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Error Bounds for Fractional Integral Inequalities with Applications

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Abstract: Fractional calculus has been a concept used to obtain new variants of some well-known integral inequalities. In this study, our main goal is to establish the new fractional Hermite–Hadamard, and Simpson's type estimates by employing a differentiable function. Furthermore, a novel class of fractional integral related to prominent fractional operator (Caputo–Fabrizio) for differentiable convex functions of first order is proven. Then, taking this equality into account as an auxiliary result, some new estimation of the Hermite–Hadamard and Simpson's type inequalities as generalization is presented. Moreover, few inequalities for concave function are obtained as well. It is observed that newly established outcomes are the extension of comparable inequalities existing in the literature. Additionally, we discuss the applications to special means, matrix inequalities, and the q-digamma function.

Keywords: Milne-type inequalities; *s*-convex function; fractional integrals; Hölder inequality; bounded function

MSC: 26D07; 26D10; 26D15

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1. Introduction

Fractional calculus indeed has wide range of applications including in mathematics as well as in various fields of the modern sciences, such as bio-engineering [1,2], biological membranes [3,4], medicine [5], geophysics [6], demography [7], economy [8], and physics [9]. The field of fractional calculus was established in order to solve differential equations with fractional order derivatives. The solutions to these problems of engineering disciplines have motivated many mathematicians to work in this new field of research. Classical derivations cannot properly model a lot of practical problems. Fractional integral and derivative operators provide solutions that, in terms of application domains, are very ideal for problems discovered in everyday life. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex on the interval I of a real number and $\Omega_1, \Omega_2 \in I$ with $\Omega_1 < \Omega_2$. The Hermite–Hadamard inequality is crucial to the study of convex functions in numerous fields of mathematics and its applications, the original version is given below [10].

$$f\left(\frac{\Omega_1+\Omega_2}{2}\right) \leq \frac{1}{\Omega_2-\Omega_1}\int_{\Omega_1}^{\Omega_2} f(x)dx \leq \frac{f(\Omega_1)+f(\Omega_2)}{2}$$

Mathematical inequalities give error bounds and uniqueness of solutions to boundary value problems as the basis of computational methods. Mathematicians have turned to more generalized and advanced inequalities as a result of their extensive application in both the field of mathematics and other present domains of research [11,12]. With enormous popularity, the Caputo and Riemann–Liouville derivatives were widely employed in physics, biology, engineering, and various other kinds of domains to show complex dynamics [13]. It is well-known that memory-impacting systems often play a role in natural events. Therefore, we continually look for the best nonlocal model with each kind of data. The use of innovative fractional operators with distinct local and nonlocal kernels has also been studied by other authors [14–16]. They have several fractional operators that distinguish the operators from one another has different singularity and locality properties, the kernel expression of the operator is demonstrated by functions like the power law, the exponential function, or a Mittag–Leffler function. For more literature reviews, see [17–20].

Basic and advanced mathematics have recently become more interested in convex analysis, which has been essential to the generalizations and extensions of inequality theory. The theory of convexity presents a really incredible exciting and intriguing field of study. This theory allows us to develop and improve the numerical tools required to explore and investigate complex mathematical subjects. The most intriguing inequality is the Hermite–Hadamard, which is the first outcome of convex mappings. It has a simple geometric explanation and several applications. See these articles [21–23] for more information on the Hermite–Hadamard type inequalities. Budak et al. has established Hermite–Hadamard type inequalities using Riemann–Liouville fractional integrals (see [24]). Set, E. et al. [25] employed the Riemann–Liouville integrals on the Ostrowski type inequality for differentiable mappings. Budak discussed the Ostrowski and Simpson's type inequalities for differentiable convex function by using the extended fractional integrals in [26]. Recently, a few generic and midpoint-shaped fractional inequalities were explored by Hyder et al. (see [27]). Using the well-known fractional operator (Caputo–Fabrizio), Xiaobin wang et al. [28] proved the Hermite–Hadamard type inequalities for modified h-convex functions. Abbasi demonstrated the novel versions of Hermite–Hadamard type inequalities for s-convex functions utilizing the (Caputo–Fabrizio integral operator). In [29], similar inequalities for strongly (s, m)-convex functions were provided by H. Yang in [30]. Akdemir et. al [31] employed the Atangana–Baleanu fractional integral operators for convex and concave functions. For further literature review also see [32–34]. We recall the definitions of R-L fractional operator and Caputo- Fabrizio fractional operator as follows:

Definition 1 ([35])**.** *Let $f \in L_1[\Omega_1, \Omega_2]$. The Riemann–Liouville fractional integrals of order $\alpha \in \mathbb{R}^+$are presented as follows:*

$$\begin{aligned} J^{\alpha}_{\Omega_1^+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_{\Omega_1}^{x} (x-\zeta)^{\alpha-1} f(\zeta) d\zeta, \quad \Omega_1 < x, \\ J^{\alpha}_{\Omega_2^-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_{x}^{\Omega_2} (\zeta - x)^{\alpha-1} f(\zeta) d\zeta, \quad \Omega_2 > x. \end{aligned}$$

Here, $\Gamma(\alpha)$ is the Gamma function and it is defined as follows:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

It is noted that $J^0_{\Omega_1^+} f(x) = J^0_{\Omega_2^-} f(x) = f(x)$.

Definition 2 ([36])**.** *Let $f \in H^1(\Omega_1, \Omega_2)$, $\Omega_1 < \Omega_2$, $\alpha \in [0,1]$. Then, the Caputo–Fabrizio fractional integral are defined by the following:*

$$\begin{aligned} \left({}^{CF}_{\Omega_1} I^{\alpha} f \right)(x) &= \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)} \int_{\Omega_1}^{x} f(\zeta) d\zeta \\ \left({}^{CF} I^{\alpha}_{\Omega_2} f \right)(x) &= \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)} \int_{x}^{\Omega_2} f(\zeta) d\zeta, \end{aligned}$$

where $\beta(\alpha) > 0$ is normalization function that satisfies $\beta(0) = \beta(1) = 1$.

