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Study on a Nonlocal Fractional Coupled System Involving (k,ψ)-Hilfer Derivatives and (k,ψ)-Riemann–Liouville Integral Operators

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Abstract: This paper deals with a nonlocal fractional coupled system of (k,ψ)-Hilfer fractional differential equations, which involve, in boundary conditions, (k,ψ)-Hilfer fractional derivatives and (k,ψ)-Riemann–Liouville fractional integrals. The existence and uniqueness of solutions are established for the considered coupled system by using standard tools from fixed point theory. More precisely, Banach and Krasnosel'skiĭ's fixed-point theorems are used, along with Leray–Schauder alternative. The obtained results are illustrated by constructed numerical examples.

Keywords: Hilfer fractional differential system; fractional integrals; nonlocal boundary conditions; existence and uniqueness; fixed-point theorems

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1. Introduction

Recently, there has been a great interest in fractional differential equations, since fractional order models are more accurate than integral models. For theoretical developments in fractional calculus and differential equations of fractional orders, see the books [1–8], while for an extensive study on fractional boundary value problems, see the monograph [9]. Usually, fractional derivative operators depend on Euler's gamma function and are defined via fractional integral operators. One can find a variety of such operators in the literature, such as Riemann–Liouville, Erdélyi-Kober, Caputo, Hadamard, Katugampola, Hilfer fractional derivatives, etc. In [10], the concept of the Riemann–Liouville fractional integral operator was generalized to a k-Riemann–Liouville fractional integral operator, with the help of the generalized Euler's k gamma function. The k-Riemann–Liouville fractional derivative was introduced in [11]. For some results on the k-Riemann–Liouville fractional derivative, we refer to [12–17] and the references cited therein. The ψ-Riemann–Liouville fractional integral and derivative were introduced in [2]. The (k,ψ)-Riemann–Liouville fractional integral and derivative were defined in [11,18]. The Hilfer fractional derivative defined in [19] extends both Riemann–Liouville and Caputo fractional derivatives. The ψ-Hilfer fractional derivative was defined in [20]. For applications of Hilfer fractional derivatives in mathematics, physics, etc., see [21–26]. For recent results in boundary value problems for fractional differential equations and inclusions with Hilfer fractional derivatives, see the survey paper by Ntouyas [27].

The (k,ψ)-Hilfer fractional derivative operator was introduced recently in [28], in which the authors studied the following (k,ψ)-Hilfer fractional nonlinear initial value problem of the form

$$\begin{cases} {}^{k,H}\mathbb{D}_{c+}^{\alpha,\beta;\psi}\ell(\omega) = f(\omega,\ell(\omega)), \ \ \omega\in(c,d],\ 0<\alpha<k,\ 0\le\beta\le1, \\ {}^{k}I^{k-\theta_k;\varphi}\ell(c) = w_c \in\mathbb{R}. \end{cases} \tag{1}$$

Here, ${}^{k,H}\mathbb{D}^{\alpha,\beta;\psi}$ is the (k,ψ)-Hilfer derivative operator of fractional order α, $0<\alpha\leq 1$, and parameter β, $0\leq\beta\leq 1$, $\theta_k=\alpha+\beta(k-\alpha)$, and $f\in C([c,d]\times\mathbb{R},\mathbb{R})$. By using Banach's fixed-point theorem, the existence of a unique solution was proven. For some recent results on (k,ψ)-Hilfer fractional derivative operators of orders in $(0,1]$, see [29,30] and references cited therein.

Boundary value problems of the (k,ψ)-Hilfer fractional derivative operator of orders in $(1,2]$ were initiated in [31] by studying the problem

$$\begin{cases} {}^{k,H}\mathbb{D}^{\alpha,\beta;\psi}\ell(\omega)=f(\omega,\ell(\omega)), \quad \omega\in(c,d], \\ \ell(c)=0, \quad \ell(d)=\sum_{i=1}^{m}\lambda_i\,\ell(\xi_i), \end{cases} \tag{2}$$

where ${}^{k,H}\mathbb{D}^{\alpha,\beta;\psi}$ is the (k,ψ)-Hilfer fractional derivative of order α, $1<\alpha<2$, and parameter β, $0\leq\beta\leq 1$; $f:[c,d]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $\lambda_i\in\mathbb{R}$, and $c<\xi_i<d, i=1,2,\ldots,m$. Banach contraction mapping principle, Krasnosel'skiĭ's fixed-point theorem, and Laray–Schauder nonlinear alternative are used to establish the existence and uniqueness results.

Nonlocal fractional order coupled systems are also significant, as such systems often occur in applications, for example in fractional dynamics [32], bio-engineering [33], financial economics [34], etc. In a series of papers [35–37], a variety of coupled systems for (k,ψ)-Hilfer differential equations of fractional order were investigated.

In [38], the authors discuss a coupled system of nonlinear fractional differential equations involving (k,ψ)-Hilfer fractional derivative operator of order α, $1<\alpha\leq 2$, and parameter β, $0\leq\beta\leq 1$, of the form

$$\begin{cases} {}^{k,H}\mathbb{D}^{\alpha,\beta;\psi}\ell(\omega)=f(\omega,\ell(\omega),z(\omega)), \quad \omega\in(c,d], \\ {}^{k,H}\mathbb{D}^{\alpha_1,\beta_1;\psi}z(\omega)=f_1(\omega,\ell(\omega),z(\omega)), \quad \omega\in(c,d], \\ \ell(c)=0, \quad \ell(d)=\sum_{i=1}^{m}\lambda_i\,z(\xi_i), \\ z(c)=0, \quad z(d)=\sum_{j=1}^{k}\mu_j\,\ell(\eta_j), \end{cases} \tag{3}$$

in which ${}^{k,H}\mathbb{D}^{\alpha,\beta;\psi}$, ${}^{k,H}\mathbb{D}^{\alpha_1,\beta_1;\psi}$ are the (k,ψ)-Hilfer derivative operators of fractional orders α,α_1, $1<\alpha,\alpha_1<2$, and parameters β,β_1, $0\leq\beta,\beta_1\leq 1$, respectively, $f,f_1\in C([c,d]\times\mathbb{R}^2,\mathbb{R})$, $\lambda_i,\mu_j\in\mathbb{R}$, and $a<\xi_i,\eta_j<b, i=1,2,\ldots,m, j=1,2,\ldots,k$. By using standard fixed-point theorems such as Banach's and Krasnosel'skiĭ's fixed-point theorems along with Leray–Schauder alternative, the existence and uniqueness results were established.

In [39], a new class of boundary value problems of sequential ψ-Hilfer-type fractional integro-differential equations of the form

$$\begin{cases} \left({}^{H}\mathbb{D}^{\alpha,\beta;\psi}+\lambda\,{}^{H}\mathbb{D}^{\alpha-1,\beta;\psi}\right)\ell(\omega)=\Pi\left(\omega,\ell(\omega),(P_1\ell)(\omega),\mathbb{I}_{a+}^{2-\gamma,\psi}\ell(\omega)\right), \ \omega\in[a,T], \\ \ell(a)=0, \ \mathbb{I}_{a+}^{2-\gamma,\psi}\ell(T)=\sum_{i=1}^{m}\lambda_{1i}\ell(\eta_i)+\sum_{i=1}^{m}\lambda_{2i}\ell'(\eta_i)+P_2(\ell(\xi)), \end{cases} \tag{4}$$

was considered, where $\Pi\in C([a,T]\times\mathbb{R}^3,\mathbb{R})$, $a<\eta_1<\ldots<\eta_m<\xi<T$, $\lambda,\lambda_{1i},\lambda_{2i}\in\mathbb{R}$ $(i=1,2,\ldots,m)$ are given constants; $P_1:C([a,T],\mathbb{R})\to C([a,T],\mathbb{R})$ is an operator (not necessarily linear); $P_2:\mathbb{R}\to\mathbb{R}$ is a continuous function with $P_2(a)=0$; ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order $1<\alpha\leq 2$, $0\leq\beta\leq 1$; and $\mathbb{I}_{a+}^{2-\gamma,\psi}$ denotes the Riemmann–Liouville fractional integral of order $2-\alpha$ with $\gamma=\alpha+\beta(2-\alpha)$. The existence and uniqueness of the solutions were investigated via Banach, Sadovski, and Krasnoselskiĭ-Schaefer fixed-point theorems.

In order to enrich the literature in the new research area of (k,ψ)-Hilfer coupled systems, in the present paper, motivated by the above cited papers, we investigate a coupled nonlinear fractional system of differential equations involving (k,ψ)-Hilfer fractional derivative operators of orders in $(1,2]$ and (k,ψ)-Riemann–Liouville integral operators of the form

$$\begin{cases} {}^{k,H}\mathbb{D}^{a_1,b_1,\psi}\ell(\omega) = \Pi_1(\omega,\ell(\omega),m(\omega)), \ \ \omega\in[0,1], \\ {}^{k,H}\mathbb{D}^{a_2,b_2,\psi}m(\omega) = \Pi_2(\omega,\ell(\omega),m(\omega)), \ \ \omega\in[0,1], \\ \ell(0)=0, \ \ {}^{k}\mathbb{I}^{2-\gamma_1,\psi}\ell(1) = \sum_{i=1}^{p}\int_0^1 {}^{k,H}\mathbb{D}^{r_i,s_i,\psi}m(s)ds + \sum_{i=1}^{p}\lambda_i m(\eta_i) + \sum_{j=1}^{p}\mu_i m'(\eta_i), \\ m(0)=0, \ \ {}^{k}\mathbb{I}^{2-\gamma_2,\psi}m(1) = \sum_{j=1}^{q}\int_0^1 {}^{k,H}\mathbb{D}^{u_j,v_j,\psi}\ell(s)ds + \sum_{j=1}^{q}\xi_j\ell(\zeta_j) + \sum_{j=1}^{q}\theta_j\ell'(\zeta_j), \end{cases} \tag{5}$$

where ${}^{k,H}\mathbb{D}^{\chi,w,\psi}$ denotes the (k,ψ)-Hilfer derivative operator of order $\chi\in\{a_1,a_2,r_i,u_j\}$ and type $w\in\{b_1,b_2,s_i,v_j\}$; $\Pi_1,\Pi_2:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions; ${}^{k}\mathbb{I}^{2-\gamma_i,\psi}$ is the (k,ψ)-Riemann–Liouville fractional integral operator of order $2-\gamma_k>0$, $\gamma_k = a_k+b_k(2-a_k), k=1,2$, $\lambda_i,\mu_i,\xi_j,\theta_j\in\mathbb{R}$, $\eta_i,\zeta_j\in(0,1)$, $i=1,2,\ldots,p$, $j=1,2,\ldots,q$. By using standard fixed-point theorems such as Banach's and Krasnosel'skiĭ's fixed-point theorems, along with the Leray–Schauder alternative existence and uniqueness, results are provided.

The rest of the paper is organized as follows. In Section 2, we collect some concepts and results used in this paper. An auxiliary result is also proven concerning a linear variant of the system (5). Section 3 presents the main results, while in Section 4, illustrative examples are presented. The results of this paper are new and enrich the literature on the new subject of coupled systems of (k,ψ)-Hilfer differential equations of fractional order. Although the used methods are standard, their configuration in the present problem is new.

2. Preliminaries

First, some essential concepts and results related to this article are presented.

