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A Note on Averaging Principles for Fractional Stochastic Differential Equations

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Abstract: Over the past few years, many scholars began to study averaging principles for fractional stochastic differential equations since they can provide an approximate analytical method to reduce such systems. However, in the most previous studies, there is a misunderstanding of the standard form of fractional stochastic differential equations, which consequently causes the wrong estimation of the convergence rate. In this note, we take fractional stochastic differential equations with Lévy noise as an example to clarify these two issues. The corrections herein have no effect on the main proofs except the two points mentioned above. The innovation of this paper lies in three aspects: (i) the standard form of the fractional stochastic differential equations is derived under natural time scale; (ii) it is first proved that the convergence interval and rate are related to the fractional order; and (iii) the presented results contain and improve some well known research achievements.

Keywords: averaging principle; fractional stochastic differential equations; time scale; convergence rate

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1. Introduction

As an example, this note considers Bogolyubov's averaging principle for the following Caputo fractional stochastic differential system with a fast oscillating right-hand side and Lévy noise:

$$\begin{cases} D_t^\alpha u(t) = f(\frac{t}{\varepsilon}, u(t)) + g(\frac{t}{\varepsilon}, u(t))\frac{dw(t)}{dt} + \frac{\int_Z h(\frac{t}{\varepsilon}, u(t), z)\tilde{N}(dt, dz)}{dt}, \\ u(0) = u_0, \end{cases} \tag{1}$$

where D_t^α denotes Caputo fractional derivative; $\alpha \in (\frac{1}{2}, 1]$, $u(t)$ denotes an n-dimensional stochastic process; and $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : [0,T] \times \mathbb{R}^n \times Z \to \mathbb{R}^{n\times d}$ are all measurable functions. $w(t)$ denotes an m-dimensional Brownian motion in a complete probability space. $\tilde{N}(dt, dz) = N(dt, dz) - v(dz)dt$ is a d-dimensional compensated jump measure, in which $N(dt, dz)$ is a real-valued Poisson counting measure with a σ-finite intensity measure $v(dz)$ on a measurable set $Z \in \mathcal{B}(\mathbb{R}^d - 0)$. u_0 is a $\mathcal{F}_0$ measurable $\mathbb{R}^n$-valued random variable satisfying $\mathbb{E}|u_0|^2 < \infty$.

Note that it is difficult to study Equation (1) directly; this is because $f(\frac{t}{\varepsilon}, \cdot), g(\frac{t}{\varepsilon}, \cdot), h(\frac{t}{\varepsilon}, \cdot, z)$ are fast time oscillating forces. It is generally known that a highly oscillating system can be "averaged" under suitable conditions. There are three different types of averaging principles, i.e., the so-called first Bogolyubov theorem, the second Bogolyubov theorem, and the global averaging principle [1]. Up to now, all the results on averaging principles of fractional stochastic differential equations are first Bogolyubov theorems, and this work is no exception.

In recent years, averaging principles for fractional stochastic differential equations (FSDEs) have attracted great attention since they can provide approximate analytical methods to reduce such systems [2]. Xu et al. [3,4] developed averaging results for various types of FSDEs early. Luo and Zhu [5] explored an averaging result for fractional stochastic delay differential equations (FSDDEs). Taking the correlated noise into account, an averaging theorem for FSDEs with fractional Brownian motion was demonstrated by Duan et al. [6]. After considering the external excitation with jumps, an averaging principle for Hilfer FSDDEs driven by Poisson noise was subsequently proven by Ahmed and Zhu [7]. Later, Guo et al. [8] obtained an averaging result for FSDEs by applying distinct techniques. After that, with the aid of an alternative processing skill, an averaging principle for FSDEs was developed by Xiao et al. [9]. Introducing the neutral term, Shen et al. [10] presented the averaging result for neutral FSDEs. Xia and Yan et al. [11] derived an averaging theorem for a new kind of FSDE. Considering the different types of fractional derivatives, Makhlouf et al. [12] showed an averaging principle for Hadarmard Itô-Doob FSDEs. Liu et al. [13] established an averaging result for Caputo–Hadamard FSDEs. Recently, Yang et al. [14] developed an averaging principle for Hilfer FSDEs in the infinite-dimensional space. Li and Wang [15] presented an averaging theorem for Caputo FSDDEs driven by Gaussian white noise. Bai et al. [16] proved an averaging result for Caputo FSDDEs with Poisson noise. Overall, the studies of averaging principles for FSDEs gained rapid development with tremendous achievements. Specifically, Caputo fractional derivative can maintain the initial value condition of the original function and remain consistent with classical differentiation when the fractional order is an integer. Consequently, the studies on the average principles of Caputo FSDEs are the most extensive. But the biggest flaw in most of the existing related research studies is that they use the same standard form of the fractional differential equation as the integer-order case, which results in incorrect estimations of the convergence rate.