Definition 3 ([36]). *Let $H^1(\Omega_1, \Omega_2)$ be the Sobolev space of order one defined as*

$$H^1(\Omega_1, \Omega_2) = \left\{ f \in L^2(\Omega_1, \Omega_2) : f' \in L^2(\Omega_1, \Omega_2) \right\},$$

where

$$L^2(\Omega_1, \Omega_2) = \left\{ f(z) : \left(\int_{\Omega_1}^{\Omega_2} f^2(z) dz \right)^{\frac{1}{2}} < \infty \right\}.$$

Let $f \in H^1(\Omega_1, \Omega_2)$, $\Omega_1 < \Omega_2$ and $\alpha \in [0,1]$, the nth notion of left derivative in the sense of Caputo–Fabrizio is defined as follows:

$$\left({}^{CFD}_{\Omega_1} D^{\alpha} f \right)(x) = \frac{\beta(\alpha)}{1-\alpha} \int_{\Omega_1}^{x} f'(\zeta) e^{\frac{-\alpha(x-\zeta)}{1-\alpha}} d\zeta,$$

$x > \alpha$ and the associated integral operator is

$$\left({}^{CF}_{\Omega_1} I^{\alpha} f \right)(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)} \int_{\Omega_1}^{x} f(\zeta) d\zeta,$$

where $\beta(\alpha) > 0$ is the normalization function satisfying $\beta(0) = \beta(1) = 1$. For $\alpha = 0, \alpha = 1$, the left derivative is defined as follows, respectively

$$\begin{aligned} \left({}^{CFD}_{\Omega_1} D^{1} f \right)(x) &= f'(x) \\ \left({}^{CFD}_{\Omega_1} D^{0} f \right)(x) &= f(x) - f(\Omega_1). \end{aligned}$$

For the right derivative operator

$$\left({}^{CFD}_{\Omega_2} D^{\alpha} f \right)(x) = \frac{\beta(\alpha)}{1-\alpha} \int_{x}^{\Omega_2} f'(\zeta) e^{\frac{-\alpha(\zeta-x)}{1-\alpha}} d\zeta,$$

$x < \Omega_2$ and the associated integral operator is

$$\left({}^{CF} I^{\alpha}_{\Omega_2} f \right)(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)} \int_{x}^{\Omega_2} f(\zeta) d\zeta.$$

The Simpson's type inequality is important in numerous fields of mathematics. The Simpson type inequality in classical form is defined as follows:

Theorem 1. *Let $f : [\Omega_1, \Omega_2] \to \mathbb{R}$ be a four times continuously differentiable function on (Ω_1, Ω_2) and $\left\| f^{(4)} \right\|_{\infty} = \sup_{x \in (\Omega_1, \Omega_2)} \left| f^{(4)}(x) \right| < \infty$, then we have the following inequality:*

$$\left| \left[\frac{1}{6} f(\Omega_1) + \frac{2}{3} f\left(\frac{\Omega_1 + \Omega_2}{2} \right) + \frac{1}{6} f(\Omega_2) \right] - \frac{1}{\Omega_2 - \Omega_1} \int_{\Omega_1}^{\Omega_2} f(x) dx \right| \leq \frac{(\Omega_2 - \Omega_1)^4}{2880} \left\| f^4 \right\|_{\infty}. \quad (1)$$

Munir et al. [37], proved the Simpson's type inequality using the Caputo–Fabrizio fractional operator as follows:

Lemma 1 ([37]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I^o (interior of I) where $\Omega_1, \Omega_2 \in I$ with $\Omega_1 < \Omega_2$, then the following equality holds:*

$$\left[\frac{f(\Omega_1)}{6} + \frac{4}{6} f\left(\frac{\Omega_1 + \Omega_2}{2} \right) + \frac{f(\Omega_2)}{6} \right]$$

$$
\begin{aligned}
& -\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left[\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k) \\
= & \ (\Omega_2-\Omega_1)\int_0^1 K(\zeta)f'(\zeta\Omega_2+(1-\zeta)\Omega_1)d\zeta,
\end{aligned}
$$

where

$$
K(\zeta)=\begin{cases} \zeta-\frac{1}{6} & \zeta\in\left[0,\frac{1}{2}\right), \\ \zeta-\frac{5}{6} & \zeta\in\left[\frac{1}{2},1\right]. \end{cases}
$$

Theorem 2 ([37]). *Let the condition of Lemma 1 hold. If $|f'|$ is (s,m)-convex on $[\Omega_1,\Omega_2]$, then the following fractional inequality holds:*

$$
\begin{aligned}
& \left|\left[\frac{f(\Omega_1)}{6}+\frac{4}{6}f\left(\frac{\Omega_1+\Omega_2}{2}\right)+\frac{f(\Omega_2)}{6}\right]\right. \\
& \left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left[\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right| \\
\leq & \ (\Omega_2-\Omega_1)\left[\frac{2^{-1-s}\times 3^{-2-s}(1-2^{2+s}\times 3^{1+s}-3^{2+s}+5^{2+s}+2^s\times 3^{1+s}s)}{(s+1)(s+2)}|f'(\Omega_2)|\right. \\
& \left.+m\frac{6^{-2-s}(2(-1+3^{2+s}-5^{2+s}+17\times 6^s)+2^s\times 3^{2+s}s+5\times 6^s\alpha^2)}{(s+1)(s+2)}|f'(\Omega_1)|\right].
\end{aligned}
$$

Proposition 1. *Let the condition of Theorem 2 hold. If we take $s=m=1$, then we have;*

$$
\begin{aligned}
& \left|\frac{1}{6}f(\Omega_1)++\frac{4}{6}f\left(\frac{\Omega_1+\Omega_2}{2}\right)+\frac{1}{6}f(\Omega_2)-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)\right| \\
\leq & \ \frac{5(\Omega_2-\Omega_1)}{72}(|f'(\Omega_1)|+|f'(\Omega_2)|). \qquad (2)
\end{aligned}
$$

Gürbüz et al. [38], showed related recent developments using the well-known operator Caputo–Fabrizio for the Hermite–Hadamard inequality.

Theorem 3 ([38]). *Let $f:[\Omega_1,\Omega_2]\to\mathbb{R}$ be a differentiable function on I^o and $|f'|$ is convex function on $[\Omega_1,\Omega_2]$, where $\Omega_1,\Omega_2\in I$ with $\Omega_1<\Omega_2$. If $f'\in L_1[\Omega_1,\Omega_2]$ and $\alpha\in[0,1]$, then the inequality as follows:*

$$
\begin{aligned}
& \left|\frac{f(\Omega_1)+f(\Omega_2)}{2}-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\alpha(\Omega_2-\Omega_1)}f(k)\right| \\
\leq & \ \frac{(\Omega_2-\Omega_1)}{8}(|f'(\Omega_1)|+|f'(\Omega_2)|). \qquad (3)
\end{aligned}
$$

