3,x_4)| &\leq a_0 + a_1|x_1| + a_2|x_2| + a_3|x_3| + a_4|x_4|, \\ |f_2(t,x_1,x_2,x_3,x_4)| &\leq b_0 + b_1|x_1| + b_2|x_2| + b_3|x_3| + b_4|x_4|. \end{aligned} \tag{64}$$

*If $(a_1 + a_3)(M_1 + M_3 + M_5 + M_7) + (b_1 + b_3)(M_2 + M_4 + M_6 + M_8) < 1$ and $(a_2 + a_4)$ $(M_1 + M_3 + M_5 + M_7) + (b_2 + b_4)(M_2 + M_4 + M_6 + M_8) < 1$, where $M_i(i = 1,2,3,4,5,6,7,8)$ are defined by (53)–(60), then (5) and (7) has at least one solution on J.*

**Proof.** We define the operator $T : \mathcal{C} \to \mathcal{C}$ by

$$T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix}, \tag{65}$$

where

$$\begin{aligned} T_1(u,v)(t) &= I^{\alpha_1} f_1(u,v)(t) + \frac{t^{\theta_1-1}}{(1-l_1l_2)} \\ &\Bigg[l_1\Bigg(\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f_1(u,v)(s)ds - I^{\alpha_2} f_2(u,v)(1)\Bigg) \\ &+ \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} f_2(u,v)(s)ds - I^{\alpha_1} f_1(u,v)(1)\Bigg], \end{aligned} \tag{66}$$

$$\begin{aligned} T_2(u,v)(t) &= I^{\alpha_2} f_2(u,v)(t) + \frac{t^{\theta_2-1}}{(1-l_1l_2)} \\ &\Bigg[l_2\Bigg(\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} f_2(u,v)(s)ds - I^{\alpha_1} f_1(u,v)(1)\Bigg) \\ &+ \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f_1(u,v)(s)ds - I^{\alpha_2} f_2(u,v)(1)\Bigg]. \end{aligned} \tag{67}$$

Also, according to (28) and (29), it is easy to see that

$$\begin{aligned} {}^H D^{\gamma_1,\beta} T_1(u,v)(t) &= I^{\alpha_1-\gamma_1} f_1(u,v)(t) + \frac{\Gamma(\theta_1) t^{\theta_1-\gamma_1-1}}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)} \\ &\Bigg[l_1\Bigg(\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f_1(u,v)(s)ds - I^{\alpha_2} f_2(u,v)(1)\Bigg) \\ &+ \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} f_2(u,v)(s)ds - I^{\alpha_1} f_1(u,v)(1)\Bigg], \end{aligned} \tag{68}$$

$$
\begin{aligned}
&{}^{H}D^{\gamma_2,\beta}T_2(u,v)(t) = I^{\alpha_2-\gamma_2}f_2(u,v)(t) + \frac{\Gamma(\theta_2)t^{\theta_2-\gamma_2-1}}{\Gamma(\theta_2-\gamma_2)(1-l_1l_2)} \\
&\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}f_2(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}f_2(u,v)(s)ds - I^{\alpha_1}f_1(u,v)(1)\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}f_1(u,v)(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}f_1(u,v)(s)ds - I^{\alpha_2}f_2(u,v)(1)\right].
\end{aligned} \tag{69}
$$

We will show that the operator $T : \mathcal{C} \to \mathcal{C}$ is completely continuous.

(a) The operator $T : \mathcal{C} \to \mathcal{C}$ is continuous.

By the continuity of the functions $f_1$ and $f_2$, the operator $T$ is continuous.

(b) The operator $T : \mathcal{C} \to \mathcal{C}$ is uniformly bounded. Let $\Omega$ be any bounded subset of $\mathcal{C}$. There exist positive constants $L_1, L_2$ such that

$$
|f_1(u,v)(t)| \le L_1, |f_2(u,v)(t)| \le L_2. \tag{70}
$$

For any $(u,v) \in \Omega$, we get

$$
\begin{aligned}
&|t^{2-\theta_1}T_1(u,v))(t)| \le I^{\alpha_1}|f_1(u,v)|(t) + \frac{t^{\theta_1-1}}{(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}|f_1(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}|f_1(u,v)(s)|ds + I^{\alpha_2}|f_2(u,v)|(1)\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}|f_2(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}|f_2(u,v)|(s)ds + I^{\alpha_1}|f_1(u,v)|(1)\right] \\
&\le \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}L_1 + \frac{t^{\theta_1-1}}{(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}L_1}{\Gamma(l_{2i})\Gamma(\alpha_1+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\alpha_1}ds + \sum_{j=1}^{n}\frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)}L_1 + \frac{L_2}{\Gamma(\alpha_2+1)}\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}L_2}{\Gamma(l_{1i})\Gamma(\alpha_2+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\alpha_2}ds + \sum_{j=1}^{n}\frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)}L_2 + \frac{L_1}{\Gamma(\alpha_1+1)}\right] \\
&\le L_1M_1 + L_2M_2,
\end{aligned} \tag{71}
$$

which yields

$$
\|T_1(u,v)\|_{C_{2-\theta_1}} \le L_1M_1 + L_2M_2, \tag{72}
$$

$$
\begin{aligned}
&|t^{2-\theta_1}\,{}^{H}D^{\gamma_1,\beta}T_1(u,v)(t)| \le I^{\alpha_1-\gamma_1}|f_1(u,v)|(t) + \frac{\Gamma(\theta_1)t^{\theta_1-\gamma_1-1}}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}|f_1(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}|f_1(u,v)|(s)ds + I^{\alpha_2}|f_2(u,v)|(1)\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}|f_2(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}|f_2(u,v)|(s)ds + I^{\alpha_1}|f_1(u,v)|(1)\right] \\
&\le \frac{t^{\alpha_1-\gamma_1}}{\Gamma(\alpha_1-\gamma_1+1)}L_1 + \frac{\Gamma(\theta_1)t^{\theta_1-\gamma_1-1}}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}L_1}{\Gamma(l_{2i})\Gamma(\alpha_1+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\alpha_1}ds + \sum_{j=1}^{n}\frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)}L_1 + \frac{L_2}{\Gamma(\alpha_2+1)}\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}L_2}{\Gamma(l_{1i})\Gamma(\alpha_2+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\alpha_2}ds + \sum_{j=1}^{n}\frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)}L_2 + \frac{L_1}{\Gamma(\alpha_1+1)}\right] \\
&\le L_1M_3 + L_2M_4,
\end{aligned} \tag{73}
$$

which yields

$$\|{}^{H}D^{\gamma_1,\beta}T_1(u,v)\|_{C_{2-\theta_1}} \leq L_1M_3 + L_2M_4. \tag{74}$$

So we can get

$$\|T_1(u,v)\|_{C_1} \leq L_1(M_1+M_3) + L_2(M_2+M_4). \tag{75}$$

Similarly, we obtain that

$$\|T_2(u,v)\|_{C_{2-\theta_2}} \leq L_1M_5 + L_2M_6, \tag{76}$$

$$\|{}^{H}D^{\gamma_2,\beta}T_2(u,v)\|_{C_{2-\theta_2}} \leq L_1M_7 + L_2M_8, \tag{77}$$

and

$$\|T_2(u,v)\|_{C_2} \leq L_1(M_5+M_7) + L_2(M_6+M_8). \tag{78}$$

Hence, for $(u,v) \in \Omega$, $T_1, T_2$ is uniformly bounded. Thus it follows from the above inequalities that the set $T\Omega$ is uniformly bounded.