Definition 1 ([18])**.** *Let $P\in L^1([c_1,c_2],\mathbb{R})$, $k>0$, and ψ is an increasing function such that $\psi'(\omega)\neq 0$ for all $\omega\in[c_1,c_2]$. The (k,ψ)-Riemann–Liouville fractional integral of the function P, of order $0<a$ $(a\in\mathbb{R})$ is given by*

$${}^{k}\mathbb{I}^{a;\psi}_{c_1^+}P(\omega) = \frac{1}{k\Gamma_k(a)}\int_{c_1^+}^{\theta}\psi'(s)(\psi(\omega)-\psi(s))^{\frac{a}{k}-1}P(s)ds,$$

where $\Gamma_k(z)=\int_0^\infty s^{z-1}e^{\frac{s^k}{k}}ds$ is the k-Gamma function defined in [40] for $z\in\mathbb{C}$ with $\Re(z)>0$ and $k\in\mathbb{R}$.

The following properties are well known:

$$\Gamma(x) = \lim_{k\to 1}\Gamma_k(x), \ \ \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \text{ and } \Gamma_k(x+k) = x\Gamma_k(x).$$

Definition 2 ([28])**.** *Let $a,k\in\mathbb{R}^+=(0,\infty)$, $b\in[0,1]$; ψ is an increasing function such that $\psi\in C^n([c_1,c_2],\mathbb{R})$, $\psi'(s)\neq 0, s\in[c_1,c_2]$ and $P\in C^n([c_1,c_2],\mathbb{R})$. Then, the (k,ψ)-Hilfer fractional derivative of the function P of order a and type b, is defined by*

$${}^{k,H}\mathcal{D}^{a,b;\psi}P(s) = \mathbb{I}^{b(nk-a);\psi}_{c_1^+}\left(\frac{k}{\psi'(s)}\frac{d}{ds}\right)^n {}^{k}\mathbb{I}^{(1-b)(nk-a);\psi}_{c_1^+}P(s), \ \ n=\left\lceil\frac{a}{k}\right\rceil, \tag{6}$$

where $\lceil\frac{a}{k}\rceil$ is the ceiling function of $\frac{a}{k}$.

For special cases of the variables involved in the above definition, ${}^{k,H}\mathcal{D}^{a,b;\psi}$ is reduced to many known fractional derivative operators; for details, see [27].

Lemma 1 ([28])**.** *Let $a, k \in \mathbb{R}^+$ and $n = \lceil \frac{a}{k} \rceil$. Assume that $P \in C^n([c_1, c_2], \mathbb{R})$ and ${}^k\mathbb{I}_{c_1+}^{nk-a;\psi} P \in C^n([c_1, c_2], \mathbb{R})$. Then,*

$$ {}^k\mathbb{I}^{a;\psi}\left({}^{k,RL}\mathbb{D}^{a;\psi} P(\omega)\right) = P(\omega) - \sum_{j=1}^{n} \frac{(\psi(\omega) - \psi(c_1))^{\frac{a}{k}-j}}{\Gamma_k(a - jk + k)} \left[\left(\frac{k}{\psi'(\omega)}\frac{d}{d\omega}\right)^{n-j} {}^k\mathbb{I}_{c_1+}^{nk-a;\psi} P(\omega)\right]_{\omega=c_1}. $$

Lemma 2 ([28])**.** *Assume that $a, k \in \mathbb{R}^+$, $a < k$, and $b \in [0, 1]$. If $\theta_k = a + b(k - a)$, then we have*

$$ {}^k\mathbb{I}^{\theta_k;\psi}\left({}^{k,RL}\mathbb{D}^{\theta_k;\psi} P\right)(\omega) = {}^k\mathbb{I}^{a;\psi}\left({}^{k,H}\mathbb{D}^{a,b;\psi} P\right)(\omega), \quad P \in C^n([c_1, c_2], \mathbb{R}). $$

Lemma 3 ([28])**.** *Suppose that $\zeta, k \in \mathbb{R}^+, \eta \in \mathbb{R}$ are such that $\frac{\eta}{k} > -1$. Then,*

$$ (i).\ {}^k\mathbb{I}^{\zeta,\psi}(\psi(s) - \psi(c_1))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k + \zeta)}(\psi(s) - \psi(c_1))^{\frac{\eta+\zeta}{k}}. $$

$$ (ii).\ {}^k\mathbb{D}^{\zeta,\psi}(\psi(s) - \psi(c_1))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k - \zeta)}(\psi(s) - \psi(c_1))^{\frac{\eta-\zeta}{k}}. $$

Lemma 4 ([2])**.** *Let $a_1, a_2, b, k \in (0, \infty)$ with $a_2 > a_1$, $k > 0$, and $b \in [0, 1]$. Then,*

$$ {}^{k,H}\mathbb{D}^{a_1,b;\varphi}\ ({}^k\mathbb{I}_{0^+}^{a_2;\psi})P(\omega) = {}^k\mathbb{I}_{0^+}^{a_2-a_1;\psi} P(\omega), \quad P \in C([c_1, c_2], \mathbb{R}). $$

Remark 1. *Note that the (k, ψ)-Hilfer derivative operator of fractional order can be defined in the form of (k, ψ)-Riemann–Liouville derivative of fractional order*

$$ \begin{aligned} {}^{k,H}\mathbb{D}^{a,b;\psi} P(\omega) &= {}^k\mathbb{I}_{c_1+}^{\theta_k-a;\psi}\left(\frac{k}{\psi'(w)}\frac{d}{d\omega}\right)^n {}^k\mathbb{I}_{c_1+}^{nk-\theta_k;\psi} P(\omega) \\ &= {}^k\mathbb{I}_{c_1+}^{\theta_k-a;\psi}\left({}^{k,RL}\mathbb{D}^{\theta_k;\psi} P\right)(\omega), \end{aligned} $$

if we use the relations $b(nk - a) = \theta_k - a$ and $(1 - b)(nk - a) = nk - \theta_k$. Moreover , we have $n - 1 < \frac{\theta_k}{k} \le n$, for $n - 1 < \frac{a}{k} \le n$ and $0 \le b \le 1$.

The following lemma is the basic tool for transforming the coupled system (5) into integral equations, and it concerns a linear variant of problem (5).

Lemma 5. *Let $k > 0$, $1 < a_1, a_2, r_i, u_j \le 2$, $b_1, b_2, s_i, v_j \in [0, 1]$, $\gamma_1 = a_1 + b_1(2k - a_1)$, $\gamma_2 = a_2 + b_2(2k - a_2)$, $\gamma_3 = r_i + s_i(2k - r_i)$, $\gamma_4 = u_j + v_j(2k - u_j)$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$, $\pi_1, \pi_2 \in C^2([0, 1], \mathbb{R})$ and $\Theta \neq 0$. Then, the unique solution of the nonlocal (k, ψ)-Hilfer system*

$$ \begin{cases} {}^{k,H}\mathbb{D}^{a_1,b_1,\psi}\ell(\omega) = \pi_1(\omega), \ \ \omega \in [0, 1], \\ {}^{k,H}\mathbb{D}^{a_2,b_2,\psi} m(\omega) = \pi_2(\omega), \ \ \omega \in [0, 1], \\ \ell(0) = 0, \quad {}^k\mathbb{I}^{2-\gamma_1,\psi}\ell(1) = \sum_{i=1}^{p}\int_0^1 {}^{k,H}\mathbb{D}^{r_i,s_i,\psi} m(s)ds + \sum_{i=1}^{p}\lambda_i m(\eta_i) + \sum_{j=1}^{p}\mu_i m'(\eta_i), \\ m(0) = 0, \quad {}^k\mathbb{I}^{2-\gamma_2,\psi} m(1) = \sum_{j=1}^{q}\int_0^1 {}^{k,H}\mathbb{D}^{u_j,v_j,\psi}\ell(s)ds + \sum_{j=1}^{q}\xi_j\ell(\zeta_j) + \sum_{j=1}^{q}\theta_j\ell'(\zeta_j) \end{cases} \tag{7} $$

is given by

$$ \begin{aligned} \ell(\omega) &= {}^k\mathbb{I}^{a_1,\psi}\pi_1(\omega) + \frac{(\psi(\omega) - \psi(0))^{\frac{\gamma_1}{k}-1}}{\Theta\Gamma_k(\gamma_1)}\Bigg[A_2\Bigg(\sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1-u_j,\psi}\pi_1(s)ds \\ &\quad + \sum_{j=1}^{q}\xi_j {}^k\mathbb{I}^{a_1,\psi}\pi_1(\zeta_j) + \sum_{j=1}^{q}\theta_j\psi(\zeta_j)^k I^{a_1-1,\psi}\pi_1(\zeta_j) - {}^k\mathbb{I}^{2-\gamma_2+a_2,\psi}\pi_2(1)\Bigg) \end{aligned} $$

$$+B_2\bigg(\sum_{i=1}^{p}\int_0^1 {}^k\mathbb{I}^{a_2-r_i,\psi}\pi_2(s)ds+\sum_{i=1}^{p}\lambda_i{}^k\mathbb{I}^{a_2,\psi}\pi_2(\eta_i)$$
$$+\sum_{i=1}^{p}\mu_i\psi(\eta_i){}^k\mathbb{I}^{a_2-1,\psi}\pi_2(\eta_i)-{}^k\mathbb{I}^{2-\gamma_1+a_1,\psi}\pi_1(1)\bigg)\bigg], \quad (8)$$

and

$$m(\omega) = {}^k\mathbb{I}^{a_2,\psi}\pi_2(\omega)+\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_2}{k}-1}}{\Theta\Gamma_k(\gamma_2)}\bigg[A_1\bigg(\sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1-u_j,\psi}\pi_1(s)ds$$
$$+\sum_{j=1}^{q}\xi_j{}^kI^{a_1,\psi}\pi_1(\zeta_j)+\sum_{j=1}^{q}\theta_j\psi(\zeta_j){}^k\mathbb{I}^{a_1-1,\psi}\pi_1(\zeta_j)-{}^k\mathbb{I}^{2-\gamma_2+a_2,\psi}\pi_2(1)\bigg)$$
$$+B_1\bigg(\sum_{i=1}^{p}\int_0^1 {}^k\mathbb{I}^{a_2-r_i,\psi}\pi_2(s)ds+\sum_{i=1}^{p}\lambda_i{}^k\mathbb{I}^{a_2,\psi}\pi_2(\eta_i)$$
$$+\sum_{i=1}^{p}\mu_i\psi(\eta_i){}^k\mathbb{I}^{a_2-1,\psi}\pi_2(\eta_i)-{}^k\mathbb{I}^{2-\gamma_1+a_1,\psi}\pi_1(1)\bigg)\bigg], \quad (9)$$

