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Article

Fundamental Matrix, Integral Representation and Stability Analysis of the Solutions of Neutral Fractional Systems with Derivatives in the Riemann—Liouville Sense

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Abstract: The paper studies a class of nonlinear disturbed neutral linear fractional systems with derivatives in the the Riemann–Liouville sense and distributed delays. First, it is proved that the initial problem for these systems with discontinuous initial functions under some natural assumptions possesses a unique solution. The assumptions used for the proof are similar to those used in the case of systems with first-order derivatives. Then, with the obtained result, we derive the existence and uniqueness of a fundamental matrix and a generalized fundamental matrix for the homogeneous system. In the linear case, via these fundamental matrices we obtain integral representations of the solutions of the homogeneous system and the corresponding inhomogeneous system. Furthermore, for the fractional systems with Riemann–Liouville derivatives we introduce a new concept for weighted stabilities in the Lyapunov, Ulam–Hyers, and Ulam–Hyers–Rassias senses, which coincides with the classical stability concepts for the cases of integer-order or Caputo-type derivatives. It is proved that the zero solution of the homogeneous system is weighted stable if and only if all its solutions are weighted bounded. In addition, for the homogeneous system it is established that the weighted stability in the Lyapunov and Ulam–Hyers senses are equivalent if and only if the inequality appearing in the Ulam–Hyers definition possess only bounded solutions. Finally, we derive natural sufficient conditions under which the property of weighted global asymptotic stability of the zero solution of the homogeneous system is preserved under nonlinear disturbances.

Keywords: fractional derivatives; neutral fractional systems; distributed delay; integral representation

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1. Introduction

Practically, it has been shown that many real-world phenomena in various fields of science can be represented more accurately through mathematical models, including fractional differential equations [1–3]. More detailed information on fractional calculus theory and fractional differential equations can be seen in the monographs of Kilbas et al. [4] and Podlubny [5]. Compared to fractional equations with Caputo-type derivatives, the fractional equations with Riemann–Liouville-type derivatives have been studied significantly less. The first obstacle was the lack of meaningful geometric and physical interpretations of the Riemann–Liouville type of integration and differentiation, which was overcome with the appearance of Ref. [6]. The other problem was to derive an appropriate formulation of the initial conditions for the initial problem. Some considerations about them are given in [7,8], but as was mentioned in [9] "The initial value problem is a subject that remains quite up-to-date".

It is well known that the existence of explicit solutions or an integral representation (variation of constants formula) of the solutions of linear fractional differential equations and/or systems (ordinary or delayed) is a main tool in executing their qualitative analysis. That is why establishing either explicit solutions (see [10]) or integral representation, for which existence of a fundamental matrix is needed (see [11]), are important tasks for stability analysis, especially in the case of equations with Riemann–Liouville derivatives. But it is surprising that there are not many articles devoted to this problem. As far as we know, a survey concerning stability results for retarded and neutral fractional differential equations with Riemann–Liouville-type derivatives does not exist. In general, partially this gap can be fulfilled by reading the overview [12] and the references therein. From the recent works devoted to the discussed themes concerning fractional differential equations with Riemann–Liouville-type derivatives we refer to [13] for equations, to [14–16] where retarded fractional differential equations are considered, and to [17,18] where the neutral case is considered. We suppose that the mentioned works with their references give a good enough picture of the studies in this area.

In the present article, we study a class of nonlinear disturbed neutral linear fractional systems with derivatives in the the Riemann–Liouville sense and distributed delay. We first study the important problem of the existence and uniqueness of the solutions of an initial problem (IP) for these systems in the case of discontinuous initial functions. As far as we know, there are no results devoted to the initial problem with discontinuous initial functions for neutral differential equations with derivatives in the Riemann–Liouville sense. Since the classically stability concepts are not directly applicable to systems with derivatives with the property that the derivative of a constant is not equal to zero (like the Riemann–Liouville fractional derivative, for example), we introduce a new concept for weighted stabilities in the Lyapunov, Ulam–Hyers, and Ulam–Hyers–Rassias senses, which coincides with the classical stability concepts for the cases of integer-order or Caputo-type derivatives.

The following abbreviations will be used in this manuscript: BV—bounded variation; GAS—globally asymptotically stable; UH—Ulam–Hyers; LAS—locally asymptotically stable; LT—Laplace transform; UHR—Ulam–Hyers–Rassias; IP—initial problem; PC—piecewise continuous; RL—Riemann–Liouville; WML—weighted Mittag–Leffler.

The paper is organized as follows: Section 2 presents the necessary definitions and properties concerning the RL and Caputo fractional derivatives, the problem statement, and the needed auxiliary definitions and facts for our exposition. Section 3 is devoted to the existence and the uniqueness of the solutions of the initial problem (IP) for the studied class of nonlinear neutral systems in the case when the initial function is discontinuous. In Section 4, as a consequence of the derived result we prove the existence and uniqueness of a fundamental matrix and a generalized fundamental matrix for the linear homogeneous system, as well as establishing an integral representation of the solutions of the IP for the homogeneous system and the corresponding inhomogeneous system. Section 5 is devoted to a new concept for weighted stabilities in the Lyapunov, UH, and UHR senses, the definition of which coincides with the classical stability concepts for the cases of integer-order or Caputo-type derivatives. It is proved that the zero solution of the homogeneous system is weighted stable if and only if all its solutions are weighted bounded. In addition, it is established that the weighted stabilities in the Lyapunov and UH senses are equivalent if and only if the inequality appearing in the UH definition possesses only bounded solutions. In Section 6, natural sufficient conditions are obtained under which the property of weighted global asymptotic stability of the zero solution of the homogeneous system is preserved under nonlinear disturbances. Finally, in Section 7 an illustrative example is presented. In the last Section 8, we summarize some conclusions concerning the obtained results and propose some open problems.

2. Preliminaries and Problem Statement

As is standard to avoid possible misunderstandings, we recall some properties concerning the RL-type derivatives, as well as the needed definitions, conditions, and auxiliary results necessary for the exposition below. For more comprehensive information on fractional calculus we refer to Refs. [4,5].

For a function $g\colon \mathbb{R} \to \mathbb{R}$, we will say that some property holds locally if it is fulfilled on every compact subinterval $[b,c] \subset \mathbb{R}$. We will use the following notation for the real linear spaces: $L_1^{loc}(\mathbb{R},\mathbb{R})$ consists of all locally Lebesgue integrable functions, $BL_1^{loc}(\mathbb{R},\mathbb{R}) \subset L_1^{loc}(\mathbb{R},\mathbb{R})$ and $BV^{loc}(\mathbb{R},\mathbb{R}) \subset BL_1^{loc}(\mathbb{R},\mathbb{R})$ are, respectively, the subspaces of all functions which are locally bounded and of all functions which have locally bounded variation.

We define the left-sided fractional RL integral operator of arbitrary order $\alpha \in (0,1)$ with lower limit $a \in \mathbb{R}$ for any $g \in L_1^{loc}(\mathbb{R},\mathbb{R})$, via the relation

$$(I_{a+}^{\alpha} g)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) ds.$$

The corresponding left-hand side RL and Caputo fractional derivatives of arbitrary order $\alpha \in (0,1)$ with lower limit a are defined for any $t > a$ by

$$\left({}^{RL}D_{a+}^{\alpha} g\right)(t) = \frac{d}{dt}(I_{a+}^{1-\alpha} g(t)),$$

$$\left({}^{C}D_{a+}^{\alpha} g\right)(t) = \left({}^{RL}D_{a+}^{\alpha} g\right)(t) - \left({}^{RL}D_{a+}^{\alpha} g\right)(a).$$

For shortness, we will use the notation $D_{a+}^{\alpha} = {}^{RL}D_{a+}^{\alpha}$.

Consider the neutral fractional nonlinear system with RL-type derivatives of order $\alpha \in (0,1)$ with lower limit $a \in \mathbb{R}$ and distributed delays in the following general form:

$$D_{a+}^{\alpha}\left(X(t) - \int_{-h}^{0} [d_\theta V(t,\theta)] X(t+\theta)\right) = \int_{-h}^{0} [d_\theta U(t,\theta)] X(t+\theta) + \mathfrak{F}(t, X_\tau^T(\theta)), \tag{1}$$

where $h > 0$, $J = [a,\infty)$, $\mathbb{R}_+ = [0,\infty)$, $J^0 = (a,\infty)$, $\mathbb{R}_+^0 = (0,\infty)$, $\langle n \rangle = \{1,2,\ldots,n\}$, $\langle m \rangle_0 = \langle m \rangle \bigcup \{0\}$, $X(t) = col(x_1(t),\ldots,x_n(t))\colon J^0 \to \mathbb{R}^n$, $\mathfrak{F}(t, X_t^T) = col(f_1(t,X_t^T),\ldots,f_n(t,X_t^T))\colon J \times PC \to \mathbb{R}^n$, (the notation col means column and superscript ^{T} denotes the transposed vector), $U(t,\theta) = \{u_k^j(t,\theta)\}_{k,j=1}^n\colon J \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $V(t,\theta) = \{v_k^j(t,\theta)\}_{k,j=1}^n\colon J \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $D_{a+}^{\alpha} X(t) = col(D_{a+}^{\alpha} x_1(t),\ldots,D_{a+}^{\alpha} x_n(t))$, and $X_t(\theta) = X(t+\theta)$ (Krasovskii-type functional) for $t \in J$ and $\theta \in [-h,0]$. In addition, $\mathbf{I}, \mathbf{\Theta} \in \mathbb{R}^{n\times n}$ denote the identity and the zero matrices, $\mathbf{0} \in \mathbb{R}^n$ denotes the zero vector, and $I_\gamma((t-a)) = (t-a)^\gamma \mathbf{I}$, $\gamma \in [-1,1]$.

For $Y\colon J \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $Y(t,\theta) = \{y_k^j(t,\theta)\}_{k,j=1}^n$, we use the norm

$$|Y(t,\theta)| = \sum_{k,j=1}^{n} |y_k^j(t,\theta)|, \quad Var_{[b,c]} Y(t,\cdot) = \{Var_{[b,c]} y_k^j(t,\cdot)\}_{k,j=1}^n,$$

$PC([-h,0],\mathbb{R}^n) = \{\Phi = (\phi_1,\ldots,\phi_n)^T\colon [-h,0] \to \mathbb{R}^n \mid \Phi \text{ is piecewise continuous}\}$ and S_Φ denotes the set of all jump points for any $\Phi(t) \in PC([-h,0],\mathbb{R}^n)$. We will use the following Banach spaces of initial functions:

$$\mathbf{PC} = \{\Phi(t) = col(\varphi_1(t),\ldots,\varphi_n(t))\colon [-h,0] \to \mathbb{R}^n \mid \Phi(t) \in PC([-h,0],\mathbb{R}^n)\},$$
$$\mathbf{PC}^* = PC \cap BV([-h,0],\mathbb{R}^n) \quad \text{and} \quad \mathbf{C}([-h,0],\mathbb{R}^n),$$

endowed with the norm $\|\Phi\| = \sum_{k\in\langle n \rangle} \sup_{s\in[-h,0]} |\phi_k(s)| < \infty$. In addition, we assume that in all spaces the functions $\Phi(t)$ are right continuous at $t \in S_\Phi$ (for $\Phi \in \mathbf{C}$ this assumption ultimately holds).

For any $\Phi \in \mathbf{PC}$ we define the initial condition for the system (1) as follows:

$$X(t) = \Phi(t-a) \ t \in [a-h,a], D_{a+}^{\alpha-1} X(a+0) = \Phi(0), \ h > 0. \tag{2}$$

More details concerning other types of initial conditions are given in [8] for the case when the initial functions are continuous.

Let us introduce the auxiliary integral system (whose system is considered in detail in Lemma 4 below) for any $\Phi \in \mathbf{PC}$ and $t \in J^0$:

$$\begin{aligned} X(t) \quad &= \Phi(0)(t-a)^{\alpha-1} + \int_{-h}^{0} [d_\theta V(t,\theta)]X(t+\theta) \\ &+ I_{-1}(\Gamma(\alpha)) \int_{a}^{t} I_{\alpha-1}(t-\tau) \left(\int_{-h}^{0} [d_\theta U(\tau,\theta)]X(\tau+\theta) + \mathfrak{F}(\tau, X_\tau^T) \right) d\tau. \end{aligned} \quad (3)$$

Let $b > a$ and $\gamma \in [0,1]$ be arbitrary.

Definition 1 ([4])**.** *The function $G(t) = (g_1(t), \ldots, g_n(t)) \in C(J^0, \mathbb{R}^n)$, $\gamma \in [0,1]$ will be called right γ-continuous at a if the function $I_\gamma(t-a)G(t) = col((t-a)^\gamma g_1(t), \ldots, (t-a)^\gamma g_n(t)) \in C([a,\infty), \mathbb{R}^n)$, i.e., the function $I_\gamma(t-a)G(t)$ is right continuous at a. By $\mathbf{C}^\gamma$ we denote the real linear space of all right γ-continuous-at-a functions and for any $b \in \mathbb{R}_+$ by $\mathbf{C}_b^\gamma$ the real Banach space*

$$\mathbf{C}_b^\gamma = \{R(t) \in C((a,b], \mathbb{R}^n) \,|R(t) = G(t)|_{(a,b]},\, G(t) \in \mathbf{C}^\gamma\}$$

with norm $\|R\|_b^\gamma = \sum_{k \in \langle n \rangle} \sup_{t \in [a,b]} (t-a)^\gamma |r_k(t)|$.

Definition 2 ([11])**.** *The vector function $col\, X(t) = (x_1(t), \ldots, x_n(t))$ is a solution of IP (1), (2) in $(a,b](J^0)$ if $X(t) \in \mathbf{C}_b^{1-\alpha}(\mathbf{C}^{1-\alpha})$ satisfies the system (1) for all $t \in (a,b](J^0)$ and the initial condition (2) too.*

Definition 3 ([11])**.** *The vector function $col\, X(t) = (x_1(t), \ldots, x_n(t))$ is a solution of IP (3), (2) in $(a,b](J^0)$ if $X(t) \in \mathbf{C}_b^{1-\alpha}(\mathbf{C}^{1-\alpha})$ satisfies system (3) for all $t \in (a,b](J^0)$ and initial condition (2) too.*

The hypotheses $(\mathfrak{S})$ stated below in Definition 4, as in the cases of systems with derivatives of integer order or Caputo-type fractional order, will play a major role in the solvability of IP (1), (2) (IP (3), (2)) (see [19–21]).

Definition 4 ([19–21])**.** *We say that for the kernels $U, V : J \times \mathbb{R} \to \mathbb{R}^{n\times n}$ the hypotheses $(\mathfrak{S})$ hold if the following conditions are fulfilled for any $i \in \langle m \rangle_0$ and $q \in \langle l \rangle$, $l, m \in \mathbb{N}$:*

- $(\mathfrak{S}1)$ *The functions $(t,\theta) \to U(t,\theta)$, $(t,\theta) \to V(t,\theta)$ are measurable in $(t,\theta) \in J \times \mathbb{R}$ and normalized so that $U(t,\theta) = 0$, $V(t,\theta) = 0$ for $\theta \geq 0$, $U(t,\theta) = U(t,-\sigma)$ for $\theta \leq -\sigma$, $V(t,\theta) = V(t,-\tau)$ for $\theta \leq -\tau$, $\sigma, \tau > 0$, $h = \max(\sigma,\tau)$ for any $t \in J$.*
- $(\mathfrak{S}2)$ *For any fixed $t \in J$ the kernels $U(t,\theta)$ and $V(t,\theta)$ are left continuous in θ on $(-\sigma, 0)$ and $(-\tau, 0)$, $U(t,\cdot), V(t,\cdot) \in BV_0^{loc}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$ and $Var_{[-h,0]} U(t,\cdot), Var_{[-h,0]} V(t,\cdot) \in BL_1^{loc}(J, \mathbb{R}_+)$, $Var_{[-h,0]} V(t,\cdot)$ is uniformly bounded in $t \in J$ and is uniformly nonatomic at zero [21] (i.e., for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $t \in J$, we have that $Var_{[-\delta,0]} V(t,\cdot) < \epsilon$).*
- $(\mathfrak{S}3)$ *For any fixed $t \in J$, the Lebesgue decomposition of the kernels $U(t,\theta)$ and $V(t,\theta)$ has the form:*

$$U(t,\theta) = U_j(t,\theta) + U_{ac}(t,\theta) + U_s(t,\theta),\ V(t,\theta) = V_j(t,\theta) + V_{ac}(t,\theta) + V_s(t,\theta),$$

$$U_j(t,\theta) = \sum_{i \in \langle m \rangle} A^i(t) H(\theta + \sigma^i(t)),\ A^i(t) = \{a_{kj}^i(t)\}_{k,j=1}^n \in BL_1^{loc}(J, \mathbb{R}^{n\times n}),$$

$$V_j(t,\theta) = \sum_{l \in \langle r \rangle} \overline{A}^l(t) H(\theta + \tau^l(t)),\ \overline{A}^l(t) = \{\overline{a}_{kj}^l(t)\}_{k,j=1}^n \in C(J, \mathbb{R}^{n\times n}),$$

$\sigma^0(t) \equiv 0$, $\sigma^i(t), \tau^l(t) \in C(J,[0,h])$, $U_{ac}(t,\cdot), V_{ac}(t,\cdot) \in AC([-h,0],\mathbb{R}^{n\times n})$, $U_s(t,\cdot)$, $V_s(t,\cdot) \in C([-h,0],\mathbb{R}^{n\times n})$ *and* $H(t)$ *is the Heaviside function.*

($\mathfrak{S}4$) *The sets* $S^i_\Phi = \{t \in J \,|\, t - \sigma^i(t) \in S_\Phi, i \in \langle m\rangle\}$, $S^l_\Phi = \{t \in J \,|\, t - \tau^l(t) \in S_\Phi, l \in \langle r\rangle\}$ *do not have limit points and the relations*

$$\lim_{t\to t_*}\int_{-h}^{0} |U^i(t,\theta) - U^i(t_*,\theta)|d\theta = 0\,, \quad \lim_{t\to t_*}\int_{-h}^{0} |V^l(t,\theta) - V^l(t_*,\theta)|d\theta = 0$$

hold for any $t_* \in J$.