Theorem 4. *[38] Let $f:[\Omega_1,\Omega_2]\to\mathbb{R}$ be a differentiable function on I^o and let $|f'|$ be a convex function on $[\Omega_1,\Omega_2]$, where $\Omega_1,\Omega_2\in I$ with $\Omega_1<\Omega_2$. If $f'\in L_1[\Omega_1,\Omega_2]$ and $\alpha\in[0,1]$, then we have the following inequality:*

$$
\begin{aligned}
& \left|\frac{f(\Omega_1)+f(\Omega_2)}{2}-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\alpha(\Omega_2-\Omega_1)}f(k)\right| \\
\leq & \ \frac{(\Omega_2-\Omega_1)}{8}(|f'(\Omega_1)|+|f'(\Omega_2)|). \qquad (4)
\end{aligned}
$$

Sahoo et al. [39], obtained new error bounds for the midpoint inequality for convex function as follows:

Theorem 5 ([39])**.** *Let $f : [\Omega_1, \Omega_2] \to \mathbb{R}$ be a differentiable function on I^o and let $|f'|$ be a convex function on $[\Omega_1, \Omega_2]$, where $\Omega_1, \Omega_2 \in I$ with $\Omega_1 < \Omega_2$. If $f' \in L_1[\Omega_1, \Omega_2]$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\frac{\Omega_1+\Omega_2}{2}^-}I^\alpha f\right)(k)+\left({}^{CF}I^\alpha_{\frac{\Omega_1+\Omega_2}{2}^+}f\right)(k)\right)-f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right.\\&\left.-\frac{(1-\alpha)}{\beta(\alpha)}(f(\Omega_1)+f(\Omega_2))\right|\le\frac{(\Omega_2-\Omega_1)}{8}\left(|f'(\Omega_1)|+|f'(\Omega_2)|\right).\end{aligned}$$

Theorem 6 ([39])**.** *Let $f : [\Omega_1, \Omega_2] \to \mathbb{R}$ be a differentiable function on I^o and let $|f'|$ be a concave function on $[\Omega_1, \Omega_2]$, where $\Omega_1, \Omega_2 \in I$ with $\Omega_1 < \Omega_2$. If $f' \in L_1[\Omega_1, \Omega_2]$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\frac{\Omega_1+\Omega_2}{2}^-}I^\alpha f\right)(k)+\left({}^{CF}I^\alpha_{\frac{\Omega_1+\Omega_2}{2}^+}f\right)(k)\right)\right.\\&\left.-f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{(1-\alpha)}{\beta(\alpha)}(f(\Omega_1)+f(\Omega_2))\right|\\\le\;&\frac{\Omega_2-\Omega_1}{8}\left[\left|f'\left(\frac{\Omega_1+2\Omega_2}{3}\right)\right|+\left|f'\left(\frac{2\Omega_1+\Omega_2}{3}\right)\right|\right].\end{aligned}\tag{5}$$

There are two sections of the current study. The first part is connected to the body of the introduction, basic concepts and theories that operate as the base for each section. The auxiliary outcomes are obtained for the well-known operator (Caputo–Fabrizio) and shown in the second section. Additionally, the main intention in this article is to develop the Hermite–Hadamard and Simpson's type integral inequalities for convex and concave functions employing the fractional operator. Furthermore, we discuss the special cases of our results compared with similar results that have been identified in the literature. Finally, we discuss the applications to special means, matrix inequalities, and the q-digamma function. We hope that the investigation will be able to inspire more research in this field.

2. Main Results

In this section, we have established a novel Caputo–Fabrizio fractional integral identity that will act as an auxiliary result.

Lemma 2. *Let $f : [\Omega_1, \Omega_2] \to \mathbb{R}$ be a differentiable function on (Ω_1, Ω_2). If $f' \in L[\Omega_1, \Omega_2]$ with $\alpha \in [0, 1]$ and $\vartheta, \psi \in (0, 1]$, then the following equality holds:*

$$\begin{aligned}&\frac{\Omega_2-\Omega_1}{8}\left[\int_0^1[(1-\zeta)-\psi]f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta+\int_0^1[\vartheta-(1-\zeta)]f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\right.\\&\left.+\int_0^1(\psi-\zeta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta+\int_0^1(\zeta-\vartheta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\right]\\=\;&\frac{(\vartheta+\psi)}{2}\left(\frac{f(\Omega_1)+f(\Omega_2)}{2}\right)+\frac{(2-\vartheta-\psi)}{2}f\left(\frac{\Omega_1+\Omega_2}{2}\right)\\&-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^\alpha f\right)(k)+\left({}^{CF}I^\alpha_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k),\end{aligned}$$

where $\beta(\alpha)$ is a normalization function.

Proof. Let

$$\int_0^1[(1-\zeta)-\psi]f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta+\int_0^1[\vartheta-(1-\zeta)]f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta$$

$$
\begin{aligned}
&+\int_0^1 (\psi-\zeta) f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta+\int_0^1(\zeta-\vartheta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\\
=\;& I_1+I_2+I_3+I_4.
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
I_1 &= \int_0^1 [(1-\zeta)-\psi] f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\\
&= \frac{-2((1-\zeta)-\psi)f\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)}{\Omega_2-\Omega_1}\Bigg|_0^1-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\\
&= \frac{2\psi}{\Omega_2-\Omega_1}f(\Omega_1)+\frac{2(1-\psi)}{\Omega_2-\Omega_1}f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\\
&= \frac{2\psi}{\Omega_2-\Omega_1}f(\Omega_1)+\frac{2(1-\psi)}{\Omega_2-\Omega_1}f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{4}{(\Omega_2-\Omega_1)^2}\int_{\Omega_1}^{\frac{\Omega_1+\Omega_2}{2}} f(u)du. \qquad (6)
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
I_2 &= \int_0^1 [\vartheta-(1-\zeta)] f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\\
&= \frac{2(\vartheta-(1-\zeta))}{\Omega_2-\Omega_1}f\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\Bigg|_0^1-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\\
&= \frac{2\vartheta}{\Omega_2-\Omega_1}f(\Omega_2)+\frac{2}{\Omega_2-\Omega_1}(1-\vartheta)f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\\
&= \frac{2\vartheta}{\Omega_2-\Omega_1}f(\Omega_2)+\frac{2}{\Omega_2-\Omega_1}(1-\vartheta)f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{4}{(\Omega_2-\Omega_1)^2}\int_{\frac{\Omega_1+\Omega_2}{2}}^{\Omega_2} f(u)du, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_0^1 (\psi-\zeta) f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta\\
&= \frac{-2(\psi-\zeta)}{\Omega_2-\Omega_1}f\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)\Bigg|_0^1-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta\\
&= \frac{-2(\psi-1)}{\Omega_2-\Omega_1}f\left(\frac{\Omega_1+\Omega_2}{2}\right)+\frac{2\psi}{\Omega_2-\Omega_1}f(\Omega_2)-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta\\
&= \frac{-2(\psi-1)}{\Omega_2-\Omega_1}f\left(\frac{\Omega_1+\Omega_2}{2}\right)+\frac{2\psi}{\Omega_2-\Omega_1}f(\Omega_2)-\frac{4}{(\Omega_2-\Omega_1)^2}\int_{\frac{\Omega_1+\Omega_2}{2}}^{\Omega_2} f(u)du, \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1 (\zeta-\vartheta) f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\\
&= \frac{2(\zeta-\vartheta)}{\Omega_2-\Omega_1}f\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)\Bigg|_0^1-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\\
&= \frac{2(1-\vartheta)}{\Omega_2-\Omega_1}f\left(\frac{\Omega_1+\Omega_2}{2}\right)+\frac{2\vartheta}{\Omega_2-\Omega_1}f(\Omega_2)-\frac{2}{\Omega_2-\Omega_1}\int_0^1 f\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\\
&= \frac{2(1-\vartheta)}{\Omega_2-\Omega_1}f\left(\frac{\Omega_1+\Omega_2}{2}\right)+\frac{2\vartheta}{\Omega_2-\Omega_1}f(\Omega_2)-\frac{4}{(\Omega_2-\Omega_1)^2}\int_{\Omega_1}^{\frac{\Omega_1+\Omega_2}{2}} f(u)du. \qquad (9)
\end{aligned}
$$