where

$$\mathbb{A}_1 = \frac{(\psi(1)-\psi(0))^{\frac{2}{k}-1}}{\Gamma_k(2)},$$
$$\mathbb{A}_2 = \sum_{i=1}^{p}\int_0^1\frac{(\psi(s)-\psi(0))^{\frac{\gamma_2-r_i}{k}-1}}{\Gamma_k(\gamma_2-r_i)}ds+\sum_{i=1}^{p}\lambda_i\frac{(\psi(\eta_i)-\psi(0))^{\frac{\gamma_2}{k}-1}}{\Gamma_k(\gamma_2)}$$
$$+\sum_{i=1}^{p}\mu_i\bigg(\frac{\gamma_2}{k}-1\bigg)\psi'(\eta_i)\frac{(\psi(\eta_i)-\psi(0))^{\frac{\gamma_2}{k}-2}}{\Gamma_k(\gamma_2)},$$
$$\mathbb{B}_1 = \sum_{j=1}^{q}\int_0^1\frac{(\psi(s)-\psi(0))^{\frac{\gamma_1-u_j}{k}-1}}{\Gamma_k(\gamma_1-u_j)}ds+\sum_{j=1}^{q}\xi_j\frac{(\psi(\zeta_j)-\psi(0))^{\frac{\gamma_1}{k}-1}}{\Gamma_k(\gamma_1)}$$
$$+\sum_{j=1}^{q}\theta_j\bigg(\frac{\gamma_1}{k}-1\bigg)\psi'(\zeta_j)\frac{(\psi(\zeta_j)-\psi(0))^{\frac{\gamma_1}{k}-2}}{\Gamma_k(\gamma_1)},$$
$$\mathbb{B}_2 = \frac{(\psi(1)-\psi(0))^{\frac{2}{k}-1}}{\Gamma_k(2)},$$

with

$$\Theta = \mathbb{A}_1\mathbb{B}_2-\mathbb{A}_2\mathbb{B}_1.$$

Proof. Assume that (ℓ, m) is a solution of the system (7). Operating ${}^k\mathbb{I}^{a_1,\psi}$ and ${}^k\mathbb{I}^{a_2,\psi}$ on both sides of the equations in (7), and applying Lemmas 1 and 2, we get

$$\ell(\omega) = {}^k\mathbb{I}^{a_1;\psi}\pi_1(\omega)+c_0\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_1}{k}-1}}{\Gamma_k(\gamma_1)}+c_1\frac{(\psi(\omega)-\varphi(0))^{\frac{\gamma_1}{k}-2}}{\Gamma_k(\gamma_1-1)}, \quad (10)$$

and

$$m(\omega) = {}^k\mathbb{I}^{a_2;\psi}\pi_2(\omega)+d_0\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_2}{k}-1}}{\Gamma_k(\gamma_2)}+d_1\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_2}{k}-2}}{\Gamma_k(\gamma_2-1)}, \quad (11)$$

where

$$c_0 = \bigg[\bigg(\frac{k}{\psi'(\omega)}\frac{d}{dt}\bigg){}^k\mathbb{I}^{2k-\gamma_1:\psi}\ell(\omega)\bigg]_{\omega=0},\ \ c_1 = \Big[{}^k\mathbb{I}^{2k-\gamma_1:\psi}\ell(\omega)\Big]_{\omega=0},$$
$$d_0 = \bigg[\bigg(\frac{k}{\psi'(\omega)}\frac{d}{dt}\bigg){}^k\mathbb{I}^{2k-\gamma_2:\psi}m(\omega)\bigg]_{\omega=0},\ \ d_1 = \Big[{}^k\mathbb{I}^{2k-\gamma_2:\psi}m(\omega)\Big]_{\omega=0}.$$

In view of $\ell(0) = 0$ and $m(0) = 0$ with (10) and (11), we get $c_1 = 0$ and $d_1 = 0$, since by Remark 1, $\frac{\gamma_1}{k} - 2 < 0$ and $\frac{\gamma_2}{k} - 2 < 0$.

On the other hand, by taking the operators ${}^{k}\mathbb{I}^{2-\gamma_1;\psi}$, ${}^{k}\mathbb{I}^{2-\gamma_2;\psi}$ in (10) and (11), and differentiating (10) and (11), we obtain

$$
\begin{aligned}
{}^{k}\mathbb{I}^{2-\gamma_1;\psi}\ell(\omega) &= {}^{k}\mathbb{I}^{2-\gamma_1+a_1;\psi}\pi_1(\omega) + c_0\frac{(\psi(\omega)-\psi(0))^{\frac{2}{k}-1}}{\Gamma_k(2)},\\
{}^{k}\mathbb{I}^{2-\gamma_2;\psi}m(\omega) &= {}^{k}\mathbb{I}^{2-\gamma_2+a_2;\psi}\pi_2(\omega) + c_0\frac{(\psi(\omega)-\psi(0))^{\frac{2}{k}-1}}{\Gamma_k(2)},\\
\ell'(\omega) &= \psi(\omega)^{k}\mathbb{I}^{a_1-1;\psi}\pi_1(\omega) + c_0\left(\frac{\gamma_1}{k}-1\right)\psi'(\omega)\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_1}{k}-2}}{\Gamma_k(\gamma_1)},\\
m'(\omega) &= \psi(\omega)^{k}\mathbb{I}^{a_2-1;\psi}\pi_2(\omega) + d_0\left(\frac{\gamma_2}{k}-1\right)\psi'(\omega)\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_2}{k}-2}}{\Gamma_k(\gamma_2)}.
\end{aligned}
$$

Hence, applying the conditions

$$
{}^{k}\mathbb{I}^{2-\gamma_1,\psi}\ell(1) = \sum_{i=1}^{p}\int_0^1 {}^{k,H}\mathbb{D}^{r_i,s_i,\psi}m(s)ds + \sum_{i=1}^{p}\lambda_i m(\eta_i) + \sum_{j=1}^{p}\mu_i m'(\eta_i)
$$

and

$$
{}^{k}\mathbb{I}^{2-\gamma_2,\psi}m(1) = \sum_{j=1}^{q}\int_0^1 {}^{k,H}\mathbb{D}^{u_j,v_j,\psi}\ell(s)ds + \sum_{j=1}^{q}\xi_j\ell(\zeta_j) + \sum_{j=1}^{q}\theta_j\ell'(\zeta_j)
$$

in (10) and (11) and using Lemmas 3 and 4, we have

$$
\begin{aligned}
&{}^{k}\mathbb{I}^{2-r_i+a_1,\psi}\pi_1(1) + c_0\frac{(\psi(1)-\psi(0))^{\frac{2}{k}-1}}{\Gamma_k(2)}\\
=\;& \sum_{i=1}^{p}\int_0^1 {}^{k}\mathbb{I}^{a_2-r_i,\psi}\pi_2(s)ds + d_0\sum_{i=1}^{p}\int_0^1\frac{(\psi(s)-\psi(0))^{\frac{\gamma_2-r_i}{k}-1}}{\Gamma_k(\gamma_2-r_i)}\\
&+\sum_{i=1}^{p}\lambda_i{}^{k}\mathbb{I}^{a_2,\psi}\pi_2(\eta_i) + d_0\sum_{i=1}^{p}\lambda_i\frac{(\psi(\eta_i)-\psi(0))^{\frac{\gamma_2}{k}-1}}{\Gamma_k(\gamma_2)} + \sum_{i=1}^{p}\mu_i\psi(\eta_i)^{k}I^{a_2-1,\psi}\pi_2(\eta_i)\\
&+d_0\sum_{i=1}^{p}\mu_i\left(\frac{\gamma_2}{k}-1\right)\psi'(\eta_i)\frac{(\psi(\eta_i)-\psi(0))^{\frac{\gamma_2}{k}-2}}{\Gamma_k(\gamma_2)},
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
&{}^{k}\mathbb{I}^{2-u_j+a_2,\psi}\pi_2(1) + d_0\frac{(\psi(1)-\psi(0))^{\frac{2}{k}-1}}{\Gamma_k(2)}\\
=\;& \sum_{j=1}^{q}\int_0^1 {}^{k}\mathbb{I}^{a_1-u_j,\psi}\pi_2(s)ds + c_0\sum_{j=1}^{q}\int_0^1\frac{(\psi(s)-\psi(0))^{\frac{\gamma_1-u_j}{k}-1}}{\Gamma_k(\gamma_1-u_j)}\\
&+\sum_{j=1}^{q}\xi_j{}^{k}\mathbb{I}^{a_1,\psi}\pi_1(\zeta_j) + c_0\sum_{j=1}^{q}\xi_j\frac{(\psi(\zeta_j)-\psi(0))^{\frac{\gamma_1}{k}-1}}{\Gamma_k(\gamma_1)} + \sum_{j=1}^{q}\theta_j\psi(\zeta_j)^{k}\mathbb{I}^{a_1-1,\psi}\pi_1(\zeta_j)\\
&+c_0\sum_{j=1}^{q}\theta_i\left(\frac{\gamma_1}{k}-1\right)\psi'(\zeta_j)\frac{(\psi(\zeta_j)-\psi(0))^{\frac{\gamma_1}{k}-2}}{\Gamma_k(\gamma_1-u_j)}.
\end{aligned}
\tag{13}
$$

By (2), (12), and (13), we obtain

$$
\begin{cases}
\mathbb{A}_1c_0 - \mathbb{A}_2d_0 = Q_1,\\
-\mathbb{B}_1c_0 + \mathbb{B}_2d_0 = Q_2,
\end{cases}
\tag{14}
$$

where

$$\begin{aligned} Q_1 &= \sum_{i=1}^{p}\int_0^1 {}^k\mathbb{I}^{a_2-r_i,\psi}\pi_2(s)ds + \sum_{i=1}^{p}\lambda_i{}^k\mathbb{I}^{a_2,\psi}\pi_2(\eta_i) \\ &\quad + \sum_{i=1}^{p}\mu_i\psi(\eta_i){}^k\mathbb{I}^{a_2-1,\psi}\pi_2(\eta_i) - {}^k\mathbb{I}^{2-\gamma_1+a_1,\psi}\pi_1(1), \\ Q_2 &= \sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1-u_j,\psi}\pi_1(s)ds + \sum_{j=1}^{q}\xi_j{}^k\mathbb{I}^{a_1,\psi}\pi_1(\zeta_j) \\ &\quad + \sum_{j=1}^{q}\theta_j\psi(\zeta_j){}^k\mathbb{I}^{a_1-1,\psi}\pi_1(\zeta_j) - {}^k\mathbb{I}^{2-\gamma_2+a_2,\psi}\pi_2(1). \end{aligned}$$

Bysolving system (14), we obtain

$$c_0 = \frac{1}{\Theta}[\mathbb{A}_2Q_2 + \mathbb{B}_2Q_1] \text{ and } d_0 = \frac{1}{\Theta}[\mathbb{A}_1Q_2 + \mathbb{B}_1Q_1].$$

Replacing c_0, d_0, c_1, d_1 in (10) and (11), we obtain the solutions (8) and (9). By using direct computations, we can prove the converse. Thus, the proof is completed. □

3. Existence and Uniqueness Results

Assume that $\mathcal{Z} = C([0,1],\mathbb{R})$ is the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$, eqipped with the norm $\|\ell\| = \max\{|\ell(\omega)|, \omega \in [0,1]\}$. Obviously, the product space $(\mathcal{Z}\times\mathcal{Z}, \|(\ell,m)\|)$ is a Banach space with norm $\|(\ell,m)\| = \|\ell\| + \|m\|$.