Definition 5 ([20,21])**.** *We say that the vector-valued functional* $\mathfrak{F}(t,Y)\colon J \times PC \to \mathbb{R}^n$ *satisfies the modified Caratheodory conditions* ($\mathfrak{C}$) *if the following conditions hold:*

($\mathfrak{C}1$) *For almost all fixed* $t \in J$ *the functional* $(t,Y) \to \mathfrak{F}(t,Y)$ *is continuous in arbitrary* $Y \in \mathbf{PC}(\mathbf{PC}^*)$ *and for each fixed function* $Y \in \mathbf{PC}(\mathbf{PC}^*)$ *the function* $\mathfrak{F}(t,Y) \in BL^{loc}_1(J,\mathbb{R}^n)$.

($\mathfrak{C}2$) *(Local Lipschitz-type condition) For any* $(t,Y) \in J \times \mathbf{PC}$ *and for some its vicinity* $O(t,Y) \subset J \times PC$, *there exists a function* $\ell(t) \in BL^{loc}_1(J,\mathbb{R}_+)$ *such that the inequalities*

$$|\mathfrak{F}(t,Y_1) - \mathfrak{F}(t,Y_2)| \le \ell(t)|Y_1(t) - Y_2(t)|$$

hold for every $(t,Y_1),(t,Y_2) \in O(t,Y)$.

In our investigations below, we will use the following auxiliary result:

Lemma 1 ([4])**.** *Let* $\alpha \in (0,1)$ *and let* $y(t)$ *be a Lebesgue measurable function on* J.

(a) *If there exists a.e. (almost everywhere) the limit* $\lim\limits_{t\to a+0}[(t-a)^{1-\alpha}y(t)] = c \in \mathbb{R}$, *then there also exists a.e. the limit* $(D^{\alpha-1}_a y)(a+0) = (I^{1-\alpha}_a y)(a+0) = \lim\limits_{t\to a+0}(I^{1-\alpha}_a y)(t) = c\Gamma(\alpha)$.

(b) *If there exist a.e. the limit* $\lim\limits_{t\to a+0}[(t-a)^{1-\alpha}y(t)]$ *and* $\lim\limits_{t\to a+0}(I^{1-\alpha}_a y)(t) = c^*$, *then we have that* $\lim\limits_{t\to a+0}[(t-a)^{1-\alpha}y(t)] = \dfrac{c^*}{\Gamma(\alpha)}$.

Definition 6 ([22])**.** *The low terminal a will be called a noncritical point (noncritical jump point) for some initial function* $\Phi \in \mathbf{PC}$ *relative to the delay* $\tau^l(t)$, $l \in \langle r\rangle$ *if the equality* $\tau^l(a) = 0$ *implies that there exists a constant* $b_0 > a$ *(eventually depending on* $\tau^l(t)$*) such that* $t - \tau^l(t) < a$ *for* $t \in (a,b_0]$.

Definition 7 ([22])**.** *The low terminal a for a function* $\Phi \in \mathbf{PC}$ *with* $a \notin S^\Phi$ $(a \in S^\Phi)$ *will be called a critical point (critical jump point) relative to some delay* $\tau^l(t)$, $l \in \langle r\rangle$ *if the equality* $\tau^l(a) = 0$ *implies that there exists a constant* $b_0 > a$ *(eventually depending on* $\tau^l(t)$*) such that* $t - a \ge \tau^l(t)$ *for* $t \in (a,b_0]$.

Lemma 2 ([22])**.** *Let the hypotheses* ($\mathfrak{S}$) *hold and* $\prod\limits_{l\in\langle r\rangle}\tau_l(a) > 0$. *Then, there exists* $b \in (a,a+h]$ *(eventually depending from* τ_l*), such that* $\Phi(t - \tau_l(t))$ *is continuous for* $t \in (a,b]$.

Lemma 3 ([22])**.** *Let the hypotheses* ($\mathfrak{S}$) *hold and* $\prod\limits_{l\in\langle r\rangle}\tau_l(a) = 0$.

Then, for any initial function $\Phi \in \mathbf{PC}$ *with* $S^\Phi = \{a\}$, *one of the following statements holds:*

(i) *The statement of Lemma 2 holds.*

(ii) *The low terminal a is a critical jump point for* Φ *relative to the kernel* $V(t,\theta)$ *for some* $l_0 \in \langle r\rangle$.

Theorem 1 ([23], Krasnosel'skii's fixed point theorem)**.** *Let $(\mathfrak{E}, \|\cdot\|_{\mathfrak{E}})$ be a Banach space, $H \subset \mathfrak{E}$ be a nonempty, closed, and convex subset of $\mathfrak{E}$ and the maps $\mathfrak{T}, \mathfrak{N}\colon H \to \mathfrak{E}$ satisfy the following conditions:*

(i) The operator $\mathfrak{T}$ is a contraction with constant $\gamma \in (0,1)$;
(ii) The operator $\mathfrak{K}$ is continuous and the set $\mathfrak{K}(\mathfrak{B})$ is contained in a compact set;
(iii) For any $x, y \in H$, we have that $\mathfrak{T}x + \mathfrak{K}y \in H$.

Then, there exists a $z \in H$ with $\mathfrak{T}z + \mathfrak{K}z = z$.

Theorem 2 ([24], Corollary 2)**.** *Suppose that $\alpha \in (0,1)$ and the following conditions hold:*

1. *The functions $a(t), g(t), y(t) \in L_1^{loc}(J, \mathbb{R}_+)$.*
2. *The functions $a(t), g(t)$ are nondecreasing, $g(t)$ is bounded on J, and $y(a) = 0$.*
3. *The inequality $y(t) \leq a(t) + g(t) \int\limits_a^t (t-s)^{\beta-1} y(s) ds$ holds for $t \in J$.*

Then, for any $t \in J$ we have that $y(t) \leq a(t) E_\alpha(g(t)\Gamma(\alpha)t^\alpha)$.

Remark 1. *The main difference between the systems (even in the linear case) with different types of fractional derivatives from the point of view of their applicability as model tools of real-world phenomena is the answer for a concrete type of derivative, whether the derivative of a constant is identically equal to zero or not. Typical representatives of these two classes are the fractional derivatives of Caputo and RL types. So, for the study of retarded and neutral systems with Caputo derivatives a lot of the ideas and techniques known from the systems with integer-order derivatives can be used, because of the continuity of the corresponding Krasovskii functional. Note that in some important special cases it is possible that the Krasovskii functional can have a jump only of the first kind at the initial point, but this obstacle can be overcome with the help of some variants of Banach's contraction principle. In contrast, for retarded and neutral systems with RL derivatives we have at least in the technical aspect several complications, mainly based on the availability of a discontinuity (jump of second kind) of the solutions at the low terminal, a fact which greatly complicates the use of the Riesz theorem for the representation of linear continuous functionals (Krasovskii functionals) on **C** via a Lebesgue–Stieltjes integral. Even for the retarded systems with RL derivatives, more sophisticated techniques such as Weissinger theorem combined with application of the Mittag–Lafleur function must be used. We emphasize that the important neutral case with RL derivatives, the study of which is the aim of this article, is essentially more complicated even in comparison with the retarded systems with the same kind of derivatives, not only in the technical aspects, but in their applications as models.*

3. The Initial Problem (1), (2) with Discontinuous Initial Functions

The next Lemma is well known for the case of systems with Caputo-type derivatives and for delayed systems with RL-type derivatives, and therefore, we will sketch only the differences appearing in the neutral case with RL-type derivatives and distributed delays.

Lemma 4. *Let the following conditions be fulfilled:*

1. *The hypotheses $\mathfrak{S}$ hold.*
2. *For any $\Phi \in$ **PC**, the vector-valued functional $\mathfrak{F}\colon J \times$ **PC** $\to \mathbb{R}^n \in BL_1^{loc}(J, \mathbb{R}^n)$.*

Then, every solution $X(t)$ of IP (1), (2) is a solution of IP (3), (2) and vice versa.

Proof. Let $\Phi \in$ **PC** be arbitrary and $X(t)$ be the corresponding solution of IP (1), (2). Then, according to formula 2.1.40 in [5], we have

$$I_{a+}^{\alpha}D_{a+}^{\alpha}\left(X(t)-\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)\right)=X(t)-\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)$$
$$-\lim_{t\to a+0}I_{a+}^{1-\alpha}\left(X(t)-\left(\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)\right)\right)=X(t)-\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)$$
$$-\Phi(0)(t-a)^{\alpha-1}+\lim_{t\to a+0}I_{a+}^{1-\alpha}\left(\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)\right).$$

Then, obviously $X(t)$ will be a solution of IP (3), (2) if we prove

$$\lim_{t\to a+0}I_{a+}^{1-\alpha}\left(\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)\right)=\mathbf{0}.$$

Condition ($\mathfrak{S}2$) implies that there exists a constant $V^h>0$ such that $|Var_{\theta\in[-h,0]}V(t,\theta)|\le V^h$ for $t\in[a,a+h]$, and since $Var_{\theta\in[-h,0]}V(t,\theta)$ is uniformly nonatomic at zero in t, then for any $\epsilon>0$ there exists $\delta\in(0,\epsilon)$ such that $|Var_{\theta\in[-h,0]}V(t,\theta)|<\dfrac{\epsilon\Gamma(\alpha)}{2\|\Phi\|}$ uniformly for $t\in[a,a+h]$. Then, for $t\in[a,a+\delta]$ we have

$$\begin{aligned}&\left|\lim_{t\to a+0}I_{a+}^{1-\alpha}\left(\int_{-h}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)\right)\right|\\&=\frac{1}{\Gamma(\alpha)}\lim_{t\to a+0}(t-a)^{1-\alpha}\left|\int_{-h}^{a-t}[d_{\theta}V(t,\theta)]\Phi(t+\theta)+\int_{a-t}^{0}[d_{\theta}V(t,\theta)]X(t+\theta)\right|\\&\le\frac{1}{\Gamma(\alpha)}\lim_{t\to a+0}(t-a)^{1-\alpha}\|\Phi\||Var_{\theta\in[-h,0]}V(t,\theta)|\\&+\frac{1}{\Gamma(\alpha)}\lim_{t\to a+0}(t-a)^{1-\alpha}\sup_{\theta\in[-\delta,0]}|X(t,\theta)|Var_{\theta\in[-\delta,0]}V(t,\theta)\end{aligned}\tag{4}$$

and, hence, for the first addend on the right-hand side of (4) the relation

$$\lim_{t\to a+0}(t-a)^{1-\alpha}\|\Phi\||Var_{\theta\in[-h,0]}V(t,\theta)|=\mathbf{0}$$

holds. Since $X(t)\in\mathbf{C}^{1-\alpha}$, then there exists $\overline{\delta}\in(0,\delta)$ such that for $t\in[a,a+\delta]$ and any $\theta\in[-h,0]$ we have that $|X(t+\theta)|\le|X(t+\theta)-X(t)|+|X(t)|\le 2|X(t)|$. Then, for the second addend on the right-hand side of (4) for $t\in[a,a+\delta]$ we obtain that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\lim_{t\to a+0}(t-a)^{1-\alpha}\sup_{\theta\in[-\delta,0]}|X(t,\theta)|Var_{\theta\in[-\delta,0]}V(t,\theta)\\&\le\frac{2}{\Gamma(\alpha)}\lim_{t\to a+0}(t-a)^{1-\alpha}|X(t)|\frac{\epsilon\Gamma(\alpha)}{2\|\Phi\|}\le 2\Phi(0)\frac{\epsilon}{2\|\Phi\|}<\epsilon,\end{aligned}$$

which implies that

$$\lim_{t\to a+0}(t-a)^{1-\alpha}\sup_{\theta\in[-\delta,0]}|X(t,\theta)|Var_{\theta\in[-\delta,0]}V(t,\theta)=\mathbf{0}.$$

Thus, $X(t)$ is a solution of IP (3), (2).

Let $X(t)$ is a solution of IP (3), (2), and then, applying the operator D_{a+}^{α} to both sides of (3) we obtain immediately the opposite statement. □

Modifying the approach in [11], for an arbitrary fixed number $\alpha \in (0,1)$ we introduce the following real linear space:

$$\mathfrak{E} = \{G : [a-h,\infty) | G(t)|_{[a-h,a]} = \Phi(t-a), \Phi \in \mathbf{PC};$$
$$G|_{J^0} \in \mathbf{C}^{1-\alpha}(J^0,\mathbb{R}^n), \lim_{t\to a+0}(t-a)^{1-\alpha}G(t) = \Phi(0)\}$$

and for any $b > a$ introduce the linear subspaces:

$$\mathfrak{E}_b = \{R : [a-h,b] | R(t) = G|_{[a-h,b]}, G \in \mathfrak{E}\}$$

endowed with the norm $\|R\|_b = \|\Phi\| + \|G|_{(a,b]}\|_b^{1-\alpha}$.

Let $\Phi_0 \in \mathbf{PC}$ with $S^{\Phi_0} = \{a\}$ be arbitrary fixed and define the nonempty, closed, and convex subset $\mathfrak{E}_b^{\Phi_0} \subset \mathfrak{E}_b$ as follows:

$$\mathfrak{E}_b^{\Phi_0} = \{R \in \mathfrak{E}_b | R(t)|_{[a-h,a]} = \Phi_0(t-a), \lim_{t\to a+0}(t-a)^{1-\alpha}R(t) = \Phi_0(0)\}.$$

It is simple to see that for any $R \in \mathfrak{E}_b$ we have that the norms $\|R\|_b = \|\Phi_0\| + \|R\|_b^{1-\alpha}$ and $\|R\|_b = \|R\|_b^{1-\alpha}$ are equivalent.

For any $b > a, t \in (a,b]$ and $R \in \mathfrak{E}_b$ define the operator $\mathfrak{R}$: $\mathfrak{E}_b^{\Phi_0} \to \mathfrak{E}_b$ as follows:

$$(\mathfrak{R}R)(t) = \Phi(0)(t-a)^{\alpha-1} + \int_{-h}^{0} [d_\theta V(t,\theta)]R(t+\theta)$$
$$+I_{-1}(\Gamma(\alpha))\left(\int_a^t I_{\alpha-1}(t-s)\int_{-h}^{0}[dU(s,\theta)]R(s+\theta)ds + \int_a^t I_{\alpha-1}(t-s)\mathfrak{F}(t,R_s)ds\right), \quad (5)$$

with the additional conditions

$$(\mathfrak{R}R)(t) = \Phi_0(t-a), t \in [a-h,a]; \lim_{t\to b-0}(\mathfrak{R}R)(t) = (\mathfrak{R}R)(b). \quad (6)$$

In addition, for arbitrary $b > h$, $t \in (a,b]$ and $R \in \mathfrak{E}_b$ we define the operators $\mathfrak{T}, \mathfrak{K}$: $\mathfrak{E}_b^{\Phi_0} \to \mathfrak{E}_b$ as follows:

$$(\mathfrak{T}R)(t) = \Phi_0(0)(t-a)^{\alpha-1} + \int_{-h}^{0}[d_\theta V(t,\theta)]R(t+\theta)$$
$$(\mathfrak{T}R)(t) = \Phi_0(t-a), t \in [a-h,a]; \lim_{t\to b-0}(\mathfrak{T}R)(t) = (\mathfrak{T}R)(b), \quad (7)$$

$$(\mathfrak{K}R)(t) = I_{-1}(\Gamma(\alpha))$$
$$\left(\int_a^t I_{\alpha-1}(t-s)\left(\int_{-h}^{0}[d_\theta U(s,\theta)]R(s+\theta)\right)ds + \int_a^t I_{\alpha-1}(t-s)\mathfrak{F}(s,R_s)ds\right)$$
$$(\mathfrak{K}R)(t) = \mathbf{0}, t \in [a-h,a]; \lim_{t\to b-0}(\mathfrak{K}R)(t) = (\mathfrak{K}R)(b). \quad (8)$$

Then, the operator $\mathfrak{R}$, defined via (5), (6), for any $R \in \mathfrak{E}_b^{\Phi_0}$ has the form

$$(\mathfrak{R}R)(t) = (\mathfrak{T}R)(t) + (\mathfrak{K}R)(t). \quad (9)$$

The next technical lemma is an immediate generalization of Theorem 4 in [25]. The lemma below is more appropriate for applications and can be used in the cases of RL derivatives too. For simplicity we will consider only the case when the initial function

$\Phi \in \mathbf{PC}$ has only one jump point, but it is simple to see that the proof provided below can be used in the case of an arbitrary but finite number of jumps too.