Adding the equalities (6)–(9), we obtain

$$
\begin{aligned}
&\int_0^1 [(1-\zeta)-\psi] f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta+\int_0^1[\vartheta-(1-\zeta)]f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta \\
&+\int_0^1(\psi-\zeta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta+\int_0^1(\zeta-\vartheta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta \\
= &\frac{2}{\Omega_2-\Omega_1}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{4}{\Omega_2-\Omega_1}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right) \\
&-\frac{8}{(\Omega_2-\Omega_1)^2}\int_{\Omega_1}^{\Omega_2} f(u)du. \qquad (10)
\end{aligned}
$$

Multiply the equality (10) with $\frac{\alpha(\Omega_2-\Omega_1)^2}{8\beta(\alpha)}$ and subtracting $\frac{2(1-\alpha)}{\beta(\alpha)}f(k)$, we obtain

$$
\begin{aligned}
&\frac{\alpha(\Omega_2-\Omega_1)^2}{8\beta(\alpha)}\left[\int_0^1 [(1-\zeta)-\psi] f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta+\int_0^1[\vartheta-(1-\zeta)]f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\right. \\
&\left.+\int_0^1(\psi-\zeta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta+\int_0^1(\zeta-\vartheta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\right]-\frac{2(1-\alpha)}{\beta(\alpha)}f(k) \\
= &\frac{2}{\Omega_2-\Omega_1}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))\frac{\alpha(\Omega_2-\Omega_1)^2}{8\beta(\alpha)}+\frac{4}{\Omega_2-\Omega_1}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\frac{\alpha(\Omega_2-\Omega_1)^2}{8\beta(\alpha)} \\
&-\frac{8}{(\Omega_2-\Omega_1)^2}\int_{\Omega_1}^{\Omega_2} f(u)du\times\frac{\alpha(\Omega_2-\Omega_1)^2}{8\beta(\alpha)}-\frac{2(1-\alpha)}{\beta(\alpha)}f(k) \\
= &\frac{\alpha(\Omega_2-\Omega_1)}{4\beta(\alpha)}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{\alpha(\Omega_2-\Omega_1)}{2\beta(\alpha)}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right) \\
&-\left(\frac{\alpha}{\beta(\alpha)}\int_{\Omega_1}^{k} f(u)du+\frac{(1-\alpha)}{\beta(\alpha)}f(k)+\frac{\alpha}{\beta(\alpha)}\int_{k}^{\Omega_2} f(u)du+\frac{(1-\alpha)}{\beta(\alpha)}f(k)\right) \\
= &\frac{\alpha(\Omega_2-\Omega_1)}{4\beta(\alpha)}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{\alpha(\Omega_2-\Omega_1)}{2\beta(\alpha)}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right) \\
&-\left(\left({}_{\Omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right). \\
&\frac{\Omega_2-\Omega_1}{8}\left[\int_0^1 [(1-\zeta)-\psi] f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta+\int_0^1[\vartheta-(1-\zeta)]f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)d\zeta\right. \\
&\left.+\int_0^1(\psi-\zeta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)d\zeta+\int_0^1(\zeta-\vartheta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)d\zeta\right] \\
= &\frac{(\psi+\vartheta)}{2}\left(\frac{f(\Omega_1)+f(\Omega_2)}{2}\right)+\frac{(2-\vartheta-\psi)}{2}f\left(\frac{\Omega_1+\Omega_2}{2}\right) \\
&-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}_{\Omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k).
\end{aligned}
$$

The proof of Lemma 2 is completed. □

Theorem 7. *Let the conditions of Lemma 2 hold. If $f' \in L^1[\Omega_1,\Omega_2]$ and $|f'|$ is a convex on $[\Omega_1,\Omega_2]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}_{\Omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right| \\
\leq &\frac{(\Omega_2-\Omega_1)}{8}\left[\frac{1}{6}\left(2-3\psi+6\psi^2-2\psi^3-3\vartheta+6\vartheta^2-2\vartheta^3\right)\left(|f'(\Omega_1)|+|f'(\Omega_2)|\right)\right. \\
&\left.+\frac{1}{6}\left(4-3\psi+2\psi^3-3\vartheta+2\vartheta^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|\right].
\end{aligned}
$$