In view of Lemma 5, an operator $\mathbb{D} : \mathcal{Z}\times\mathcal{Z} \to \mathcal{Z}\times\mathcal{Z}$ is defined by

$$\mathbb{D}(\ell,m)(\omega) = \begin{pmatrix} \mathbb{D}_1(\ell,m)(\omega) \\ \mathbb{D}_2(\ell,m)(\omega) \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} &\mathbb{D}_1(\ell,m)(\omega) \\ = &{}^k\mathbb{I}^{a_1,\psi}\Pi_1(\omega,\ell(\omega),m(\omega)) \\ &+ \frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_1}{k}-1}}{\Theta\Gamma_k(\gamma_1)}\Bigg[\mathbb{A}_2\Bigg(\sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1-u_j,\psi}\Pi_1(s,\ell(s),m(s))ds \\ &+ \sum_{j=1}^{q}\xi_j{}^k\mathbb{I}^{a_1,\psi}\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j)) + \sum_{j=1}^{q}\theta_j\psi(\zeta_j){}^k\mathbb{I}^{a_1-1,\psi}\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j)) \\ &- {}^k\mathbb{I}^{2-\gamma_2+a_2,\psi}\Pi_2(1,\ell(1),m(1))\Bigg) + \mathbb{B}_2\Bigg(\sum_{i=1}^{p}\int_0^1 {}^k\mathbb{I}^{a_2-r_i,\psi}\Pi_2(s,\ell(s),m(s))ds \\ &+ \sum_{i=1}^{p}\lambda_i{}^k\mathbb{I}^{a_2,\psi}\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i)) + \sum_{i=1}^{p}\mu_i\psi(\eta_i){}^k\mathbb{I}^{a_2-1,\psi}\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i)) \\ &- {}^k\mathbb{I}^{2-\gamma_1+a_1,\psi}\Pi_1(1,\ell(1),m(1))\Bigg)\Bigg], \ \omega \in [0,1], \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\mathbb{D}_2(\ell,m)(\omega) \\ &= {}^k\mathbb{I}^{a_2,\psi}\Pi_2(\omega,\ell(\omega),m(\omega)) \\ &+ \frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_2}{k}-1}}{\Theta\Gamma_k(\gamma_2)}\Bigg[\mathbb{A}_1\Bigg(\sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1-u_j,\psi}\Pi_1(s,\ell(s),m(s))ds \\ &+ \sum_{j=1}^{q}\xi_j{}^k\mathbb{I}^{a_1,\psi}\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j)) + \sum_{j=1}^{q}\theta_j\psi(\zeta_j){}^k\mathbb{I}^{a_1-1,\psi}\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j)) \end{aligned}$$

$$-{}^{k}\mathbb{I}^{2-\gamma_2+a_2,\psi}\Pi_2(1,\ell(1),m(1))\Big)+\mathbb{B}_1\Big(\sum_{i=1}^{p}\int_0^1 {}^{k}\mathbb{I}^{a_2-r_i,\psi}\Pi_2(s,\ell(s),m(s))ds$$
$$+\sum_{i=1}^{p}\lambda_i{}^{k}\mathbb{I}^{a_2,\psi}\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i))+\sum_{i=1}^{p}\mu_i\psi(\eta_i){}^{k}\mathbb{I}^{a_2-1,\psi}\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i))$$
$$-{}^{k}\mathbb{I}^{2-\gamma_1+a_1,\psi}\Pi_1(1,\ell(1),m(1))\Big)\Big],\ \ \omega\in[0,1]. \tag{17}$$

Now, the following constants are introduced for convenience.

$$\begin{aligned}
\mathbb{G}_1 &= \frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\Bigg[\mathbb{A}_2\Bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}\\
&\quad+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}\Bigg)\\
&\quad+\mathbb{B}_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}\Bigg],\\
\mathbb{G}_1^* &= \mathbb{G}_1-\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)},\\
\mathbb{G}_2 &= \frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\Bigg[\mathbb{A}_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}+\mathbb{B}_2\Bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}\\
&\quad+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}+\sum_{i=1}^{p}|\mu_i|\psi(1)\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}\Bigg)\Bigg],\\
\mathbb{G}_3 &= \frac{(\psi(1)-\psi(0))^{\frac{\gamma_2}{k}-1}}{|\Theta|\Gamma_k(\gamma_2)}\Bigg[\mathbb{A}_1\Bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}\\
&\quad+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}\Bigg)+\mathbb{B}_1\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}\Bigg],\\
\mathbb{G}_4 &= \frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_2}{k}-1}}{|\Theta|\Gamma_k(\gamma_2)}\Bigg[\mathbb{A}_1\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}\\
&\quad+\mathbb{B}_1\Bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}\\
&\quad+\sum_{i=1}^{p}|\mu_i|\psi(1)\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}\Bigg)\Bigg],\\
\mathbb{G}_4^* &= \mathbb{G}_4-\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}.
\end{aligned} \tag{18}$$

In the next theorem, the first result concerning the existence and uniqueness of solutions of the coupled system (5) is proved via Banach's fixed-point theorem [41].

Theorem 1. *Assume that:*

(H_1) *There exist constants* $s_i, t_i, i = 1, 2$ *such that,*

$$|\Pi_1(\omega,\ell_1,\ell_2)-\Pi_1(\omega,m_1,m_2)|\le s_1|\ell_1-m_1|+s_2|\ell_2-m_2|,$$
$$|\Pi_2(\omega,\ell_1,\ell_2)-\Pi_2(\omega,m_1,m_2)|\le t_1|\ell_1-m_1|+t_2|\ell_2-m_2|,$$

for $\omega\in[0,1]$ *and* $\ell_i, m_i\in\mathbb{R}, i=1,2.$

Then, the (k,ψ)-Hilfer fractional nonlocal coupled system (5) has on $[0,1]$ a unique solution, provided that

$$(\mathbb{G}_1+\mathbb{G}_3)(s_1+s_2)+(\mathbb{G}_2+\mathbb{G}_4)(t_1+t_2)<1, \tag{19}$$

where $\mathbb{G}_i, i=1,2,3,4$ are given in (18).

Proof. We will give the proof by considering the following two steps:

(i) $\mathbb{D}(\mathbb{B}_x)\subseteq\mathbb{B}_x$, in which $\mathbb{B}_x=\{(\ell,m)\in\mathcal{Z}\times\mathcal{Z}:\|(\ell,m)\|\le x\}$ with

$$x\ge\frac{(\mathbb{G}_1+\mathbb{G}_3)\mathcal{M}+(\mathbb{G}_2+\mathbb{G}_4)\mathcal{N}}{1-[(\mathbb{G}_1+\mathbb{G}_3)(s_1+s_2)+(\mathbb{G}_2+\mathbb{G}_4)(t_1+t_2)]},$$

$\mathcal{M}=\sup_{\omega\in[0,1]}\Pi_1(\omega,0,0)<\infty$, $\mathcal{N}=\sup_{\omega\in[0,1]}\Pi_2(\omega,0,0)<\infty$.

(ii) $\mathbb{D}$ is a contraction.

To verify (i), let $(\ell,m)\in\mathbb{B}_x$ and $\omega\in[0,1]$. Then, we obtain

$$\begin{aligned}
&|\mathbb{D}_1(\ell,m)(\omega)|\\
\le\;& {}^k\mathbb{I}^{a_1,\psi}(|\Pi_1(\omega,\ell(\omega),m(\omega))-\Pi_1(\omega,0,0)|+|\Pi_1(\omega,0,0)|)\\
&+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\int_0^1{}^k\mathbb{I}^{a_1-u_j,\psi}|\Pi_1(s,\ell(s),m(s))-\Pi_1(s,0,0)|\\
&+|\Pi_1(s,0,0)|)ds+\sum_{j=1}^{q}|\xi_j|{}^k\mathbb{I}^{a_1,\psi}(|\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j))-\Pi_1(\zeta_j,0,0)|+|\Pi_1(\zeta_j,0,0)|)\\
&+\sum_{j=1}^{q}|\theta_j||\psi(1)|{}^k\mathbb{I}^{a_1-1,\psi}(|\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j))-\Pi_1(\zeta_j,0,0)|+|\Pi_1(\zeta_j,0,0)|)\\
&+{}^k\mathbb{I}^{2-\gamma_2+a_2,\psi}(|\Pi_2(1,\ell(1),m(1))-\Pi_2(1,0,0)|+|\Pi_1(1,0,0)|)\bigg)\\
&+\mathbb{B}_2\bigg(\sum_{i=1}^{p}\int_0^1{}^k\mathbb{I}^{2-r_i,\psi}(|\Pi_2(s,\ell(s),m(s))-\Pi_2(s,0,0)|+|\Pi_2(s,0,0)|)ds\\
&+\sum_{i=1}^{p}|\lambda_i|{}^k\mathbb{I}^{a_2,\psi}(|\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i))-\Pi_2(\eta_i,0,0)|+|\Pi_2(\eta_i,0,0)|)\\
&+\sum_{i=1}^{p}|\mu_i|\psi(1){}^k\mathbb{I}^{a_2-1,\psi}(|\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i))-\Pi_2(\eta_i,0,0)|+|\Pi_2(\eta_i,0,0)|)\\
&+{}^k\mathbb{I}^{2-\gamma_1+a_1,\psi}(|\Pi_1(1,\ell(1),m(1))-\Pi_1(1,0,0)|+|\Pi_1(1,0,0)|)\bigg)\bigg]\\
\le\;&\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}(s_1\|\ell\|+s_2\|m\|+\mathcal{M})+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_{\gamma_1})}\\
&\times\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}(s_1\|\ell\|+s_2\|m\|+\mathcal{M})\\
&+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}(s_1\|\ell\|+s_2\|m\|+\mathcal{M})\\
&+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}(s_1\|\ell\|+s_2\|m\|+\mathcal{M})\\
&+\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}(t_1\|\ell\|+t_2\|m\|+\mathcal{N})\bigg)
\end{aligned}$$

$$
\begin{aligned}
& +\mathbb{B}_2\bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}(t_1\|\ell\|+t_2\|m\|+\mathcal{N}) \\
& +\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}(t_1\|\ell\|+t_2\|m\|+\mathcal{N}) \\
& +\sum_{i=1}^{p}|\mu_i||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}(t_1\|\ell\|+t_2\|m\|+\mathcal{N}) \\
& +\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}(s_1\|\ell\|+s_2\|m\|+\mathcal{M})\bigg)\bigg] \\
\leq\; & (s_1\|\ell\|+s_2\|m\|+\mathcal{M})\bigg\{\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)} \\
& \times\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)} \\
& +\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}\bigg)+\mathbb{B}_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}\bigg] \\
& +(t_1\|\ell\|+t_2\|m\|+\mathcal{N})\bigg\{\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\bigg[\mathbb{A}_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)} \\
& +\mathbb{B}_2\bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)} \\
& +\sum_{i=1}^{p}|\mu_i|\psi(1)\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}\bigg)\bigg]\bigg\} \\
=\; & (s_1\|\ell\|+s_2\|m\|+\mathcal{M})\mathbb{G}_1+(t_1\|\ell\|+t_2\|m\|+\mathcal{N})\mathbb{G}_2 \\
=\; & (s_1\mathbb{G}_1+t_1\mathbb{G}_2)\|\ell\|+(s_2\mathbb{G}_1+t_2\mathbb{G}_2)\|m\|+\mathbb{G}_1\mathcal{M}+\mathbb{G}_2\mathcal{N} \\
\leq\; & (s_1\mathbb{G}_1+t_1\mathbb{G}_2+s_2\mathbb{G}_1+t_2\mathbb{G}_2)x+\mathbb{G}_1\mathcal{M}_1+\mathbb{G}_2\mathcal{N}.
\end{aligned}
$$