Accordingly, taking fractional stochastic differential equations with Lévy noise as an example, this note elaborates the standard form of fractional stochastic differential equations and the estimation of the convergence rate. Furthermore, the relationship between the averaging principles of fractional stochastic differential equations and integer-order stochastic differential equations is commented.

2. Preliminaries

Definition 1 ([4])**.** *Suppose that f is a Lebesgue integrable function, the α-order integral of f is given by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds, \ t \in [t_0, \infty),$$

where $0 < \alpha < 1$ and Γ is a gamma function.

Definition 2 ([4])**.** *The α-order Caputo-type fractional derivative of f is determined by*

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(s)}{(t-s)^{\alpha}} ds, \ t \in [t_0, \infty)$$

for arbitrary given $0 < \alpha < 1$.

Next, we show the time scale change property for Caputo fractional derivative, which is a key tool to obtain the standard form of Caputo fractional differential equations.

Lemma 1. *(Time scale change property). Let the time scale $t = \beta\tau$, then*

$$D_{\tau}^{\alpha} f(\beta\tau) = \beta^{\alpha} D_t^{\alpha} f(t).$$

Proof. Based on Definition 2, one can obtain

$$
\begin{aligned}
D_\tau^\alpha f(\beta\tau) =& \frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{f'(\beta s)}{(\tau-s)^\alpha} ds \\
\underline{\underline{u=\beta s}}& \frac{\beta^\alpha}{\Gamma(1-\alpha)} \int_0^{\beta\tau} \frac{\beta f'(u)}{(\beta\tau-u)^\alpha} \frac{1}{\beta} du \\
=& \frac{\beta^\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{f'(u)}{(t-u)^\alpha} du \\
=& \beta^\alpha D_t^\alpha f(t),
\end{aligned}
$$

and then the assertion of Lemma 1 is proved. □

Suppose the following general assumptions hold throughout this paper.

Condition 1. *Linear growth condition:*

$$|f(t,u)|^2 + |g(t,u)|^2 + \int_Z |h(t,u,z)|^2 v(dz) \le K(1+|u|^2),$$

where K is a positive constant.

Condition 2. *The Lipschitz condition: there exists a constant $L > 0$ such that*

$$|f(t,u) - f(t,x)|^2 + |g(t,u) - g(t,x)|^2 + \int_Z |h(t,u,z) - h(t,x,z)|^2 v(dz) \le L|u-x|^2.$$

Then, following Ref. [17], we have no problem giving the existence and uniqueness result of system (1).

Theorem 1. *Under Conditions 1–2, there exists a unique solution to system (1) on* $[0,T]$.

At the end of this section, we recall the generalized Gronwall inequality proposed in Ref. [18], which is the key tool to prove our main result.

Lemma 2. *Assume that $\alpha > 0$ and $J = [0,S), S \le \infty$. Let function $a(t)$ be non-negative and locally integrable on J, while function $b(t)$ be non-negative, nondecreasing, continuous and bounded on J. In case $y(t)$ is non-negative and locally integrable, the following inequality holds:*

$$y(t) \le a(t) + b(t) \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

then,

$$y(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right] ds, t \in J.$$

Furthermore, if $a(t)$ is nondecreasing on J, then

$$y(t) \le a(t) \mathrm{E}_\alpha (b(t)\Gamma(\alpha) t^\alpha),$$

where $\mathrm{E}_\alpha(\cdot)$ is the Mittag–Leffler function.