Proof. By using Lemma 2 and convexity, we obtain

$$
\left|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right.
$$

$$
\begin{aligned}
& -\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\Big| \\
= & \frac{(\Omega_2-\Omega_1)}{8}\left[\left|\int_0^1[(1-\zeta)-\psi]f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|d\zeta\right. \\
& +\left|\int_0^1[\vartheta-(1-\zeta)]f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|d\zeta \\
& \left.+\left|\int_0^1(\psi-\zeta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)\right|d\zeta+\left|\int_0^1(\zeta-\vartheta)f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)\right|d\zeta\right] \\
\leq & \frac{(\Omega_2-\Omega_1)}{8}\left[\int_0^1|[(1-\zeta)-\psi]|\left|f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|d\zeta\right. \\
& +\int_0^1|[\vartheta-(1-\zeta)]|\left|f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|d\zeta \\
& \left.+\int_0^1|(\psi-\zeta)|\left|f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)\right|d\zeta+\int_0^1|(\zeta-\vartheta)|\left|f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)\right|d\zeta\right] \\
\leq & \frac{(\Omega_2-\Omega_1)}{8}\left[\int_0^1|[(1-\zeta)-\psi]|\left(\zeta|f'(\Omega_1)|+(1-\zeta)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|\right)d\zeta\right. \\
& +\int_0^1|[\vartheta-(1-\zeta)]|\left(\zeta|f'(\Omega_2)|+(1-\zeta)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|\right)d\zeta \\
& +\int_0^1|(\psi-\zeta)|\left(\zeta\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|+(1-\zeta)|f'(\Omega_2)|\right)d\zeta \\
& \left.+\int_0^1|(\zeta-\vartheta)|\left(\zeta\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|+(1-\zeta)|f'(\Omega_1)|d\zeta\right)\right] \\
\leq & \frac{(\Omega_2-\Omega_1)}{8}\left[\frac{1}{6}\left(1-3\psi+6\psi^2-2\psi^3\right)|f'(\Omega_1)|+\frac{1}{6}\left(2-3\psi+2\psi^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|\right. \\
& \frac{1}{6}\left(1-3\vartheta+6\vartheta^2-2\vartheta^3\right)|f'(\Omega_2)|+\frac{1}{6}\left(2-3\vartheta+2\vartheta^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right| \\
& \frac{1}{6}\left(2-3\psi+2\psi^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|+\frac{1}{6}\left(1-3\psi+6\psi^2-2\psi^3\right)|f'(\Omega_2)| \\
& \left.\frac{1}{6}\left(2-3\vartheta+2\vartheta^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|+\frac{1}{6}\left(1-3\vartheta+6\vartheta^2-2\vartheta^3\right)|f'(\Omega_1)|\right] \\
\leq & \frac{(\Omega_2-\Omega_1)}{8}\left[\frac{1}{6}\left(2-3\psi+6\psi^2-2\psi^3-3\vartheta+6\vartheta^2-2\vartheta^3\right)(|f'(\Omega_1)|+|f'(\Omega_2)|)\right. \\
& \left.+\frac{1}{6}\left(4-3\psi+2\psi^3-3\vartheta+2\vartheta^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|\right].
\end{aligned}
$$

This completes the proof. □

Corollary 1. *We use the further convexity of $|f'|$ in Theorem 7, we have*

$$
\begin{aligned}
& \left|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right. \\
& \left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right| \\
\leq & \frac{(\Omega_2-\Omega_1)}{8}\left[1-\vartheta+\vartheta^2-\psi+\psi^2\right]\left[|f'(\Omega_1)|+|f'(\Omega_2)|\right].
\end{aligned}
$$

Remark 1. *If we choose $\psi=\vartheta=1$ in Corollary 1, then we have*

$$
\begin{aligned}
& \left|\frac{f(\Omega_1)+f(\Omega_2)}{2}-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right| \\
\leq & \frac{(\Omega_2-\Omega_1)}{8}\left[|f'(\Omega_1)|+|f'(\Omega_2)|\right],
\end{aligned}
$$

which was proved by Gürbüz et al. in [38].

Remark 2. *If we choose $\psi=\vartheta=0$ in Corollary 1, then we have*

$$
\left|f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|
$$

$$\leq \frac{(\Omega_2-\Omega_1)}{8}\left[\left|f'(\Omega_1)\right|+\left|f'(\Omega_2)\right|\right].$$

which was proved by Sahoo et al. in [39].

Remark 3. *If we choose $\psi=\vartheta=\frac{1}{3}$ in Corollary 1, then we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(\Omega_1)++4f\left(\frac{\Omega_1+\Omega_2}{2}\right)+f(\Omega_2)\right]\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\ \leq\;&\frac{5(\Omega_2-\Omega_1)}{72}\left[\left|f'(\Omega_1)\right|+\left|f'(\Omega_2)\right|\right].\end{aligned}$$

which was proved by Munir et al. in [37].

Corollary 2. *If we choose $\psi=\vartheta=\frac{1}{2}$ in Corollary 1, then we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(\Omega_1)+f(\Omega_2)}{2}+f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right]\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\ \leq\;&\frac{(\Omega_2-\Omega_1)}{16}\left[\left|f'(\Omega_1)\right|+\left|f'(\Omega_2)\right|\right].\end{aligned}$$

Corollary 3. *If we choose $\psi=\vartheta=\frac{2}{3}$ in Corollary 1, then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[f(\Omega_1)+f(\Omega_2)+f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right]\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\ \leq\;&\frac{5(\Omega_2-\Omega_1)}{72}\left[\left|f'(\Omega_1)\right|+\left|f'(\Omega_2)\right|\right].\end{aligned}$$

Theorem 8. *Let the conditions of Lemma 2 hold. If $f'\in L[\Omega_1,\Omega_2]$ and $|f'|^q$, $q\geq 1$, is a convex on $[\Omega_1,\Omega_2]$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\ \leq\;&\frac{(\Omega_2-\Omega_1)}{8}\left[\frac{1}{2}\left(1-2\vartheta+2\vartheta^2\right)+\frac{1}{2}\left(1-2\psi+2\psi^2\right)\right]^{1-\frac{1}{q}}\\&\times\left[\left\{\frac{1}{6}\left(2-3\vartheta+6\vartheta^2-2\vartheta^3-3\psi+6\psi^2-2\psi^3\right)\left|f'(\Omega_1)\right|^q\right.\right.\\&\left.\frac{1}{6}\left(4-3\vartheta+2\vartheta^3-3\psi+2\psi^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right\}\\&+\left\{\frac{1}{6}\left(2-3\vartheta+6\vartheta^2-2\vartheta^3-3\psi+6\psi^2-2\psi^3\right)\left|f'(\Omega_2)\right|^q\right.\\&\left.\left.\frac{1}{6}\left(4-3\vartheta+2\vartheta^3-3\psi+2\psi^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right\}\right].\end{aligned}$$

