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Evolving Patterns in Irrational Numbers Using Waiting Times between Digits

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Abstract: There is an increasing interest in determining if there exist observable patterns or structures within the digits of irrational numbers. We extend this search by investigating the interval in position between two consecutive occurrences of the same digit, a kind of waiting time statistics. We characterise these by the burstiness measure which distinguishes if the inter-event times are periodic, bursty, or Poisson processes. Furthermore, the complexity–entropy plane was used to determine if the intervals are stochastic or chaotic. We analyse sequences of the first 1 million digits of the numbers π, e, $\sqrt{2}$, and ϕ. We find that the intervals between single, double, and triple digits are Poisson processes with a burstiness measure in the range $-0.05 \leq B \leq 0.05$ for the four numbers studied. This result is supported by a complexity–entropy plane analysis, which shows that the time intervals have the same characteristics as Gaussian noise. The four irrational numbers have identical degrees of complexity and burstiness in their inter-event analysis.

Keywords: inter-event times; irrational numbers; permutation entropy; burstiness

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1. Introduction

The number system has evolved significantly from the first numbering system.

Apart from use in communication, numbers have found applications in various affairs of humanity. Several simulation techniques such as Monte Carlo rely on the special properties of numbers. Another use of number system is in gambling [1] and cryptography [2]. Random numbers have been employed in geographic information systems for assessing the impact of error and aggregation within data sets and developing simulated data [3]. The successful application of numbers in these different scenarios depends on the inherently complex nature of the numbers. The use of numbers in gambling and cryptography relies on the randomness in a group of numbers. Generating random numbers has been the subject of several studies. According to Knuth [4], "Random numbers should not be generated with a method chosen at random". The significance and necessity of dependable random number sequences have spurred the development of various methods for their generation. Some methods of generating random numbers include physical processes and computational algorithms [5]. The use of chaos theory [6,7], quantum random number generators [8], and machine learning algorithms [9] for the generation of random numbers has also been proposed.

One potential approach for generation of random numbers is using the digits of irrational numbers. Irrational numbers are numbers that cannot be explicitly represented as a simple fraction in the form h/k, where h, k are integers and $k \neq 0$. Certain irrational numbers arise from the roots of polynomials with integer coefficients, although numerous others do not. Some common irrational numbers include the golden ratio ($0.5 \times (1 + \sqrt{5})$), pi ($\pi$), Euler's number ($e$), and many others. The golden ratio is regarded as the most irrational of all irrational numbers because it has the slowest convergence of the optimal approximation by a sequence of rational numbers which is given by the continued fraction expansion [10,11]. Pi (π) expressed as 3.141592653589793 . . . is an important number in science, especially geometry and trigonometry. Euler's number (e), expressed as 2.7182818284 . . ., has applications in complex number theory, probability distribution, and differential equations. The square root of 2 ($\sqrt{2}$) is defined as the length of a diagonal across a square with sides of one unit of length. $\sqrt{2}$ has the sequence 1.41421356237309 . . .. It was used in the ground plan of the Norwich Cathedral [12] and the design of paper sizes [13]. The golden ratio (ϕ) is expressed as $(0.5 \times (1 + \sqrt{5})$. It has the form 1.618033988749894 . . .. The sequence has applications in arts, flowering plants, and construction of the great pyramid [14]. One of the practical applications of irrational numbers is in the field of cryptography. Irrational numbers have been used to improve on the efficiency of the Data Encryption Standard [2]. Liu et al. [15] proposed a new method to generate a random phase mask for information encoding using the decimal expansion of irrational numbers. It has been reported that transforming the digits of irrational number to binary sequences can generate random sequences with applications to cryptography [5]. Statistical tests have shown that π can be used as a random number generator [16]. The application of irrational numbers depends on whether they are stochastic or chaotic.

Stochastic and chaotic systems share similar properties that make them difficult to distinguish [17]. There are several tools that have been developed to differentiate between random and chaotic systems. Chaotic systems are sensitive to initial conditions, which makes long-term prediction difficult. Some of the tests developed to investigate chaos in natural time series include: Lyapunov exponent, recurrence analysis, 0-1 test, information theory, and correlation dimension. One of the few tests designed specifically to distinguish chaos from noise is the complexity–entropy plane [17]. The coexistence of both stochasticity and chaos in a single system has been propounded [18].