(c) The operator $T\Omega$ is equicontinous.

For any $(u,v) \in \Omega$ and $t_1, t_2 \in J$ such that $t_1 < t_2$, we have

$$\begin{aligned}
&|t_2^{2-\theta_1}T_1(u,v)(t_2) - t_1^{2-\theta_1}T_1(u,v)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha_1)}\left|\int_0^{t_1}\left[t_2^{2-\theta_1}(t_2-s)^{\alpha_1-1} - t_1^{2-\theta_1}(t_1-s)^{\alpha_1-1}\right]f_1(u,v)(s)ds + \int_{t_1}^{t_2} t_2^{2-\theta_1}(t_2-s)^{\alpha_1-1}f_1(u,v)(s)ds\right| + \frac{t_2-t_1}{(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}|f_1(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}|f_1(u,v)(s)|ds + I^{\alpha_2}|f_2(u,v)|(1)\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}|f_2(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}|f_2(u,v)|(s)ds + I^{\alpha_1}|f_1(u,v)|(1)\right] \\
&\leq \frac{t_2^{\alpha_1}-t_1^{\alpha_1}}{\Gamma(\alpha_1+1)}L_1 + \frac{t_2-t_1}{(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}L_1}{\Gamma(l_{2i})\Gamma(\alpha_1+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\alpha_1}ds + \sum_{j=1}^{n}\frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)}L_1 + \frac{L_2}{\Gamma(\alpha_2+1)}\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}L_2}{\Gamma(l_{1i})\Gamma(\alpha_2+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\alpha_2}ds + \sum_{j=1}^{n}\frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)}L_2 + \frac{L_1}{\Gamma(\alpha_1+1)}\right] \\
&\to 0 \ \ as\ t_2 \to t_1,
\end{aligned} \tag{79}$$

and

$$\begin{aligned}
&|t_2^{2-\theta_1}\,{}^{H}D^{\gamma_1,\beta}T_1(u,v)(t_2) - t_1^{2-\theta_1}\,{}^{H}D^{\gamma_1,\beta}T_1(u,v)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha_1-\gamma_1)}\left|\int_0^{t_1}\left[t_2^{2-\theta_1}(t_2-s)^{\alpha_1-\gamma_1-1} - t_1^{2-\theta_1}(t_1-s)^{\alpha_1-\gamma_1-1}\right]f_1(u,v)(s)ds + \int_{t_1}^{t_2} t_2^{2-\theta_1}(t_2-s)^{\alpha_1-\gamma_1-1}f_1(u,v)(s)ds\right| \\
&+\frac{\Gamma(\theta_1)(t_2^{1-\gamma_1}-t_!^{1-\gamma_1})}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(l_{2i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}\int_0^s(s-\tau)^{\alpha_1-1}|f_1(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{2j}}{\Gamma(\alpha_1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_1-1}|f_1(u,v)|(s)ds + I^{\alpha_2}|f_2(u,v)|(1)\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(l_{1i})}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}\int_0^s(s-\tau)^{\alpha_2-1}|f_2(u,v)|(\tau)d\tau ds + \sum_{j=1}^{n}\frac{b_{1j}}{\Gamma(\alpha_2)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_2-1}|f_2(u,v)|(s)ds + I^{\alpha_1}|f_1(u,v)|(1)\right] \\
&\leq \frac{t_2^{\alpha_1-\gamma_1}-t_1^{\alpha_1-\gamma_1}}{\Gamma(\alpha_1-\gamma_1+1)}L_1 + \frac{\Gamma(\theta_1)(t_2^{1-\gamma_1}-t_!^{1-\gamma_1})}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)} \\
&\left[l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}L_1}{\Gamma(l_{2i})\Gamma(\alpha_1+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\alpha_1}ds + \sum_{j=1}^{n}\frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)}L_1 + \frac{L_2}{\Gamma(\alpha_2+1)}\right)\right. \\
&\left.+\sum_{i=1}^{m}\frac{\lambda_{1i}L_2}{\Gamma(l_{1i})\Gamma(\alpha_2+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\alpha_2}ds + \sum_{j=1}^{n}\frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)}L_2 + \frac{L_1}{\Gamma(\alpha_1+1)}\right] \\
&\to 0 \ \ as\ t_2 \to t_1.
\end{aligned} \tag{80}$$

Therefore the set $T_1\Omega$ is equicontinous for all $(u,v) \in \Omega$. Similarly, we can get the set $T_2\Omega$ is equicontinous for all $(u,v) \in \Omega$. As a consequence, the set $T\Omega$ is equicontinous for all $(u,v) \in \Omega$. By applying the Arzelá–Ascoli theorem, the set $T\Omega$ is relative compact which implies that the operator T is completely continuous.

Lastly, we shall show that the set $\xi = \{(u,v) \in \mathcal{C} : (u,v) = \lambda T(u,v), 0 \le \lambda \le 1\}$ is bounded. Let any $(u,v) \in \xi$, then $(u,v) = \lambda T(u,v)$. For any $t \in J$, we have

$$\begin{aligned} u(t) &= \lambda T_1(u,v)(t), \\ v(t) &= \lambda T_2(u,v)(t). \end{aligned} \tag{81}$$

Then, we get

$$\begin{aligned} \|u\|_{C_1} &\le \big[a_0 + (a_1+a_3)\|u\|_{C_1} + (a_2+a_4)\|v\|_{C_2}\big](M_1+M_3) + \big[b_0 + (b_1+b_3)\|u\|_{C_1} + (b_2+b_4)\|v\|_{C_2}\big](M_2+M_4), \\ \|v\|_{C_2} &\le \big[a_0 + (a_1+a_3)\|u\|_{C_1} + (a_2+a_4)\|v\|_{C_2}\big](M_5+M_7) + \big[b_0 + (b_1+b_3)\|u\|_{C_1} + (b_2+b_4)\|v\|_{C_2}\big](M_6+M_8), \end{aligned} \tag{82}$$

which imply that

$$\begin{aligned} \|u\|_{C_1} + \|v\|_{C_2} \le{}& (M_1+M_3+M_5+M_7)a_0 + (M_2+M_4+M_6+M_8)b_0 \\ &+ [(a_1+a_3)(M_1+M_3+M_5+M_7) + (b_1+b_3)(M_2+M_4+M_6+M_8)]\|u\|_{C_1} \\ &+ [(a_2+a_4)(M_1+M_3+M_5+M_7) + (b_2+b_4)(M_2+M_4+M_6+M_8)]\|v\|_{C_2}. \end{aligned} \tag{83}$$

Thus, we obtain

$$\|(u,v)\|_{\mathcal{C}} \le \frac{(M_1+M_3+M_5+M_7)a_0 + (M_2+M_4+M_6+M_8)b_0}{M^*}, \tag{84}$$

where $M^* = \min\{1-(a_1+a_3)(M_1+M_3+M_5+M_7) - (b_1+b_3)(M_2+M_4+M_6+M_8)$, $1-(a_2+a_4)(M_1+M_3+M_5+M_7) - (b_2+b_4)(M_2+M_4+M_6+M_8)\}$, which shows that the set $\xi$ is bounded. Therefore, by applying Lemma 11, the operator T has at least one fixed point. Therefore, we deduce that problem (5) and (7) has at least one solution on J.