Lemma 5. *Let the following conditions be fulfilled:*

1. *The hypotheses* $(\mathfrak{S})$ *hold and* $\Phi \in \mathbf{PC}$ *with* $S^{\Phi} = \{t^{jump}\}, t^{jump} \in (a-h, a]$ *are arbitrary.*
2. *The low terminal* a *is a jump point for* Φ *relative to the delay* $\tau^{l_0}(t)$ *for some* $l_0 \in \langle r \rangle$.

   Then, $\lim_{t \to a+0} \overline{A}^{l_0}(t) = \Theta, l_0 \in \langle r \rangle$.

Proof. We will consider the more complicated case when $S^{\Phi} = \{t^{jump} = a\}$ and a is a critical jump point. Then, since $\tau^l(t) \in C(J, [0,h])$ for all $l \in \langle r \rangle$ we have that $\lim_{t \to a+0} t - \tau^{l_0}(t) = a$. Let assume that there exists a sequence $\{t_i\}_{i=1}^{\infty} \subset (a, b_0)$ such that $t_i - \tau^{l_0}(t_i) = a$, and hence, the set S^{Φ} has a limit point which contradicts condition $(\mathfrak{S}4)$. Then, there exists $b_0 \in (a, a+h)$ such that $t - \tau^{l_0}(t) > a$ for any $t \in (a, b_0]$. Let us assume that $\left|\overline{A}^{l_0}(a)\right| > 0$, $\epsilon \in \left(0, \frac{\left|\overline{A}^{l_0}(a)\right|}{4}\right)$ and $\delta \in (o, \epsilon)$ is the number existing according to condition $(\mathfrak{S}2)$ such that $Var_{[-\delta,0]} V(t, \cdot) < \epsilon$ for any $t \in J$. Then, for any $t \in J$ $\theta \in (0, \delta]$ with $t + \theta \geq a$

$$\begin{aligned}\left|\overline{A}^{l_0}(a)H(a)\right| &= \left|\overline{A}^{l_0}(a)\right| \leq |V_d(t,\theta)| \leq |V_d(t,0)| + \left|Var_{\theta \in [-\delta,0]} V_d(t,\theta)\right| \\ &= \left|Var_{\theta \in [-\delta,0]} V_d^l(t,\theta)\right| < \epsilon < \frac{\left|\overline{A}^{l_0}(a)\right|}{4},\end{aligned}$$

which is impossible. Thus, we have that $\left|\overline{A}^{l_0}(a)\right| = 0$, and hence, $\lim_{t \to a+0} = \left|\overline{A}^{l_0}(t)\right| = 0$.

The proofs of the cases when a is a noncritical jump point and when $t^{jump} \neq a$ are simpler versions of the proof in the case when $t^{jump} = a$ (a is a critical jump point), and therefore, will be omitted. □

It is clear that without loss of generality it is possible to renumber all concentrated delays in the jump part $V_j(t,\theta) = \sum_{l \in \langle r \rangle} \overline{A}^l(t) H(\theta + \tau^l(t))$ and give the first $\bar{l}$ numbers, $0 \leq \bar{l} \leq r$ of these delays, relative to which a is a critical jump point. Everywhere below we will assume that this renumbering is made.

Theorem 3. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 4 hold.*
2. $\sup_{\theta \in [-h,0]} |\partial V_{\theta}(t,\theta)|, \sup_{\theta \in [-h,0]} |\partial U_{\theta}(t,\theta)| \in BL_1^{loc}(J, \mathbb{R}^{n \times n})$, *where* $\partial V_{\theta}, \partial U_{\theta}$ *denote the partial derivatives with respect to* θ.
3. *The condition* $(\mathfrak{C})$ *holds.*

Then, for any $\Phi_0 \in \mathbf{PC}$ *with* $S^{\Phi} = \{t^{jump}\}, t^{jump} \in (a-h, a]$ *there exists* $b_0 > a$ *such that IP (3), (2) has at least one solution* $X(t) \in \mathfrak{E}_b^{\Phi_0}$ *with an interval of existence* $(a, b_0]$.

Proof. As above, we will consider the more complicated case when $\Phi \in \mathbf{PC}$ with $S^{\Phi} = \{a\}$. Let $b \in (a, a+h)$ is arbitrary. As a first step we will prove that $\mathfrak{T}(\mathfrak{E}_b^{\Phi_0}) \subseteq \mathfrak{E}_b^{\Phi_0}$.

For any $R \in \mathfrak{E}_b^{\Phi_0}$ from (6) and Lemmas 2 and 3 it follows that

$$(\mathfrak{T}R)(t) = \Phi_0(0)(t-a)^{\alpha-1} + \int_{-h}^{a-t} [d_\theta V(t,\theta)]\Phi_0(t-a+\theta) + \int_{a-t}^{0} [d_\theta V(t,\theta)]R(t+\theta)$$

$$= \Phi_0(0)(t-a)^{\alpha-1} + \int_{-h}^{a-t} [d_\theta V(t,\theta)]\Phi_0(t-a+\theta)$$

$$+ \sum_{l=\bar{l}+1}^{r} \overline{A}^{l}(t)\Phi_0(t-\tau^l(t)-a) + \sum_{l=1}^{\bar{l}} \overline{A}^{l}(t)R(t-\tau^l(t)) + \int_{a-t}^{0} [d_\theta V_c(t,\theta)]R(t+\theta), \tag{10}$$

where $V_c(t,\theta) = V_{ac}(t,\theta) + V_s(t,\theta)$, and $t \in (a,b]$.

The first addend on the right-hand side of (10) is continuous for any $t \in J^0$, and hence, belongs to $\mathbf{C}^{1-\alpha}$. Since

$$\lim_{t\to a+0} (t-a)^{1-\alpha}\Phi_0(0)(t-a)^{\alpha-1} = \Phi_0(0),$$

we can prolong $\Phi_0(0)(t-a)^{\alpha-1}$ as a function from $\mathfrak{E}_b^{\Phi_0}$. According to Lemma 1 in [11] and Lemma 4 in [22], since the initial function $\Phi_0(t-a)$ is a continuous function for $t \in [a-h,a)$, the second and the third addends on the right-hand side of (10) are continuous functions at $t \in (a,b]$. Taking into account that a is a critical jump point relative to the delays with numbers $l \in \langle \bar{l} \rangle$, we conclude that the fourth addend is a continuous function at $t \in (a,b]$ too. For the fifth addend, via the substitution $t+\theta = s$ we obtain

$$\int_{a-t}^{0} [d_\theta V_c(t,\theta)]R(t+\theta) = \int_{a}^{t} [d_s V_c(t,s-t)]R(s). \tag{11}$$

Because $V_c(t,s-t)$ and $R(s)$ for any $t \in (a,b]$ are continuous at $s \in (a,t]$, then the fifth addend is a continuous function at $t \in (a,b]$ by virtue of lemma 1 in [19]. Since the right-hand side of (10) is a continuous function at $t = b$, then the second additional relation in (7) holds too, and hence, from (7) it follows that $\mathfrak{T}(\mathfrak{E}_b^{\Phi_0}) \in \mathfrak{E}_b^{\Phi_0}$. Thus, $\mathfrak{T}(\mathfrak{E}_b^{\Phi_0}) \subseteq \mathfrak{E}_b^{\Phi_0}$.

As a second step we will prove that the operator $\mathfrak{T}$ is a contraction. Denote $L_b = \sup_{t\in[a,b]} \ell(t)$,

$$V_b = \max\left(\sup_{\substack{t\in[a,b],\\ \theta\in[-h,0]}} |\partial_\theta V(t,\theta)|, \sup_{t\in[a,b]} \left| \underset{\theta\in[-h,0]}{Var} V(t,\theta) \right| \right)$$

and

$$U_b = \max\left(\sup_{t\in[a,a+b]} \left| \underset{\theta\in[-h,0]}{Var} U(t,\theta) \right|, \sup_{\substack{t\in[a,b],\\ \theta\in[-h,0]}} |\partial_\theta U(t,\theta)| \right).$$

Then, for arbitrary $R, \overline{R} \in \mathfrak{E}_b^{\Phi_0}$ from (7) and Lemmas 2 and 3 it follows that

$$\left|(\mathfrak{T}(R))(t) - (\mathfrak{T}(\overline{R}))(t)\right| = \left| \int_{-h}^{0} [d_\theta V(t,\theta)](R(t+\theta) - \overline{R}(t+\theta)) \right|$$

$$\leq \left| \int_{-h}^{0} [d_\theta V_j(t,\theta)](R(t+\theta) - \overline{R}(t+\theta)) \right| + \left| \int_{-h}^{0} [d_\theta V_c(t,\theta)](R(t+\theta) - \overline{R}(t+\theta)) \right|$$

$$\leq \left|\sum_{l=1}^{\bar{l}} \overline{A}^l(t)(R(t-\tau^l(t)) - \overline{R}(t-\tau^l(t)))\right|$$
$$+\left|\sum_{l=\bar{l}}^{l=r} \overline{A}^l(t)(\Phi_0(t-a-\tau^l(t)) - \overline{\Phi_0}(t-a-\tau^l(t)))\right|$$
$$+\left|\int_{a-t}^{0} [d_\theta V_c(t,\theta)](R(t+\theta) - \overline{R}(t+\theta))\right|$$
$$\leq \left|\sum_{l=1}^{\bar{l}} \overline{A}^l(t)(R(t-\tau^l(t)) - \overline{R}(t-\tau^l(t)))\right| \overset{t+\theta=s}{+} \left|\int_a^t [d_s V(t,s-t)](R(s) - \overline{R}(s))\right|. \tag{12}$$

For the first addend on the right-hand side of (12), since $\tau^l(a) = 0$ and the low terminal a is a critical jump point for Φ relative to the delays $\tau^l(t), l \in \langle \bar{l} \rangle$, by virtue of Lemma 5 there exists $b_1 \in (a,b]$ such that for any $t \in (a,b_1]$ we have the estimation

$$\left|\sum_{l=1}^{\bar{l}} \overline{A}^l(t)(R(t-\tau^l(t)) - \overline{R}(t-\tau^l(t)))\right|$$
$$\leq (t-a)^{\alpha-1}\left|\sum_{l=1}^{\bar{l}} \overline{A}^l(t)(t-a)^{1-\alpha}(R(t-\tau^l(t)) - \overline{R}(t-\tau^l(t)))\right|$$
$$\leq \frac{(t-a)^{1-\alpha}}{6} \sup_{s\in(a,t]} (s-a)^{1-\alpha}\left|R(s) - \overline{R}(s)\right|. \tag{13}$$

Hypothesis $(\mathfrak{S}2)$ implies that for $\epsilon = \dfrac{1}{6}$ there exists $\delta \in (0,\epsilon)$ such that

$$\left|\operatorname*{Var}_{\theta\in[-\delta,0]} V_c(t,\theta)\right| < \frac{1}{6}$$

for any $t \in (a,b_2]$, where $b_2 = \min(b_1, a+\delta)$, we obtain

$$|V(t,s-t)| \leq |V(t,0)| + \left|Var_{s-t\in[-\delta,0]} V_c(t,s-t)\right| \leq \frac{1}{6}.$$

Then, the second addend in (12) for any $t \in (a,b_2]$ has the estimation

$$\left|\int_a^t [d_s V(t,s-t)](R(s) - \overline{R}(s))\right|$$
$$= \left|\int_a^t [\partial_s V(t,s-t)](s-a)^{\alpha-1}(s-a)^{1-\alpha}(R(s) - \overline{R}(s))ds\right|$$
$$\leq \frac{(t-a)^\alpha V_b}{\alpha} \sup_{s\in(a,t]} (s-a)^{1-\alpha}\left|R(s) - \overline{R}(s)\right|$$

and, hence, for $t \in (a,b_3]$, where $b_3 \in \left(a, \min\left(1, b_2, a + \left(\frac{\alpha}{6V_b}\right)^{\frac{1}{\alpha}}\right)\right]$ we obtain that

$$\left|\int_a^t [d_s V(t,s-t)](R(s) - \overline{R}(s))\right| \leq \frac{1}{6} \sup_{s\in[a,t]} (s-a)^{1-\alpha}\left|R(s) - \overline{R}(s)\right|. \tag{14}$$

Multiplying both sides of (12) by $(t-a)^{1-\alpha}$ and then from (12), (13), and (14) it follows that for any $t \in (a, b_3]$ we obtain that $\|\mathfrak{T}(R) - \mathfrak{T}(\overline{R})\|_{b_3}^{\Phi_0} \leq \frac{1}{3}\|R - \overline{R}\|_{b_3}^{\Phi_0}$ and, hence, $\mathfrak{T}$ is a contraction operator in $\mathfrak{E}_{b_3}^{\Phi_0}$.

For arbitrary $R, \overline{R} \in \mathfrak{E}_{b_3}^{\Phi_0}$, from (7) we obtain that

$$
\begin{aligned}
&\left|(\mathfrak{K}R)(t) - (\mathfrak{K}\overline{R})(t)\right| \leq \frac{n}{\Gamma(\alpha)}\left|\int_a^t I_{\alpha-1}(t-s)\left(\int_{-h}^{0}[d_\theta U(s,\theta)](R(s+\theta) - \overline{R}(s+\theta))\right)ds\right| \\
&+\frac{n}{\Gamma(\alpha)}\left|\int_a^t I_{\alpha-1}(t-s)\left(\mathfrak{F}(s,R_s) - \mathfrak{F}(s,\overline{R}_s)\right)ds\right| \\
&\leq \frac{n^2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left|\int_{a-s}^{0}[d_\theta U(s,\theta)](R(s+\theta) - \overline{R}(s+\theta))\right|ds \\
&+\frac{n^2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left|\mathfrak{F}(s,R_s) - \mathfrak{F}(s,\overline{R}_s)\right|ds \\
&\leq \frac{n^2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\int_{a-s}^{0}|\partial_\theta U(s,\theta)||R(s+\theta) - \overline{R}(s+\theta)|ds \\
&+\frac{n^2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\ell(s)\left|R(s) - \overline{R}(s)\right|ds \\
&\leq \frac{n^2 U_b}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{\alpha-1}\left(\int_a^s |R(\eta) - \overline{R}(\eta)|d\eta\right)ds + L_b\int_a^t (t-s)^{\alpha-1}|R(s) - \overline{R}(s)|ds\right).
\end{aligned} \tag{15}
$$

For the first addend on the right-hand side of (15), for any $t \in (a, b_3]$ the following estimation holds:

$$
\begin{aligned}
&\frac{n^2 U_b}{\Gamma(\alpha)}\left|\int_a^t (t-s)^{\alpha-1}\left(\int_a^s |R(\eta) - \overline{R}(\eta)|d\eta\right)ds\right| = \frac{n^2 U_b}{\Gamma(\alpha)}\left|\int_a^t \left(\int_a^s |R(\eta) - \overline{R}(\eta)|d\eta\right)d(t-s)^\alpha\right| \\
&\frac{n^2 U_b}{\Gamma(\alpha)}\left|\int_a^t (t-s)^\alpha |R(s) - \overline{R}(s)|ds\right| \\
&\overset{t-s=y-a}{=} \frac{n^2 U_b}{\Gamma(\alpha)}\left|\int_a^t (y-a)^\alpha (y-a)^{\alpha-1}(y-a)^{1-\alpha}|R(t-y+a) - \overline{R}(t-y+a)|dy\right| \\
&\leq (2\alpha)^{-1}\frac{n^2 U_b}{\Gamma(\alpha)}(t-a)^{2\alpha}\sup_{s\in[a,t]}(s-a)^{1-\alpha}|R(s) - \overline{R}(s)| \\
&\leq \frac{n^2 U_b}{2\Gamma(1+\alpha)}\sup_{s\in[a,t]}(s-a)^{1-\alpha}|R(s) - \overline{R}(s)|
\end{aligned} \tag{16}
$$

and analogously, for the second one we obtain

$$\frac{n^2 L_b}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|R(s)-\overline{R}(s)|ds = \frac{n^2 L_b}{\Gamma(1+\alpha)}\int_a^t |R(s)-\overline{R}(s)|d(t-s)^{\alpha}$$
$$\leq \frac{n^2 L_b}{\Gamma(1+\alpha)}\int_a^t (t-s)^{\alpha} d|R(s)-\overline{R}(s)|$$
$$\leq \frac{n^2 L_b (t-a)^{2\alpha-1}}{\Gamma(1+\alpha)} \sup_{s\in[a,t]} (s-a)^{1-\alpha}|R(s)-\overline{R}(s)|. \tag{17}$$

Multiplying both sides of (15) by $(t-a)^{1-\alpha}$, and then, from (15), (16), and (17), it follows that for any $t \in (a, b_3]$ we obtain that

$$\left\|\mathfrak{K}(R)-\mathfrak{K}(\overline{R})\right\|_{b_3}^{\Phi_0} \leq \frac{n^2}{2\Gamma(1+\alpha)}(U_b + 2L_b(b_3-a)^{\alpha})\left\|R-\overline{R}\right\|_{b_3}^{\Phi_0} \tag{18}$$

and, hence, $\mathfrak{K}\colon \mathfrak{E}_{b_3}^{\Phi_0} \to \mathfrak{E}_{b_3}$ is a continuous operator.