3. Main Results

This section is dedicated to presenting the deduction procedure of the correct standard form, the corresponding convergence interval and rate of Caputo fractional stochastic differential equations. At the beginning, we demonstrate the standard form of system (1).

Let $\frac{t}{\varepsilon} = \tau$, and with the aid of Lemma 1, system (1) can be rewritten as:

$$\begin{cases} \varepsilon^{-\alpha} D_\tau^\alpha u(\varepsilon\tau) = f(\tau, u(\varepsilon\tau)) + g(\tau, u(\varepsilon\tau)) \frac{dw(\varepsilon\tau)}{\varepsilon d\tau} + \frac{\int_Z h(\tau, u(\varepsilon\tau), z) \tilde{N}(\varepsilon d\tau, dz)}{\varepsilon d\tau}, \\ u(0) = u_0, \end{cases}$$

note that $dw(\varepsilon\tau) = \sqrt{\varepsilon} dw(\tau)$; denote by $u(\varepsilon\tau) = u_\varepsilon(\tau)$, then,

$$\begin{cases} D_\tau^\alpha u_\varepsilon(\tau) = \varepsilon^\alpha f(\tau, u_\varepsilon(\tau)) + \varepsilon^{\alpha-\frac{1}{2}} g(\tau, u_\varepsilon(\tau)) \frac{dw(\tau)}{d\tau} + \varepsilon^{\alpha-\frac{1}{2}} \frac{\int_Z h(\tau, u_\varepsilon(\tau), z) \tilde{N}(d\tau, dz)}{d\tau}, \\ u_\varepsilon(0) = u_0, \end{cases}$$

without loss of generality, one can denote $\tau := t$. In a word, with the natural time scaling $\frac{t}{\varepsilon} \mapsto t$ and Lemma 1, the standard form of fractional stochastic differential equations is given as

$$\begin{aligned} D_t^\alpha u_\varepsilon(t) =& \varepsilon^\alpha f(t, u_\varepsilon(t)) + \varepsilon^{\alpha-\frac{1}{2}} g(t, u_\varepsilon(t)) \frac{dw(t)}{dt} + \varepsilon^{\alpha-\frac{1}{2}} \frac{\int_Z h(t, u_\varepsilon(t), z) \tilde{N}(dt, dz)}{dt}, \\ u_\varepsilon(0) =& u_0; \end{aligned} \tag{2}$$

then, we can rewrite Equation (2) in its integral form as

$$\begin{aligned} u_\varepsilon(t) =& u_0 + \frac{\varepsilon^\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_\varepsilon(s)) ds \\ &+ \frac{\varepsilon^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_\varepsilon(s)) dw(s) \\ &+ \frac{\varepsilon^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_Z h(s, u_\varepsilon(s), z) \tilde{N}(ds, dz). \end{aligned} \tag{3}$$

Remark 1. *The standard form of Caputo fractional stochastic differential equations under natural time scale should be obtained by the above derivation. It should not be the same as the case of integer-order stochastic differential equations unless* $\alpha = 1$.

To Equation (1), we associate the following averaged system:

$$\begin{aligned} D_t^\alpha x_\varepsilon(t) =& \varepsilon^\alpha \overline{f}(x_\varepsilon(t)) + \varepsilon^{\alpha-\frac{1}{2}} \overline{g}(x_\varepsilon(t)) \frac{dw(t)}{dt} + \varepsilon^{\alpha-\frac{1}{2}} \frac{\int_Z \overline{h}(x_\varepsilon(t), z) \tilde{N}(dt, dz)}{dt}, \\ x_\varepsilon(0) =& u_0, \end{aligned} \tag{4}$$

where the functions $\overline{f}(\cdot), \overline{g}(\cdot), \overline{h}(\cdot)$ are all measurable functions meeting the hypotheses below:

Hypothesis 1 (H1).

$$\frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left| f(s,u) - \overline{f}(u) \right|^2 ds \le \gamma_1(t)(1+|u|^2),$$