The proof is completed. □

### *3.2. The Existence Results of the Coupled Hilfer Fractional Differential Inclusions*

$X$ is a real (or complex) separable Banach space with a norm $\|\cdot\|$, defined by $\|u\| = \sup_{t\in J}|u(t)|$, $\mathcal{P}(X)$ is the family of all nonempty subsets of $X$. For a normed space $(X, \|\cdot\|)$, let $Y$ be a subset of $X$. We denote

(*i*) $\mathcal{P}$(X) = $\{Y \subseteq$ X: $Y \neq \emptyset\}$;
(*ii*) $\mathcal{P}_{cl}$(X) = $\{Y \in \mathcal{P}(X)$: $Y$ closed$\}$;
(*iii*) $\mathcal{P}_{b}$(X) = $\{Y \in \mathcal{P}(X)$: $Y$ bounded$\}$;
(*iv*) $\mathcal{P}_{cp}$(X) = $\{Y \in \mathcal{P}(X)$: $Y$ compact$\}$;
(*v*) $\mathcal{P}_{cv}$(X) = $\{Y \in \mathcal{P}(X)$: $Y$ convex$\}$;
(*vi*) $\mathcal{P}_{cp,cv}$(X) = $\{Y \in \mathcal{P}(X)$: $Y$ compact, convex$\}$;

$F, G$: $J \times X^4 \to \mathcal{P}(X)$, $i = 1,2$, are given multivalued maps. When $F, G$ are convex valued, to complete our result we need the following assumtions:

(H1) $F, G : J \times X^4 \to \mathcal{P}_{cp,cv}(X)$ are $L^1$-Carathéodory multivalued maps;
(H2) There exist $m_1, m_2 \in C(J, X^+)$ and $\phi_1, \phi_2, \psi_1, \psi_2, \varrho_1, \varrho_2, \rho_1, \rho_2 : [0,\infty) \to (0,\infty)$ continuous, nondecreasing such that

$$\|F(t,u,v,w,z)\| = sup\{|f| : f \in F(t,u,v,w,z)\} \le m_1(t)(\phi_1(|u|) + \psi_1(|v|) + \varrho_1(|w|) + \rho_1(|z|)), \tag{85}$$

and

$$\|G(t,u,v,w,z)\| = sup\{|g| : g \in F(t,u,v,w,z)\} \le m_2(t)(\phi_2(|u|) + \psi_2(|v|) + \varrho_2(|w|) + \rho_2(|z|)), \tag{86}$$

for $u, v, w, z \in X$ and a.e.$t \in J$.

**Lemma 15.** *[28] Let $X$ be a Banach space. Let $F : J \times X^4 \to \mathcal{P}_{cp,cv}(X)$ be an $L^1$-Carathéodory multivalued map and $T$ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$T \circ S_F : C(J,X) \to \mathcal{P}_{cp,cv}(C(J,X)), \quad y \mapsto (T \circ S_F)(y) = T(S_F, y), \tag{87}$$

*is a closed graph operator in $C(J,X) \times C(J,X)$.*

**Theorem 2.** *Assume that (H1), (H2) are satisfied and there exists $K > 0$ such that*

$$\begin{aligned} K > {} & (M_1+M_3+M_5+M_7)\|m_1\|(\phi_1(\|(u,v)\|_{\mathcal{C}})+\psi_1(\|(u,v)\|_{\mathcal{C}})+\varrho_1(\|(u,v)\|_{\mathcal{C}})+\rho_1(\|(u,v)\|_{\mathcal{C}})) \\ & +(M_2+M_4+M_6+M_8)\|m_2\|(\phi_2(\|(u,v)\|_{\mathcal{C}})+\psi_2(\|(u,v)\|_{\mathcal{C}})+\varrho_2(\|(u,v)\|_{\mathcal{C}})+\rho_2(\|(u,v)\|_{\mathcal{C}})), \end{aligned}$$

*where $M_i(i = 1,2,3,4,5,6,7,8)$ are defined by (53)–(60), then the coupled system (6) and (7) has at least one solution on $J$.*

**Proof.** For each $(u,v) \in \mathcal{C}$, define the sets of selections of $F, G$ by

$$S_{F,(u,v)} = \left\{ f(t) \in L^1(J,E) : f(t) \in F(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t)) \text{ for a.e. } t \in J \right\}, \tag{88}$$

and

$$S_{G,(u,v)} = \left\{ g(t) \in L^1(J,E) : g(t) \in G(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t)) \text{ for a.e. } t \in J \right\}. \tag{89}$$

Define the multivalued operators $N_1 : C_1 \to \mathcal{P}(C_1)$ and $N_2 : C_2 \to \mathcal{P}(C_2)$ by

$$N_1(u,v) = \left\{ h_1 \in C_1 : \text{there exist } f \in S_{F,(u,v)} \text{ and } g \in S_{G,(u,v)} \text{ such that } h_1(u,v)(t) = A_1(t,u,v) \right\}, \tag{90}$$

and

$$N_2(u,v) = \left\{ h_2 \in C_2 : \text{there exist } f \in S_{F,(u,v)} \text{ and } g \in S_{G,(u,v)} \text{ such that } h_2(u,v)(t) = A_2(t,u,v) \right\}, \tag{91}$$

where

$$\begin{aligned} A_1(t,u,v) = {} & I^{\alpha_1} f(t) + \frac{t^{\theta_1-1}}{(1-l_1 l_2)} \\ & \Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f(s) ds - I^{\alpha_2} g(1) \Bigg) \\ & + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} g(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} g(s) ds - I^{\alpha_1} f(1) \Bigg], \end{aligned} \tag{92}$$

$$\begin{aligned} A_2(t,u,v) = {} & I^{\alpha_2} g(t) + \frac{t^{\theta_2-1}}{(1-l_1 l_2)} \\ & \Bigg[ l_2 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} g(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} g(s) ds - I^{\alpha_1} f(1) \Bigg) \\ & + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f(s) ds - I^{\alpha_2} g(1) \Bigg]. \end{aligned} \tag{93}$$

Consider the continuous operator $N : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ defined by

$$N(u,v) = \{(h_1,h_2) \in \mathcal{C} : h_1 \in N_1(u,v), h_2 \in N_2(u,v)\}. \tag{94}$$

Clearly, the fixed points of $N$ are solutions of the system (6) and (7).

Step 1. $N(u,v)$ is convex valued.