Analogously, we have

$$
|\mathbb{D}_2(\ell,m)(\omega)|\leq(s_1\mathbb{G}_3+t_1\mathbb{G}_4+s_2\mathbb{G}_3+t_2\mathbb{G}_4)x+\mathbb{G}_3\mathcal{M}+\mathbb{G}_4\mathcal{N}.
$$

Consequently, we obtain

$$
\begin{aligned}
& \|\mathbb{D}(\ell,m)\|=\|\mathbb{D}_1(\ell,m)\|+\|\mathbb{D}_2(\ell,m)\| \\
\leq\; & [(\mathbb{G}_1+\mathbb{G}_3)(s_1+s_2)+(\mathbb{G}_2+\mathbb{G}_4)(t_1+t_2)]x+(\mathbb{G}_1+\mathbb{G}_3)\mathcal{M}+(\mathbb{G}_2+\mathbb{G}_4)\mathcal{N} \\
\leq\; & x.
\end{aligned}
$$

Hence, we obtain $\mathbb{D}(\mathbb{B}_x)\subseteq\mathbb{B}_x$.
Next, we prove (ii). For $(\ell,m)\in\mathbb{B}_x$ and $\omega\in[0,1]$, we have

$$
\begin{aligned}
& |\mathbb{D}_1(\ell_2,m_2)(\omega)-\mathbb{D}_1(\ell_1,m_1)(\omega)| \\
\leq\; & {}^kI^{a_1,\psi}|\Pi_1(\omega,\ell_2(\omega),m_2(\omega))-\Pi_1(\omega,\ell_1(\omega),m_1(\omega))| \\
& +\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\int_0^1{}^k\mathbb{I}^{a_1-u_j,\psi}|\Pi_1(s,\ell_2(s),m_2(s))-\Pi_1(s,\ell_1(s),m_1(s))|)ds \\
& +\sum_{j=1}^{q}|\xi_j|^k\mathbb{I}^{a_1,\psi}(|\Pi_1(\zeta_j,\ell_2(\zeta_j),m_2(\zeta_j))-\Pi_1(\zeta_j,\ell_1(\zeta_j),m_1(\zeta_j))|) \\
& +\sum_{j=1}^{q}|\theta_j||\psi(1)|^k\mathbb{I}^{a_1-1,\psi}(|\Pi_1(\zeta_j,\ell_2(\zeta_j),m_2(\zeta_j))-\Pi_1(\zeta_j,\ell_1(\zeta_j),m_1(\zeta_j))|)
\end{aligned}
$$

$$
\begin{aligned}
&+{}^{k}\mathbb{I}^{2-\gamma_2+a_2,\psi}(|\Pi_2(1,\ell_2(1),m_2(1))-\Pi_2(1,\ell_1(1),m_1(1))|)\Big)\\
&+\mathbb{B}_2\Big(\sum_{i=1}^{p}\int_0^1 {}^{k}\mathbb{I}^{2-r_i,\psi}(|\Pi_2(s,\ell_2(s),m_2(s))-\Pi_2(s,\ell_1(s),m_1(s))|)ds\\
&+\sum_{i=1}^{p}|\lambda_i|{}^{k}\mathbb{I}^{a_2,\psi}(|\Pi_2(\eta_i,\ell_2(\eta_i),m_2(\eta_i))-\Pi_2(\eta_i,\ell_1(\eta_i),m_1(\eta_i))|)\\
&+\sum_{i=1}^{p}|\mu_i|\psi(1){}^{k}\mathbb{I}^{a_2-1,\psi}(|\Pi_2(\eta_i,\ell_2(\eta_i),m_2(\eta_i))-\Pi_2(\eta_i,\ell_1(\eta_i),m_1(\eta_i))|)\\
&+{}^{k}\mathbb{I}^{2-\gamma_1+a_1,\psi}(|\Pi_1(1,\ell_2(1),m_2(1))-\Pi_1(1,\ell_1(1),m_1(1))|)\Big)\Big]\\
\leq\;& \frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_{\gamma_1})}\\
&\times\Big[\mathbb{A}_2\Big(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\\
&+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\\
&+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\\
&+\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\Big)\\
&+\mathbb{B}_2\Big(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\\
&+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\\
&+\sum_{i=1}^{p}|\mu_i|\times|\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\\
&+\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\Big)\\
=\;&(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\mathbb{G}_1+(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\mathbb{G}_2\\
=\;&(s_1\mathbb{G}_1+t_1\mathbb{G}_2)\|\ell_2-\ell_1\|+(s_2\mathbb{G}_1+t_2\mathbb{G}_2)\|m_2-m_1\|.
\end{aligned}
$$

Thus, we have

$$
\|\mathbb{D}_1(\ell_2,m_2)-\mathbb{D}_1(\ell_1,m_1)\|\leq(s_1\mathbb{G}_1+t_1\mathbb{G}_2+s_2\mathbb{G}_1+t_2\mathbb{G}_2)(\|\ell_2-\ell_1\|+\|m_2-m_1\|). \quad (20)
$$

Analogously, we have

$$
\|\mathbb{D}_2(\ell_2,m_2)-\mathbb{D}_2(\ell_1,m_1)\|\leq(s_1\mathbb{G}_3+t_1\mathbb{G}_4+s_2\mathbb{G}_3+t_2\mathbb{G}_4)(\|\ell_2-\ell_1\|+\|m_2-m_1\|). \quad (21)
$$

Consequently, by (20) and (21), we conclude that

$$
\begin{aligned}
&\|\mathbb{D}(\ell_2,m_2)(\omega)-\mathbb{D}(\ell_1,m_1)(\omega)\|\\
\leq\;&\Big[(\mathbb{G}_1+\mathbb{G}_3)(s_1+s_2)+(\mathbb{G}_2+\mathbb{G}_4)(t_1+t_2)\Big](\|\ell_2-\ell_1\|+\|m_2-m_1\|).
\end{aligned}
$$

Thus, by (19), the contraction property of the operator $\mathbb{D}$ is obtained, and the Banach's contraction mapping principle implies that system (5) has on $[0,1]$ a unique solution, which completes the proof. □

Now, via Leray–Schauder alternative [42], the first existence result is proved.

Theorem 2. *Let $\Pi_1, \Pi_2 : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous functions. Assume that*

(H_2) *There exist $\phi_i, \epsilon_i \geq 0$ for $i = 1,2$ and $\phi_0, \epsilon_0 > 0$ such that*

$$|\Pi_1(\omega, \ell_1, \ell_2| \leq \phi_0 + \phi_1|\ell_1| + \phi_2|\ell_2|,$$
$$|\Pi_1(\omega, \ell_1, \ell_2)| \leq \epsilon_0 + \epsilon_1|\ell_1| + \epsilon_2|\ell_2|,$$

for all $\omega \in [0,1]$ and $\ell_1, \ell_2 \in \mathbb{R}$.

If

$$(\mathbb{G}_1 + \mathbb{G}_3)\phi_1 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_1 < 1, \quad (\mathbb{G}_1 + \mathbb{G}_3)\phi_2 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_2 < 1,$$

where G_i, for $i = 1,2,3,4$, is given by (18), then the the system (5) has on $[0,1]$ at least one solution.

Proof. By the continuity of the function Π_1 and Π_2, we conclude that the operator $\mathbb{D}$ is continuous. Now, it is proven that $\mathbb{D}(\mathbb{B}_x)$ is uniformly bounded, where

$$\mathbb{B}_x = \{(\ell, m) \in \mathcal{Z} \times \mathcal{Z}; \|(\ell, m)\| \leq x\}.$$

From (H_2), for all $(\ell, m) \in \mathbb{B}_x$, we have

$$\begin{aligned}
|\Pi_1(\omega, \ell(\omega), m(\omega)| &\leq \phi_0 + \phi_1|\ell(\omega)| + \phi_2|m(\omega)| \\
&\leq \phi_0 + \phi_1\|\ell\| + \phi_2\|m\| \\
&\leq \phi_0 + (\phi_1 + \phi_2)(\|\ell\| + \|m\|) \\
&\leq \phi_0 + (\phi_1 + \phi_2)x := P_1,
\end{aligned}$$

and analogously,

$$|\Pi_2(\omega, \ell(\omega), m(\omega))| \leq \epsilon_0 + \epsilon_1\|\ell\| + \epsilon_2\|m\| \leq \epsilon_0 + (\epsilon_1 + \epsilon_2)x := P_2.$$

Hence, for all $(\ell, m) \in \mathbb{B}_x$, we have

$$\begin{aligned}
&|\mathbb{D}_1(\ell, m)(\omega)| \\
\leq\ & {}^k\mathbb{I}^{a_1,\psi}(|\Pi_1(\omega, \ell(\omega), m(\omega))|) \\
&+ \frac{(\psi(1) - \psi(0))^{\frac{\gamma_1}{k} - 1}}{|\Theta|\Gamma_k(\gamma_1)}\Bigg[\mathbb{A}_2\Bigg(\sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1 - u_j,\psi}|\Pi_1(s, \ell(s), m(s))|)ds \\
&+ \sum_{j=1}^{q}|\xi_j|{}^k\mathbb{I}^{a_1,\psi}(|\Pi_1(\gamma_j, \ell(\zeta_j), m(\zeta_j))|) + \sum_{j=1}^{q}|\theta_j||\psi(1)|{}^k\mathbb{I}^{a_1 - 1,\psi}(|\Pi_1(\zeta_j, \ell(\zeta_j), m(\zeta_j))|) \\
&+ {}^k\mathbb{I}^{2 - \gamma_2 + a_2,\psi}(|\Pi_2(1, \ell(1), m(1))|)\Bigg) + \mathbb{B}_2\Bigg(\sum_{i=1}^{p}\int_0^1 {}^k\mathbb{I}^{2 - r_i,\psi}(|\Pi_2(s, \ell(s), m(s))|)ds \\
&+ \sum_{i=1}^{p}|\lambda_i|{}^k\mathbb{I}^{a_2,\psi}(|\Pi_2(\eta_i, \ell(\eta_i), m(\eta_i))|) + \sum_{i=1}^{p}|\mu_i|\psi(1){}^k\mathbb{I}^{a_2 - 1,\psi}(|\Pi_2(\eta_i, \ell(\eta_i), m(\eta_i))|) \\
&+ {}^k\mathbb{I}^{2 - \gamma_1 + a_1,\psi}(|\Pi_1(1, \ell(1), m(1))|)\Bigg)\Bigg] \\
\leq\ & \frac{(\psi(1) - \psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1 + k)}P_1 + \frac{(\psi(1) - \psi(0))^{\frac{\gamma_1}{k} - 1}}{|\Theta|\Gamma_k(\gamma_1)}\mathbb{A}_2\Bigg(\sum_{j=1}^{q}\frac{(\psi(1) - \psi(0))^{\frac{a_1 - u_j}{k}}}{\Gamma_k(a_1 - u_j + k)}P_1
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=1}^{q} |\xi_j| \frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)} P_1 + \sum_{j=1}^{q} |\theta_j||\psi(1)| \frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)} P_1 \\
& + \frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)} P_2 \bigg) + \mathbb{B}_2 \bigg(\sum_{i=1}^{p} \frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)} P_2 \\
& + \sum_{i=1}^{p} |\lambda_i| \frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)} P_2 + \sum_{i=1}^{p} |\mu_i||\psi(1)| \frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)} P_2 \\
& + \frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)} P_1 \bigg) \bigg] \\
\leq \;& P_1 \bigg\{ \frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)} + \frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)} \bigg[\mathbb{A}_2 \bigg(\sum_{j=1}^{q} \frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)} \\
& + \sum_{j=1}^{q} |\xi_j| \frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)} + \sum_{j=1}^{q} |\theta_j||\psi(1)| \frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)} \bigg) \\
& + \epsilon_2 \frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)} \bigg] \bigg\} + P_2 \bigg\{ \frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)} \bigg[\mathbb{A}_2 \frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)} \\
& + \mathbb{B}_2 \bigg(\sum_{i=1}^{p} \frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)} + \sum_{i=1}^{p} |\lambda_i| \frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)} \\
& + \sum_{i=1}^{p} |\mu_i| \psi(1) \frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)} \bigg) \bigg] \bigg\},
\end{aligned}
$$