Let $r_0 \in \mathbb{R}_+^0$ and $R_0 \in \mathfrak{E}_{b_3}^{\Phi_0}$ be arbitrary fixed and consider the ball

$$\mathfrak{B}(R_0, r_0) = \{R \in \mathfrak{E}_{b_3}^{\Phi_0} \mid \|R_0 - R\|_{b_3}^{\Phi_0} \leq r_0\} \subset \mathfrak{E}_{b_3}^{\Phi_0}.$$

For any $R \in \mathfrak{B}(R_0, r_0)$, from (18) it follows that

$$\|\mathfrak{K}(R)\|_{b_3}^{\Phi_0} \leq \|\mathfrak{K}(R_0)\|_{b_3}^{\Phi_0} + \|\mathfrak{K}(R)-\mathfrak{K}(R_0)\|_{b_3}^{\Phi_0}$$
$$\leq \frac{n^2}{2\Gamma(1+\alpha)}(U_b + 2L_b(b_3-a)^{\alpha})\|R-R_0\|_{b_3}^{\Phi_0} + \|\mathfrak{N}(R_0)\|_{b_3}^{\Phi_0}$$
$$\leq \frac{n^2 r_0}{2\Gamma(1+\alpha)}(U_b + 2L_b(b_3-a)^{\alpha}) + \|\mathfrak{N}(R_0)\|_{b_3}^{\Phi_0} = C(r_0, b_3) \tag{19}$$

and, hence, the set $\mathfrak{K}(\mathfrak{B}(R_0, r_0))$ is uniformly bounded.

To apply Theorem 1 we must prove that the set $\mathfrak{N}(\mathfrak{B}(R_0, r_0))$ is equicontinuous, and hence, it is at least relative compact. For any $R \in \mathfrak{B}(R_0, r_0)$ and arbitrary $t_1, t_2 \in (a, b_3]$ we have the estimation

$$|(\mathfrak{K}R)(t_2) - (\mathfrak{K}R)(t_1)|$$
$$\leq \frac{n^2}{\Gamma(\alpha)}\left|\int_a^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})\left(\int_{-h}^{0} [d_\theta U(s,\theta)]R(s+\theta) + \mathfrak{F}(s, R_s)\right) ds\right|$$
$$+\frac{n^2}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\left(\int_{-h}^{0} [d_\theta U(s,\theta)]R(s+\theta) + \mathfrak{F}(s, R_s)\right) ds\right|$$
$$\leq \frac{n^2}{\Gamma(\alpha)}\int_a^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|\left|\int_{-h}^{0} [d_\theta U(s,\theta)]R(s+\theta) + \mathfrak{F}(s, R_s)\right| ds$$
$$+\frac{n^2}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\left|\int_{-h}^{0} [d_\theta U(s,\theta)]R(s+\theta) + \mathfrak{F}(s, R_s)\right| ds. \tag{20}$$

Since for $t \in (a, b_3]$ from conditions $(\mathfrak{C})$ and (19) we have that

$$\left|\int_{-h}^{0} [d_\theta U(s,\theta)]R(s+\theta) + \mathfrak{F}(s, R_s)\right| \leq (U_b + L_b)C(r_0, b_3), \tag{21}$$

then from (20) and (21) for any $t \in (a, b_3]$ we obtain

$$\begin{aligned}
&|(\mathfrak{K}R)(t_2) - (\mathfrak{K}R)(t_1)| \\
&\leq \frac{n^2 C(r_0, b_3)(U_b + L_b)}{\Gamma(\alpha)} \left(\int_a^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right) \\
&\leq \frac{n^2 C(r_0, b_3)(U_b + L_b)}{\alpha\Gamma(\alpha)} (2(t_2 - t_1)^\alpha + (t_1 - a)^\alpha - (t_2 - a)^\alpha) \\
&\leq \frac{n^2 C(r_0, b_3)(U_b + L_b)}{\Gamma(1+\alpha)} (2(t_2 - t_1)^\alpha + |(t_1 - a)^\alpha - (t_2 - a)^\alpha|).
\end{aligned} \tag{22}$$

Then, since $(t-a)^\alpha$ is uniformly continuous at $t \in [a, a+h]$, for any $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that if $|t_2 - t_1| < \delta$ we have that

$$|(t_1 - a)^\alpha - (t_2 - a)^\alpha| < \epsilon \frac{\Gamma(1+\alpha)}{n^2 C(r_0, b_3)(U_b + L_b)}$$

and, hence, from (22) it follows that the set $\mathfrak{K}(\mathfrak{B}(R_0, r_0))$ is equicontinuous. Thus, by virtue of the Arzella–Ascoli theorem, $\mathfrak{K}(\mathfrak{B}(R_0, r_0))$ is at least relative compact. For arbitrary $R, \overline{R} \in \mathfrak{E}_{b_3}^{\Phi_0}$ we consider the sum $(\mathfrak{T}R)(t) + (\mathfrak{K}\overline{R})(t)$. Taking into account that $\lim_{t \to a+0} (\mathfrak{K}\overline{R})(t) = \mathbf{0}$ and can be prolonged as $\mathbf{0}$ for any $t \in [a-h, a]$, from (7) it follows that $(\mathfrak{T}R)(t) + (\mathfrak{K}\overline{R})(t) = \Phi_0(t-a)$. Since $\lim_{t \to a+0} (\mathfrak{K}\overline{R})(t) = \mathbf{0}$, then $\lim_{t \to a+0} (t-a)^{1-\alpha} (\mathfrak{K}\overline{R})(t) = \mathbf{0}$ too. Considering the fact that

$$\lim_{t \to a+0} (t-a)^{1-\alpha} (\mathfrak{T}R)(t) = \Phi_0(0) + \lim_{t \to a+0} \left((t-a)^{1-\alpha} \int_{-h}^{0} [d_\theta V(t, \theta)] R(t+\theta) \right) = \Phi_0(0),$$

then, we obtain that

$$\lim_{t \to a+0} (t-a)^{1-\alpha} ((\mathfrak{T}R)(t) + (\mathfrak{K}\overline{R})(t)) = \Phi_0(t-a)$$

and, hence, $(\mathfrak{T}R)(t) + (\mathfrak{K}\overline{R})(t) \in \mathfrak{E}_{b_3}^{\Phi_0}$. Applying Theorem 1, we obtain that there exists at least one fixed point $R^* \in \mathfrak{E}_{b_0}^{\Phi_0}$, i.e., $R^*(t) = (\mathfrak{T}R^*)(t) + (\mathfrak{K}R^*)(t)$, $t \in (a, b_0], b_0 = b_3$. □

Theorem 4. *Let the conditions of Theorem 3 be fulfilled.*

Then, for any $\Phi_0 \in \mathbf{PC}$ with $S^\Phi = \{t^{jump}\}, t^{jump} \in (a-h, a]$ the solution $X(t) \in \mathfrak{E}_{b_0}^{\Phi_0}$ of IP (3), (2) is unique in the interval of its existence $(a, b_0]$.

Proof. Assume the contrary, that IP (3), (2) possess two different solutions $X^1(t), X^2(t) \in \mathfrak{E}_b^{\Phi_0}$. Then, from (3) for any $t \in (a, b_0]$ we obtain that

$$\begin{aligned}
&\left| X^1(t) - X^2(t) \right| = |Y(t)| \leq |(\mathfrak{T}Y)(t) + (\mathfrak{K}Y)(t)| \leq |(\mathfrak{T}Y)(t)| + |(\mathfrak{K}Y)(t)| \leq q|Y(t)| \\
&+ \frac{n^2}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left(\left| \int_{-h}^{0} \partial_\theta U(\tau, \theta) Y(\tau+\theta) d\theta \right| + \left| \mathfrak{F}(\tau, X_\tau^{1,T}) - \mathfrak{F}(\tau, X_\tau^{2,T}) \right| \right) d\tau \\
&\leq q|Y(t)| + \frac{n^2 (b_0 - a)^{1-\alpha} U_{b_0}}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left((\sup_{s \in [a,\tau]} |Y(s)| + \ell(\tau)|Y(s)|) d\tau \right).
\end{aligned} \tag{23}$$

Since (2) implies that $|Y(a)| = 0$, then from (23) it follows that

$$\sup_{s\in[a,t]} |Y(s)| \le \frac{n^2(b_0-a)^{1-\alpha}U_{b_0}}{\Gamma(\alpha)(1-q)} \int_a^t (t-\tau)^{\alpha-1}\left((\sup_{s\in[a,\tau]} |Y(s)| + \ell(\tau)|Y(s)|)d\tau \right)$$
$$\le \frac{n^2(b_0-a)^{1-\alpha}U_{b_0}(1+L_{b_0})}{\Gamma(\alpha)(1-q)} \int_a^t (t-\tau)^{\alpha-1} \sup_{s\in[a,\tau]} |Y(s)|d\tau.$$

Denoting $g(t) = g_0 = \dfrac{n^2(b_0-a)^{1-\alpha}U_{b_0}(1+L_{b_0})}{\Gamma(\alpha)(1-q)}$, $a(t) \equiv 0$ and applying Theorem 2, we obtain for $t \in (a, b_0]$ the estimation $\sup_{s\in[a,t]} |Y(s)| \le a(t)E_\alpha(g_0\Gamma(\alpha)t^\alpha) \equiv 0$ which contradicts our assumption. The proof of the case when $t^{jump} \neq a$ is almost the same as the proof in the case when $t^{jump} = a$, and therefore, will be omitted. □

Definition 8. *We say that the solution of IP (3), (2) $X^2(t) \in \mathfrak{E}_{b_2}^{\Phi_0}$ is a continuation of a solution $X^1(t) \in \mathfrak{E}_{b_1}^{\Phi_0}$ of IP (3), (2) if $(a, b_1] \subset (a, b_2]$ and $X^2(t) \equiv X^1(t)$ on $(a, b_1]$. The solution $X^{Max}(t) \in \mathfrak{E}_{b_{Max}}^{\Phi_0}$ of IP (3), (2) which coincides with all of its continuations will be called the maximal solution.*

Theorem 5. *Let the conditions of Theorem 3 be fulfilled.*

Then, for any $\Phi_0 \in \mathbf{PC}$ with $S^\Phi = \{t^{jump}\}$, $t^{jump} \in (a-h, a]$ the unique solution $X(t) \in \mathfrak{E}_{b_0}^{\Phi_0}$ with interval of existence $(a, b_0]$ of IP (3), (2) can be continued as a unique solution $X(t) \in \mathfrak{E}$ of IP (3), (2) with interval of existence J^0.

Proof. Assume the contrary, that there exists a maximal solution $X^{Max}(t) \in \mathfrak{E}_{b_{Max}}^{\Phi_0}$ of IP (3), (2) with the interval of its existence $(a, b_{Max}]$ (which is closed from the right), $b_{Max} < \infty$, and define the operator $\Re\colon \mathfrak{E}_b^{\Phi_0} \to \mathfrak{E}_b$ with $b > b_{Max}$ via (5) with the additional conditions $(\Re R)(t) = X^{Max}(t), t \in [a-h, a]$; $\lim_{t\to b-0} (\Re R)(t) = (\Re R)(b)$. Then, we can prove, fully analogous to the proof of Theorem 3, that for any $\Phi_0 \in \mathbf{PC}$ with $S^\Phi = \{t^{jump}\}$, $t^{jump} \in (a-h, a]$ there exists $\bar{b} > b_{Max}$ such that IP (3), (2) has at least one solution $X^{\bar{b}}(t) \in \mathfrak{E}_{\bar{b}}^{\Phi_0}$ with interval of existence $(a, \bar{b}]$, and obviously, $X^{\bar{b}}(t) \equiv X^{Max}(t)$ for any $t \in (a-h, b_{Max}]$. Thus, we obtain that $X^{\bar{b}}(t)$ is a continuation of $X^{Max}(t)$, which is impossible.

Let assume that that there exists a maximal solution $X^{Max}(t)$ of IP (3), (2) and its interval of existence is open and finite, i.e., (a, b_{Max}), $b_{Max} < \infty$. Then, since $X^{Max}(t)$ satisfies (3) for any $t \in (a, b_{Max})$ and the additional condition $\lim_{t\to b_{Max}-0} (\Re R)(t) = (\Re R)(b)$ too, we conclude that $X^{Max}(t)$ satisfies (3) for any $t \in (a, b_{Max}]$, which contradicts our assumption. Thus, we have that $b_{Max} = \infty$, which completes the proof. □

Remark 2. *We note that from Theorems 3–5 it follows that the requirement in Theorem 2 [11], that in the Lebesgue decomposition of $U(t,\theta)$ a singular part does not exist, is unnecessary. So, the results proved in Theorems 3–5 generalize the statement of Theorem 2 [11] even in the retarded case.*

4. The Linear Case $\mathfrak{F}(t, X_t^T) = F(t)$ – Fundamental Matrices and Integral Representation of the Solutions

In this section, we establish the existence of two different types of fundamental matrices and of their bases; we obtain two types integral representations for the solutions of IP (24), (2) for different kinds of initial functions. In the case when $\mathfrak{F}(t, X_t^T) = F(t)$, system (1) becomes an inhomogeneous linear system and has the following form:

$$D_{a+}^{\alpha}\left(Y(t)-\int_{-h}^{0}[d_{\theta}V(t,\theta)]Y(t+\theta)\right)=\int_{-h}^{0}[d_{\theta}U(t,\theta)]Y(t+\theta)+F(t), \tag{24}$$

where $Y(t)=(y_1(t),\ldots,y_n(t))^T: J\to\mathbb{R}^n$ and $F(t)=(f_1(t),\ldots,f_n(t))^T\in BL_1^{loc}(J,\mathbb{R}^n)$. Consider the corresponding homogeneous linear system of (1) and (24):

$$D_{a+}^{\alpha}\left(Y(t)-\int_{-h}^{0}[d_{\theta}V(t,\theta)]Y(t+\theta)\right)=\int_{-h}^{0}[d_{\theta}U(t,\theta)]Y(t+\theta), \tag{25}$$

and following [11] introduce for any $j\in\langle n\rangle$ and $\bar{s}\in J$ the initial function

$$Y^j(t,\bar{s})=\mathbf{0}, t\in[\bar{s}-h,\bar{s}); \quad Y(t,\bar{s})=I^j, t=\bar{s}, \tag{26}$$

where I^j denotes the j-th column of the identity matrix $\mathbf{I}$.

From Theorem 5 it follows that for any $j\in\langle n\rangle$ and $\bar{s}\in J$ IP (25), (2) has a unique solution $\mathfrak{H}^j(t,s)$, which satisfies (25) for any $t\in(\bar{s},\infty)$ and the initial condition (2) with the initial function (26) too. Then, the matrix $\mathfrak{H}(t,s)=(\mathfrak{H}^1(t,s),\ldots,\mathfrak{H}^n(t,s))$ will be called fundamental matrix of the system (25).

Additionally, we introduce the following initial function for any $j\in\langle n\rangle$ and $\bar{s}\in[a-h,a]$:

$$Y^j(t,\bar{s})=I^j, a-h\le\bar{s}\le t\le a; \quad Y(t,\bar{s})=\mathbf{0}, t<\bar{s}; Y(t,\bar{s})=\mathbf{0}, \bar{s}<a-h. \tag{27}$$

As above, for any $j\in\langle n\rangle$ and $\bar{s}\in J$ we can conclude that IP (24), (2) has a unique solution $\mathfrak{Q}^j(t,s)$ which satisfies (24) for any $t\in J^0$ and the initial condition (2) with the initial function (27) too. The matrix $\mathfrak{Q}(t,s)=(\mathfrak{Q}^1(t,s),\ldots,\mathfrak{Q}^n(t,s))$ we will call the generalized fundamental matrix.

Both matrices play crucial roles in the construction of different kinds of integral representations of the solutions of IP (24), (2) and IP (25), (2), having a lot of applications in the qualitative theory of the fractional systems.

It is clear that $\mathfrak{H}(t,a)\equiv\mathfrak{Q}(t,a)$ if and only if $s=a$. First, we will establish some analytical properties of the matrices $\mathfrak{Q}(t,a)$ and $\mathfrak{H}(t,a)$, needed to obtain appropriate integral representations of the solutions of (24) and (25).

As in the case of Caputo-type derivatives [26], for any $\Phi\in\mathbf{PC}^*$ we define the function

$$\tilde{X}(t)=\int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s), \tag{28}$$

where $\tilde{\Phi}(s-a)=\Phi(s-a), s\in(a-h,a]$, and $\tilde{\Phi}(-h)=\mathbf{0}, s=a-h$. It is necessary to establish for any $j\in\langle n\rangle$ that the solutions $\mathfrak{Q}^j(t,s)$ to IP (25), (2) are Lebesgue integrable in s on $s\in[a-h,a]$ for any fixed $t\in J^0$.