Suppose $(h_i, \widetilde{h}_i) \in (N_1, N_2)$ $(i = 1, 2)$. Then there exist $f_i \in S_{F,(u,v)}$, $g_i \in S_{G,(u,v)}$ $(i = 1, 2)$ such that for any $t \in J, t = 1, 2$, we have

$$h_i(t) = I^{\alpha_1} f_i(t) + \frac{t^{\theta_1 - 1}}{(1 - l_1 l_2)} \left[ l_1 \left( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i} - 1)} \int_0^s (s - \tau)^{\alpha_1 - 1} f_i(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_i(s) ds - I^{\alpha_2} g_i(1) \right) + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i} - 1)} \int_0^s (s - \tau)^{\alpha_2 - 1} g_i(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} g_i(s) ds - I^{\alpha_1} f_i(1) \right], \tag{95}$$

$$\widetilde{h}_i(t) = I^{\alpha_2} g_i(t) + \frac{t^{\theta_2 - 1}}{(1 - l_1 l_2)} \left[ l_2 \left( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i} - 1)} \int_0^s (s - \tau)^{\alpha_2 - 1} g_i(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} g_i(s) ds - I^{\alpha_1} f_i(1) \right) + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i} - 1)} \int_0^s (s - \tau)^{\alpha_1 - 1} f_i(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_i(s) ds - I^{\alpha_2} g_i(1) \right]. \tag{96}$$

Let $0 \leq \lambda \leq 1$. Then, for any $t \in J$, we have

$$\begin{aligned} (\lambda h_1 + (1 - \lambda) h_2)(t) = {} & I^{\alpha_1} (\lambda f_1 + (1 - \lambda) f_2)(t) + \frac{t^{\theta_1 - 1}}{(1 - l_1 l_2)} \\ & \left[ l_1 \left( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i} - 1)} \int_0^s (s - \tau)^{\alpha_1 - 1} (\lambda f_1 + (1 - \lambda) f_2)(\tau) d\tau ds \right.\right. \\ & \left. + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} (\lambda f_1 + (1 - \lambda) f_2)(s) ds - I^{\alpha_2} (\lambda g_1 + (1 - \lambda) g_2)(1) \right) \\ & + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i} - 1)} \int_0^s (s - \tau)^{\alpha_2 - 1} (\lambda g_1 + (1 - \lambda) g_2)(\tau) d\tau ds \\ & \left. + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} (\lambda g_1 + (1 - \lambda) g_2)(s) ds - I^{\alpha_1} (\lambda f_1 + (1 - \lambda) f_2)(1) \right], \end{aligned} \tag{97}$$

and

$$\begin{aligned} (\lambda \widetilde{h}_1 + (1 - \lambda) \widetilde{h}_2)(t) = {} & I^{\alpha_2} (\lambda g_1 + (1 - \lambda) g_2)(t) + \frac{t^{\theta_2 - 1}}{(1 - l_1 l_2)} \\ & \left[ l_2 \left( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i} - 1)} \int_0^s (s - \tau)^{\alpha_2 - 1} (\lambda g_1 + (1 - \lambda) g_2)(\tau) d\tau ds \right.\right. \\ & \left. + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} (\lambda g_1 + (1 - \lambda) g_2)(s) ds - I^{\alpha_1} (\lambda f_1 + (1 - \lambda) f_2)(1) \right) \\ & + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i} - 1)} \int_0^s (s - \tau)^{\alpha_1 - 1} (\lambda f_1 + (1 - \lambda) f_2)(\tau) d\tau ds \\ & \left. + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} (\lambda f_1 + (1 - \lambda) f_2)(s) ds - I^{\alpha_2} (\lambda g_1 + (1 - \lambda) g_2)(1) \right]. \end{aligned} \tag{98}$$

Since $F$ and $G$ are convex valued, we infer that $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are convex. Obviously, $\lambda h_1 + (1 - \lambda) h_2 \in N_1$, $\lambda \widetilde{h}_1 + (1 - \lambda) \widetilde{h}_2 \in N_2$. Therefore, $\lambda (h_1, \widetilde{h}_1) + (1 - \lambda)(h_2, \widetilde{h}_2) \in N$.

Step 2. $N$ maps bounded sets into bounded sets in $\mathcal{C}$.

Let $r > 0$, $B_r = \{(u,v) \in \mathcal{C} : \|(u,v)\|_{\mathcal{C}} \leq r\}$ be a bounded subset of $\mathcal{C}$, $(h_1, h_2) \in N(u,v)$ and $(u,v) \in B_r$. Then there exist $f \in S_{F,(u,v)}$ and $g \in S_{G,(u,v)}$ such that for any $t \in J$,

$$h_1(u,v)(t) = I^{\alpha_1} f_1(t) + \frac{t^{\theta_1 - 1}}{(1 - l_1 l_2)} \Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} f_1(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_1(s) ds - I^{\alpha_2} f_2(1) \Bigg) + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} f_2(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} f_2(s) ds - I^{\alpha_1} f_1(1) \Bigg], \tag{99}$$

$$h_2(u,v)(t) = I^{\alpha_2} f_2(t) + \frac{t^{\theta_2 - 1}}{(1 - l_1 l_2)} \Bigg[ l_2 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} f_2(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} f_2(s) ds - I^{\alpha_1} f_1(1) \Bigg) + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} f_1(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_1(s) ds - I^{\alpha_2} f_2(1) \Bigg]. \tag{100}$$

We have

$$\begin{aligned} |t^{2-\theta_1} h_1(u,v)| \leq{} & \frac{1}{\Gamma(\alpha_1+1)} \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + \frac{1}{(1 - l_1 l_2)} \\ & \Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i} \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r))}{\Gamma(l_{2i}) \Gamma(\alpha_1 + 1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\alpha_1} ds \\ & + \sum_{j=1}^{n} \frac{b_{2j} \eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)} \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + \frac{\|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r))}{\Gamma(\alpha_2+1)} \Bigg) \\ & + \sum_{i=1}^{m} \frac{\lambda_{1i} \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r))}{\Gamma(l_{1i}) \Gamma(\alpha_2 + 1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\alpha_2} ds \\ & + \sum_{j=1}^{n} \frac{b_{1j} \eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)} \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r)) + \frac{\|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r))}{\Gamma(\alpha_1+1)} \Bigg] \\ \leq{} & M_1 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + M_2 \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r)), \end{aligned} \tag{101}$$

which yields

$$\|h_1(u,v)\|_{C_{2-\theta_1}} \leq M_1 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + M_2 \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r)), \tag{102}$$

$$\begin{aligned} |t^{2-\theta_1} {}^H D^{\gamma_1, \beta} h_1(u,v)(t)| \leq{} & \frac{1}{\Gamma(\alpha_1 - \gamma_1 + 1)} M_1 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + \frac{\Gamma(\theta_1)}{\Gamma(\theta_1 - \gamma_1)(1 - l_1 l_2)} \\ & \Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i} M_1 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r))}{\Gamma(l_{2i}) \Gamma(\alpha_1 + 1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} s^{\alpha_1} ds \\ & + \sum_{j=1}^{n} \frac{b_{2j} \eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)} M_1 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + \frac{\|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r))}{\Gamma(\alpha_2+1)} \Bigg) \\ & + \sum_{i=1}^{m} \frac{\lambda_{1i} \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r))}{\Gamma(l_{1i}) \Gamma(\alpha_2 + 1)} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} s^{\alpha_2} ds \\ & + \sum_{j=1}^{n} \frac{b_{1j} \eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)} \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r)) + \frac{M_1 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r))}{\Gamma(\alpha_1+1)} \Bigg] \\ \leq{} & M_3 \|m_1\| (\phi_1(r) + \psi_1(r) + \varrho_1(r) + \rho_1(r)) + M_4 \|m_2\| (\phi_2(r) + \psi_2(r) + \varrho_2(r) + \rho_2(r)), \end{aligned} \tag{103}$$

which yields

$$\|{}^{H}D^{\gamma_1,\beta}h_1(u,v)\|_{C_{2-\theta_1}} \leq M_3\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + M_4\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)). \quad (104)$$