which yields

$$\|\mathbb{D}_1(\ell,m)\| \leq P_1\mathbb{G}_1 + P_2\mathbb{G}_2.$$

Likewise, one can get

$$\|\mathbb{D}_2(\ell,m)\| \leq P_1\mathbb{G}_3 + P_2\mathbb{G}_4.$$

Hence, we get

$$\|\mathbb{D}(\ell,m)\| = \|\mathbb{D}_2(\ell,m)\| + \|\mathbb{D}_2(\ell,m)\| \leq (\mathbb{G}_1+\mathbb{G}_3)P_1 + (\mathbb{G}_2+\mathbb{G}_3)P_2,$$

and, hence, the operator $\mathbb{D}$ is uniformly bounded.

Now, it is indicated that the operator $\mathbb{D}$ is equicontinuous. Let $\omega_1, \omega_2 \in [0,1]$, with $\omega_1 < \omega_2$. Thus, we have

$$
\begin{aligned}
& |\mathbb{D}_1(\ell(\omega_2), m(\omega_2)) - \mathbb{D}_1(\ell(\omega_1), m(\omega_1))| \\
\leq \;& \bigg| \frac{1}{\Gamma_k(a_1)} \int_0^{\omega_1} [\psi'(s)(\psi(\omega_2)-\psi(s))^{\frac{a_1}{k}-1} - (\psi(\omega_1)-\psi(s))^{\frac{a_1}{k}-1}] ds \\
& + \int_{\omega_1}^{\omega_2} \psi'(s)(\psi(\omega_2)-\psi(s))^{\frac{a_1}{k}-1} ds \bigg| \\
& + \frac{(\psi(\omega_2)-\psi(s))^{\frac{\gamma_1}{k}-1} - (\psi(\omega_1)-\psi(s))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)} \bigg\{ P_1 \bigg[\mathbb{A}_2 \bigg(\sum_{j=1}^{q} \frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)} \\
& + \sum_{j=1}^{q} |\xi_j| \frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)} + \sum_{j=1}^{q} |\theta_j||\psi(1)| \frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)} \bigg) \\
& + \frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)} \bigg] + P_2 \bigg[\phi_2 \frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{B}_2\bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}\\
&+\sum_{i=1}^{p}|\mu_i|\psi(1)\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(\phi_2-1+k)}\bigg)\bigg]\bigg\}\\
\leq\;& \frac{P_1}{\Gamma_k(a_1+k)}[2(\psi(\omega_2)-\psi(\omega_1))^{\frac{a_1}{k}}+|(\psi(\omega_2)-\psi(0))^{\frac{a_1}{k}}-(\psi(\omega_1)-\psi(0))^{\frac{a_1}{k}}|]\\
&+\frac{(\psi(\omega_2)-\psi(s))^{\frac{\gamma_1}{k}-1}-(\psi(\omega_1)-\psi(s))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\bigg\{P_1\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}\\
&+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}\bigg)\\
&+\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}\bigg]+P_2\bigg[\phi_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}\\
&+\mathbb{B}_2\bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}\\
&+\sum_{i=1}^{p}|\mu_i|\psi(1)\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}\bigg)\bigg]\bigg\},
\end{aligned}
$$

which tends to zero as $\omega_2-\omega_1 \longrightarrow 0$ and is independent of (ℓ,m). Thus, $\mathbb{D}_1(\ell,m)$ is equicontinuous. Similarly, the equicontinuous property of the operator $\mathbb{D}_2(\ell,m)$ is obtained. Hence, the operator $\mathbb{D}(\ell,m)$ is equicontinuous, and by the Arzelá–Ascoli theorem, it is completely continuous.

Finally, the boundedness property of the set

$$\Xi=\Big\{(\ell,m)\in\mathcal{Z}\times\mathcal{Z};(\ell,m)=\lambda\mathbb{D}(\ell,m),\;\;0\leq\lambda<1\Big\}$$

is shown. Let $(\ell,m)\in\Xi$. Thus, $(\ell,m)=\lambda\mathbb{D}(\ell,m)$, and for all $\omega\in[0,1]$, we have

$$\ell(\omega)=\lambda\mathbb{D}_1(\ell,m)(\omega),\quad m(\omega)=\lambda\mathbb{D}_2(\ell,m)(\omega).$$

Thus,

$$
\begin{aligned}
&|\ell(\omega)|\\
\leq\;&(\phi_0+\phi_1|\ell(\omega)|+\phi_2|m(\omega)|)\bigg\{\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}+\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\\
&\times\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}\\
&+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}\bigg)+\mathbb{B}_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}\bigg]\bigg\}\\
&+(\epsilon_0+\epsilon_1|\ell(\omega)|+\epsilon_2|m(\omega)|)\bigg\{\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_1)}\bigg[\mathbb{A}_2\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}\\
&+\mathbb{B}_2\bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}\\
&+\sum_{i=1}^{p}|\mu_i|\psi(1)\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}\bigg)\bigg]\bigg\}.
\end{aligned}
$$

Thus, we have

$$\|\ell\| \le (\phi_0 + \phi_1\|\ell\| + \phi_2\|m\|)\mathbb{G}_1 + (\epsilon_0 + \epsilon_1\|\ell\| + \epsilon_2\|m\|)\mathbb{G}_2.$$

Similarly, we have

$$\|m\| \le (\phi_0 + \phi_1\|\ell\| + \phi_2\|m\|)\mathbb{G}_3 + (\epsilon_0 + \epsilon_1\|\ell\| + \epsilon_2\|m\|)\mathbb{G}_4.$$

Hence,

$$\begin{aligned}\|\ell\| + \|m\| \quad &\le (\mathbb{G}_1 + \mathbb{G}_3)\phi_0 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_0 + ((\mathbb{G}_1 + \mathbb{G}_3)\phi_1 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_1)\|\ell\| \\ &+((\mathbb{G}_1 + \mathbb{G}_3)\phi_2 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_2)\|m\|.\end{aligned}$$

Therefore,

$$\|(\ell, m)\| \le \frac{(\mathbb{G}_1 + \mathbb{G}_3)\phi_0 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_0}{M_0},$$

in which M_0 is defined by

$$M_0 = \min\Big\{1 - [\mathbb{G}_1 + \mathbb{G}_3)\phi_1 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_1], 1 - [(\mathbb{G}_1 + \mathbb{G}_3)\phi_2 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_2]\Big\}.$$

Thus, Ξ is bounded, and by applying Leray–Schauder alternative [42], the system (5) has at least one solution on $[0, 1]$. □

In the following, our final existence result is presented via Krasnosel'skiĭ's fixed-point theorem [43].

Theorem 3. *Let* $\Pi_1, \Pi_2 : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be two continuous functions satisfying* (H_1). *Moreover, assume that*

(H_3) *There exist* $\beta_1, \beta_2 \in C([0, 1], \mathbb{R})$ *such that*

$$|\Pi_1(\omega, \ell, m)| \le \beta_1(\omega), \quad |\Pi_2(\omega, \ell, m)| \le \beta_2(\omega),$$

for each $\omega \in [0, 1]$ *and* $\ell, m \in \mathbb{R}$.

Then, the system (5) has at least one solution on $[0, 1]$, *provided that*

$$(\mathbb{G}_1^* + \mathbb{G}_3)(s_1 + s_2) + (\mathbb{G}_2 + \mathbb{G}_4^*)(t_1 + t_2) < 1, \tag{22}$$

where $\mathbb{G}_i, i = 2, 3$ *and* $\mathbb{G}_i^*, i = 1, 4$ *are given in (18).*

Proof. We divide the operator $\mathbb{D}$ into four operators $\mathbb{D}_{1,1}, \mathbb{D}_{1,2}, \mathbb{D}_{2,1}, \mathbb{D}_{2,2}$ as follows:

$$\begin{aligned}\mathbb{D}_{1,1}(\ell, m)(\omega) \quad &= \quad {}^k\mathbb{I}^{a_1,\psi}\Pi_1(\omega, \ell(\omega), m(\omega)), \quad \omega \in [0, 1], \\ &\mathbb{D}_{1,2}(\ell, m)(\omega) \\ = \quad &\frac{(\psi(\omega) - \psi(0))^{\frac{\gamma_1}{k} - 1}}{\Theta\Gamma_k(\gamma_1)}\Bigg[\mathbb{A}_2\Bigg(\sum_{j=1}^{q}\int_0^1 {}^k\mathbb{I}^{a_1 - u_j,\psi}\Pi_1(s, \ell(s), m(s))ds \\ &+\sum_{j=1}^{q}\xi_j{}^k\mathbb{I}^{a_1,\psi}\Pi_1(\zeta_j, \ell(\zeta_j), m(\zeta_j)) + \sum_{j=1}^{q}\theta_j\psi(\zeta_j)^k\mathbb{I}^{a_1 - 1,\psi}\Pi_1(\zeta_j, \ell(\zeta_j), m(\zeta_j)) \\ &-{}^k\mathbb{I}^{2-\gamma_2+a_2,\psi}\Pi_2(1, \ell(1), m(1))\Bigg) + \mathbb{B}_2\Bigg(\sum_{i=1}^{p}\int_0^1 {}^k\mathbb{I}^{a_2 - r_i,\psi}\Pi_2(s, \ell(s), m(s))ds \\ &+\sum_{i=1}^{p}\lambda_i{}^k\mathbb{I}^{a_2,\psi}\Pi_2(\eta_i, \ell(\eta_i), m(\eta_i)) + \sum_{i=1}^{p}\mu_i\psi(\eta_i)^k\mathbb{I}^{a_2 - 1,\psi}\Pi_2(\eta_i, \ell(\eta_i), m(\eta_i))\end{aligned}$$

$$-{}^{k}\mathbb{I}^{2-\gamma_1+a_1,\psi}\Pi_1(1,\ell(1),m(1))\bigg)\bigg],\ \ \omega\in[0,1],$$

$$\mathbb{D}_{2,1}(\ell,m)(\omega) = {}^{k}\mathbb{I}^{a_2,\psi}\Pi_2(\omega,\ell(\omega),m(\omega)),\ \ \omega\in[0,1],$$

$$\begin{aligned} &D_{2,2}(\ell,m)(\omega)\\ =\ &\frac{(\psi(\omega)-\psi(0))^{\frac{\gamma_2}{k}-1}}{\Theta\Gamma_k(\gamma_2)}\bigg[\mathbb{A}_1\bigg(\sum_{j=1}^{q}\int_0^1 {}^{k}\mathbb{I}^{a_1-u_j,\psi}\Pi_1(s,\ell(s),m(s))ds\\ &+\sum_{j=1}^{q}\xi_j{}^{k}\mathbb{I}^{a_1,\psi}\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j))+\sum_{j=1}^{q}\theta_j\psi(\zeta_j)^{k}\mathbb{I}^{a_1-1,\psi}\Pi_1(\zeta_j,\ell(\zeta_j),m(\zeta_j))\\ &-{}^{k}\mathbb{I}^{2-\gamma_2+a_2,\psi}\Pi_2(1,\ell(1),m(1))\bigg)+\mathbb{B}_1\bigg(\sum_{i=1}^{p}\int_0^1 {}^{k}\mathbb{I}^{a_2-r_i,\psi}\Pi_2(s,\ell(s),m(s))ds\\ &+\sum_{i=1}^{p}\lambda_i{}^{k}\mathbb{I}^{a_2,\psi}\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i))+\sum_{i=1}^{p}\mu_i\psi(\eta_i)^{k}\mathbb{I}^{a_2-1,\psi}\Pi_2(\eta_i,\ell(\eta_i),m(\eta_i))\\ &-{}^{k}\mathbb{I}^{2-\gamma_1+a_1,\psi}\Pi_1(1,\ell(1),m(1))\bigg)\bigg],\ \ \omega\in[0,1]. \end{aligned}$$