There exists the need to further investigate irrational numbers for applications in random number generation and cryptography. Specifically, it is expedient to determine if irrational numbers are governed by stochastic, chaotic, or both stochastic and chaotic dynamics. However, the direct investigation of irrational numbers has been difficult due to the limited digits (0–9) repeating at intervals in the sequence. A suitable approach is to use the difference in the positional occurrence of the digits to characterise irrational numbers. Thus, irrational numbers can be investigated for periodicity, burstiness, or randomness within the digits directly. If the numbers occur at regular intervals in the sequence, then an expression for generating the numbers can be proposed. The non-random occurrence of the intervals will suggest applications in random number generation and cryptography. An understanding of the distribution of irrational numbers will help in predicting potential weaknesses in their applications. The objectives of this study are to (1) determine whether the inter-event times of irrational numbers are periodic, stochastic, or chaotic processes, and (2) evaluate the statistical differences in irrational numbers using a complex system approach.

2. Methods

2.1. Irrational Numbers

In this study, four irrational numbers were considered for analysis. In Table 1, the approximation and expression of these numbers are presented. We chose two algebraic numbers ($\sqrt{2}$, ϕ) and two transcendental numbers (e,π) for this study. For each of the irrational numbers, the inter-event time was estimated using the position of the digits in the irrational numbers (Figure 1). A total of 1,000,000 digits for each of the irrational numbers were analysed.

Table 1. List of irrational numbers, their symbol, abbreviation and first few digits.

Number	Symbol	Approximations	Expression
Euler number	e	$\sum_{n=0}^{\infty} \frac{1}{n!}$	2.781828
Pi	π	$\frac{1}{\pi} = \sum_{k=0}^{\infty} s(k) \frac{Ak+B}{C^k}$ [19]	3.141592
Square root of 2	$\sqrt{2}$	$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$	1.414213
Golden ratio	ϕ	$\frac{1+\sqrt{5}}{2}$	1.618033

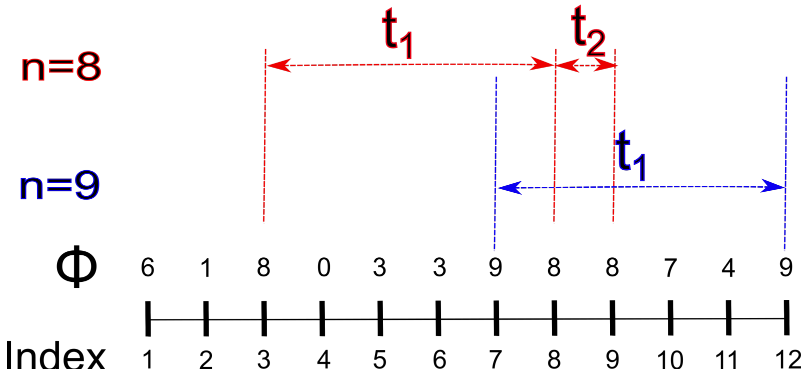

Figure 1. Schematic diagram to demonstrate the concept of inter-event time using the mantissa of ϕ. The inter-event times for two digits 8 (red) and 9 (blue) in the mantissa of ϕ are used as examples in the figure.

2.2. Burstiness Index

Historically, the coefficient of variation (r) is used to measure irregularity in inter-event times. It is defined as:

$$r = \frac{\sigma}{\mu} \tag{1}$$

where σ and μ are the standard deviation and mean value of inter-event train $x_i = t_{i+1} - t_i$. The spike train is regarded as a Poisson process if $r = 1$ and a regular process if $r = 0$. This approach has been used for inter-spike trains in neurons [20] and earthquake analysis [21]. Goh and Barabási [22] extended this definition to introduce a bounded parameter called burstiness, B. It is defined as:

$$B = \frac{r-1}{r+1} \tag{2}$$

The values of B are $-1, 0, 1$ for regular, Poisson, and bursty time series, respectively. A time series is called *bursty* if $\sigma \gg \mu$, i.e., in the limit $r \to \infty$.

However, it has been shown that the burstiness measure (Equation (2)) is susceptible to the finite-size effect. In this study, the modified Burstiness index introduced by Kim and Jo [23] was considered. It is defined as:

$$B = \frac{\sqrt{n+1}r - \sqrt{n+1}}{(\sqrt{n+1}-2)r + \sqrt{n-1}} \tag{3}$$

where n is the length of the time series. It has the same interpretation as in Equation (2).