Remark 3. *Note that to prove the statement that $\mathfrak{Q}^j(t,s), j\in\langle n\rangle$ are Lebesgue integrable in s on $s\in[a-h,a]$ for any fixed $t\in J^0$ is not trivial, in contrast with the Caputo case where this fact can be proved via elementary application of a generalized Bellman–Gronwall inequality (see Theorem 2). The main tool in all these proofs of theorems devoted to integral representations of the solutions of fractional systems with Caputo-type derivatives is the Fubini theorem. Thus, from our point of view it is important to justify its correct application. The greatest obstacle in the neutral case with RL derivatives is the problem with the integrability of the integrands relative to the product measures, which are defined as the products of two Lebesgue–Stieltjes measures. Note that this problem arises in our case, since the integrand is with a singularity of order $1-\alpha$ at the point a in contrast to the case of neutral systems with Caputo-type fractional derivatives, in which case the integrand is a Lebesgue measurable and locally bounded function. This difference leads at least to some additional technical complications, but not only that.*

To overcome this obstacle, we need the statements of the next two lemmas.

Lemma 6. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 2 of Theorem 3 hold.*
2. $\sup_{t\in J^0}\left|Var_{\theta\in[-h,0]}V(t,\theta)\right| < 1$

Then, for any compact interval $[\bar{a},b]\subset J^0, t\in[\bar{a},b]$, *and* $j\in\langle n\rangle$ *we have that*

$$\mathfrak{Q}^j(t,\cdot)\in BL_1^{loc}([a-h,a],\mathbb{R}^n).$$

Proof. Let $t\in J^0$ and $j\in\langle n\rangle$ be arbitrary fixed. Then, from (25), since $\mathfrak{F}(\tau,X_\tau^T)\equiv\mathbf{0}$ we have

$$|\mathfrak{Q}^j(t,s)| \le |I^j\mathbf{1}_{[s,a]}(t-a)|(t-a)^{\alpha-1} + \left|\int_{-h}^{0}[d_\theta V(t,\theta)]\mathfrak{Q}^j(t,s)\right|$$
$$+\frac{n^2}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\left|\int_{-h}^{0}[d_\theta U(\tau,\theta)]\mathfrak{Q}^j(\tau+\theta,s)\right|d\tau$$
$$|I^j\mathbf{1}_{[s,a]}(t-a)|(t-a)^{\alpha-1} + V\sup_{\eta\in(a,t]}\left|\mathfrak{Q}^j(t,s)\right|$$
$$=\frac{n^2U_b}{\Gamma(\alpha)(1-V)}\left|\int_a^t (t-\tau)^{\alpha-1}\sup_{\eta\in(a,\tau]}\left|\mathfrak{Q}^j(\eta,s)\right|\right|d\tau$$

and, hence, by virtue of Theorem 2 we obtain

$$\sup_{s\in[-h,0]}\left(\sup_{\eta\in(a,t]}\left|\mathfrak{Q}^j(\eta,s)\right|\right) \le |I^j\mathbf{1}_{[s,a]}(t-a)|\frac{(t-a)^{\alpha-1}}{1-V}$$
$$+\frac{n^2U_b}{\Gamma(\alpha)(1-V)}\left|\int_a^t (t-\tau)^{\alpha-1}\sup_{s\in[-h,0]}\left(\sup_{\eta\in(a,\tau]}\left|\mathfrak{Q}^j(\eta,s)\right|\right)\right|d\tau$$
$$\le |I^j\mathbf{1}_{[s,a]}(t-a)|\frac{(t-a)^{\alpha-1}}{1-V}E_\alpha\left(\frac{n^2U_b(t-a)^\alpha}{(1-V)}\right), \tag{29}$$

where in (29) $E_\alpha(z)=\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$ is the one-parameter Mittag–Laffler function. This completes the proof. □

By virtue of Lemma 6 we can state that the integral on the right-hand side of (28) is correctly defined.

Corollary 1. *Let the conditions of Lemma 6 hold.*

Then, for arbitrary fixed $\bar{s}\in[-h,0)$, $\mathfrak{Q}(t,s)$ *is continuous in* t *for* $t-a\in[-h,\bar{s})\cup(\bar{s},0)$ *or when* $\bar{s}=0$ *for* $t-a\in[-h,0)$. *At* $t-a=\bar{s}$, $\mathfrak{Q}(t,s)$ *has a first-kind jump.*

Lemma 7. *Let the conditions of Lemma 6 hold.*

Then, for any $t\in J^0$ *we have that* $\mathfrak{Q}(t,\cdot)\in BV([a-h,a],\mathbb{R}^{n\times n})$.

Proof. Let $\Phi^*(s)=(\varphi_1^*(s),\ldots,\varphi_n^*(s))^T\in\mathbf{PC}^*$ with $\inf_{s\in[a-h,a]}\varphi_j^*(s)\ge c_0>0, j\in\langle n\rangle$. Then, integrating by parts formally the integral in (28) we have that

$$\int_{a-h}^{a} [d_s\tilde{\Phi}^*(s-a)]\mathfrak{Q}(t,s)$$
$$= \tilde{\Phi}^*(0)\mathfrak{Q}(t,a) - \tilde{\Phi}(-h)]\mathfrak{Q}(t,a-h) - \int_{a-h}^{a} [d_s\mathfrak{Q}(t,s)]\tilde{\Phi}^*(s-a), \quad (30)$$

and since the integral on the left-hand side of (30) exists, then the integral on the right-hand side exists too (see [27], point 5, page 229). Let $t \in J^0$ be arbitrary, $\epsilon > 0$ and $\prod_q = \{s_0 = a-h, s_1, \dots, s_q = a\} \subset [a-h,a]$ be an arbitrary partition. Then, there exists a number $q_\epsilon \in \mathbb{N}$ such that for any $q \geq q_\epsilon$ we have that

$$\left| \int_{a-h}^{a} [d_s\mathfrak{Q}(t,s)]\tilde{\Phi}^*(s-a) - \sum_{k=1}^{q} \tilde{\Phi}^*(s_k-a)\left|\mathfrak{Q}^j(t,s_k) - \mathfrak{Q}^j(t,s_{k-1})\right| \right| < \epsilon,$$

and hence,

$$0 \leq c_0 \sum_{k=1}^{q} \left|\mathfrak{Q}^j(t,s_k) - \mathfrak{Q}^j(t,s_{k-1})\right| \leq \inf_{s\in[a-h,a]} \left|\tilde{\Phi}^*(s-a)\right| \sum_{k=1}^{q} \left|\mathfrak{Q}^j(t,s_k) - \mathfrak{Q}^j(t,s_{k-1})\right|$$
$$\leq \sum_{k=1}^{q} \tilde{\Phi}^*(s_k)\left|\mathfrak{Q}^j(t,s_k) - \mathfrak{Q}^j(t,s_{k-1})\right| \leq \int_{a-h}^{a} [d_s\mathfrak{Q}(t,s)]\tilde{\Phi}^*(s-a) + \epsilon.$$

Thus, $\mathfrak{Q}(t,\cdot) \in BV([a-h,a],\mathbb{R}^{n\times n})$. □

Theorem 6. *Let the conditions of Lemma 6 hold.*

Then, for arbitrary initial function $\Phi \in \mathbf{PC}^*$ the function $\tilde{X}(t)$, defined via (28), is the unique solution of IP (25), (2) with interval of existence $t \in J^0$.

Proof. The proof is based on ideas used in [22]. Thereupon, we will emphasize in detail those differences that arise from the influence on the neutral system of the Riemann–Liouville-type derivatives.

Theorem 5 implies that $\mathfrak{Q}(\cdot,s) \in C(J^0,\mathbb{R}^{n\times n})$ for any $s \in [a-h,a]$. Then, for the function $\tilde{X}(t)$, defined via (28), by virtue of Lemma 1 in [22] we have that $\tilde{X}(t) \in C(J^0,\mathbb{R}^n)$ too.

More concretely, for arbitrary fixed $t \in J^0$ the kernels $V(t,\theta)$ of $U(t,\theta)$ define two Lebesgue–Stieltjes measures $\mu_V^\theta = \mu_V((\theta_1,\theta_2]) = V(t,\theta_2) - V(t,\theta_1)$ and $\mu_U^\theta = \mu_U((\theta_1,\theta_2]) = U(t,\theta_2) - U(t,\theta_1)$ for any $(\theta_1,\theta_2] \subset [-h,0]$, as well as the function $\overline{\Phi}(s)$ defining a measure $\mu_{\overline{\Phi}}^s = \mu_{\overline{\Phi}}((s_1,s_2]) = \overline{\Phi}(s_1) - \overline{\Phi}(s_2)$ for any $(s_1,s_2] \subset [a-h,a]$. We introduce the product measures

$$(\mu_V^\theta \times \mu_{\overline{\Phi}}^s)((\theta_1,\theta_2] \times (s_1,s_2]) = \mu_V((\theta_1,\theta_2]\mu_{\overline{\Phi}}((s_1,s_2])$$

and

$$(\mu_U^\theta \times \mu_{\overline{\Phi}}^s)((\theta_1,\theta_2] \times (s_1,s_2]) = \mu_U((\theta_1,\theta_2]\mu_{\overline{\Phi}}((s_1,s_2])$$

of the rectangles in $\mathbf{P} = [-h,0] \times [a-h,a]$. Then, to use the proposition 5.15 in [28] we need to prove that the relations $\left|\iint_P \mathfrak{Q}(t+\theta,s)\mu_U^\theta \times \mu_{\overline{\Phi}}^s\right| < \infty$ and $\left|\iint_P \mathfrak{Q}(t+\theta,s)\mu_V^\theta \times \mu_{\overline{\Phi}}^s\right| < \infty$ hold. Since according to Lemmas 5 and 6 we have $\mathfrak{Q}(t,\cdot) = BV([a-h,a],\mathbb{R}^{n\times n})$ and $\mathfrak{Q}(t+\theta,\cdot) = L_1^{loc}([a-h,\infty),\mathbb{R}^{n\times n})$ for any $t \in J^0$ and $\theta \in [-h,0]$ concerning the first argument, then taking into account that $\mu_V^\theta \times \mu_{\overline{\Phi}}^s = \mu_{\overline{\Phi}}^s \times \mu_V^\theta$ and $\mu_U^\theta \times \mu_{\overline{\Phi}}^s = \mu_{\overline{\Phi}}^s \times \mu_U^\theta$ we obtain

$$\left|\iint_P \mathfrak{Q}(t+\theta,s)\mu_U^\theta \times \mu_{\tilde{\Phi}}^s\right| = \left|\int_{-h}^{0}[d_\theta U(t,\theta)]\left(\int_{a-h}^{a}[d_s\Phi(s-a)]\mathfrak{Q}(t+\theta,s)\right)\right| < \infty,$$
$$\left|\iint_P \mathfrak{Q}(t+\theta,s)\mu_V^\theta \times \mu_{\tilde{\Phi}}^s\right| = \left|\int_{-h}^{0}[d_\theta V(t,\theta)]\left(\int_{a-h}^{a}[d_s\Phi(s-a)]\mathfrak{Q}(t+\theta,s)\right)\right| < \infty.$$

Then, applying proposition 5.15 in [28] we have for $t \in J^0$ that

$$\int_{-h}^{0}[d_\theta V(t,\theta)]\tilde{X}(t+\theta) = \int_{-h}^{0}[d_\theta V(t,\theta)]\left(\int_{a-h}^{a}[d_s\Phi(s-a)]\mathfrak{Q}(t+\theta,s)\right)$$
$$= \int_{a-h}^{a}[d_s\Phi(s-a)]\left(\int_{-h}^{0}[d_\theta V(t,\theta)]\mathfrak{Q}(t+\theta,s)\right) \tag{31}$$

and in the same way we obtain

$$\int_{-h}^{0}[d_\theta U(t,\theta)]\tilde{X}(t+\theta) = \int_{a-h}^{a}[d_s\Phi(s-a)]\left(\int_{-h}^{0}[d_\theta U(t,\theta)]\mathfrak{Q}(t+\theta,s)\right). \tag{32}$$

From (25) and (31) via the Fubini theorem it follows that

$$D_{a+}^{\alpha}\left(\tilde{X}^j(t) - \int_{-h}^{0}[d_\theta V(t,\theta)]\tilde{X}^j(t+\theta)\right)$$
$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\eta)^{-\alpha}\left(\int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(\eta,s)\right)d\eta$$
$$-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\eta)^{-\alpha}\int_{-h}^{0}[d_\theta V(\eta,\theta)]\left(\int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(\eta+\theta,s)\right)d\eta$$
$$= \int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]\left(\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\eta)^{-\alpha}\mathfrak{Q}(\eta,s)d\eta\right)$$
$$-\int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]\left(\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\eta)^{-\alpha}\left(\int_{-h}^{0}[d_\theta V(\eta,\theta)]\mathfrak{Q}(\eta+\theta,s)\right)d\eta\right)$$
$$= \int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]D^{\alpha}\left(\mathfrak{Q}(t,s) - \int_{-h}^{0}[d_\theta V(t,\theta)]\mathfrak{Q}(t+\theta,s)\right). \tag{33}$$

Hence, from (25), (32), and (33) it follows that $\tilde{X}(t)$, defined via (28), satisfies (25) for any $t \in J^0$.

Let $\tilde{s} \in [a-h,a]$ be an arbitrary number, and then, from (28) when $t = \tilde{s}$, it follows that

$$\tilde{X}(\tilde{s}) = \int_{a-h}^{a}[d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s) = \int_{a-h}^{\tilde{s}}[d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s) + \int_{\tilde{s}}^{a}[d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s)$$
$$= \int_{a-h}^{\tilde{s}}[d_s\tilde{\Phi}(s-a)]\mathbf{I} = \tilde{\Phi}(\tilde{s}) - \tilde{\Phi}(a-h) = \tilde{\Phi}(\tilde{s}),$$

i.e., $\tilde{X}(t)$ satisfies the initial condition (2), and hence, it is the unique solution of the IP (25), (2) with interval of existence $t \in J^0$. □

Following the idea in [22], we introduce

$$\mathfrak{X}(t) = \int_a^t \mathfrak{H}(t,s) D_{a+}^{1-\alpha} F^0(s) ds, \tag{34}$$

where $F^0(t) \equiv F(t), t \in J^0$, and $F^0(0) = \mathbf{0}$.

Theorem 7. *Let the conditions of Lemma 6 hold.*

Then, the function $\mathfrak{X}(t)$ introduced via (34) is the unique solution of IP (24), (2) for the initial function $\Phi(t-a) \equiv \mathbf{0}, t \in [a-h, a]$ and interval of existence $t \in J^0$.

Proof. As in the theorem above, the proof is based on some ideas used in [15]. Therefore, we will only sketch the similar parts and we will emphasize in detail those differences that arise from the influence on the neutral system of the Riemann–Liouville-type derivatives.

For any $t \in J^0$ and $\theta \in [-h, 0]$, since $\mathfrak{H}(t,s) = 0$ when $t < s$, we have that

$$\int_{t+\theta}^t \mathfrak{H}(t+\theta,s) D_{a+}^{1-\alpha} F^0(s) ds = \mathbf{0}. \tag{35}$$

Then, substituting $\mathfrak{X}(t)$ on the left-hand side of (24) for any $t \in J^0$ via the Fubini theorem and using formula (2.211) in [5] and (35) we obtain that

$$\begin{aligned}
&D_{a+}^{\alpha}\left(\mathfrak{X}(t) - \int_{-h}^{0} [d_\theta V(t,\theta)]\mathfrak{X}(t+\theta)\right) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\eta)^{-\alpha} \left(\int_a^\eta \mathfrak{H}(\eta,s) \left(D_{a+}^{1-\alpha} F^0\right)(s) ds \right) d\eta \\
&- \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\eta)^{-\alpha} \left(\int_{-h}^{0} [d_\theta V(\eta,\theta)] \left(\int_a^{\eta+\theta} \mathfrak{H}(\eta+\theta,s) \left(D_{a+}^{1-\alpha} F^0\right)(s) ds \right) \right) d\eta \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(\int_s^t (t-\eta)^{-\alpha} \mathfrak{H}(\eta,s) d\eta \right) \left(D_{a+}^{1-\alpha} F^0\right)(s) ds \\
&- \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\eta)^{-\alpha} \left(\int_{-h}^{0} [d_\theta V(\eta,\theta)] \left(\int_a^{\eta} \mathfrak{H}(\eta+\theta,s) \left(D_{a+}^{1-\alpha} F^0\right)(s) ds \right) \right) d\eta \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(\int_a^t (t-\eta)^{-\alpha} \mathfrak{H}(\eta,s) d\eta \right) \left(D_{a+}^{1-\alpha} F^0\right)(s) ds \\
&- \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(\int_a^t (t-\eta)^{-\alpha} \left(\int_{-h}^{0} [d_\theta V(\eta,\theta)] \mathfrak{H}(\eta+\theta,s) \right) d\eta \right) \left(D_{a+}^{1-\alpha} F^0\right)(s) ds \\
&= \int_a^t \left(D_{a+}^{1-\alpha} F^0\right)(s) D_{a+}^{\alpha}(\mathfrak{H}(t,s)) ds - \int_a^t \left(D_{a+}^{1-\alpha} F^0\right)(s) D_{a+}^{\alpha} \left(\int_{-h}^{0} [d_\theta V(t,\theta)] \mathfrak{H}(t+\theta,s) \right) ds \\
&+ \lim_{s \to t-0} {}^{RL}D_{a+}^{\alpha-1} \left(\left(D_{a+}^{1-\alpha} F^0\right)(s) \mathfrak{H}(t,s) \right) \\
&- \lim_{s \to t-0} {}^{RL}D_{a+}^{\alpha-1} \left(\left(D_{a+}^{1-\alpha} F^0\right)(s) \left(\int_{-h}^{0} [d_\theta V(t,\theta)] \mathfrak{H}(t+\theta,s) \right) \right).
\end{aligned} \tag{36}$$