Proof. By using Lemma 2 and power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}_{\Omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\ & \frac{(\Omega_2-\Omega_1)}{8}\left[\left(\int_0^1|(1-\zeta)-\psi|d\zeta\right)^{1-\frac{1}{q}}\right.\\
&\left\{\int_0^1|(1-\zeta)-\psi|\left|f'\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|^q d\zeta\right\}^{\frac{1}{q}}\\
&+\left(\int_0^1|\vartheta-(1-\zeta)|d\zeta\right)^{1-\frac{1}{q}}\left\{\int_0^1|\vartheta-(1-\zeta)|\left|f'\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|^q d\zeta\right\}^{\frac{1}{q}}\\
&+\left(\int_0^1|\psi-\zeta|d\zeta\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1|\psi-\zeta|\right)\left|f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)\right|^q d\zeta\right\}^{\frac{1}{q}}\\
&\left.+\left(\int_0^1|\zeta-\vartheta|d\zeta\right)^{1-\frac{1}{q}}\left\{\int_0^1|\zeta-\vartheta|\left|f'\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)\right|^q\right\}^{\frac{1}{q}}d\zeta\right]\\
\leq\ & \frac{(\Omega_2-\Omega_1)}{8}\left[\left(\int_0^1|(1-\zeta)-\psi|d\zeta\right)^{1-\frac{1}{q}}\right.\\
&\left\{\int_0^1|(1-\zeta)-\psi|\left(\zeta|f'(\Omega_1)|^q+(1-\zeta)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right)d\zeta\right\}^{\frac{1}{q}}\\
&+\left(\int_0^1|\psi-\zeta|d\zeta\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1|\psi-\zeta|\right)\left(\zeta|f'(\Omega_2)|^q+(1-\zeta)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right)d\zeta\right\}^{\frac{1}{q}}\\
&+\left(\int_0^1|\psi-\zeta|d\zeta\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1|\psi-\zeta|\right)\left(\zeta\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q+(1-\zeta)|f'(\Omega_2)|^q\right)d\zeta\right\}^{\frac{1}{q}}\\
&\left.+\left(\int_0^1|\zeta-\vartheta|d\zeta\right)^{1-\frac{1}{q}}\left\{\int_0^1|\zeta-\vartheta|\left(\zeta\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q+(1-\zeta)|f'(\Omega_1)|^q\right)\right\}^{\frac{1}{q}}d\zeta\right]\\
\leq\ & \frac{(\Omega_2-\Omega_1)}{8}\left[\frac{1}{2}\left(1-2\vartheta+2\vartheta^2\right)+\frac{1}{2}\left(1-2\psi+2\psi^2\right)\right]^{1-\frac{1}{q}}\\
&\left[\left\{\frac{1}{6}\left(2-3\vartheta+6\vartheta^2-2\vartheta^3-3\psi+6\psi^2-2\psi^3\right)|f'(\Omega_1)|^q\right.\right.\\
&\left.\frac{1}{6}\left(4-3\vartheta+2\vartheta^3-3\psi+2\psi^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right\}\\
&+\left\{\frac{1}{6}\left(2-3\vartheta+6\vartheta^2-2\vartheta^3-3\psi+6\psi^2-2\psi^3\right)|f'(\Omega_2)|^q\right.\\
&\left.\left.\frac{1}{6}\left(4-3\vartheta+2\vartheta^3-3\psi+2\psi^3\right)\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right\}\right].
\end{aligned}$$

This completes the proof. □

Corollary 4. *If we choose $\psi=\vartheta=1$ in Theorem 8, then we have*

$$\left|\frac{f(\Omega_1)+f(\Omega_2)}{2}-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}_{\Omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$

$$\leq \frac{(\Omega_2-\Omega_1)}{8}\left[\left(\frac{2}{3}|f'(\Omega_1)|^q+\frac{1}{3}\left|f'\left(\frac{\Omega_1\,+\,\Omega_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{2}{3}|f'(\Omega_2)|^q+\frac{1}{3}\left|f'\left(\frac{\Omega_1\,+\,\Omega_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5. *If we choose $\psi=\vartheta=0$ in Theorem 8, then we have*

$$\left|f\left(\frac{\Omega_1+\Omega_2}{2}\right)-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^\alpha f\right)(k)\;+\;\left({}^{CF}I^\alpha_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq \frac{(\Omega_2-\Omega_1)}{8}\left[\left(\frac{1}{3}|f'(\Omega_1)|^q\;+\;\frac{2}{3}\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}|f'(\Omega_2)|^q\;+\;\frac{2}{3}\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 6. *If we choose $\psi=\vartheta=\frac{1}{3}$ in Theorem 8, then we have*

$$\left|\frac{1}{6}\left[f(\Omega_1)++4f\left(\frac{\Omega_1+\Omega_2}{2}\right)+f(\Omega_2)\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^\alpha f\right)(k)\;+\;\left({}^{CF}I^\alpha_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq \frac{5(\Omega_2-\Omega_1)}{72}\left[\left(\frac{61|f'(\Omega_1)|^q+29|f'(\Omega_2)|^q}{90}\right)^{\frac{1}{q}}+\left(\frac{29|f'(\Omega_1)|^q+61|f'(\Omega_2)|^q}{90}\right)^{\frac{1}{q}}\right].$$

Corollary 7. *If we choose $\psi=\vartheta=\frac{1}{2}$ in Theorem 8, then we have*

$$\left|\frac{1}{2}\left[\frac{f(\Omega_1)+f(\Omega_2)}{2}+f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^\alpha f\right)(k)+\left({}^{CF}I^\alpha_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq \frac{(\Omega_2-\Omega_1)}{16}\left[\left(\frac{3|f'(\Omega_1)|^q+|f'(\Omega_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(\Omega_1)|^q+3|f'(\Omega_2)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Corollary 8. *If we choose $\psi=\vartheta=\frac{2}{3}$ in Theorem 8, then we have*

$$\left|\frac{1}{3}\left[f(\Omega_1)+f(\Omega_2)+f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^\alpha f\right)(k)+\left({}^{CF}I^\alpha_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq \frac{5(\Omega_2-\Omega_1)}{72}\left[\left(\frac{37|f'(\Omega_1)|^q+8|f'(\Omega_2)|^q}{45}\right)^{\frac{1}{q}}+\left(\frac{8|f'(\Omega_1)|^q+37|f'(\Omega_2)|^q}{45}\right)^{\frac{1}{q}}\right].$$

Theorem 9. *Let the conditions of Lemma 2 hold. If $|f'|^q$ is a concave function on $[\Omega_1,\Omega_2]$, then we have the following inequality:*

$$\left|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right)\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^\alpha f\right)(k)+\left({}^{CF}I^\alpha_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq \frac{(\Omega_2-\Omega_1)}{8}\left[\left\{I_{19}\left|f'\left(\frac{I_{11}(\Omega_1)+I_{12}\left(\frac{\Omega_1+\Omega_2}{2}\right)}{I_{19}}\right)\right|+I_{20}\left|f'\left(\frac{I_{13}(\Omega_2)+I_{14}\left(\frac{\Omega_1+\Omega_2}{2}\right)}{I_{20}}\right)\right|\right\}\right.$$

$$+\left\{I_{21}\left|f'\left(\frac{I_{15}\left(\frac{\Omega_1+\Omega_2}{2}\right)+I_{16}(\Omega_2)}{I_{21}}\right)\right|+I_{22}\left|f'\left(\frac{I_{17}\left(\frac{\Omega_1+\Omega_2}{2}\right)+I_{18}(\Omega_1)}{I_{22}}\right)\right|\right\}\Bigg],$$