Thus

$$\|h_1(u,v)\|_{C_1} \leq (M_1+M_3)\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + (M_2+M_4)\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)). \quad (105)$$

In a similar manner, we have

$$\|h_2(u,v)\|_{C_{2-\theta_2}} \leq M_5\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + M_6\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)), \quad (106)$$

$$\|{}^{H}D^{\gamma_2,\beta}h_2(u,v)\|_{C_{2-\theta_2}} \leq M_7\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + M_8\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)), \quad (107)$$

and

$$\|h_2(u,v)\|_{C_2} \leq (M_5+M_7)\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + (M_6+M_8)\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)). \quad (108)$$

Hence we have

$$\begin{aligned} \|(h_1,h_2)\|_{\mathcal{C}} &= \|h_1(u,v)\|_{C_1} + \|h_2(u,v)(t)\|_{C_2} \\ &\leq (M_1+M_3+M_5+M_7)\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) \\ &\quad + (M_2+M_4+M_6+M_8)\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)). \end{aligned} \quad (109)$$

Step 3. $N$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.
Let $B_r$ be a bounded set of $\mathcal{C}$ as in step 2. Let $0 \leq t_1 \leq t_2 \leq 1$ and $(u,v) \in B_r$.

$$\begin{aligned} &|t_2^{2-\theta_1}h_1(u,v)(t_2) - t_1^{2-\theta_1}h_1(u,v)(t_1)| \leq \frac{t_2^{\alpha_1}-t_1^{\alpha_1}}{\Gamma(\alpha_1+1)}\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + \frac{t_2-t_1}{(1-l_1l_2)} \\ &\Bigg[l_1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{2i}\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r))}{\Gamma(l_{2i})\Gamma(\alpha_1+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\alpha_1}ds \\ &+\sum_{j=1}^{n}\frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)}\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + \frac{\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r))}{\Gamma(\alpha_2+1)}\Bigg) \\ &+\sum_{i=1}^{m}\frac{\lambda_{1i}\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r))}{\Gamma(l_{1i})\Gamma(\alpha_2+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\alpha_2}ds \\ &+\sum_{j=1}^{n}\frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)}\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)) + \frac{\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r))}{\Gamma(\alpha_1+1)}\Bigg] \\ &\to 0 \ \ as\ t_2 \to t_1, \end{aligned} \quad (110)$$

$$\begin{aligned} &|t_2^{2-\theta_1}\,{}^{H}D^{\gamma_1,\beta}h_1(u,v)(t_2) - t_1^{2-\theta_1}\,{}^{H}D^{\gamma_1,\beta}h_1(u,v)(t_1)| \leq \frac{t_2^{\alpha_1-\gamma_1}-t_1^{\alpha_1-\gamma_1}}{\Gamma(\alpha_1-\gamma_1+1)}\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) \\ &+\frac{\Gamma(\theta_1)(t_2^{1-\gamma_1}-t_1^{1-\gamma_1})}{\Gamma(\theta_1-\gamma_1)(1-l_1l_2)}\Bigg[l_1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{2i}\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r))}{\Gamma(l_{2i})\Gamma(\alpha_1+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{2i}-1)}s^{\alpha_1}ds \\ &+\sum_{j=1}^{n}\frac{b_{2j}\eta_j^{\alpha_1}}{\Gamma(\alpha_1+1)}\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r)) + \frac{\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r))}{\Gamma(\alpha_2+1)}\Bigg) \\ &+\sum_{i=1}^{m}\frac{\lambda_{1i}\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r))}{\Gamma(l_{1i})\Gamma(\alpha_2+1)}\int_0^{\xi_i}(\xi_i-s)^{(l_{1i}-1)}s^{\alpha_2}ds \\ &+\sum_{j=1}^{n}\frac{b_{1j}\eta_j^{\alpha_2}}{\Gamma(\alpha_2+1)}\|m_2\|(\phi_2(r)+\psi_2(r)+\varrho_2(r)+\rho_2(r)) + \frac{\|m_1\|(\phi_1(r)+\psi_1(r)+\varrho_1(r)+\rho_1(r))}{\Gamma(\alpha_1+1)}\Bigg] \\ &\to 0 \ \ as\ t_2 \to t_1. \end{aligned} \quad (111)$$

Analogously, we can obtain

$$\|h_2(u,v)(t_2) - h_2(u,v)(t_1)\| \to 0 \ \ as\ t_2 \to t_1, \tag{112}$$

$$\|{}^H D^{\gamma_2,\beta} h_2(u,v)(t_2) - {}^H D^{\gamma_2,\beta} h_2(u,v)(t_1)\| \to 0 \ \ as\ t_2 \to t_1. \tag{113}$$

Therefore, the operator $N(u,v)$ is equicontinuous. By the Arzelá-Ascoli theorem, we infer that the operator $N(u,v)$ is completely continuous.

Step 4. $N$ has a closed graph.

Let $(u_n,v_n) \to (u_*,v_*)$, $(h_n,\overline{h}_n) \in N(u_n,v_n)$ and $(h_n,\overline{h}_n) \to (h_*,\overline{h}_*)$, we need to proof $(h_*,\overline{h}_*) \in N(u_*,v_*)$. $(h_n,\overline{h}_n) \in N(u_n,v_n)$ implies that there exist $f_n \in S_{F,(u_n,v_n)}$ and $g_n \in S_{G,(u_n,v_n)}$ such that for all $t \in J$,

$$\begin{aligned} h_n(u_n,v_n)(t) &= I^{\alpha_1} f_n(t) + \frac{t^{\theta_1-1}}{(1-l_1l_2)} \\ &\Bigg[ l_1 \Bigg( \sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f_n(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f_n(s) ds - I^{\alpha_2} g_n(1) \Bigg) \\ &+ \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} g_n(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} g_n(s) ds - I^{\alpha_1} f_n(1) \Bigg], \end{aligned} \tag{114}$$

$$\begin{aligned} \overline{h}_n(u_n,v_n)(t) &= I^{\alpha_2} g_n(t) + \frac{t^{\theta_2-1}}{(1-l_1l_2)} \\ &\Bigg[ l_2 \Bigg( \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} g_n(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} g_n(s) ds - I^{\alpha_1} f_n(1) \Bigg) \\ &+ \sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f_n(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f_n(s) ds - I^{\alpha_2} g_n(1) \Bigg]. \end{aligned} \tag{115}$$

Let us consider the continuous linear operators $\Phi_1, \Phi_2 : L^1(J,C) \to C(J,C)$ given by

$$\begin{aligned} \Phi_1(u,v)(t) &= I^{\alpha_1} f(t) + \frac{t^{\theta_1-1}}{(1-l_1l_2)} \\ &\Bigg[ l_1 \Bigg( \sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f(s) ds - I^{\alpha_2} g(1) \Bigg) \\ &+ \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} g(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} g(s) ds - I^{\alpha_1} f(1) \Bigg], \end{aligned} \tag{116}$$

$$\begin{aligned} \Phi_2(u,v)(t) &= I^{\alpha_2} g(t) + \frac{t^{\theta_2-1}}{(1-l_1l_2)} \\ &\Bigg[ l_2 \Bigg( \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2-1} g(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2-1} g(s) ds - I^{\alpha_1} f(1) \Bigg) \\ &+ \sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1-1} f(\tau) d\tau ds + \sum_{j=1}^n \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1-1} f(s) ds - I^{\alpha_2} g(1) \Bigg]. \end{aligned} \tag{117}$$