2.3. Complexity–Entropy Plane

The complexity–entropy plane is an approach to distinguish noise from chaos. It consists of two parameters—statistical complexity and entropy measure, in representation space. The statistical complexity is defined as

$$C = H \times D \tag{4}$$

where D is the "disequilibrium", as defined by Lamberti et al. [24] and H is an entropic measure [17]. The entropic measure is the Shannon entropy normalised by $\ln N$, where N is the number of possible values in the series. The disequilibrium used in this study is defined in terms of the Jensen–Shannon divergence as

$$Q_J[P, P_e] = Q_0\{S[(P+P_e)/2] - S[P]/2 - S[P_e]/2\} \tag{5}$$

where $S[P]$ is the Shannon entropy of the probability distribution P and $S[P_e]$ is the Shannon entropy of a uniform distribution. We will use as a specific case of a Shannon entropy the permutation entropy in a given embedding dimension m. This means that the probabilities in the Shannon entropy are the probabilities that different permutations of the ranks of m successive digits occur. Since there are $m!$ different possible permutations, the corresponding uniform distribution is given by $m!$ identical values.

3. Results

3.1. Autocorrelation and Burstiness Analysis

The auto-correlation function for all the digits of the four irrational numbers, as presented in Figure 2, did not show any distinctive pattern. The lack of discernible pattern in the autocorrelation plots for the four irrational numbers implies that there is no systematic relationship between the digits of the irrational numbers and their lagged values. This is a signature of randomness in the time series. It means that the current digit in any of the irrational number is not dependent on past digits. Furthermore, the autocorrelation values were found to be statistically significant at 95% confidence interval, except for lag 3 and 5 in $\sqrt{2}$.

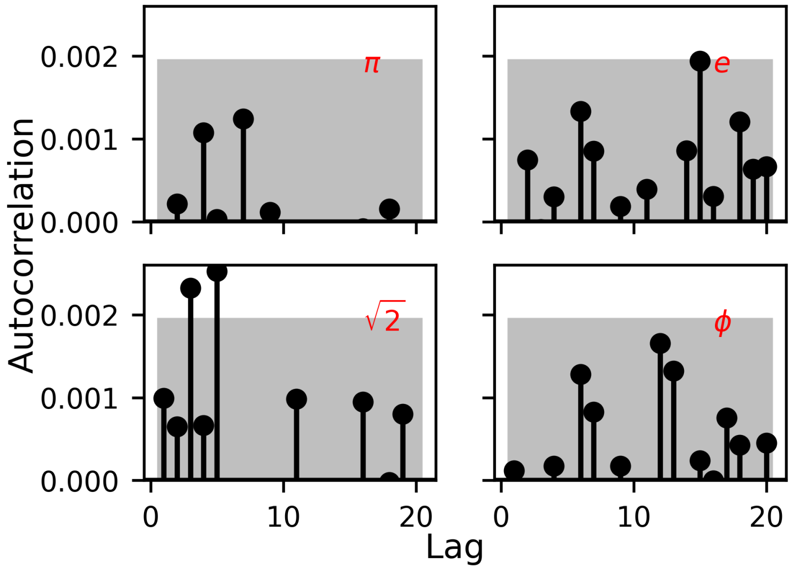

Figure 2. Autocorrelation plot for the digits of the four irrational numbers. Patterns within the autocorrelation plot reveal randomness within the digits. There were no distinctive patterns in the autocorrelation plots of the four digits.

The inter-event time, as described in Section 2.1, was for each digit in each of the four irrational numbers under consideration. In Figure 3, the burstiness measure for each of the digits is presented. The burstiness values for all the digits considered were found to be in the range $-0.030 \le B \le -0.020$. We report $B \approx 0$, which implies that the inter-event times for the irrational numbers are largely Poisson processes. This means that the time intervals between the occurrence of a single digit within the four irrational numbers are independent of each other. This result supports the position of randomness as revealed by the autocorrelation function of the entire irrational number sequences. While the Burstiness values were within the same range $-0.030 \le B \le -0.020$, there exist slight variations in their values.