Hypothesis 2 (H2).

$$\frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} |g(s,u) - \overline{g}(u)|^2 ds \le \gamma_2(t)(1+|u|^2),$$

Hypothesis 3 (H3).

$$\frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \int_Z \left| h(s,u,z) - \overline{h}(u,z) \right|^2 v(dz) ds \le \gamma_3(t)(1+|u|^2),$$

where functions $\gamma_i(t)$ are positive and bounded with $\lim_{t\to\infty} \gamma_i(t) = 0, i = 1, 2, 3.$

Remark 2. *When $\alpha = 1$, hypotheses $(H1)$–$(H3)$ degenerate into general assumptions of the case of integer order [19].*

In this way, we can prove that when the oscillating frequency speeds up ($\varepsilon \to 0^+$), the original solution $u_\varepsilon(t)$ converges to the solution $x_\varepsilon(t)$ of system (4).

Remark 3. *The conditions $\lim_{t\to\infty} \gamma_j(t) = 0$ guarantee the existence and uniqueness of the solution to the averaged system (4). This is because it can ensure the functions $\overline{f}(\cdot), \overline{g}(\cdot), \overline{h}(\cdot)$ satisfy the Lipschitz condition and the linear growth condition.*

For example, for $\forall u, x \in \mathbb{R}^n$, by Condition 2 and (H1), one can reach

$$
\begin{aligned}
\left|\overline{f}(u) - \overline{f}(x)\right|^2 \leq & \frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left|\overline{f}(u) - \overline{f}(x)\right|^2 ds \\
= & \frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left|\overline{f}(u) - f(s,u) + f(s,u) - f(s,x) + f(s,x) - \overline{f}(x)\right|^2 ds \\
\leq & \frac{3}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left|f(s,u) - \overline{f}(u)\right|^2 ds \\
& + \frac{3}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} |f(s,u) - f(s,x)|^2 ds \\
& + \frac{3}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left|f(s,x) - \overline{f}(x)\right|^2 ds \\
\leq & 3\gamma_1(t)(1+|u|^2) + \frac{3L}{2\alpha-1}|u-x|^2 + 3\gamma_1(t)(1+|x|^2),
\end{aligned}
$$

taking $t \to \infty$, we demonstrate that $\overline{f}$ meets the Lipschitz condition.

Moreover, Condition 1 and (H2) give that

$$
\begin{aligned}
\left|\overline{f}(u)\right|^2 \leq & \frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left|\overline{f}(u)\right|^2 ds \\
\leq & \frac{2}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left|f(s,u) - \overline{f}(u)\right|^2 ds + \frac{2}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} |f(s,u)|^2 ds \\
\leq & 2\gamma_1(t)(1+|u|^2) + \frac{2K}{2\alpha-1}(1+|u|^2);
\end{aligned}
$$

as $t \to \infty$, it is shown that $\overline{f}$ meets the linear growth condition. Similarly, we can prove that $\overline{g}(\cdot), \overline{h}(\cdot)$ also meet the Lipschitz condition and the linear growth condition. All the above assertions guarantee the existence and uniqueness of the averaged system (4).

Based on Definition 1, one can give the integral form for system (4):

$$
\begin{aligned}
x_\varepsilon(t) = & u_0 + \frac{\varepsilon^\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(x_\varepsilon(s)) ds \\
& + \frac{\varepsilon^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{g}(x_\varepsilon(s)) dw(s) \\
& + \frac{\varepsilon^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_Z \overline{h}(x_\varepsilon(s), z) \tilde{N}(ds, dz).
\end{aligned}
\tag{5}
$$

At this point, Bogolyubov's averaging principle for the Caputo fractional stochastic differential system with a fast oscillating right-hand side and Lévy noise is determined.

Theorem 2. *Under Conditions 1, 2 and (H1)–(H3), there exist $P > 0, 0 < \varepsilon_1 \ll 1$ and $\gamma \in (0, 2\alpha - 1)$ such that for all $\varepsilon \in (0, \varepsilon_1], t \in [0, P\varepsilon^{-\gamma}] \subseteq [0, T]$,*

$$\mathbb{E} \sup_{0 \le t \le P\varepsilon^{-\gamma}} |u_\varepsilon(t) - x_\varepsilon(t)|^2 \le \rho,$$

for arbitrary $\rho > 0$.