From Lemma 15, we know that $(\Phi_1,\Phi_2) \circ (S_F,S_G)$ is a closed graph operator. Moreover, we get $(h_n,\overline{h}_n) \in (\Phi_1,\Phi_2) \circ (S_{F,(u_n,v_n)}, S_{G,(u_n,v_n)})$ for all $n$. Since $(u_n,v_n) \to (u_*,v_*)$, $(h_n,\overline{h}_n) \to (h_*,\overline{h}_*)$, it follows the existence of $f_* \in S_{F,(u_*,v_*)}$ and $g_* \in S_{G,(u_*,v_*)}$ such that

$$h_*(u_*,v_*)(t) = I^{\alpha_1} f_*(t) + \frac{t^{\theta_1 - 1}}{(1 - l_1 l_2)}$$
$$\Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} f_*(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_*(s) ds - I^{\alpha_2} g_*(1) \Bigg) \tag{118}$$
$$+ \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} g_*(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} g_*(s) ds - I^{\alpha_1} f_*(1) \Bigg],$$

$$\overline{h}_*(u_*,v_*)(t) = I^{\alpha_2} g_*(t) + \frac{t^{\theta_2 - 1}}{(1 - l_1 l_2)}$$
$$\Bigg[ l_2 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} g_*(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} g_*(s) ds - I^{\alpha_1} f_*(1) \Bigg) \tag{119}$$
$$+ \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} f_*(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f_*(s) ds - I^{\alpha_2} g_*(1) \Bigg],$$

that is, $(h_*, \overline{h}_*) \in N(u_*, v_*)$.

Step 5. A priori bounds on solutions.

Let $(u,v) \in \lambda N(u,v)$ for some $\lambda \in (0,1)$. Then there exist $f \in S_{F,(u,v)}$ and $g \in S_{G,(u,v)}$ such that for all $t \in J$,

$$u(t) = \lambda I^{\alpha_1} f(t) + \frac{\lambda t^{\theta_1 - 1}}{(1 - l_1 l_2)}$$
$$\Bigg[ l_1 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} f(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f(s) ds - I^{\alpha_2} g(1) \Bigg) \tag{120}$$
$$+ \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} g(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} g(s) ds - I^{\alpha_1} f(1) \Bigg],$$

and

$$v(t) = \lambda I^{\alpha_2} g(t) + \frac{\lambda t^{\theta_2 - 1}}{(1 - l_1 l_2)}$$
$$\Bigg[ l_2 \Bigg( \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(l_{1i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{1i}-1)} \int_0^s (s-\tau)^{\alpha_2 - 1} g(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{1j}}{\Gamma(\alpha_2)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_2 - 1} g(s) ds - I^{\alpha_1} f(1) \Bigg) \tag{121}$$
$$+ \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(l_{2i})} \int_0^{\xi_i} (\xi_i - s)^{(l_{2i}-1)} \int_0^s (s-\tau)^{\alpha_1 - 1} f(\tau) d\tau ds + \sum_{j=1}^{n} \frac{b_{2j}}{\Gamma(\alpha_1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha_1 - 1} f(s) ds - I^{\alpha_2} g(1) \Bigg].$$

With the same arguments as in Step 2 of our proof, for each $(u,v) \in \mathcal{C}$, we obtain

$$\begin{aligned} \|u\|_{C_1} \le\, & (M_1 + M_3)\|m_1\|(\phi_1(\|(u,v)\|_{\mathcal{C}}) + \psi_1(\|(u,v)\|_{\mathcal{C}}) + \varrho_1(\|(u,v)\|_{\mathcal{C}}) + \rho_1(\|(u,v)\|_{\mathcal{C}})) \\ & + (M_2 + M_4)\|m_2\|(\phi_2(\|(u,v)\|_{\mathcal{C}}) + \psi_2(\|(u,v)\|_{\mathcal{C}}) + \varrho_2(\|(u,v)\|_{\mathcal{C}}) + \rho_2(\|(u,v)\|_{\mathcal{C}})), \end{aligned} \tag{122}$$

and

$$\begin{aligned} \|v\|_{C_2} \le\, & (M_5 + M_7)\|m_1\|(\phi_1(\|(u,v)\|_{\mathcal{C}}) + \psi_1(\|(u,v)\|_{\mathcal{C}}) + \varrho_1(\|(u,v)\|_{\mathcal{C}}) + \rho_1(\|(u,v)\|_{\mathcal{C}})) \\ & + (M_6 + M_8)\|m_2\|(\phi_2(\|(u,v)\|_{\mathcal{C}}) + \psi_2(\|(u,v)\|_{\mathcal{C}}) + \varrho_2(\|(u,v)\|_{\mathcal{C}}) + \rho_2(\|(u,v)\|_{\mathcal{C}})). \end{aligned} \tag{123}$$

Hence we have

$$\begin{aligned}\|(u,v)\|_{\mathcal{C}} &= \|u\|_{C_1} + \|v\|_{C_2} \\ &\leq (M_1+M_3+M_5+M_7)\|m_1\|(\phi_1(\|(u,v)\|_{\mathcal{C}}) + \psi_1(\|(u,v)\|_{\mathcal{C}}) + \varrho_1(\|(u,v)\|_{\mathcal{C}}) + \rho_1(\|(u,v)\|_{\mathcal{C}})) \\ &+ (M_2+M_4+M_6+M_8)\|m_2\|(\phi_2(\|(u,v)\|_{\mathcal{C}}) + \psi_2(\|(u,v)\|_{\mathcal{C}}) + \varrho_2(\|(u,v)\|_{\mathcal{C}}) + \rho_2(\|(u,v)\|_{\mathcal{C}})).\end{aligned} \tag{124}$$

Now we set $U = \{(u,v) \in \mathcal{C} : \|(u,v)\|_{\mathcal{C}} < K\}$. Clearly, $U$ is an open subset of $\mathcal{C}$ and $(0,0) \in U$. As a consequence of Steps 1–4, together with the Arzelá-Ascoli theorem, we can conclude that $N : \overline{U} \to \mathcal{P}_{cp,cv}(C_1) \times \mathcal{P}_{cp,cv}(C_2)$ is upper semicontinuous and completely continuous. From the choice of $U$, there is no $(u,v) \in \partial U$ such that $(u,v) \in \lambda N(u,v)$ for some $\lambda \in (0,1)$. Therefore, by Lemma 12, we deduce that $N$ has a fixed point $(u,v) \in \overline{U}$, which is a solution of the coupled system (6) and (7).