This study also considered the distribution of the inter-event statistics across different segments of the irrational numbers. To achieve this, the 1,000,000 digits were divided into segments of 100,000 digits each and burstiness analysis was conducted for each digit within the segments for the four irrational numbers. The result is shown in Figure 4. In the four irrational numbers studied, there were no discernible patterns in Burstiness across the 10 segments for each of the digits. All the values were found to be close to zero, which confirms that the distribution of the inter-event times across the different intervals are Poisson processes. This confirms the independence and time homogeneity in the inter-event times. The burstiness values were found to be in the range $-0.037 \le B \le -0.017$, $-0.041 \le B \le -0.014$, $-0.037 \le B \le -0.015$, and $-0.038 \le B \le -0.017$ for π, e, $\sqrt{2}$, and ϕ, respectively. The lack of symmetry in the burstiness measures across the different segments also suggests that the four irrational numbers considered have digits with different dynamical structures.

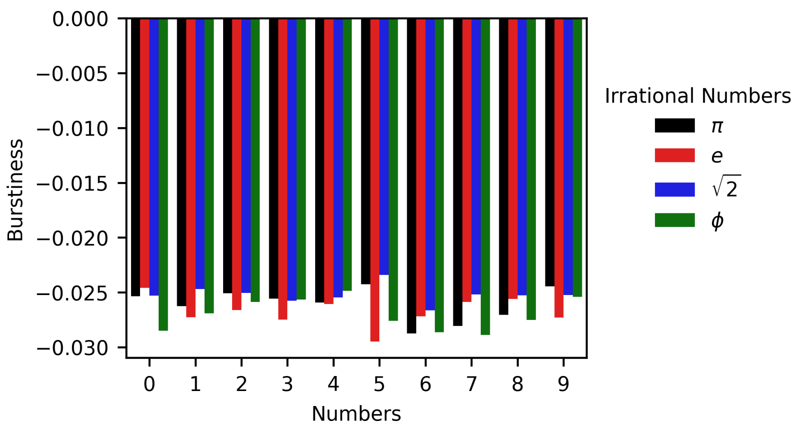

Figure 3. Burstiness value of each digit of the four irrational numbers to determine their statistical characteristics.