where

$$\begin{aligned}
I_{11} &= \int_0^1 |(1-\zeta)-\psi|\zeta d\zeta = \frac{1}{6}\left(1-3\psi+6\psi^2-2\psi^3\right),\ I_{12} = \int_0^1 |(1-\zeta)-\psi|(1-\zeta)d\zeta = \frac{1}{6}\left(2-3\psi+2\psi^3\right) \\
I_{13} &= \int_0^1 |\vartheta-(1-\zeta)|\zeta d\zeta = \frac{1}{6}\left(1-3\vartheta+6\vartheta^2-2\vartheta^3\right),\ I_{14} = \int_0^1 |\vartheta-(1-\zeta)|(1-\zeta)d\zeta = \frac{1}{6}\left(2-3\vartheta+2\vartheta^3\right) \\
I_{15} &= \int_0^1 |\psi-\zeta|\zeta d\zeta = \frac{1}{6}\left(2-3\psi+2\psi^3\right), \quad I_{16} = \int_0^1 |\psi-\zeta|(1-\zeta)d\zeta = \frac{1}{6}\left(1-3\psi+6\psi^2-2\psi^3\right) \\
I_{17} &= \int_0^1 |\zeta-\vartheta|\zeta d\zeta = \frac{1}{6}\left(2-3\vartheta+2\vartheta^3\right), \quad I_{18} = \int_0^1 |\zeta-\vartheta|(1-\zeta)d\zeta = \frac{1}{6}\left(1-3\vartheta+6\vartheta^2-2\vartheta^3\right) \\
I_{19} &= \int_0^1 |(1-\zeta)-\psi|d\zeta = \frac{1}{2}\left(1-2\vartheta+2\vartheta^2\right),\ I_{20} = \int_0^1 |\vartheta-(1-\zeta)|d\zeta = \frac{1}{2}\left(1-2\psi+2\psi^2\right) \\
I_{21} &= \int_0^1 |\psi-\zeta|d\zeta = \frac{1}{2}\left(1-2\psi+2\psi^2\right), \quad I_{21} = \int_0^1 |\zeta-\vartheta|d\zeta = \frac{1}{2}\left(1-2\vartheta+2\vartheta^2\right).
\end{aligned}$$

Proof. By the concavity and Jensen integral inequality, we obtain

$$\begin{aligned}
&\Bigg|\frac{1}{4}(\psi+\vartheta)(f(\Omega_1)+f(\Omega_2))+\frac{1}{2}(2-\vartheta-\psi)f\left(\frac{\Omega_1+\Omega_2}{2}\right) \\
&-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\Bigg| \\
\leq\ &\frac{(\Omega_2-\Omega_1)}{8}\Bigg[\left(\int_0^1|(1-\zeta)-\psi|d\zeta\right)\left|f'\left(\frac{\int_0^1((1-\zeta)-\psi)\left|\left(\zeta\Omega_1+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|d\zeta}{\int_0^1((1-\zeta)-\psi)d\zeta}\right)\right| \\
&+\left(\int_0^1|\vartheta-(1-\zeta)|d\zeta\right)\left|f'\left(\frac{\int_0^1(\vartheta-(1-\zeta))\left|\left(\zeta\Omega_2+(1-\zeta)\frac{\Omega_1+\Omega_2}{2}\right)\right|d\zeta}{\int_0^1(\vartheta-(1-\zeta))d\zeta}\right)\right| \\
&+\left(\int_0^1|\psi-\zeta|d\zeta\right)\left|f'\left(\frac{\int_0^1(\psi-\zeta)\left|\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_2\right)\right|d\zeta}{\int_0^1(\psi-\zeta)d\zeta}\right)\right| \\
&+\left(\int_0^1|\zeta-\vartheta|d\zeta\right)\left|f'\left(\frac{\int_0^1(\zeta-\vartheta)\left|\left(\zeta\frac{\Omega_1+\Omega_2}{2}+(1-\zeta)\Omega_1\right)\right|d\zeta}{\int_0^1(\zeta-\vartheta)d\zeta}\right)\right|\Bigg] \\
\leq\ &\frac{(\Omega_2-\Omega_1)}{8}\Bigg[\left\{I_{19}\left|f'\left(\frac{I_{11}(\Omega_1)+I_{12}\left(\frac{\Omega_1+\Omega_2}{2}\right)}{I_{19}}\right)\right|+I_{20}\left|f'\left(\frac{I_{13}(\Omega_2)+I_{14}\left(\frac{\Omega_1+\Omega_2}{2}\right)}{I_{20}}\right)\right|\right\} \\
&+\left\{I_{21}\left|f'\left(\frac{I_{15}\left(\frac{\Omega_1+\Omega_2}{2}\right)+I_{16}(\Omega_2)}{I_{21}}\right)\right|+I_{22}\left|f'\left(\frac{I_{17}\left(\frac{\Omega_1+\Omega_2}{2}\right)+I_{18}(\Omega_1)}{I_{22}}\right)\right|\right\}\Bigg].
\end{aligned}$$

This completes the proof. □

Remark 4. *If we choose $\psi=\vartheta=\frac{1}{3}$ in Theorem 9, then we have*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(\Omega_1)++4f\left(\frac{\Omega_1+\Omega_2}{2}\right)+f(\Omega_2)\right] \\
&-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left(\left({}^{CF}_{\Omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\Omega_2}f\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\Bigg|
\end{aligned}$$

$$\leq \frac{5(\Omega_2-\Omega_1)}{72}\left[\left|f'\left(\frac{16\Omega_1+29\Omega_2}{45}\right)\right|+\left|f'\left(\frac{29\Omega_1+16\Omega_2}{45}\right)\right|\right].$$

which was proved by Munir et al. in [37].