Remark 4. *It must be pointed out that the order of convergence interval is $\varepsilon^{-\gamma}$, but $\gamma \in (0, 2\alpha - 1)$ rather than $\gamma \in (0, 1)$. This is because the the standard form of the Caputo fractional differential equations has been modified.*

Based on Refs. [1,17], the following lemma can be achieved.

Lemma 3. *Let the conditions of Theorem 2 hold; there exists a positive constant C such that*

$$\mathbb{E} \sup_{0 \le t \le T} |x_\varepsilon(t)|^2 \le C,$$

for any $T < \infty$.

Next, we briefly show the proof process of Theorem 2, and then demonstrate the impact of incorrect standard form on the convergence interval and rate.

Proof. By Equations (3) and (5) and the elementary inequality, one can easily obtain:

$$\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le \delta} |u_\varepsilon(t) - x_\varepsilon(t)|^2 \\
&\le \frac{3\varepsilon^{2\alpha}}{\Gamma^2(\alpha)} \mathbb{E} \sup_{0 \le t \le \delta} \left| \int_0^t (t-s)^{\alpha-1} [f(s, u_\varepsilon(s)) - \overline{f}(x_\varepsilon(s))] ds \right|^2 \\
&+ \frac{3\varepsilon^{2\alpha-1}}{\Gamma^2(\alpha)} \mathbb{E} \sup_{0 \le t \le \delta} \left| \int_0^t (t-s)^{\alpha-1} [g(s, u_\varepsilon(s)) - \overline{g}(x_\varepsilon(s))] dw(s) \right|^2 \\
&+ \frac{3\varepsilon^{2\alpha-1}}{\Gamma^2(\alpha)} \mathbb{E} \sup_{0 \le t \le \delta} \left| \int_0^t (t-s)^{\alpha-1} \int_Z [h(s, u_\varepsilon(s), z) - \overline{h}(x_\varepsilon(s), z)] \tilde{N}(ds, dz) \right|^2 \\
&:= \frac{3\varepsilon^{2\alpha}}{\Gamma^2(\alpha)} I_1 + \frac{3\varepsilon^{2\alpha-1}}{\Gamma^2(\alpha)} I_2 + \frac{3\varepsilon^{2\alpha-1}}{\Gamma^2(\alpha)} I_3,
\end{aligned} \tag{6}$$

where $\delta \in [0, T]$, I_1, I_2 and I_3 will be processed separately.

Firstly, the elementary inequality, Cauchy–Schwarz inequality, Condition 2, (H1) and Lemma 3 yield that

$$\begin{aligned}
I_1 \le& 2\mathbb{E} \sup_{0 \le t \le \delta} \left| \int_0^t (t-s)^{\alpha-1} [f(s, u_\varepsilon(s)) - f(s, x_\varepsilon(s))] ds \right|^2 \\
&+ 2\mathbb{E} \sup_{0 \le t \le \delta} \left| \int_0^t (t-s)^{\alpha-1} [f(s, x_\varepsilon(s)) - \overline{f}(x_\varepsilon(s))] ds \right|^2 \\
\le& 2\delta L \int_0^\delta (\delta - s)^{2\alpha-2} (\mathbb{E} \sup_{0 \le \theta \le s} |u_\varepsilon(\theta) - x_\varepsilon(\theta)|^2) ds \\
&+ 2\mathbb{E} \sup_{0 \le t \le \delta} t^{2\alpha} \left(\frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2\alpha-2} \left| f(s, x_\varepsilon(s)) - \overline{f}(x_\varepsilon(s)) \right|^2 ds \right) \\
\le& 2\delta L \int_0^\delta (\delta - s)^{2\alpha-2} (\mathbb{E} \sup_{0 \le \theta \le s} |u_\varepsilon(\theta) - x_\varepsilon(\theta)|^2) ds + 2\delta^{2\alpha} (\sup_{0 \le t \le \delta} \gamma_1(t))(1 + \mathbb{E} \sup_{0 \le t \le \delta} |x_\varepsilon(t)|^2) \\
\le& 2\delta L \int_0^\delta (\delta - s)^{2\alpha-2} (\mathbb{E} \sup_{0 \le \theta \le s} |u_\varepsilon(\theta) - x_\varepsilon(\theta)|^2) ds + 2C_{11}\delta^{2\alpha},
\end{aligned} \tag{7}$$