Obviously, $\mathbb{D}_1=\mathbb{D}_{1,1}+\mathbb{D}_{1,2}$ and $\mathbb{D}_2=\mathbb{D}_{2,1}+\mathbb{D}_{2,2}$. Let $\mathbb{B}_\delta=\{(\ell,m)\in\mathcal{Z}\times\mathcal{Z};\|(\ell,m)\|\le\delta\}$ with $\delta\ge(\mathbb{G}_1+\mathbb{G}_3)\|\beta_1\|+(\mathbb{G}_2+\mathbb{G}_4)\|\beta_2\|$. As in Theorem 2, we obtain

$$|\mathbb{D}_{1,1}(\ell_1,m_1)(\omega)+\mathbb{D}_{1,2}(\ell_2,m_2)(\omega)|\le\mathbb{G}_1\|\beta_1\|+\mathbb{G}_2\|\beta_2\|.$$

Analogously, we have

$$|\mathbb{D}_{2,1}(\ell_2,m_2)(\omega)+\mathbb{D}_{2,2}(\ell_2,m_2)(\omega)|\le\mathbb{G}_3\|\beta_1\|+\mathbb{G}_4\|\beta_2\|.$$

Thus, we have

$$|\mathbb{D}_1(\ell_1,m_1)+\mathbb{D}_2(\ell_2,m_2)|\le(\mathbb{G}_1+\mathbb{G}_3)\|\beta_1\|+(\mathbb{G}_2+\mathbb{G}_4)\|\beta_2\|<\delta,$$

which shows that $\mathbb{D}_1(\ell_1,m_1)+\mathbb{D}_2(\ell_2,m_2)\in\mathbb{B}_\delta$.

Now, it is proven that the operator $(\mathbb{D}_{1,2},\mathbb{D}_{2,2})$ is a contraction. Let $(\ell_1,m_1),(\ell_2,m_2)\in B_\delta$. Then, similar to the proof of Theorem 1, we obtain

$$\begin{aligned} &|\mathbb{D}_{1,2}(\ell_2,m_2)(\omega)-\mathbb{D}_{1,2}(\ell_1,m_1)(\omega)|\\ \le\ &\frac{(\psi(1)-\psi(0))^{\frac{\gamma_1}{k}-1}}{|\Theta|\Gamma_k(\gamma_{\gamma_1})}\bigg[\mathbb{A}_2\bigg(\sum_{j=1}^{q}\frac{(\psi(1)-\psi(0))^{\frac{a_1-u_j}{k}}}{\Gamma_k(a_1-u_j+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\\ &+\sum_{j=1}^{q}|\xi_j|\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\\ &+\sum_{j=1}^{q}|\theta_j||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_1-1}{k}}}{\Gamma_k(a_1-1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\\ &+\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_2+a_2}{k}}}{\Gamma_k(2-\gamma_2+a_2+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\bigg)\\ &+\mathbb{B}_2\bigg(\sum_{i=1}^{p}\frac{(\psi(1)-\psi(0))^{\frac{a_2-r_i}{k}}}{\Gamma_k(a_2-r_i+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\\ &+\sum_{i=1}^{p}|\lambda_i|\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}|\mu_i||\psi(1)|\frac{(\psi(1)-\psi(0))^{\frac{a_2-1}{k}}}{\Gamma_k(a_2-1+k)}(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\\
&+\frac{(\psi(1)-\psi(0))^{\frac{2-\gamma_1+a_1}{k}}}{\Gamma_k(2-\gamma_1+a_1+k)}(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\Big]\\
=\;&(s_1\|\ell_2-\ell_1\|+s_2\|m_2-m_1\|)\mathbb{G}_1^*+\mathbb{G}_2(t_1\|\ell_2-\ell_1\|+t_2\|m_2-m_1\|)\\
=\;&(s_1\mathbb{G}_1^*+t_1\mathbb{G}_2)\|\ell_2-\ell_1\|+(s_2\mathbb{G}_1^*+t_2\mathbb{G}_2)\|m_2-m_1\|, \qquad (23)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&|\mathbb{D}_{2,2}(\ell_2,m_2)(\omega)-\mathbb{D}_{2,2}(\ell_1,m_1)(\omega)|\\
\le\;&(s_1\mathbb{G}_3+t_1\mathbb{G}_4^*)\|\ell_2-\ell_1\|+(s_2\mathbb{G}_3+t_2\mathbb{G}_4^*)\|m_2-m_1\|. \qquad (24)
\end{aligned}
$$

From(23) and (24), we obtain

$$
\begin{aligned}
&\|(\mathbb{D}_{1,2},\mathbb{D}_{2,2})(\ell_1,m_1)-(\mathbb{D}_{1,2},\mathbb{D}_{2,2})(\ell_2,m_2)\|\\
\le\;&\Big\{(\mathbb{G}_1^*+\mathbb{G}_3)(s_1+s_2)+(\mathbb{G}_2+\mathbb{G}_4^*)(t_1+t_2)\Big\}(\|\ell_2-\ell_1\|+\|m_2-m_1\|),
\end{aligned}
$$

which shows that $(\mathbb{D}_{1,2},\mathbb{D}_{2,2})$ is a contraction by (22).

By continuity of the functions Π_1 and Π_2, we conclude that the operator $(\mathbb{D}_{1,1},\mathbb{D}_{2,1})$ is also continuous. On the other hand, since

$$\|\mathbb{D}_{1,1}(\ell,m)\|\le\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}\|\beta_1\|,$$

and

$$\|\mathbb{D}_{2,1}(\ell,m)\|\le\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}\|\beta_2\|,$$

we conclude that

$$\|(\mathbb{D}_{1,1},\mathbb{D}_{2,1})(\ell,m)\|\le\frac{(\psi(1)-\psi(0))^{\frac{a_1}{k}}}{\Gamma_k(a_1+k)}\|\beta_1\|+\frac{(\psi(1)-\psi(0))^{\frac{a_2}{k}}}{\Gamma_k(a_2+k)}\|\beta_2\|,$$

which shows that the set $(\mathbb{D}_{1,1},\mathbb{D}_{2,1})B_\delta$ is uniformly bounded. Next, we prove that the set $(\mathbb{D}_{1,1},\mathbb{D}_{2,1})B_\delta$ is equicontinuous. For all $(\ell,m)\in\mathbb{R}\times\mathbb{R}$ and $\omega\in[0,1]$, we have

$$
\begin{aligned}
&|\mathbb{D}_{1,1}(\ell,m)(\omega_2)-\mathbb{D}_{1,1}(\ell,m)(\omega_1)|\\
\le\;&\bigg|\frac{1}{\Gamma_k(a_1)}\int_0^{\omega_1}\psi'(s)(\psi(\omega_2)-\psi(s))^{\frac{a_1}{k}-1}-(\psi(\omega_1)-\psi(s))^{\frac{a_1}{k}-1}]\Pi_1(s,\ell(s),m(s))ds\\
&+\int_{\omega_1}^{\omega_2}\psi'(s)(\psi(\omega_2)-\psi(s))^{\frac{a_1}{k}-1}\Pi_1(s,\ell(s),m(s))ds\bigg|\\
\le\;&\frac{\|\beta_1\|}{\Gamma_k(a_1+k)}[2(\psi(\omega_2)-\psi(\omega_1))^{\frac{a_1}{k}}+|(\psi(\omega_2)-\psi(0))^{\frac{a_1}{k}}-(\psi(\omega_1)-\psi(0))^{\frac{a_1}{k}}|].
\end{aligned}
$$

The right hand of the above inequality tends to zero as $\omega_1\longrightarrow\omega_2$, independently of $(\ell,m)\in\mathbb{B}_\delta$. Similarly, we can indicate that $|\mathbb{D}_{2,1}(\ell,m)(\omega_2)-\mathbb{D}_{2,1}(\ell,m)(\omega_1)|\to 0$ as $\omega_1\longrightarrow\omega_2$ independently of $(\ell,m)\in\mathbb{B}_\delta$. Hence, $|(\mathbb{D}_{1,1},\mathbb{D}_{2,1})(\ell,m)(\omega_2)-(D_{1,1},D_{2,1})(\ell,m)(\omega_1)|\to 0$ as $\omega_1\to\omega_2$. Consequently, the operator $(\mathbb{D}_{1,1},\mathbb{D}_{2,1})$ is equicontinuous, and the Arzelá–Ascoli theorem implies that the operator $(\mathbb{D}_{1,1},\mathbb{D}_{2,1})$ is compact on $\mathbb{B}_\delta$.