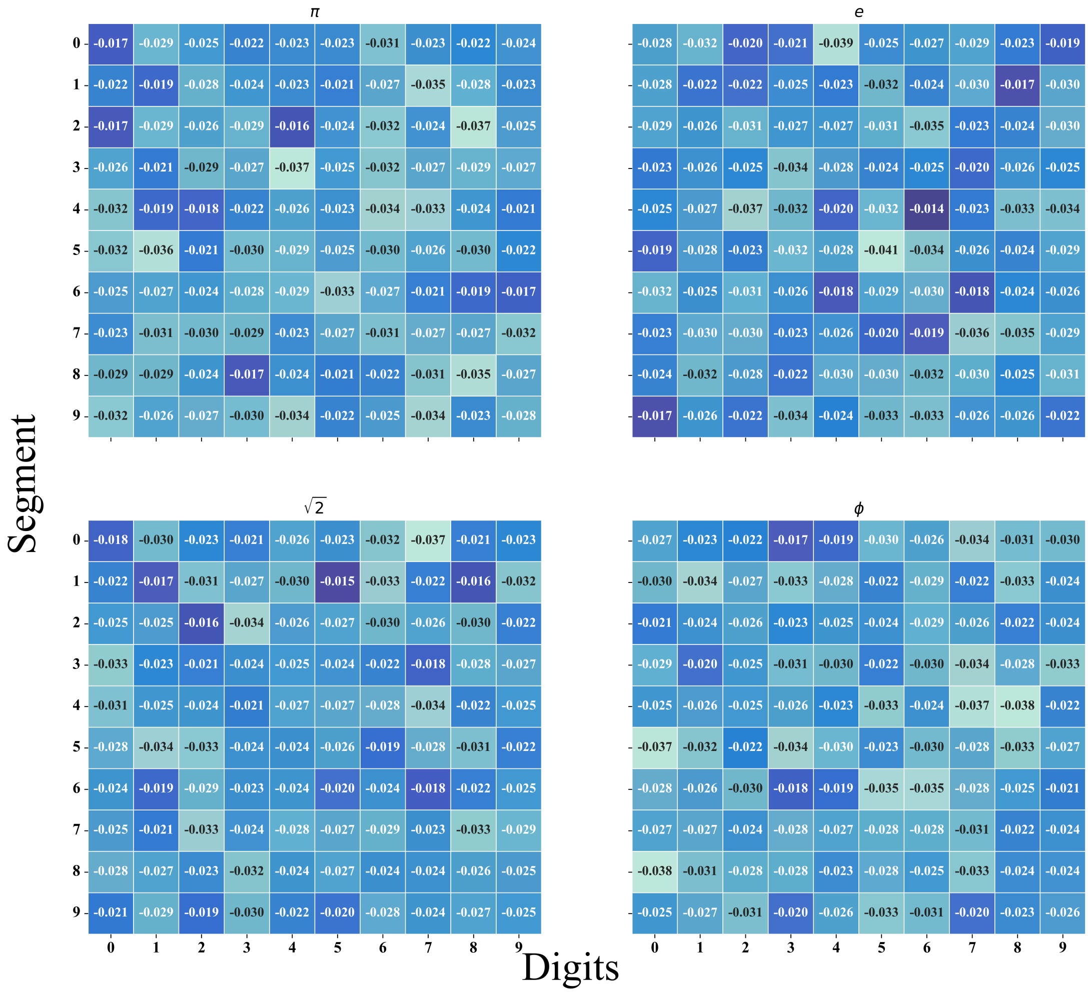

Figure 4. Burstiness value for different segments (100,000 digits per segment) of the four irrational numbers. Variations within a given segment will show unique appearance of the digits.

The study was also extended to the positioning of two and three consecutive digits within the irrational numbers. In Figure 5, the burstiness measure for the interval between two consecutive numbers was presented. The result showed low values ($B \approx 0$) for all the possible combinations of digits. Certain conspicuous features could be observed in the results. First, the diagonals showed higher values compared to the off-diagonal values. This implies that the intervals between consecutive numbers of the same digits have different dynamics compared to other combinations of digits in irrational numbers. The diagonals have values in the range $0.03 \leq B \leq 0.04$ for both *pi* and $\sqrt{2}$, and $0.03 \leq B \leq 0.05$ for both e and ϕ. Secondly, certain combinations of digits yield $-0.004 \leq 0.004$, which makes them more random than the other combinations. This analysis was extended to triple consecutive digits; however, this was limited to identical digits such as 111, 222, and 999 (Figure 6). It was also observed that the inter-event times for the triple sequences also showed evidence of Poisson processes ($0 \leq B \leq 0.05$). Except for the triple digits 6 and 7, the triple digits of π showed the lowest burstiness value among the four irrational numbers considered. It is pertinent to note that compared to the burstiness of single digits (Figure 3), the values obtained for the triple sequence showed positive values.