where $C_{11} = (1+C)\sup\limits_{0\le t\le\delta}\gamma_1(t)$ is a constant.

For I_2, with the virtue of the Burkholder–Davis–Gundy inequality, the elementary inequality, Condition 2, (H2) and Lemma 3, one has

$$\begin{aligned} I_2 \le& 4\mathbb{E}\int_0^\delta (\delta-s)^{2\alpha-2}|g(s,u_\varepsilon(s))-\overline{g}(x_\varepsilon(s))|^2 ds \\ \le& 8\mathbb{E}\int_0^\delta (\delta-s)^{2\alpha-2}|g(s,u_\varepsilon(s))-g(s,x_\varepsilon(s))|^2 ds \\ &+8\mathbb{E}\int_0^\delta (\delta-s)^{2\alpha-2}|g(s,u_\varepsilon(s))-\overline{g}(x_\varepsilon(s))|^2 ds \\ \le& 8L\int_0^\delta (\delta-s)^{2\alpha-2}(\mathbb{E}\sup_{0\le\theta\le s}|u_\varepsilon(\theta)-x_\varepsilon(\theta)|^2)ds + 8C_{21}\delta^{2\alpha-1}, \end{aligned} \tag{8}$$

where $C_{21} = (1+C)\sup\limits_{0\le t\le\delta}\gamma_2(t)$ is a constant.

For the last term I_3, Doob's martingale inequality, Hölder's inequality, Condition 2, $(H3)$ and Lemma 3 give that

$$\begin{aligned} I_3 \le& 2\mathbb{E}\left|\int_0^\delta (\delta-s)^{\alpha-1}\int_Z [h(s,u_\varepsilon(s),z)-\overline{h}(x_\varepsilon(s),z)]\tilde{N}(ds,dz)\right|^2 \\ \le& 4\mathbb{E}\int_0^\delta (\delta-s)^{2\alpha-2}\int_Z |h(s,u_\varepsilon(s),z)-h(s,x_\varepsilon(s),z)|^2 v(dz)ds \\ &+4\mathbb{E}\int_0^\delta (\delta-s)^{2\alpha-2}\int_Z \left|h(s,x_\varepsilon(s),z)-\overline{h}(x_\varepsilon(s),z)\right|^2 v(dz)ds \\ \le& 4L\int_0^\delta (\delta-s)^{2\alpha-2}(\mathbb{E}\sup_{0\le\theta\le s}|u_\varepsilon(\theta)-x_\varepsilon(\theta)|^2)ds + 4C_{31}\delta^{2\alpha-1}, \end{aligned} \tag{9}$$

where $C_{31} = (1+C)\sup\limits_{0\le t\le\delta}\gamma_3(t)$ is a constant.

By substituting inequalities (7)–(9) into inequality (6), it can be concluded that

$$\begin{aligned} \mathbb{E}\sup_{0\le t\le\delta}|u_\varepsilon(t)-x_\varepsilon(t)|^2 \le& \frac{6L\varepsilon^{2\alpha}\delta+36L\varepsilon^{2\alpha-1}}{\Gamma^2(\alpha)}\int_0^\delta (\delta-s)^{2\alpha-2}(\mathbb{E}\sup_{0\le\theta\le s}|u_\varepsilon(\theta)-x_\varepsilon(\theta)|^2)ds \\ &+\frac{6C_{11}\varepsilon^{2\alpha}\delta^{2\alpha}+24C_{21}\varepsilon^{2\alpha-1}\delta^{2\alpha-1}+12C_{31}\varepsilon^{2\alpha-1}\delta^{2\alpha-1}}{\Gamma^2(\alpha)}. \end{aligned}$$