For the third addend on the right-hand side of (36), we have that

$$\begin{aligned}
&\lim_{s\to t-0} {}^{RL}D^{\alpha-1}_{a+}\left(\left(D^{1-\alpha}_{a+}F^0\right)(s)(\mathfrak{H}(t,s))\right)\\
&=\lim_{s\to t-0}\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{-\alpha}\mathfrak{H}(\tau,s)\left(D^{1-\alpha}_{a+}F^0\right)(s)d\tau\\
&=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{-\alpha}\lim_{s\to t-0}\left(\mathfrak{H}(\tau,s)\left(D^{1-\alpha}_{a+}F^0\right)(s)\right)d\tau\\
&=\left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{-\alpha}\mathfrak{H}(\tau,\tau)\left(D^{1-\alpha}_{a+}F^0\right)(\tau)d\tau\right|\\
&=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{-\alpha}\mathbf{I}\left(D^{1-\alpha}_{a+}F^0\right)(\tau)d\tau\\
&=\left({}^{RL}D^{\alpha-1}_{a+}\,{}^{RL}D^{1-\alpha}_{a+}F^0\right)(t)=F^0(t)-F^0(0)=F^0(t).
\end{aligned}\tag{37}$$

Since $V(\tau,0)=\Theta$ and $\mathfrak{H}(t+\theta,t)=\mathbf{0}$, for any $t\in J^0$ and $\theta\in[-h,0)$ we obtain

$$\begin{aligned}
&\lim_{s\to t-0} {}^{RL}D^{\alpha-1}_{a+}\left(\left(D^{1-\alpha}_{a+}F^0\right)(s)\int_{-h}^0 [d_\theta V(\tau,\theta)]\mathfrak{H}(\tau+\theta,s)\right)\\
&=\lim_{s\to t-0}\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{-\alpha}\left(\left(D^{1-\alpha}_{a+}F^0\right)(s)\int_{-h}^0 [d_\theta V(\tau,\theta)]\mathfrak{H}(\tau+\theta,s)\right)d\tau\\
&=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{-\alpha}\left(D^{1-\alpha}_{a+}F^0\right)(\tau)\int_{-h}^0 [d_\theta V(\tau,\theta)]\mathfrak{H}(\tau+\theta,\tau)d\tau=\mathbf{0},
\end{aligned}\tag{38}$$

and hence, from (36), (37), and (38) it follows that

$$\begin{aligned}
&D^\alpha_{a+}\left(\mathfrak{X}(t)-\int_{-h}^0 [d_\theta V(t,\theta)]\mathfrak{X}(t+\theta)\right)\\
&=\int_a^t\left(D^{1-\alpha}_{a+}F^0\right)(s)D^\alpha_{a+}\left(\mathfrak{H}(t,s)-\int_{-h}^0 [d_\theta V(t,\theta)]\mathfrak{H}(t+\theta,s)\right)ds+F^0(t).
\end{aligned}\tag{39}$$

Similarly as above, for the right-hand side of (24) we obtain

$$\begin{aligned}
&\int_{-h}^0 [d_\theta U(t,\theta)]\mathfrak{X}(t+\theta)+F^0(t)\\
&=\int_{-h}^0 [d_\theta U(t,\theta)]\left(\int_a^{t+\theta}\mathfrak{H}(t+\theta,s)\left(D^{1-\alpha}_{a+}F^0\right)(s)ds\right)+F^0(t)\\
&=\int_a^t\left(D^{1-\alpha}_{a+}F^0\right)(s)\int_{-h}^0 ([d_\theta U(t,\theta)]\mathfrak{H}(t,s))ds+F^0(t),
\end{aligned}\tag{40}$$

and since $\mathfrak{H}(t,s)$ is a fundamental matrix, then the statement of the theorem follows from (39) and (40). □

Corollary 2. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 6 hold.*
2. *For any initial function $\Phi \in PC^*$ and $F(t) \in BL_1^{loc}(J, \mathbb{R}^n)$, the functions $\tilde{X}(t)$ and $\mathfrak{X}(t)$ are defined via (28) and (34), respectively.*

Then, the function $X(t) = \tilde{X}(t) + \mathfrak{X}(t)$ is the unique solution of IP (24), (2) and has the following representation for any $t \in J^0$:

$$X(t) = \int_a^t \mathfrak{H}(t,s) D_{a+}^{1-\alpha} F(s) ds + \int_{a-h}^a \mathfrak{Q}(t,s) d_s \tilde{\Phi}(s-a), \quad (41)$$

where $\tilde{\Phi}(s-a) \equiv \Phi(s-a)$ for $s \in [a-h, a)$ and $\tilde{\Phi}(a-h) = \mathbf{0}$.

Proof. It follows from the superposition principle and Theorems 5 and 6. □

5. Weighted Stabilities

It is worth noting that the standard definitions for stability in the Lyapunov or Ulam–Hyers senses introduced for the systems with integer-order derivatives (without or with delays) are applicable directly for systems with fractional derivatives only when the fractional derivative of a constant is equal to zero (as Caputo-type derivatives, etc.). As we have mentioned previously [11], in general when the fractional derivative of a constant is not equal to zero (as for RL-type derivatives), then the standard definitions are not applicable since the solutions of the systems with RL-type derivatives have singularity at $a+0$ (the low terminal) of power order $\alpha - 1$. That is why new types of definitions for the different kinds of stabilities applicable to systems with RL-type derivatives are needed. The aim of this section is to introduce definitions of weighted stability in the Lyapunov sense, as well as UH and UHR weighted stabilities for fractional systems (equations) with RL-type derivatives and to study the relations with the classical definitions and with the concept "stability in time" in the Lyapunov sense, introduced in [9].

Let assume that $\mathfrak{F}(t, \mathbf{0}) \equiv \mathbf{0}$ for any $t \in J$. Below, we recall the classical definitions for Lyapunov stability, applicable only for the case when the fractional derivatives are, in the Caputo sense, of order $\alpha \in (0, 1]$ (and not applicable for RL derivatives):

Definition 9 ([29])**.** *The zero solution of IP (1), (2) (with derivatives in the Caputo sense!) is said to be:*

(*a*) *Stable if for a given low terminal $a \in \mathbb{R}$ and any $\epsilon > 0$ there is a $\delta(\epsilon, a) > 0$ such that for each initial function $\Phi \in \mathbf{PC}(\mathbf{PC}^*)$ with $\|\Phi\| < \delta$ the corresponding solution $X(t)$ satisfies for each $t \in J$ the inequality $|X(t)| \leq \epsilon$.*

(*b*) *Stable (uniformly) if for a given low terminal $a \in \mathbb{R}$ and any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for every initial function $\Phi \in \mathbf{PC}(\mathbf{PC}^*)$, with $\|\Phi\| < \delta$ the corresponding solution $X(t)$ satisfying for each $t \in J$ the inequality $|X(t)| \leq \epsilon$.*

(*c*) *Locally asymptotically stable (LAS) if for a given low terminal $a \in \mathbb{R}$ there is a $\Delta(a) > 0$ such that for every initial function $\Phi \in \mathbf{PC}(\mathbf{PC}^*)$ with $\|\Phi\| < \Delta$ the corresponding solution $X(t)$ is stable and $\lim_{t\to\infty} |X(t)| = 0$.*

(*d*) *Globally asymptotically stable (GAS) if for every for every initial function $\Phi \in \mathbf{PC}(\mathbf{PC}^*)$ the corresponding solution $X(t)$ is stable and $\lim_{t\to\infty} |X(t)| = 0$.*

Both definitions below are applicable not only for initial problems with Caputo-type derivatives, but also for initial problems with fractional derivatives in the RL sense.

Define for any $\Phi \in \mathbf{PC}$ the weighted function $w(t) \colon J \to \mathbb{R}$ as follows:

$$w(t) = \begin{cases} 1, & t = a, \\ (t-a)^{1-\alpha}, & t \in (a, a+1], \\ 1, & t > a+1. \end{cases}$$

Definition 10. *The zero solution of IP* (1), (2) *is said to be weighted stable (W-stable), weighted uniformly stable (W-uniformly stable), weighted locally asymptotically stable (W-LAS stable), or weighted globally asymptotically stable (W-GAS stable) if for the corresponding solution $X(t)$ the product $w(t)X(t)$ satisfies the conditions (a), (b), (c), or (d) from Definition 9.*

Definition 11 ([9])**.** *The zero solution of IP* (1), (2) *is said to be:*

(i) *Stable in time (Lyapunov in time stable) if for an arbitrary $\epsilon > 0$ there exists a point $t_\epsilon \in J^0$ and number $\delta(\epsilon, t_\epsilon) > 0$ such that for any initial function $\Phi \in \mathbf{PC}(\mathbf{PC}^*)$ with $\|\Phi\| < \delta$, the corresponding solution $X(t)$ satisfies for each $t \geq t_\epsilon$ the inequality $|X(t)| \leq \epsilon$.*

(ii) *Asymptotically stable in time if for any initial function $\Phi \in \mathbf{PC}(\mathbf{PC}^*)$ the corresponding solution $X(t)$ is stable in time and additionally $\lim_{t\to\infty} |X(t)| = 0$.*

Remark 4. *It is simple to see that if $\alpha = 1$ (the order of differentiation) then Definition 10 for the weighted stability coincides with the classical one. In addition, when a solution is weighted stable then we can take $t_\epsilon = a + 1$ for any $\epsilon > 0$ in Definition 11, and hence, the weighted stability implies stability in time, as well as the weighted global asymptotically stability implying asymptotic stability in time.*

Let $\epsilon > 0$ be an arbitrary number and consider the inequality

$$\left| D^\alpha_{a+}\left(X(t) - \int_{-h}^{0} [d_\theta V(t,\theta)]X(t+\theta) \right) \int_{-h}^{0} [d_\theta U(t,\theta)]X(t+\theta) + \mathfrak{F}(t, X^T_\tau(\theta)) \right| \leq \epsilon. \tag{42}$$

Definition 12. *The function $Y(t)\colon [a-h,\infty) \to \mathbb{R}^n$ is a solution of IP* (42), (2) *in J^0 if $Y(t)|_{t\in J^0} \in \mathbf{C}^{1-\alpha}$ satisfies the inequality* (42) *for $t \in J^0$ and the condition* (2) *for $t \in [a-h, a]$, with initial function $Y(t-a) = \Phi^Y(t-a) \in \mathbf{PC}(\mathbf{PC}^*)$.*

Definition 13. *The system* (1) *is said to be Ulam–Hyers weighted stable in J^0 if for any $\epsilon > 0$ there exists a number $\delta \in (0, \epsilon)$ and constant $C_\epsilon > 0$ such that for any solution $Y(t)$ of* (42) *there exists an initial function $\Phi(t) \in \mathbf{PC}(\mathbf{PC}^*)$, with $|\Phi^Y(t-a) - \Phi(t-a)| \leq \delta$ for $t \in [a-h, a]$ and for the corresponding unique solution $X(t)$ of IP* (1), (2)*, with initial function $\Phi(t)$, the inequality*

$$w(t)\big|Y(t)|_J - X(t)\big| \leq C_\epsilon \epsilon \tag{43}$$

holds for any $t \in J^0$.

The first result in this section clarifies the relations between the boundedness of all solutions and the weighted stabilities in the Lyapunov sense.

Theorem 8. *Let the conditions of Lemma 6 hold.*

Then, the zero solution of the system (25) *is W-stable in the Lyapunov sense if and only if for all solutions $X(t)$ of IP* (25), (2) *with initial function $\Phi(t) \in \mathbf{PC}$, the product $w(t)X(t)$ is bounded in J^0.*

Proof. Necessity: Let us assume that the zero solution is W-stable in the Lyapunov sense. Assume the contrary, that there exists $\overline{\Phi}(t) \in \mathbf{PC}$ such that for the existing, according to Theorem 3, unique solution $\overline{X}(t)$ of IP (25), (2) we have that $w(t)\overline{X}(t)$ is unbounded in J^0. Since the zero solution is W-stable, then for any $\epsilon > 0$ there exists a number $\delta \in (0, \epsilon)$ such that for any $\Phi(t) \in \mathbf{PC}$ with $\|\Phi\| < \delta$ we have that the corresponding solution $X(t)$ satisfies the inequality $|w(t)X(t)| \le \epsilon$ for any $t \in J^0$. Consider the following initial function $\Phi^* = \dfrac{\delta\overline{\Phi}}{2\|\overline{\Phi}\|}$ with norm $\|\Phi^*\| = \left\|\dfrac{\delta\overline{\Phi}}{2\|\overline{\Phi}\|}\right\| = \dfrac{\delta}{2} < \delta$. Then, since the system (25) is a linear system the function $X^*(t) = \dfrac{\delta}{2\|\overline{\Phi}\|}\overline{X}(t)$ is the corresponding solution of IP (25) (2) for the initial function $\Phi^*(t) \in \mathbf{PC}$, and hence, for $t \in J^0$ we have that $|w(t)X^*(t)| = \left|\dfrac{\delta}{2\|\overline{\Phi}\|}w(t)X(t)\right| \le \epsilon$, which implies that $|w(t)\overline{X}(t)| \le \dfrac{2\|\overline{\Phi}\|\epsilon}{\delta}$ for $t \in J^0$, which contradicts our assumption.

Sufficiency: Let for any $\Phi(t) \in \mathbf{PC}$ the product $w(t)X(t)$ be bounded in J^0, where $X(t)$ is the corresponding solution of IP (25), (2) with this initial function, and hence, according to Lemma 4, we have that $X(t)$ satisfies IP (3), (2) with $\mathfrak{F}(t, X_t^T) \equiv \mathbf{0}$ for $t \in J$ and vice versa. Let $t^* \in J^0$ be arbitrary and let $b > a^* > a$ be fixed numbers with $t^* \in [a^*, b]$. Then, for arbitrary initial functions $\overline{\Phi}, \Phi \in \mathbf{PC}$, their corresponding unique solutions of IP (3), (2) are $\overline{X}(t)$, $X(t)$, respectively, and for any $t^* \in [a^*, b]$ we have

$$
\begin{aligned}
|\overline{X}(t) - X(t)| &\le \big(\overline{\Phi}(0) - \Phi(0)\big)(t-a)^{\alpha-1} + \left|\int_{-h}^{0} [d_\theta V(t,\theta)]\big(\overline{X}(t+\theta) - X(t+\theta)\big)\right| \\
&+ \frac{n}{\Gamma(\alpha)}\left|\int_a^t (t-\tau)^{1-\alpha}\left(\int_{-h}^{0}[d_\theta U(\tau,\theta)]\big(\overline{X}(\tau+\theta) - X(\tau+\theta)\big)\right)d\tau\right| \\
&\le \|\overline{\Phi} - \Phi\|(t-a)^{\alpha-1} + V \sup_{s\in[a,t]} |\overline{X}(s) - X(s)| + V\|\overline{\Phi} - \Phi\| \\
&+ \frac{n}{\Gamma(\alpha)}\left|\int_a^t (t-\tau)^{\alpha-1}\left(\int_{-h}^{0}[\partial_\theta U(\tau,\theta)]\big(\overline{X}(\tau+\theta) - X(\tau+\theta)\big)d\theta\right)d\tau\right| \\
&\le V \sup_{s\in[a,t]} |\overline{X}(s) - X(s)| + \|\overline{\Phi} - \Phi\|\Big((t-a)^{\alpha-1} + V\Big) \\
&+ \frac{nU_b}{\Gamma(1+\alpha)}\int_a^t (t-\tau)^{\alpha-1} \sup_{\eta\in[a,\tau]} |\overline{X}(\eta) - X(\eta)|d\tau + \frac{nU_b(t-a)^\alpha}{\Gamma(1+\alpha)}\|\overline{\Phi} - \Phi\| \\
&\le V \sup_{s\in[a,t]} |\overline{X}(s) - X(s)| + \|\overline{\Phi} - \Phi\|\left((t-a)^{\alpha-1} + \frac{nU_b(t-a)^\alpha}{\Gamma(1+\alpha)} + V\right) \\
&+ \frac{nU_b}{\Gamma(1+\alpha)}\int_a^t (t-\tau)^{\alpha-1} \sup_{\eta\in[a,\tau]} |\overline{X}(\eta) - X(\eta)|d\tau. \qquad (44)
\end{aligned}
$$

From (44), for any $t \in [a^*, b]$ it follows that

$$
\begin{aligned}
&\sup_{s\in[a,t]} w(t)|\overline{X}(s) - X(s)| \\
&\le \|\overline{\Phi} - \Phi\|\left(\frac{\sup\limits_{s\in[a,t]} w(s)(s-a)^{\alpha-1}}{1-V} + \frac{nU_b b^\alpha}{(1-V)\Gamma(1+\alpha)} + \frac{V}{(1-V)}\right) \\
&+ \frac{nU_b}{\Gamma(1+\alpha)}\int_a^t (t-\tau)^{\alpha-1} \sup_{\eta\in[a-h,\tau]} |\overline{X}(\eta) - X(\eta)|d\tau. \qquad (45)
\end{aligned}
$$