3. Applications to Matrix Inequalities

Example 1. *We denote the set of all $n\times n$ complex matrices by C^n, and we denote M_n to be the algebra of all $n\times n$ complex matrices, and by M_n^+, we mean the strictly positive matrices in M_n. That is, $\Omega_1\in M_n^+$ if $\langle A_{\Omega_1},\Omega_1\rangle>0$ for all nonzero $\Omega_1\in C^n$. In [40], Sababheh proved that the function $\psi(\theta)=\left\|A^\theta XB^{1-\theta}+A^{1-\theta}XB^\theta\right\|$, $\Omega_1,\Omega_2\in M_n^+$, $X\in M_n$ is convex for all $\theta\in[0,1]$. Then by using Corollary 2, we have*

$$\begin{aligned}&\left\|A^{\frac{\Omega_1+\Omega_2}{2}}XB^{1-\frac{\Omega_1+\Omega_2}{2}}+A^{1-\frac{\Omega_1+\Omega_2}{2}}XB^{\frac{\Omega_1+\Omega_2}{2}}\right\|-\frac{\beta(\alpha)}{\alpha(\Omega_2-\Omega_1)}\left[{}^{CF}_{\Omega_1}I^{\alpha}\left\|A^kXB^{1-k}+A^{1-k}XB^k\right\|\right.\\&+{}^{CF}I^{\alpha}_{\Omega_2}\left\|A^kXB^{1-k}+A^{1-k}XB^k\right\|\Big\}\Big]+\frac{2(1-\alpha)}{\alpha(\Omega_2-\Omega_1)}\left\|A^kXB^{1-k}+A^{1-k}XB^k\right\|\\ \leq\ &\frac{(\Omega_2-\Omega_1)}{8}\Big[\Big\{\left\|A^{\Omega_1}XB^{1-\Omega_1}+A^{1-\Omega_1}XB^{\Omega_1}\right\|\\&\Big|+\left\|A^{\Omega_2}XB^{1-\Omega_2}+A^{1-\Omega_2}XB^{\Omega_2}\right\|\Big\}\Big].\end{aligned}$$

Applications to Special Means

We recall some special means

(1) The arithmetic mean:

$$A(\Omega_1,\Omega_2)=\frac{\Omega_1\ +\ \Omega_2}{2}\qquad \Omega_1,\Omega_2\in\mathbb{R}.$$

(2) The logarithmic mean:

$$L(\Omega_1,\Omega_2)=\frac{\Omega_2-\Omega_1}{\ln|\Omega_2|-\ln|\Omega_1|}\qquad |\Omega_1|\neq|\Omega_2|\,\Omega_1,\Omega_2\neq0\,\Omega_1,\Omega_2\in\mathbb{R}.$$

(3) The generalized logarithmic mean:

$$L_r(\Omega_1,\Omega_2)=\left[\frac{(\Omega_2)^{r+1}\ -\ (\Omega_1)^{r+1}}{(r+1)(\Omega_2-\Omega_1)}\right]^{\frac{1}{r}},\quad r\in\mathbb{R}\backslash\{-1,0\}\,\Omega_1,\Omega_2>0.$$

Proposition 2. *Let $q\geq1$ and $\Omega_1,\Omega_2\in\mathbb{R}$ such that $0<\Omega_1<\Omega_2$, then the following inequality holds:*

$$\begin{aligned}&|A(\Omega_1,\Omega_2)-L_r^r(\Omega_1,\Omega_2)|\\ \leq\ &\frac{r(\Omega_2-\Omega_1)}{8}\left[\left(\frac{2}{3}|\Omega_1|^{q(r-1)}+\frac{1}{3}\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^{q(r-1)}\right)^{\frac{1}{q}}+\left(\frac{2}{3}|\Omega_2|^{q(r-1)}+\frac{1}{3}\left|f'\left(\frac{\Omega_1+\Omega_2}{2}\right)\right|^{q(r-1)}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The assertion follows from Corollary 4 for the function $f(x)=x^r$, $\alpha=1$, $\beta(0)=\beta(1)=1$. □

Proposition 3. *Let $q\geq1$ and $\Omega_1,\Omega_2\in\mathbb{R}$ such that $0<\Omega_1<\Omega_2$, then the following inequality holds:*

$$\begin{aligned}&\left|A^{-1}(\Omega_1,\Omega_2)-L^{-1}(\Omega_1,\Omega_2)\right|\\ \leq\ &\frac{(\Omega_2-\Omega_1)}{8}\left[\left(\frac{1}{3}|\Omega_1|^q+\frac{2}{3}\left|A^{-2}(\Omega_1,\Omega_2)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}|\Omega_2|^q+\frac{2}{3}\left|A^{-2}(\Omega_1,\Omega_2)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The assertion follows from Corollary 5 for the function $f(x) = \frac{1}{x}$, $\alpha = 1$, $\beta(0) = \beta(1) = 1$. □

4. q-Digamma Function

Let $0 < q < 1$, the q-digamma function φ_q, is the q-analog of the digamma function φ are stated as [41]

$$\begin{aligned} \varphi_q &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \\ &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{kx}}{1-q^{kx}} \end{aligned}$$

For $q > 1$ and $x > 0$, q-digamma function φ_q is discussed as

$$\begin{aligned} \varphi_q &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1-q^{-(k+x)}} \right] \\ &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-kx}}{1-q^{-kx}} \right] \end{aligned}$$

Proposition 4. *Let $q \geq 1$ and Ω_1, Ω_2 be a real number such that $0 < \Omega_1 < \Omega_2$, then the following inequality holds:*

$$\begin{aligned} & \left| A(\varphi_q(\Omega_1), \varphi_q(\Omega_2)) - \frac{1}{\Omega_2 - \Omega_1} \int_{\Omega_1}^{\Omega_2} \varphi_q(u) du \right| \\ \leq & \frac{\Omega_2 - \Omega_1}{8} \left[\left(\frac{2}{3} \left| \varphi_q'(\Omega_1) \right|^q + \frac{1}{3} \left| \varphi_q'\left(\frac{\Omega_1 + \Omega_2}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\frac{2}{3} \left| \varphi_q'(\Omega_2) \right|^q + \frac{1}{3} \left| \varphi_q'\left(\frac{\Omega_1 + \Omega_2}{2} \right) \right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The assertion follows from Corollary 4 for the function $f(\varepsilon) = \varphi_q(\varepsilon)$ and $\varepsilon > 0$, $f'(\varepsilon) = \varphi_q'(\varepsilon)$ is convex $(0, +\infty)$, $\alpha = 1$, $\beta(0) = \beta(1) = 1$. □

5. Conclusions and Future Work

Fractional calculus is an interesting subject with many applications in the modeling of natural phenomena. At the moment, we need to strengthen and improve our ability to generalize several recent results directly related to the topic of fractional calculus. Many mathematicians have generalized a variety of fractional operators using the techniques and operators of fractional calculus. Regarding the fractional integral of Caputo–Fabrizio, the current fractional integral has been used to establish a few new Hermite–Hadamard and Simpson's type inequalities for differentiable mapping for convex functions. Additionally, we have discussed some applications to matrix and special means with the help of newly established inequalities. It is a fascinating and new problem that the forthcoming scientists can obtain calculus inequalities for fractional operators in future work. In the future, we intend to generalize the theory of inequality for concepts, such as interval valued analysis, quantum calculus, fuzzy interval-valued calculus, and time-scale calculus.

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