Then, by Lemma 2, one can obtain

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le\delta}|u_\varepsilon(t)-x_\varepsilon(t)|^2 \\ &\le \frac{6C_{11}\varepsilon^{2\alpha}\delta^{2\alpha}+24C_{21}\varepsilon^{2\alpha-1}\delta^{2\alpha-1}+12C_{31}\varepsilon^{2\alpha-1}\delta^{2\alpha-1}}{\Gamma^2(\alpha)}\mathrm{E}_{2\alpha-1}\left(\frac{6L\varepsilon^{2\alpha}\delta^{2\alpha}+36L\varepsilon^{2\alpha-1}\delta^{2\alpha-1}}{\Gamma^2(\alpha)}\Gamma(2\alpha-1)\right), \end{aligned}$$

where $\mathrm{E}_{2\alpha-1}(\cdot)$ is the Mittag–Leffler function.

Thus, there exist $P>0$ and $\gamma\in(0,2\alpha-1)$ such that

$$\mathbb{E}\sup_{0\le t\le P\varepsilon^{-\gamma}}|u_\varepsilon(t)-x_\varepsilon(t)|^2 \le Q\varepsilon^{2\alpha-1-\gamma},$$

where

$$\begin{aligned} Q =& \frac{6C_{11}P^{2\alpha}\varepsilon^{1+\gamma-2\alpha\gamma}+24C_{21}\varepsilon^{2\alpha-1}P^{2\alpha-1}\varepsilon^{2\gamma(1-\alpha)}+12C_{31}P^{2\alpha-1}\varepsilon^{2\gamma(1-\alpha)}}{\Gamma^2(\alpha)} \\ &\times\mathrm{E}_{2\alpha-1}\left(\frac{6LP^{2\alpha}\varepsilon^{2\alpha(1-\gamma)}+36LP^{2\alpha-1}\varepsilon^{(2\alpha-1)(1-\gamma)}}{\Gamma^2(\alpha)}\Gamma(2\alpha-1)\right) \end{aligned}$$

is a constant. This completes the proof. □

Remark 5. *When $\alpha = 1$, the obtained result is in agreement with the well-known result of Ref. [19]. From this perspective, the averaging principle for fractional stochastic differential equations contains the case of integer-order stochastic differential equations, which is a natural result. However, it must be noted that the convergence rate is $\varepsilon^{2\alpha-1-\gamma}$ instead of $\varepsilon^{1-\gamma}$ unless $\alpha = 1$. In other words, the convergence rate is related to the fractional order; this point is ignored in the existing literature due to the misuse of the standard form of fractional stochastic differential equations [3–7,9,10,15,16].*

Remark 6. *In this note, Hypotheses (H1)–(H3) in the fractional sense were used to derive the estimations to avoid the integral involving singular kernel. Obviously, these three conditions can also be replaced by the following forms:*

$(H1')$

$$\frac{1}{t}\int_0^t \left|f(s,u) - \overline{f}(u)\right|^2 ds \leq \gamma_1(t)(1+|u|^2),$$

$(H2')$

$$\frac{1}{t}\int_0^t |g(s,u) - \overline{g}(u)|^4 ds \leq \gamma_2(t)(1+|u|^4),$$

$(H3')$

$$\frac{1}{t}\int_0^t \int_Z \left|h(s,u,z) - \overline{h}(u,z)\right|^4 v(dz)ds \leq \gamma_3(t)(1+|u|^4),$$

where functions $\gamma_i(t)$ are positive and bounded with $\lim_{t\to\infty} \gamma_i(t) = 0, i = 1,2,3$. In this way, hypotheses $(H1')$, $(H2')$ and $(H3')$ are consistent with those in Ref. [9].