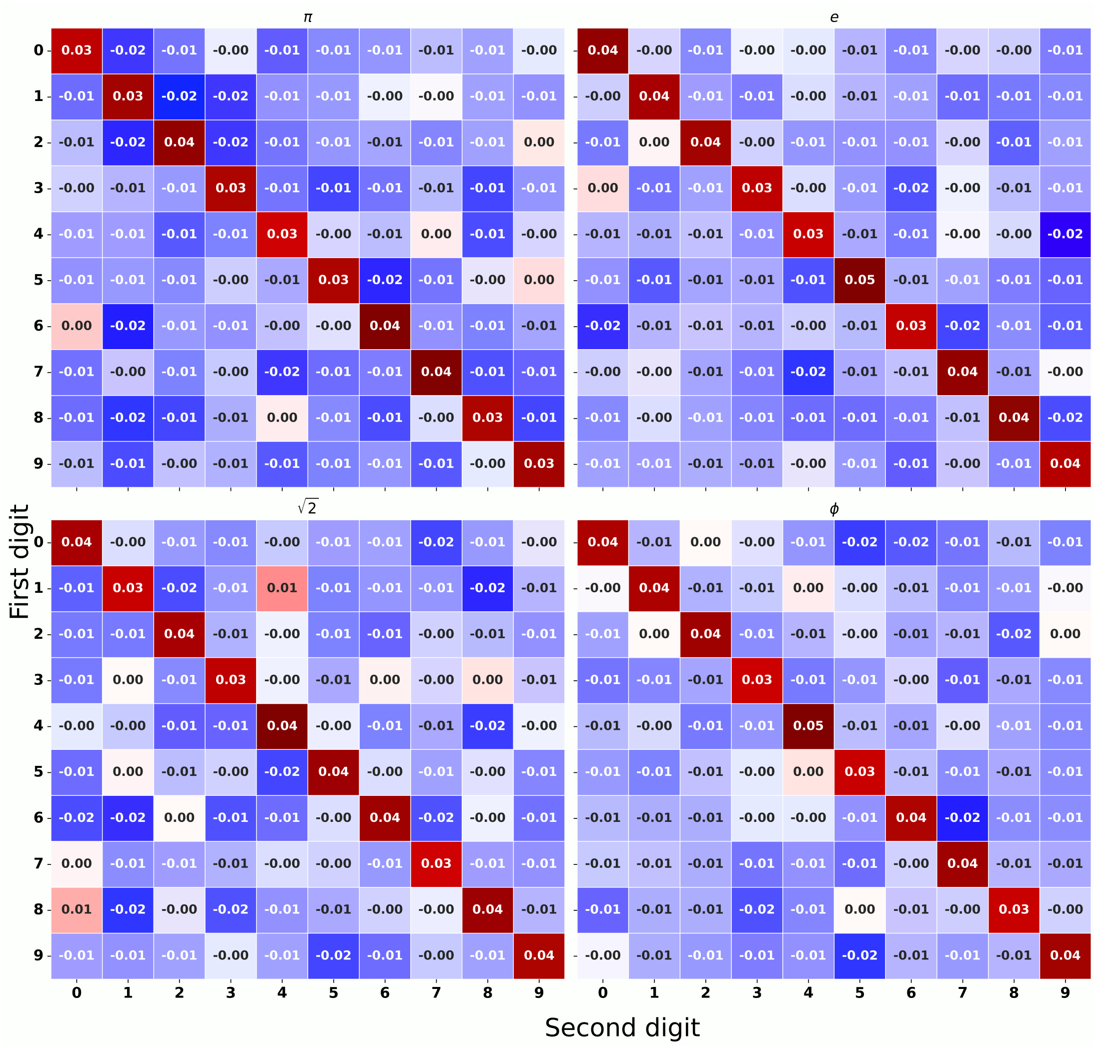

Figure 5. Burstiness value for sequences of two consecutive integer values, that is, the appearance of 11, 22, ...

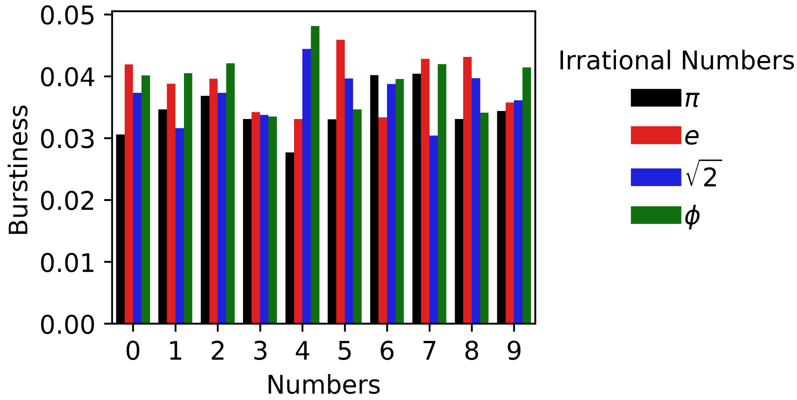

Figure 6. Burstiness value for triple consecutive digits within the four irrational numbers.

3.2. Complexity–Entropy Plane Analysis

The complexity–entropy plane analysis was conducted on the time series of the four irrational numbers at different embedding dimensions for the permutation entropy (Figure 7). It is pertinent to note that there were no significant differences between the complexity–entropy planes for the four irrational numbers at various embedding dimensions. The statistical complexity values were found to be low (0.00–0.025) and permutation entropy was high (0.98–0.995). The statistical complexity increases while the permutation entropy decreases with increasing embedding dimensions. The complexity–entropy values for the four irrational numbers at different embedding dimensions were compared to those of Gaussian noise, as a reference. It was observed that the complexity of the four irrational numbers is not different from that of a Gaussian noise.