This completes the proof. □

**Example 1.** *For $t \in J$, consider the following fractional differential system:*

$$\begin{cases} {}^HD^{2,0.5}u(t) = f_1(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t)), \\ {}^HD^{2,0.5}v(t) = f_2(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t)), \end{cases} \tag{125}$$

*with the coupled integral and discrete mixed boundary conditions:*

$$\begin{cases} u(0) = 0, u(1) = \sum_{i=1}^{2} \lambda_{1i} I^{l_{1i}} v(\xi_i) + \sum_{j=1}^{2} b_{1j} v(\eta_j), \\ v(0) = 0, v(1) = \sum_{i=1}^{2} \lambda_{2i} I^{l_{2i}} u(\xi_i) + \sum_{j=1}^{2} b_{2j} u(\eta_j), \end{cases} \tag{126}$$

*where $\lambda_{11} = 0.45$, $\lambda_{21} = 0.37$, $l_{11} = 1.1$, $l_{21} = 1.5$, $\xi_1 = 0.32$, $b_{11} = 0.47$, $b_{21} = 0.19$, $\eta_1 = 0.65$, $\lambda_{12} = 0.55$, $\lambda_{22} = 0.25$, $l_{12} = 1.7$, $l_{22} = 1.6$, $\xi_2 = 0.75$, $b_{12} = 0.36$, $b_{22} = 0.28$, $\eta_2 = 0.45$, the nonlinear functions $f_1$ and $f_2$ are defined by*

$$\begin{aligned} f_1 &= 0.7t^2 + 0.09tu(t) + 0.01t^2v(t) + 0.08t^HD^{0.1,0.5}u(t) + 0.06t^{2H}D^{0.1,0.5}v(t), \\ f_2 &= 0.5t + 0.04t^2u(t) + 0.07tv(t) + 0.06t^{2H}D^{0.1,0.5}u(t) + 0.09t^HD^{0.1,0.5}v(t), \end{aligned} \tag{127}$$

*By calculation we get $l_1 = 0.5468591$, $l_2 = 0.2877841$, $M_1 = 1.1425206$, $M_2 = 0.5024695$, $M_3 = 1.1921522$, $M_4 = 0.5224447$, $M_5 = 0.2606166$, $M_6 = 1.1446027$, $M_7 = 0.2709772$, and $M_8 = 1.2174674$.*

*From (127), we can get*

$$\begin{aligned} |f_1(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t))| &\leq 0.7 + 0.09u(t) + 0.01v(t) + 0.08^HD^{0.1,0.5}u(t) + 0.06^HD^{0.1,0.5}v(t), \\ |f_2(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t))| &\leq 0.5 + 0.04u(t) + 0.07v(t) + 0.06^HD^{0.1,0.5}u(t) + 0.09^HD^{0.1,0.5}v(t). \end{aligned} \tag{128}$$

*Let $a_0 = 0.7$, $a_1 = 0.09$, $a_2 = 0.01$, $a_3 = 0.08$, $a_4 = 0.06$, $b_0 = 0.5$, $b_1 = 0.04$, $b_2 = 0.07$, $b_3 = 0.06$ and $b_4 = 0.09$, Theorem 3.1 can be applied to problem (3.73) and (3.74). We find $(a_1 + a_3)(M_1 + M_3 + M_5 + M_7) + (b_1 + b_3)(M_2 + M_4 + M_6 + M_8) = 0.8260 < 1$ and $(a_2 + a_4)(M_1 + M_3 + M_5 + M_7) + (b_2 + b_4)(M_2 + M_4 + M_6 + M_8) = 0.7426 < 1$. Therefore, the conclusion of Theorem 3.1 implies that problem (3.73) and (3.74) have at least one solution $(u,v)$ on $J$.*

**Example 2.** *For $t \in J$, consider the following fractional differential inclusions:*

$$\begin{cases} {}^HD^{2,0.5}u(t) \in F(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t)), \\ {}^HD^{2,0.5}v(t) \in G(t,u(t),v(t),{}^H D^{\gamma_1,\beta}u(t),{}^H D^{\gamma_2,\beta}v(t)), \end{cases} \tag{129}$$

*with the coupled integral and discrete mixed boundary conditions:*

$$\begin{cases} u(0)=0, u(1)=\sum_{i=1}^{2}\lambda_{1i}I^{l_{1i}}v(\xi_i)+\sum_{j=1}^{2}b_{1j}v(\eta_j), \\ v(0)=0, v(1)=\sum_{i=1}^{2}\lambda_{2i}I^{l_{2i}}u(\xi_i)+\sum_{j=1}^{2}b_{2j}u(\eta_j), \end{cases} \tag{130}$$

*where* $\lambda_{11}=0.45$, $\lambda_{21}=0.37$, $l_{11}=1.1$, $l_{21}=1.5$, $\xi_1=0.32$, $b_{11}=0.47$, $b_{21}=0.19$, $\eta_1=0.65$, $\lambda_{12}=0.55$, $\lambda_{22}=0.25$, $l_{12}=1.7$, $l_{22}=1.6$, $\xi_2=0.75$, $b_{12}=0.36$, $b_{22}=0.28$, $\eta_2=0.45$, *and* $F, G$: $J\times X^4\to\mathcal{P}(X)$, $i=1,2$, $F, G: J\times X^4\to\mathcal{P}(X)$ *are multivalued maps given by*

$$\begin{aligned} &F(t,u(t),v(t),^H D^{\gamma_1,\beta}u(t),^H D^{\gamma_2,\beta}v(t))= \\ &\Big\{f\in X: 0\le f\le 0.7t^2+0.09tu(t)+0.01t^2v(t)+0.08t^H D^{0.1,0.5}u(t)+0.06t^{2H}D^{0.1,0.5}v(t)\Big\}, \\ &G(t,u(t),v(t),^H D^{\gamma_1,\beta}u(t),^H D^{\gamma_2,\beta}v(t))= \\ &\Big\{g\in X: 0\le g\le 0.5t+0.04t^2u(t)+0.07tv(t)+0.06t^{2H}D^{0.1,0.5}u(t)+0.09t^H D^{0.1,0.5}v(t)\Big\}. \end{aligned} \tag{131}$$

*From (131), we know that* $F, G$ *are* $L^1$*-Carathéodory and have convex values satisfying*

$$\begin{aligned} &\|F(t,u,v,w,z)\|=\sup\{|f|: f\in F(t,u,v,w,z)\}\le 5,\ for\ each(t,u,v,w,z)\in J\times X^4, \\ &\|G(t,u,v,w,z)\|=\sup\{|g|: g\in G(t,u,v,w,z)\}\le 6,\ for\ each(t,u,v,w,z)\in J\times X^4, \end{aligned} \tag{132}$$

*with* $m_1(t)=m_2(t)=\phi_1(|u|)=\phi_2(|u|)=\psi_1(|v|)=\psi_2(|v|)=\varrho_1(|w|)\equiv 1$, $\rho_1(|z|)=\varrho_2(|w|)=\rho_2(|z|)\equiv 2$.