Then, applying corollary 2 in [24] to (45), we obtain for any $t \in [a^*, b]$ the estimation

$$\sup_{s\in[a,t]} |\overline{X}(s) - X(s)| \leq \|\overline{\Phi} - \Phi\| \left(\frac{\sup_{s\in[a,t]} w(s)(s-a)^{\alpha-1}}{1-V} + \frac{nU_b(b-a)^\alpha}{(1-V)\Gamma(1+\alpha)} + \frac{V}{(1-V)} \right) (E_\alpha(nU_b t^\alpha)). \quad (46)$$

Let us define for any $t \in J^0$ a family of functionals $\aleph_t \colon \mathbf{PC} \to \mathbb{R}^n$ via the equality $\aleph_t(\Phi) = w(t)X(t)$, where $X(t)$ denotes the corresponding solution of IP (25),(2) for initial function $\Phi(t) \in \mathbf{PC}$. From (46), it follows that $\aleph_t(\Phi)$ is a continuous functional at any $\Phi(t) \in \mathbf{PC}$ for any $t \in J^0$ i.e., for every $\Phi(t) \in \mathbf{PC}$ we have that $|\aleph_t(\Phi)| \leq \|\aleph_t\| \, \|\Phi\|$, where $\|\aleph_t\|$ denotes the norm of the functional. Then applying the Banach–Steinhaus theorem we obtain that the norms of all functionals are uniformly bounded, i.e., there exists a constant $\|\aleph\| > 0$ such that $\|\aleph_t\| \leq \|\aleph\|$ for any $t \in J$. Let $\epsilon > 0$ be arbitrary and choose $\delta = \frac{\epsilon}{2\|\aleph\|} > 0$. Then, for any $\Phi(t) \in \mathbf{PC}$ with $\|\Phi\| < \delta$ for the corresponding solution $X(t)$ we have that $|w(t)X(t)| = |\aleph_t(\Phi)| \leq \|\aleph\| \, \|\Phi\| = \|\aleph\| \frac{\epsilon}{2\|\aleph\|} < \epsilon$ for any $t \in J^0$, which implies that the zero solution is W-stable in the Lyapunov sense. □

Remark 5. *It is worth emphasizing that the result from the application of any kind of stability definitions concerning the differential equation (system) with all types of derivatives (integer or fractional order) essentially depends on the functional type of the set of all solutions of the studied object. As an example, for equations (systems) with first-order derivatives the solutions are usually either continuous differentiable or absolutely continuous functions. So, the conclusions could be true for all solutions which are absolutely continuous or only for these solutions which are continuous differentiable. For retarded or neutral equations (systems) the situation is more complicated. The space of the initial functions plays a leading role in its functional type, because the analytical type of the solutions essentially depends on them. This is very important for the stability in the Ulam–Hyers sense for the following reason: the functional type of the initial function and the solutions of system* (1) *must be the same for the corresponding inequality* (42)*, since all solutions of* (1) *are solutions of inequality* (42) *too.*

As a consequence, from the proved results, we obtain two necessary conditions for W-stability of the zero solution of the system (25).

Theorem 9. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 6 hold.*
2. *The zero solution of the system* (25) *is W-stable in the Lyapunov sense.*

Then, the following relations hold:

$$\mathfrak{Q}^\infty = \sup_{t\in J^0} |w(t)\mathfrak{Q}(t)| < \infty, \mathfrak{H}^\infty = \sup_{t\in J^0} |\mathfrak{H}(t)| < \infty,$$

where $\mathfrak{Q}(t) = Var_{s\in[a-h,a]}\mathfrak{Q}(t,s)$ *and* $\mathfrak{H}(t) = \sup_{s\in[a,t]} |w(t)\mathfrak{H}(t,s)|$.

Proof. Let us, as above for $\Phi(t) \in \mathbf{PC}^*$ and any $t \in J^0$, define a family of functionals $\aleph_t \colon \mathbf{PC} \to \mathbb{R}^n$ via the equality $\aleph_t(\Phi) = w(t)X(t)$, where $X(t)$ denotes the corresponding solution of IP (25), (2) for this initial function. Then, according Theorem 6 and Lemma 6, for any $t \in J^0$ we have that

$$|\aleph_t(\Phi)| = |w(t)X(t)| = \left| w(t) \int\limits_{a-h}^{a} [d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s) \right|$$
$$\leq \sup_{s\in[-h,0]} \mathfrak{Q}(t,s)\|\Phi\| \leq \mathfrak{Q}(t)\|\Phi\| \leq Var_{s\in[-h,0]}|\mathfrak{Q}(t,s)|\|\Phi\|,$$

and hence, $\aleph_t(\Phi)$ is a continuous functional at any $\Phi(t) \in \mathbf{PC}^*$ for any $t \in J^0$ with norm $\|\aleph_t\| = Var_{s\in[-h,0]}|\mathfrak{Q}(t,s)|$. Then, by virtue of the Banach-Steinhaus theorem we have that there exists a constant $\|\aleph\| > 0$ such that $\|\aleph_t\| \leq \|\aleph\|$ for any $t \in J^0$. Thus, we have that the W-boundedness ($\sup_{t\in J^0}(Var_{s\in[-h,0]}|\mathfrak{Q}(t,s)|) < \infty$) of the matrix $w(t)\mathfrak{Q}(t)$ is a necessary condition for the W-stability of the zero solution of system (25).

By virtue of Theorem 8 we have that for any $s \in J^0$ and $j \in \langle n \rangle$, the relation $\sup_{s\in[a,t]} |w(t)\mathfrak{H}^j(t,s)| < \infty$ holds, and taking into account that $|w(t)\mathfrak{H}^j(t,s)| = 0$ when $t < s$, we can conclude that $\mathfrak{H}^\infty = \sup_{t\in J^0} |\mathfrak{H}(t)| < \infty$. □

Corollary 3. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 6 hold.*
2. *The zero solution of the system (25) is W-GAS stable in the Lyapunov sense.*

Then, the following relations

$$\lim_{t\to\infty} |\mathfrak{Q}(t)| = 0, \lim_{t\to\infty} |\mathfrak{H}(t)| = 0$$

hold, where $\mathfrak{H}(t) = \sup_{s\in[a,t]} |w(t)\mathfrak{H}(t,s)|$.

Proof. The statement follows immediately from Theorem 9, since according to condition 2 of Corollary 3, for any $j \in \langle n \rangle$ we have $\lim_{t\to\infty} |\mathfrak{Q}^j(t,s)| = \mathbf{0}$ for $s \in [a-h,a]$ and $\lim_{t\to\infty} |\mathfrak{H}^j(t,s)| = \mathbf{0}$ for any $s \in [a,t]$. □

Remark 6. *Note that the differences in the proofs of Theorems 8 and 9 are caused from the differences in the spaces of the initial functions* $\Phi(t) \in \mathbf{PC}$ *and* $\Phi(t) \in \mathbf{PC}^*$*, respectively.*

The next theorem makes clear the relation between the W-stability in the Lyapunov sense and Ulam–Hyers W-stability.

Theorem 10. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 6 hold.*
2. *The system (25) is Ulam–Hyers W-stable for any initial functions* $\Phi(t) \in \mathbf{PC}$.

Then, the zero solution of the system (25) is stable in the Lyapunov sense if and only if for any $\Phi(t) \in \mathbf{PC}$ *and arbitrary* $\epsilon > 0$ *IP (42), (2) has only W-bounded solutions.*

Proof. Necessity: Let the zero solution of the system (25) be stable in the Lyapunov sense and assume that IP (42), (2) has a W-unbounded solution $Y(t)|_{t\in J^0} \in \mathbf{C}^{1-\alpha}$ for some $\Phi^Y \in \mathbf{PC}$ and $\epsilon > 0$. Then, since (25) is Ulam–Hyers W-stable for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ and $\Phi^*(t) \in \mathbf{C}$ with $|\Phi^*(t) - \Phi^Y(t)| < \delta$ for $t \in [a-h,a]$ such that for $t \in J^0$ inequality (43) holds for $Y(t)|_{t\in J^0}$ and $X^*(t)$, where $X^*(t)$ is the corresponding solution of IP (25), (3) with initial function $\Phi^*(t)$. On the other hand, since according to Theorem 8 the function $X^*(t)$ is a W-bounded solution of IP (25), (3), and since $Y(t)|_{t\in J^0}$ is a W-unbounded solution of (42) according to our assumption, we conclude that inequality (43) is impossible to be fulfilled for $Y(t)|_{t\in J^0}$ and $X^*(t)$, which contradicts our assumption. Thus, all solutions of the inequality (43) are W-bounded.

Sufficiency: Let IP (42), (2) have only bounded solutions. Then, IP (25), (2) has, for any $\Phi \in \mathbf{PC}$, only W-bounded solutions. Then, by virtue of Theorem 8, the zero solution of system (25) is W-stable in the Lyapunov sense. □

6. On the Preservation of the Stability Properties

The aim of this section is, based on the obtained results in the previous sections, to make clear the preservation of the W-stability properties of the system (25) under nonlinear disturbances.

Definition 14. *The function $X \in \mathbf{C}^{1-\alpha}$ will be called weighted Mittag–Leffler bounded (WML-bounded) of order $\omega \in \mathbb{R}_+$ if there exist $t_0 \in J^0$, a constant $C_X > 0$, and a function $a(t) \in PC(J^0, \mathbb{R}_+)$ with $a(t) = O(t^\alpha)$ such that $|w(t)X(t)| \le a(t)E_\omega(C_X\Gamma(\omega)t^\omega)$ for $t \ge t_0$. By WML_ω, we denote the subset of all WML-bounded functions of order ω in $\mathbf{C}^{1-\alpha}$.*

Definition 15. *We say that the vector-valued functional $\mathfrak{F}\colon J \times PC \to \mathbb{R}^n$ is a damper of order ω for system (1) if for any $X(t) \in WML_\omega$ for the function $F^X(t) = (F_1^X(t), \dots, F_n^X(t))^T = \mathfrak{F}(t, X_t^T)$ there exist a point $t_0 \in J^0$ and a constant $C^0 = C^0(t_0, X) > 0$ such that the following estimation $|F^X(t)| \le C^0 t^{-(\omega+1)}$ holds for $t \ge t_0$.*

The next theorem establishes an a priori estimate of the solutions of IP (1), (2) with initial functions $\Phi(t) \in \mathbf{PC}$.

Theorem 11. *Let the following conditions be fulfilled.*

1. *The conditions of Lemma 6 hold.*
2. *The conditions $\mathfrak{C}$ hold.*
3. *$\mathfrak{F}(t, \mathbf{0}) = \mathbf{0}$ and $\sup_{t\in J} |\ell(t)| = L^\infty < \infty$.*

Then, any solution $X \in \mathbf{C}^{1-\alpha}$ of IP (3), (2) with initial function $\Phi(t) \in \mathbf{PC}$ is WML-bounded of order α.

Proof. Let $\Phi(t) \in \mathbf{PC}$ be an arbitrary initial function and $X \in \mathbf{C}^{1-\alpha}$ be the corresponding unique solution of IP (3), (2). Then, for $t \in J^0$ from (3) we obtain that

$$
\begin{aligned}
|X(t)| &\le \|\Phi\|(t-a)^{\alpha-1} + \left|\int_{-h}^{0} [d_\theta V(t,\theta)]X(t+\theta)\right| \\
&+ \frac{n}{\Gamma(\alpha)} \int_a^t (t-\tau)^{1-\alpha} \left|\int_{-h}^{0} [d_\theta U(\tau,\theta)]X(t+\theta) + \mathfrak{F}(\tau, X_\tau^T)\right| d\tau \\
&\le \|\Phi\|(t-a)^{\alpha-1} + V \sup_{s\in[a,t]} |X(s)| + V\|\Phi\| \\
&+ \frac{n}{\Gamma(\alpha)} \left|\int_a^t (t-\tau)^{\alpha-1} \left(\left|\int_{-h}^{0} [\partial_\theta U(\tau,\theta)]X(t+\theta)d\theta\right| + \left|\mathfrak{F}(\tau, X_\tau^T)\right| \right) d\tau\right| \\
&\le V \sup_{s\in[a,t]} |X(s)| + \|\Phi\|((t-a)^{\alpha-1} + V) + \frac{n(U_b + L^\infty)}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \sup_{s\in[a-h,\tau]} |X(s)| d\tau \\
&\le V \sup_{s\in[a,t]} |X(s)| + \|\Phi\|((t-a)^{\alpha-1} + V) \\
&+ \frac{n(U_b + L^\infty)}{\Gamma(\alpha)} \left(\int_a^t (t-\tau)^{\alpha-1} \sup_{s\in[a,\tau]} |X(s)| d\tau (t-a)^\alpha \|\Phi\| \right) \\
&\le V \sup_{s\in[a,t]} |X(s)| + \|\Phi\| \left((t-a)^{\alpha-1} + \frac{n(U_b + L^\infty)(t-a)^\alpha}{\Gamma(\alpha)} + V \right) \\
&+ \frac{n(U_b + L^\infty)}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \sup_{s\in[a,\tau]} |X(s)| d\tau. \qquad (47)
\end{aligned}
$$

Without loss of generality, we can assume that $\dfrac{\sup\limits_{s\in[a,a+1]} X(s)}{\alpha\|\Phi\|} = C^* \geq 1$ when $\|\Phi\| > 0$. Then, from (47) we have

$$
\begin{aligned}
&|X(t)| \leq \|\Phi\|\left(\frac{(t-a)^{\alpha-1}}{(1-V)} + \frac{V\Gamma(1+\alpha) + n(U_b+L^\infty)(t-a)^\alpha}{(1-V)\Gamma(\alpha)}\right) \\
&= \frac{n(U_b+L^\infty)}{(1-V)\Gamma(\alpha)}\int_a^{a+1}(t-\tau)^{\alpha-1}\sup_{s\in[a,\tau]}|X(s)|d\tau \\
&+ \int_{a+1}^{t}(t-\tau)^{\alpha-1}\sup_{s\in[a,a+1]}|X(s)|d\tau + \int_{a+1}^{t}(t-\tau)^{\alpha-1}\sup_{s\in[a+1,\tau]}|X(s)|d\tau \\
&\leq |\Phi\|\left(\frac{(t-a)^{\alpha-1}}{(1-V)} + \frac{V\Gamma(1+\alpha) + n(U_b+L^\infty)(t-a)^\alpha}{(1-V)\Gamma(\alpha)}\right) \\
&+ \frac{n(U_b+L^\infty)}{(1-V)\Gamma(\alpha)}\left(\alpha^{-1}\sup_{s\in[a,a+1]}X(s)(t-a)^\alpha + \int_{a+1}^{t}(t-\tau)^{\alpha-1}\sup_{s\in[a+1,\tau]}|X(s)|d\tau\right) \\
&\leq |\Phi\|\left(\frac{(t-a)^{\alpha-1}}{(1-V)} + \frac{n(U_b+L^\infty)(1+C^*)(t-a)^\alpha + V\Gamma(1+\alpha)}{(1-V)\Gamma(\alpha)}\right) \\
&+ \frac{n(U_b+L^\infty)}{(1-V)\Gamma(\alpha)}\int_{a+1}^{t}(t-\tau)^{\alpha-1}\sup_{s\in[a+1,\tau]}|X(s)|d\tau.
\end{aligned}
$$

Since $(t-a)^{\alpha-1}$ is monotone decreasing for $t \in (a, a+1]$, then $\sup\limits_{s\in[a+1,t]}|X^*(s)| \leq \sup\limits_{s\in[a,t]} w(s)|X^*(s)|$, and hence,

$$
\begin{aligned}
&\sup_{s\in[a+1,t]}|X(s)| \leq \|\Phi\|\left(\frac{\sup_{t\geq a+1}(t-a)^{\alpha-1}}{(1-V)} + \frac{n(U_b+L^\infty)(1+C^*)(t-a)^\alpha + V\Gamma(\alpha)}{(1-V)\Gamma(\alpha)}\right) \\
&+ \frac{n(U_b+L^\infty)}{(1-V)\Gamma(\alpha)}\int_{a+1}^{t}(t-\tau)^{\alpha-1}\sup_{s\in[a+1,\tau]}|X(s)|d\tau \\
&\leq \|\Phi\|\left(\frac{1+V}{(1-V)} + \frac{n(U_b+L^\infty)(1+C^*)(t-a)^\alpha}{(1-V)\Gamma(\alpha)}\right) \\
&+ \frac{n(U_b+L^\infty)}{(1-V)\Gamma(\alpha)}\int_{a+1}^{t}(t-\tau)^{\alpha-1}\sup_{s\in[a+1,\tau]}|X(s)|d\tau \\
&\sup_{s\in[a+1,t]}|X(s)| \leq \|\Phi\|\left(\frac{1+V}{1-V} + \frac{n(U_b+L^\infty)(1+C^*)(t-a)^\alpha}{(1-V)\Gamma(\alpha)}\right)E_\alpha\left(\frac{n(U_b+L^\infty)}{(1-V)}t^\alpha\right).
\end{aligned}
\tag{48}
$$

Applying to (48) Theorem 2, we obtain that

$$\sup_{s\in[a+1,t]}|X(s)| \leq \|\Phi\|\left(\frac{1+V}{(1-V)} + \frac{n(U_b+L^\infty)(1+C^*)(t-a)^\alpha}{(1-V)\Gamma(\alpha)}\right)E_\alpha\left(\frac{n(U_b+L^\infty)}{(1-V)}t^\alpha\right),$$

which completes the proof. □

Remark 7. *We note that Theorem 11 establishes that for Lipschitz-type nonlinear disturbances, all solutions of IP (3), (2) with initial functions $\Phi(t) \in$ **PC** are WML-bounded of order α.*

The proof of the next theorem is based on the following practical variant of the well-known Barbalat's lemma.