Consequently, by Krasnosel'skiĭ's fixed-point theorem, the nonlocal fractional coupled system (5) has at least one solution on $[0,1]$. The proof is finished. □

4. Examples

In this section, by considering the following coupled fractional nonlocal system involving (k,ψ)-Hilfer derivatives and (k,ψ)-Riemann–Liouville integral operators, we show some numerical examples by varying nonlinear functions in the first two equations as

$$\begin{cases} {}^{\frac{17}{16},H}\mathbb{D}^{\frac{15}{8},\frac{9}{20},\sqrt{\omega+1}}\ell(\omega) = \Pi_1(\omega,\ell(\omega),m(\omega)), \ \ \omega\in[0,1], \\ {}^{\frac{17}{16},H}\mathbb{D}^{\frac{7}{4},\frac{7}{20},\sqrt{\omega+1}}m(\omega) = \Pi_2(\omega,\ell(\omega),m(\omega)), \ \ \omega\in[0,1], \\ \ell(0)=0, \quad m(0)=0, \\ {}^{\frac{17}{16}}\mathbb{I}^{\frac{1}{80},\sqrt{\omega+1}}\ell(1) = \int_0^1 {}^{\frac{17}{16},H}\mathbb{D}^{\frac{13}{8},\frac{3}{10},\sqrt{\omega+1}}m(s)ds + \int_0^1 {}^{\frac{17}{16},H}\mathbb{D}^{\frac{3}{2},\frac{1}{4},\sqrt{\omega+1}}m(s)ds \\ \qquad +\frac{1}{13}m\left(\frac{2}{9}\right) + \frac{2}{17}m\left(\frac{7}{9}\right) + \frac{3}{19}m'\left(\frac{2}{9}\right) + \frac{4}{23}m'\left(\frac{7}{9}\right), \\ {}^{\frac{17}{16}}\mathbb{I}^{\frac{19}{160},\sqrt{\omega+1}}m(1) = \int_0^1 {}^{\frac{17}{16},H}\mathbb{D}^{\frac{11}{8},\frac{1}{5},\sqrt{\omega+1}}\ell(s)ds + \int_0^1 {}^{\frac{17}{16},H}\mathbb{D}^{\frac{5}{4},\frac{3}{20},\sqrt{\omega+1}}\ell(s)ds \\ \qquad + \int_0^1 {}^{\frac{17}{16},H}\mathbb{D}^{\frac{9}{8},\frac{1}{10},\sqrt{\omega+1}}m(s)ds + \frac{5}{29}\ell\left(\frac{1}{9}\right) + \frac{6}{31}\ell\left(\frac{5}{9}\right) \\ \qquad + \frac{7}{37}\ell\left(\frac{8}{9}\right) + \frac{8}{41}\ell'\left(\frac{1}{9}\right) + \frac{9}{43}\ell'\left(\frac{5}{9}\right) + \frac{10}{47}\ell'\left(\frac{8}{9}\right). \end{cases} \tag{25}$$

Now, we choose $k = 17/16$, $a_1 = 15/8$, $a_2 = 7/4$, $b_1 = 9/20$, $b_2 = 7/20$, $\psi(\omega) = \sqrt{\omega+1}$, $\gamma_1 = 159/80$, $\gamma_2 = 301/160$, $p = 2$, $q = 3$, $r_1 = 13/8$, $r_2 = 3/2$, $s_1 = 3/10$, $s_2 = 1/4$, $\lambda_1 = 1/13$, $\lambda_2 = 2/17$, $\mu_1 = 3/19$, $\mu_2 = 4/23$, $\eta_1 = 2/9$, $\eta_2 = 7/9$, $u_1 = 11/8$, $u_2 = 5/4$, $u_3 = 9/8$, $v_1 = 1/5$, $v_2 = 3/20$, $v_3 = 1/10$, $\xi_1 = 5/29$, $\xi_2 = 6/31$, $\xi_3 = 7/37$, $\theta_1 = 8/41$, $\theta_2 = 9/43$, $\theta_3 = 10/47$, $\zeta_1 = 1/9$, $\zeta_2 = 5/9$, $\zeta_3 = 8/9$. From the relation $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$, we can find that $\Gamma_k(2) \approx 1.008364771$, $\Gamma_k(\gamma_1) \approx 1.003624884$, $\Gamma_k(\gamma_2) \approx 0.9680918124$, $\Gamma_k(\gamma_2 - r_1) \approx 3.596657165$, $\Gamma_k(\gamma_2 - r_2) \approx 2.386550829$, $\Gamma_k(\gamma_1 - u_1) \approx 1.506571034$, $\Gamma_k(\gamma_1 - u_2) \approx 1.283446586$, $\Gamma_k(\gamma_1 - u_3) \approx 1.138205485$, $\Gamma_k(a_1 + k) \approx 1.811742648$, $\Gamma_k(a_2 + k) \approx 1.637420075$, $\Gamma_k(a_1 - 1 + k) \approx 0.9858518843$, $\Gamma_k(a_2 - 1 + k) \approx 0.9495440171$, $\Gamma_k(a_1 - u_1 + k) \approx 0.9112678992$, $\Gamma_k(a_1 - u_2 + k) \approx 0.9246374757$, $\Gamma_k(a_1 - u_3 + k) \approx 0.9495440171$, $\Gamma_k(a_2 - r_1 + k) \approx 0.9512325926$, $\Gamma_k(a_2 - r_2 + k) \approx 0.9226252531$, $\Gamma_k(2 - \gamma_1 + a_1 + k) \approx 1.830779875$, $\Gamma_k(2 - \gamma_2 + a_2 + k) \approx 1.802339039$. From these details, we can compute that $\mathbb{A}_1 = \mathbb{B}_2 \approx 0.4556580880$, $\mathbb{A}_2 \approx 4.130828938$, $\mathbb{B}_1 \approx 4.705900135$, $|\Theta| \approx 19.23164417$, $\mathbb{G}_1 \approx 0.3589435401$, $\mathbb{G}_1^* \approx 0.2424191099$, $\mathbb{G}_2 \approx 0.03495243265$, $\mathbb{G}_3 \approx 0.4163895731$, $\mathbb{G}_4 \approx 0.04520950990$, and $\mathbb{G}_4^* \approx 0.2733731807$.

(i) If the given nonlinear functions are expressed as

$$\begin{cases} \Pi_1(\omega,\ell,m) = \frac{1}{2(\omega+2)}\left(\frac{\ell^2+2|\ell|}{1+|\ell|}\right) + \frac{3}{2\omega+5}\sin m + \frac{1}{4}, \\ \Pi_2(\omega,\ell,m) = \frac{2}{4\omega+3}\tan^{-1}|\ell| + \frac{1}{\omega+4}\left(\frac{2m^2+3|m|}{1+|m|}\right) + \frac{1}{5}, \end{cases} \tag{26}$$

then, we can check that the Lipschitz properties hold, as

$$|\Pi_1(\omega,\ell_1,m_1) - \Pi_1(\omega,\ell_2,m_2)| \le \frac{1}{2}|\ell_1-\ell_2| + \frac{3}{5}|m_1-m_2|$$

and

$$|\Pi_2(\omega,\ell_1,m_1) - \Pi_1(\omega,\ell_2,m_2)| \le \frac{2}{3}|\ell_1-\ell_2| + \frac{3}{4}|m_1-m_2|,$$

with Lipschitz constants $s_1 = 1/2$, $s_2 = 3/5$, $t_1 = 2/3$, and $t_2 = 3/4$. Therefore, the functions Π_1 and Π_2 satisfy condition (H_1) in Theorem 1. In addition, we can find that

$$(\mathbb{G}_1+\mathbb{G}_3)(s_1+s_2) + (\mathbb{G}_2+\mathbb{G}_4)(t_1+t_2) \approx 0.9664291764 < 1,$$

which means that the inequality in (19) is fulfilled. Hence, the system (25), with functions Π_1, Π_2 given by (26), has a unique solution on $[0,1]$.

(ii) Now, let the functions Π_1 and Π_2 be presented as

$$\begin{cases} \Pi_1(\omega,\ell,m) = \dfrac{1}{11(\omega+1)} + \dfrac{13}{22}\left(\dfrac{(\omega+1)\ell^{2024}}{1+|\ell|^{2023}}\right)e^{-m^2} + \dfrac{3}{11}(\omega+3)m\cos^{16}\ell, \\ \Pi_2(\omega,\ell,m) = \dfrac{1}{13} + \dfrac{5(\omega+2)}{13}\left(\dfrac{|\ell|^{2021}}{1+\ell^{2020}}\right)\sin^8 m + \dfrac{8(\omega+3)}{13\pi}\left(\dfrac{m^2}{1+|m|}\right)\tan^{-1}\ell^4. \end{cases} \tag{27}$$

Observe that these nonlinear functions in (27) do not satisfy the Lipschitz condition. However, we can find the bounds as

$$|\Pi_1(\omega,\ell,m)| \le \frac{1}{11} + \frac{13}{11}|\ell| + \frac{12}{11}|m| \quad \text{and} \quad |\Pi_2(\omega,\ell,m)| \le \frac{1}{13} + \frac{15}{13}|\ell| + \frac{16}{13}|m|.$$

Then, by setting $\phi_0 = 1/11$, $\phi_1 = 13/11$, $\phi_2 = 12/11$, $\epsilon_0 = 1/13$, $\epsilon_1 = 15/13$, and $\epsilon_2 = 16/13$, we can find that

$$(\mathbb{G}_1 + \mathbb{G}_3)\phi_1 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_1 \approx 0.9984146601 < 1$$

and

$$(\mathbb{G}_1 + \mathbb{G}_3)\phi_2 + (\mathbb{G}_2 + \mathbb{G}_4)\epsilon_2 \approx 0.9963920897 < 1.$$

Then, for problem (25), with functions Π_1, Π_2 given by (26), all assumptions of Theorem 2 are satisfied. Consequently, the coupled fractional nonlocal system (25)–(27) has at least one solution on $[0,1]$.

(iii) Let Π_1 and Π_2 be given nonlinear functions defined by

$$\begin{cases} \Pi_1(\omega,\ell,m) = f(\omega) + \dfrac{|\ell|}{p+|\ell|} + \dfrac{|m|}{p+|m|}, \\ \Pi_2(\omega,\ell,m) = g(\omega) + \dfrac{|\ell|}{p+|\ell|} + \dfrac{|m|}{p+|m|}, \end{cases} \tag{28}$$

where f, g are arbitrary functions and $p > 0$ is a constant.

We have

$$|\Pi_1(\omega,\ell,m)| \le |f(\omega)| + 2 := \beta_1(\omega) \quad \text{and} \quad |\Pi_1(\omega,\ell,m)| \le |g(\omega)| + 2 := \beta_1(\omega),$$

which means that the condition (H_3) in Theorem 3 is fulfilled. Moreover, the condition (H_1) is satisfied with $s_1 = s_2 = t_1 = t_2 = 1/p$. Then, if $p > 1.191908466$, the inequality (22) holds. This implies that system (25), with functions Π_1, Π_2 given by (28), has at least one solution in $[0,1]$. In addition, if $p > 1.710990111$, the inequality in (19) is true, and we can say that the nonlocal boundary value problem of system (25), with functions Π_1, Π_2 given by (28), has a unique solution in the interval $[0,1]$.

5. Conclusions

In this paper, we establish existence and uniqueness results for a coupled system of (k,ψ)-Hilfer fractional differential equations, involving, in boundary conditions, (k,ψ)-Hilfer fractional derivatives and (k,ψ)-Riemann–Liouville fractional integrals. Standard fixed-point theorems, such as Banach contraction mapping principle, Leray–Schaude alternative, and Krasnosel'skiĭ's fixed-point theorem, are applied to derive our main results. Numerical examples are constructed to illustrate the obtained theoretical results. Our results are new and contribute significantly to enriching the existing results on (k,ψ)-Hilfer coupled systems in the literature.

It is noteworthy that a common characteristic of boundary value problems and coupled systems of (k,ψ)-Hilfer fractional differential operators of order in $(1,2]$, is the zero initial condition, which is necessary to obtain a well-defined solution. Thus, we cannot study some classes of Hilfer fractional boundary value problems or coupled systems, including, for example, anti-periodic boundary conditions and separated or non-separated boundary conditions. As a future plan, we will try to overcome this difficulty and study Hilfer fractional boundary value problems or systems subject to boundary conditions with nonzero initial conditions as the above-mentioned boundary conditions.

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