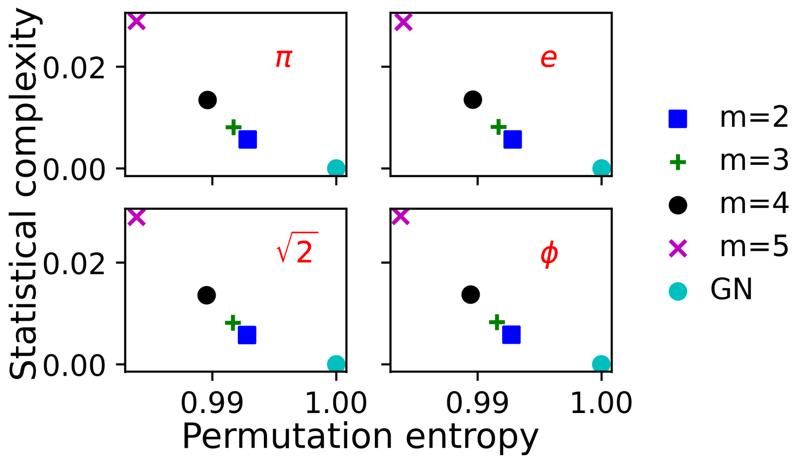

Figure 7. Complexity–entropy plane for the four irrational numbers alongside Gaussian noise at different embedding dimensions.

Temporal dependence structure and degree of complexity in inter-event times of the four irrational numbers and their shuffled versions were determined using permutation entropy (Figure 8). It was observed that there were no significant differences between the original and shuffled versions of the inter-event times of each digit in the four irrational numbers. In some instances, the permutation entropy of the shuffled data was found to be slightly higher than that of the original data. This implies that shuffling the inter-event times increased the complexity of the time series. Furthermore, there were no differences in the permutation entropy values of the inter-event times for each of the digits in the four irrational numbers. That is, the values of permutation entropy were in the same range for all the irrational numbers. Hence, it is not sufficient to differentiate the level of complexity among the digits or the numbers. Variations in the original inter-event times for each digit of the four irrational numbers were found to show no trend or pattern except for ϕ. For ϕ, the permutation entropy increases from 0 to a peak at 3 before decreasing to a minimum value at 9.

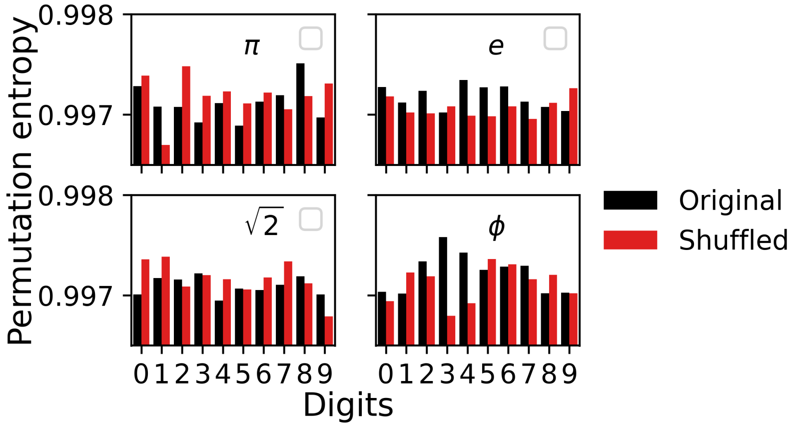

Figure 8. Permutation entropy for the four irrational numbers and their shuffled version.

The inter-event times for integers in the four irrational numbers were also examined in the complexity–entropy plane (Figure 9). The complexity–entropy plane for these digits is characterised by high permutation entropy ($\geq$0.996) and low statistical complexity ($\leq$0.0031). These values are associated with Gaussian processes [17]. Clusterings were observed in the complexity–entropy planes. In the complexity–entropy plane for π, there are five distinct groups. The first group consisting of 5, 3, and 9 have the highest values of statistical complexities. In the second group lies 2, 1, 4, 8. The third, fourth, and fifth group consists of one digit each—7, 0, and 8, respectively. The clustering of these digits suggests identical dynamical characteristics for their inter-event times. In the complexity–entropy plane for e, two distinct groups could be observed. The first group consists of 3, 9, 8, 7, and 1, while the second group has 0, 2, 6, 5, and 4. There were no clear clusters in the complexity–entropy plane for $\sqrt{2}$; hence, it is posited that all the digits showed the same level of complexity. Four clusters could be observed in the complexity–entropy plane for ϕ. There is the first cluster with the highest statistical complexity consisting of 0, 1, 8, and 9 while the second group has 2, 5, 6, and 7. The third and fourth cluster has one member—4 and 3, respectively.