Lemma 8 ([30])**.** *Let $f \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a bounded function and $\lim\limits_{t\to\infty} \int\limits_0^t f(s)ds < \infty$. Then, the relation $\lim\limits_{t\to\infty} f(t) = 0$ holds.*

Theorem 12. *Let the following conditions be fulfilled:*

1. *The conditions of Theorem 11 hold.*
2. *The zero solution of system (25) is W-GAS.*
3. *There exists $\omega > \alpha$ such that the vector-valued functional $\mathfrak{F}\colon J \times PC \to \mathbb{R}^n$ is a damper of order ω.*

Then, the zero solution of IP (1), (2) (or IP (3), (2)) with initial function $\Phi(t) \in \mathbf{PC}^*$ is W-GAS.

Proof. Let $\Phi^*(t) \in \mathbf{PC}^*$ be arbitrary and $X^*(t)$ be the corresponding unique solution of IP (1), (2) (or (3), (2)). By virtue of Theorem 11 we have that $X^*(t)$ is WML-bounded of order α. We introduce the function $F^*(t) \equiv \mathfrak{F}(t, X_t^{*T})$, $t \in J^0$, and using (28) and (34) define the functions $\tilde{X}(t)$ and $\mathfrak{X}(t)$ as follows:

$$\tilde{X}(t) = \int\limits_{a-h}^{a} [d_s\tilde{\Phi}^*(s-a)]\mathfrak{Q}(t,s), \tag{49}$$

$$\mathfrak{X}(t) = \int\limits_{a}^{t} \mathfrak{H}(t,s) D_{a+}^{1-\alpha} F^*(s)ds. \tag{50}$$

Then, from Theorem 6 it follows that $\tilde{X}(t)$ is the unique solution of IP (25), (2) with initial function $\Phi^*(t)$, and by virtue of Theorem 7 the function $\mathfrak{X}(t)$ is the unique solution of IP (24), (2) with $F(t) \equiv F^*(t)$ and initial function $Z(t-a) \equiv 0$, $t \in [a-h, a]$. Thus, $\tilde{X}(t) + \mathfrak{X}(t)$ is a solution of IP (2), (1) (IP (3), (2)), and hence, we have that

$$X^*(t) = \tilde{X}(t) + \mathfrak{X}(t) = \int\limits_{a-h}^{a} [d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s) + \int\limits_{a}^{t} \mathfrak{H}(t,s) D_{a+}^{1-\alpha} F^*(s)ds. \tag{51}$$

From (49) and Corollary 3 it follows that

$$\lim_{t\to\infty} |\tilde{X}(t)| = \lim_{t\to\infty} \left| \int\limits_{a-h}^{a} [d_s\tilde{\Phi}(s-a)]\mathfrak{Q}(t,s) \right| \le \lim_{t\to\infty} (\|\tilde{\Phi}\|)|\mathfrak{Q}(t)| \le \|\Phi\| \lim_{t\to\infty} |\mathfrak{Q}(t)| = 0. \tag{52}$$

Consideringthe function

$$G(t) = \int\limits_{a}^{t} \left(\int\limits_{a}^{\tau} \mathfrak{H}(\tau,s) D_{a+}^{1-\alpha} F^*(s)ds \right) d\tau, \tag{53}$$

obviously the integrand $\mathfrak{X}(\tau) = \int\limits_{a}^{\tau} \mathfrak{H}(\tau,s) D_{a+}^{1-\alpha} F^*(s)ds$ is a continuous function at any $\tau \in J^0$, and hence, $\frac{d}{dt}G(t) = \mathfrak{X}(t)$ for $t \in J^0$ too. From (53) it follows that

$$
\begin{aligned}
&|\mathfrak{X}(\tau)| \le \left| \int_b^\tau \mathfrak{H}(\tau,s) D_{a+}^{1-\alpha} F^*(s) ds \right| \le \Gamma^{-1}(\alpha)|\mathfrak{H}(t)| \left| \int_a^\tau \left(\frac{d}{ds} \int_a^s (s-\eta)^{\alpha-1} F^*(\eta) d\eta \right) ds \right| \\
&= \Gamma^{-1}(\alpha)|\mathfrak{H}(\tau)| \left| \int_a^\tau (\tau-\eta)^{\alpha-1} F^*(\eta) d\eta \right| \le (\alpha\Gamma(\alpha)^{-1})|\mathfrak{H}(\tau)| \int_a^\tau |F^*(\eta)| d(\tau-\eta)^\alpha \\
&\le \Gamma^{-1}(1+\alpha)|\mathfrak{H}(\tau)| \sup_{s\in[a,\tau]} (|F^*(s)|s^\alpha) \\
&\le \Gamma^{-1}(1+\alpha)|\mathfrak{H}(\tau)| \left(\sup_{s\in[a,b]} (|F^*(s)|s^\alpha) + \sup_{s\in[b,\tau]} (|F^*(s)|s^\alpha) \right) \\
&\le \Gamma^{-1}(1+\alpha)|\mathfrak{H}(\tau)| \left(\sup_{s\in[a,b]} (|F^*(s)|s^\alpha) + \sup_{s\in[b,\tau]} (C^0 s^{-(\omega+1)} s^\alpha) \right).
\end{aligned} \tag{54}
$$

Since $\omega > \alpha$ and the zero solution of the system (25) is W-GAS, then by virtue of Theorem 9 we have that $\mathfrak{H}^\infty = \sup_{t\in J^0} |\mathfrak{H}(t)| < \infty$, and hence, $|g(\tau)|$ is bounded on J^0. Then, from (53) and (54) it follows that

$$
\Gamma^{-1}(1+\alpha)|\mathfrak{H}(\tau)| \left(\sup_{s\in[a,b]} (|F^*(s)|s^\alpha) + \sup_{s\in[b,\tau]} (C^0 s^{-(\omega+1)} s^\alpha) \right)
$$

$$
\begin{aligned}
|G(t)| &= \left| \int_a^t \left(\int_b^\tau \mathfrak{H}(\tau,s) D_{a+}^{1-\alpha} F^*(s) ds \right) d\tau \right| \le \Gamma^{-1}(1+\alpha) \int_a^t |\mathfrak{H}(\tau)| \sup_{s\in[a,\tau]} (|s^\alpha F^*(s)|) d\tau \\
&\le \Gamma^{-1}(1+\alpha) b^\alpha \sup_{s\in[a,b]} |\mathfrak{H}(\tau)||F^*(s)|(b-a) + \Gamma^{-1}(1+\alpha) \int_b^t |\mathfrak{H}(\tau)| \sup_{s\in[b,\tau]} (C^0 s^{-(\omega+1)} s^\alpha) d\tau \\
&\le \Gamma^{-1}(1+\alpha) b^\alpha \sup_{s\in[a,b]} |\mathfrak{H}(\tau)||F^*(s)|(b-a) + \Gamma^{-1}(1+\alpha)\mathfrak{H}^\infty C^0 t^{-(\omega+1)+\alpha+1}
\end{aligned}
$$

and, hence, $\lim_{t\to\infty} |G(t)| < \infty$. Then, by virtue of Lemma 8 we have that $\lim_{t\to\infty} |\mathfrak{X}(\tau)| = 0$, and thus, from (51) and (52) it follows that $\lim_{t\to\infty} |X^*(t)| = 0$, which completes the proof. □

7. An Example

Example 1. *Let $n = 1$, the lower terminal $a = 0$, $\alpha \in (0,1)$, $h > 0$, $p, b, c \in \mathbb{R}$, $U(t,\theta) = aH(t) + bH(t-h)$, $V(t,\theta) = pH(t-h)$, and $\phi \in PC([-h,0])$. Then, IP (25), (2) obtains the following form:*

$$
D_{a+}^\alpha (x(t) + px(t-h)) = cx(t) + bx(t-h),\ t \in (0,\infty), \tag{55}
$$

$$
x(t) = \phi(t),\ t \in [-h,0],\ D_{a+}^{\alpha-1} x(0+) = \phi(0). \tag{56}
$$

According to the results obtained by us in Section 3, IP (55), (56) has a unique solution, which we will find an explicit representation of in this example.

To obtain an explicit representation of the solution we will use the approach based on the Laplace transform (LT) introduced in [31] for the case of Caputo-type derivatives. Below, we assume that $|p| < 1$, and then, by virtue of theorem 3 in [17] the LT can be applied correctly to Equation (55). Denote the LT of $x(t)$ by

$$
\hat{x}(s) = L(x(t);s) = \int_0^\infty e^{-st} x(t) dt,\ \hat{\phi}(s) = \int_{-h}^0 e^{-st} \phi(t) dt.
$$

Applying the LT to both sides of (55) *and taking into account* (56) *we have*

$$s^{\alpha}\hat{x}(s)+s^{\alpha}e^{-sh}p(\hat{x}(s)+\hat{\phi}(s))-(\phi(0)+p\phi(-h))=c\hat{x}(s)+be^{-sh}(\hat{x}(s)+\hat{\phi}(s)),$$

and hence,

$$\hat{x}(s)\left(1-\frac{b-ps^{\alpha}}{s^{\alpha}-c}e^{-sh}\right)=\frac{\phi(0)+p\phi(-h)}{s^{\alpha}-c}+\frac{b-ps^{\alpha}}{s^{\alpha}-c}e^{-sh}\hat{\phi}(s). \tag{57}$$

For any $s\in\mathbb{C}$ with $|s|^{\alpha}>|c|$, we have the series expansion

$$\left(1-\frac{b-ps^{\alpha}}{s^{\alpha}-c}e^{-sh}\right)^{-1}=\sum_{k=0}^{\infty}\frac{(b-ps^{\alpha})^{k}}{(s^{\alpha}-c)^{k}}e^{-ksh}$$

and then, from (57) *it follows that*

$$\hat{x}(s)=(\phi(0)+p\phi(-h))\sum_{k=0}^{\infty}\frac{(b-ps^{\alpha})^{k}}{(s^{\alpha}-c)^{k+1}}e^{-ksh}+\hat{\phi}(s)\sum_{k=0}^{\infty}\frac{(b-ps^{\alpha})^{k}}{(s^{\alpha}-c)^{k}}e^{-ksh}. \tag{58}$$

Substituting into (58) *the expression $(b-ps^{\alpha})^{k}=\sum_{i=0}^{k}\binom{k}{i}(-1)^{i}p^{i}s^{i\alpha}b^{k-i}$ we obtain that the LT of $\hat{x}(s)$ has the following form:*

$$\hat{x}(s)=(\phi(0)+p\phi(-h))\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}(-p)^{i}b^{k-i}\frac{s^{i\alpha}}{(s^{\alpha}-c)^{k+1}}\right)$$
$$+\hat{\phi}(s)\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}(-p)^{i}b^{k-i}\frac{s^{i\alpha}}{(s^{\alpha}-c)^{k}}\right). \tag{59}$$

For simplicity we will assume that $\phi(t)\equiv\phi^{0}\in\mathbb{R}$, $t\in[-h,0]$, and then, we have that

$$\hat{\phi}(s)=\phi^{0}s^{-1}(e^{-sh}-1). \tag{60}$$

Consider the three-parameter Mittag–Leffler (Prabhakar) function

$$E^{\gamma}_{\alpha,\beta}(z)=\Gamma^{-1}(\gamma)\sum_{j=0}^{\infty}\frac{\Gamma(j+\gamma)z^{j}}{j!\Gamma(\alpha j+\beta)}$$

introduced in [32]*, where $\beta,\gamma\in\mathbb{R}$. For $z=ct^{\alpha}$, $t\geq 0$, $Re\,s>0$ with $|s|^{\alpha}>|c|$ the LT of the function $e^{\gamma}_{\alpha,\beta}(t;c)=t^{\beta-1}E^{\gamma}_{\alpha,\beta}(ct^{\alpha})$ is $L(e^{\gamma}_{\alpha,\beta}(t;c);s)=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-c)^{\gamma}}$ and when $h\geq 0$ (see theorem 1.31 in* [33]*) we have that the following relation holds:*

$$L^{-1}\left(\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-c)^{\gamma}}e^{-sh},t\right)=\begin{cases}e^{\gamma}_{\alpha,\beta}(t-h;c), & t\geq h,\\ 0, & t<h.\end{cases} \tag{61}$$

Denoting as usual by $[t]$ the greatest integer number less or equal to t for any $t\in\mathbb{R}$, from (59)*,* (60)*, and* (61)*, it follows that the solution $x(t)$ of IP* (55)*,* (56) *possess the representation*

$$x(t)=\phi^{0}(1+c)\sum_{k=0}^{[\frac{t}{h}]}\left(\sum_{i=0}^{k}\binom{k}{i}(-p)^{i}b^{k-i}e^{k+1}_{\alpha,\alpha(k-i)+1}(t-hk;c)\right)$$
$$-\phi^{0}\sum_{k=0}^{[\frac{t}{h}]}\left(\sum_{i=0}^{k}\binom{k}{i}(-p)^{i}b^{k-i}e^{k}_{\alpha,\alpha(k-i)+1}(t-hk;c)\right)$$
$$+\phi^{0}\sum_{k=0}^{[\frac{t}{h}]+1}\left(\sum_{i=0}^{k}\binom{k}{i}(-p)^{i}b^{k-i}e^{k}_{\alpha,\alpha(k-i)+1}(t-hk+h;c)\right). \tag{62}$$

Remark 8. *Note that the approach used by the presentation of the solution $x(t)$ of IP* (55), (56) *can be extended to obtain a presentation of the solutions with step functions (with one jump of the first kind in the interval $[-h, 0]$) as initial functions, and hence, as a consequence, a presentation of the generalized fundamental matrix $\mathfrak{Q}(t,s)$ can be obtained. But this idea, as well as a numerical simulation via the methods developed in [34–37], is beyond the area of this article and can be a theme for future research.*

8. Conclusions and Comments

In this paper, we have studied a general class of nonlinear disturbed neutral linear fractional systems with derivatives in the Riemann–Liouville sense and distributed delays. As motivation for this study we can refer to the meaningful physical interpretations of models with these Riemann–Liouville fractional derivatives presented in [7]. Mainly, we have studied the most important, from the point of view of applications, and the technically more complicated case, when the lower terminal of the Riemann–Liouville derivatives coincides with the end point of the initial interval. First, it was proved that the initial problem for these systems with discontinuous initial functions possesses a unique solution under some natural assumptions. It is worth mentioning that the assumptions used to derive this result are similar to those used in the case of systems with first-order derivatives, and as far as we know this is the first result on this theme. Then, as a consequence of the obtained result, we have proved the existence and uniqueness of a fundamental matrix and a generalized fundamental matrix for the studied neutral linear homogeneous system. The existence of the fundamental matrices have allowed us to establish an integral representation for the solutions of the initial problem for the homogeneous system and also for the corresponding inhomogeneous system. Furthermore, for fractional systems with Riemann–Liouville derivatives we have introduced a new concept for weighted stabilities in the Lyapunov, Ulam–Hyers and Ulam–Hyers–Rassias senses. Note that the introduced concept coincides with the classical stability concept for the cases of integer-order or Caputo-type derivatives. To prove the applicability of the introduced concept it was proved that the zero solution of the homogeneous system is weighted stable if and only if all its solutions are weighted bounded; this result is well known for systems with first-order derivatives and is also established for the case of systems with Caputo-type derivatives in our former works. In addition, for the homogeneous system it was established that the weighted stability in the Lyapunov sense and weighted stability in the Ulam–Hyers sense are equivalent if and only if the inequality appearing in the Ulam–Hyers definition possesses only bounded solutions. Finally, we have derived natural sufficient conditions under which the property of weighted global asymptotic stability of the zero solution of the homogeneous system is preserved under appropriate nonlinear disturbances.

In our point of view the main contributions in the article can be highlighted as follows:

- A new concept was introduced for weighted stabilities in the Lyapunov, Ulam–Hyers and Ulam–Hyers–Rassias senses; it coincides with the classical stability concept for the cases of integer-order or Caputo-type fractional derivatives.
- Sufficient conditions have been obtained which guarantee that the weighted stability in the Lyapunov sense and the weighted stability in the Ulam–Hyers sense are equivalent.
- Sufficient conditions have been given under which the property of weighted global asymptotic stability of the zero solution of the homogeneous system is preserved under appropriate nonlinear disturbances.

As a future perspective for research we think that it will be important from the point of view of applications to establish explicit-type sufficient conditions which guarantee weighted stability in the Lyapunov or Ulam–Hyers senses.

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