*Let K be any number satisfying*

$$\begin{aligned} K>&(M_1+M_3+M_5+M_7)\|m_1\|(\phi_1(\|(u,v)\|_{\mathcal{C}})+\psi_1(\|(u,v)\|_{\mathcal{C}})+\varrho_1(\|(u,v)\|_{\mathcal{C}})+\rho_1(\|(u,v)\|_{\mathcal{C}})) \\ &+(M_2+M_4+M_6+M_8)\|m_2\|(\phi_2(\|(u,v)\|_{\mathcal{C}})+\psi_2(\|(u,v)\|_{\mathcal{C}})+\varrho_2(\|(u,v)\|_{\mathcal{C}})+\rho_2(\|(u,v)\|_{\mathcal{C}})) \\ &=34.6532. \end{aligned} \tag{133}$$

*Clearly, all the conditions of Theorem 2 are satisfied. So, there exists at least one solution to problem (129) and (130) on J.*

## 4. Conclusions

In this paper, we study the nonlinear coupled system of Hilfer fractional differential equations and inclusions with multi-strip and multi-point mixed boundary conditions. The existence results can be derived using tools such as the Leray-Schauder alternative, the Arzelá-Ascoli theorem, etc. It is very significant to study Hilfer fractional differential equations and inclusions for biological models and physical phenomena. However, for the Hilfer fractional differential inclusion, we assume a convex function. After that, we will continue studying the conclusions in the non-convex case.

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## References

1. Drapaca, C.S.; Sivaloganathan, S. A Fractional Model of Continuum Mechanics. *J. Elast.* **2012**, *107*, 105–123. [CrossRef]
2. Yin, B.L.; Liu, Y.; Li, H.; Zhong, Z.M. Two families of second-order fractional numerical formulas and applications to fractional differential equations. *Fract. Calc. Appl. Anal.* **2023**, *26*, 1842–1867. [CrossRef]

3. Mehta, P.P.; Pang, G.; Song, F.; Karniadakis, G.E. Discovering a universal variable-order fractional model for turbulent Couette flow using a physics-informed neural network. *Fract. Calc. Appl. Anal.* **2019**, *22*, 1675–1688. [CrossRef]
4. Machado, J.A.; Lopes, A.M. Analysis of Natural and Artificial Phenomena Using Signal Processing and Fractional Calculus. *Fract. Calc. Appl. Anal.* **2015**, *18*, 459–478. [CrossRef]
5. Gou, H.; Li, Y. A study on controllability of impulsive fractional evolution equations via resolvent operators. *Bound. Value Probl.* **2021**, *2021*, 25. [CrossRef]
6. Tarasov, V.E. On history of mathematical economics: Application of fractional calculus. *Mathematics* **2019**, *7*, *6*, 509. [CrossRef]
7. Bhangale, N.; Kachhia, K.B.; Gomez-Aguilar, J.F. Fractional viscoelastic models with Caputo generalized fractional derivative. *Math. Meth. Appl. Sci.* **2023**, *46*, 7835–7846. [CrossRef]
8. Hilfer, R. Fractional Time Evolution. In *Applications of Fractional Calculus in Physics*; World Scientific Publishing: River Edge, NJ, USA, 2000; pp. 87–130.
9. Hilfer, R.; Luchko, Y.; Tomovski, Z. Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. *Fract. Calc. Appl. Anal.* **2009**, **12**, 299–318.
10. Furati, K.M.; Kassim, M.D.; Tatar, N.E. Existence and uniqueness for a problem involving Hilfer fractional derivative. *Comput. Math. Appl.* **2012**, *64*, 1616–1626. [CrossRef]
11. Dhawan, K.; Vats, R.K.; Agarwal, R.P. Qualitative Analysis of Coupled Fractional Differential Equations involving Hilfer Derivative. *Analele Stiint. Univ. Ovidius C* **2022**, *30*, 191–217. [CrossRef]
12. Ahmad, B.; Alsaedi, A.; Aljahdali, A.S.; Ntouyas, S.K. A study of coupled nonlinear generalized fractional differential equations with coupled nonlocal multipoint Riemann-Stieltjes and generalized fractional integral boundary conditions. *AIMS Math.* **2024**, *9*, 1576–1594. [CrossRef]
13. Pradeesh, J.; Vijayakumar, V. Investigating the Existence Results for Hilfer Fractional Stochastic Evolution Inclusions of Order $1 < p < 2$. *Qual. Theor. Dyn. Syst.* **2024**, *23*, 46.
14. Zhao, X.; Liu, Y.; Pang, H. Iterative positive solutions to a coupled fractional differential system with the multistrip and multipoint mixed boundary conditions. *Adv. Differ. Equ.* **2019**, *2019*, 389. [CrossRef]
15. Guezane-Lakoud, A.; Ramdane, S. Existence of solutions for a system of mixed fractional differential equations. *J. Taibah Univ. Sci.* **2018**, *12*, 421–426. [CrossRef]
16. Jankowski, T. Systems of nonlinear fractional differential equations. *Fract. Calc. Appl. Anal.* **2015**, *18*, 122–132. [CrossRef]
17. Zi, Y.M.; Wang, Y. Positive solutions for Caputo fractional differential system with coupled boundary conditions. *Adv. Differ. Equ.* **2019**, *2019*, 80. [CrossRef]
18. Alsaedi, A.; Ntouyas, S.K.; Agarwal, R.P.; Ahmad, B. On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Adv. Differ. Equ.* **2015**, *2015*, 33. [CrossRef]
19. Xie, J.; Duan, L. Existence of solutions for a coupled system of fractional differential equations by means of topological degree theory. *Adv. Differ. Equ.* **2021**, *2021*, 309. [CrossRef]
20. Ahmad, B.; Alblewi, M.; Ntouyas, S.K.; Alsaedi, A. Existence results for a coupled system of nonlinear multi-term fractional differential equations with anti-periodic type coupled nonlocal boundary conditions. *Math. Meth. Appl. Sci.* **2021**, *44*, 8739–8758. [CrossRef]
21. Qiao, Y. The solution of local fractional diffusion equation involving Hilfer fractional derivative. *Therm. Sci.* **2019**, *23*, 809–814. [CrossRef]
22. Garg, M.; Sharma, A.; Manohar, P. Linear space-time fractional reaction-diffusion equation with composite fractional derivative in time. *J. Fract. Calc. Appl.* **2014**, *5*, 114–121.
23. Mainardi, F.; Pagnini, G.; Saxena, R.K. Fox H functions in fractional diffusion. *J. Comput. Appl. Math.* **2005**, *178*, 321–331 [CrossRef]
24. Adly, S.; Haddad, T.; Thibault, L. Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities. *Math. Program. Ser. B* **2014**, *148*, 5–47. [CrossRef]
25. Korda, M.; Henrion, D.; Jones, C.N. Convex computation of the maximum controlled invariant set for polynomial control systems. *SIAM J. Control Optim.* **2014**, *52*, 2944–2969. [CrossRef]
26. Wang, J.R.; Ibrahim, A.; O'Regan, G.D. Hilfer-type fractional differential switched inclusions with noninstantaneous impulsive and nonlocal conditions. *Nonlinear Anal. Model Control* **2018**, *23*, 921–941. [CrossRef]
27. Zhang, W.; Liu, W.B. Lyapunov-type inequalities for sequential fractional boundary value problems using Hilfer's fractional derivative. *J. Inequal. Appl.* **2019**, *2019*, 98. [CrossRef]
28. Wongcharoen, A.; Ntouyas, S.K.; Tariboon, J. On Coupled Systems for Hilfer Fractional Differential Equations with Nonlocal Integral Boundary Conditions. *J. Math.* **2020**, *2020*, 2875152. [CrossRef]
29. Jin, N.; Sun, S.R. On a coupled system of fractional compartmental models for a biological system. *Adv. Differ. Equ.* **2017**, *2017*, 146. [CrossRef]