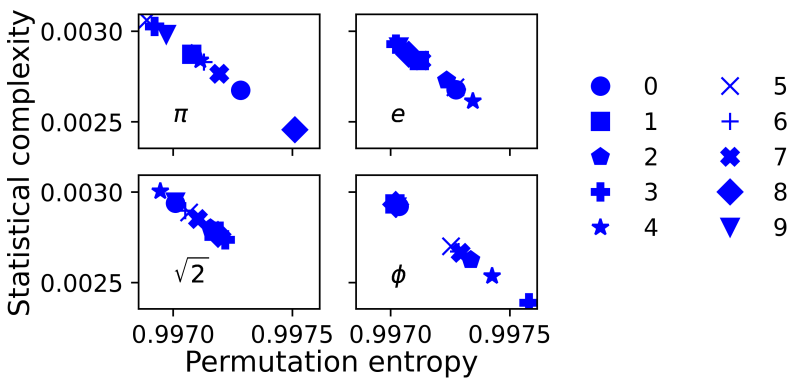

Figure 9. Complexity–entropy plane for the digits of the irrational numbers at $m = 3$.

4. Discussion

The true nature of irrational numbers is important for several of their real life application. It is necessary to determine if irrational numbers are periodic, bursty, chaotic, or stochastic. A periodic pattern within the irrational numbers will imply the possibility of better algorithms for the numbers. A chaotic series means long-term prediction of the terms of the irrational numbers will be difficult. If the irrational numbers are stochastic in nature, it implies they are useful in random number generations and similar applications. There have been attempts to determine structures within irrational numbers. Revealing or understanding these structures will improve the application of irrational numbers in real life. Due to the limited number of digits repeating within irrational numbers, it is difficult to analyse. In this study, this difficulty was addressed by looking at the position of the digits within the irrational numbers. Then, the interval between the two consecutive digits was analysed as a time series. To classify the irrational numbers, two approaches were used—burstiness and information theory. In the burstiness approach, we aim to know if the interval between two consecutive digits of an irrational number occur independent of previous intervals, occur in regular intervals, or occur in groups. Our results showed that for intervals of single, double, and triple digits, intervals of digits in irrational numbers are Poisson processes—that is, the current interval is independent of previous intervals. In the second approach, the complexity–entropy plane was applied to the inter-event times. It was found that the irrational numbers occur in the same region as fractional Gaussian noise. We have shown that the digits of four irrational numbers and their inter-event times are stochastic processes and not chaotic processes. We also showed that the statistical properties of the irrational numbers are identical. This is in agreement with results from other studies using different approaches [25,26]. Lai and Danca [25] showed that ten different irrational numbers have similar fractal dimensions of $\approx$1.38. This was also confirmed by using a weighted Shannon entropy on encoded irrational number digits [26]. Croll [26] showed that the irrational numbers have the same characteristics as commercially available random number generators. Results obtained showed that the time interval between single, double, and triple digits of irrational numbers are Poisson processes. Zhao et al. [27] showed that pairs of digits in irrational numbers are random numbers using information theory. This is in agreement with the results obtained in this study for pairs of digits in the four irrational numbers. The result obtained in [27] was extended to three digits in this present study. Using runs statistics and the strong law of large numbers, it has been shown that the first 100,000 digits of π, e are random numbers [28]. The values of permutation entropy obtained in this study were found to be similar to those reported for 10,000 sequences of π and e [29].

5. Conclusions

There are still so many questions about irrational numbers that have not been answered. Scientists are interested in the most irrational numbers, whether there are patterns in irrational numbers, and how complex the digits of irrational numbers are. The answer to these questions will not only enhance their present application to real-life scenarios but extend it. In this study, patterns in the digits of four irrational numbers were investigated using the interval between the digits analysed. First, results obtained showed that the digits and their intervals are Poisson processes. Second, it was shown that the statistical properties of the four irrational numbers investigated are identical. Third, we found that the properties of the irrational numbers are not different from fractional Gaussian noise.

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Data Availability Statement: The irrational numbers investigated in this study can be obtained from http://www.numberworld.org/constants.html (accessed on 5 November 2023).

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