Remark 7. *Although the proof of the main theorem is not new enough, the presented results are crucial to averaging principles for fractional differential equations. The time scale change property of Caputo fractional derivative and the standard form of the fractional differential equations were derived in detail. Furthermore, its difference from the integer-order case was discussed.*

4. Conclusions

This note demonstrated the Bogolyubov's averaging principle for the Caputo fractional stochastic differential system with a fast oscillating right-hand side and Lévy noise. But it must be emphasized that system (1) is just an illustrative example; the purpose here is to correct the standard form of Caputo fractional stochastic differential equations and the estimation of the convergence interval and rate. In this sense, most of the existing results on averaging principles for Caputo fractional differential equations can be improved by the presented techniques.

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References

1. Cheng, M.Y.; Liu, Z.X.; Röckner, M. Averaging principle for stochastic complex Ginzburg-Landau equations. *J. Differ. Equ.* **2023**, *368*, 58–104. [CrossRef]
2. Liu, J.K.; Wei, W.; Xu, W. An averaging principle for stochastic fractional differential equations driven by fBm involving impulses. *Fractal. Fract.* **2022**, *6*, 256. [CrossRef]
3. Xu, W.J.; Xu, W.; Zhang, S. The averaging principle for stochastic differential equations with Caputo fractional derivative. *Appl. Math. Lett.* **2019**, *93*, 79–84. [CrossRef]
4. Xu, W.J.; Duan, J.Q.; Xu, W. An averaging principle for fractional stochastic differential equations with lévy noise. *Chaos* **2020**, *30*, 083126. [CrossRef] [PubMed]
5. Luo, D.F.; Zhu, Q.X.; Luo, Z.G. An averaging principle for stochastic fractional differential equations with time-delays. *Appl. Math. Lett.* **2020**, *105*, 106290. [CrossRef]
6. Duan, P.J.; Hao, L.; Li, J.; Zhang, P. Averaging principle for Caputo fractional stochastic differentialequations driven by fractional Brownian motion with delays. *Complexity* **2021**, *2021*, 6646843. [CrossRef]
7. Ahmed, H.M.; Zhu, Q.X. The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps. *Appl. Math. Lett.* **2021**, *105*, 106755. [CrossRef]
8. Guo, Z.K.; Hu, J.H.; Yuan, C.G. Averaging principle for a type of Caputo fractional stochastic differential equations. *Chaos* **2021**, *31*, 053123. [CrossRef]
9. Xiao, G.L.; Fečkan, M.; Wang, J.R. On the averaging principle for stochastic differential equations involving Caputo fractional derivative. *Chaos* **2022**, *32*, 101105. [CrossRef]
10. Shen, G.J.; Wu, J.L.; Xiao, R.D.; Yin, H. An averaging principle for neutral stochastic fractional order differential equations with variable delays driven by Lévy noise. *Stoch. Dynam.* **2022**, *22*, 2250009. [CrossRef]
11. Xia, X.Y.; Chen, Y.M.; Yan, L.T. Averaging principle for a class of time-fractal-fractional stochastic differential Equations. *Fractal. Fract.* **2022**, *6*, 558. [CrossRef]
12. Makhlouf, A.B.; Mchiri, L.; Arfaoui, H.; Dhahri, S.; El-Hady, E.S.; Cherif, B. Hadamard Itô-Doob Stochastic Fractional Order Systems Equations. *Discrete. Cont. Dynam. Syst.* **2023**, *16*, 2060–2074. [CrossRef]
13. Liu, J.K.; Wei, W.; Wang, J.B.; Xu, W. Limit behavior of the solution of Caputo-Hadamard fractional stochastic differential equations. *Appl. Math. Lett.* **2023**, *140*, 108586. [CrossRef]
14. Yang, M.; Lv, T.; Wang, Q.R. The averaging principle for Hilfer fractional stochastic evolution equations with Lévy noise. *Fractal. Fract.* **2023**, *7*, 701. [CrossRef]
15. Li, M.M.; Wang, J.R. The existence and averaging principle for Caputo fractional stochastic delay differential systems. *Fract. Calc. Appl. Anal.* **2023**, *26*, 893–912. [CrossRef]
16. Bai, Z.Y.; Bai, C.Z. The existence and averaging principle for Caputo fractional stochastic delay differential systems with Poisson jumps. *Axioms* **2024**, *13*, 68. [CrossRef]
17. Umamaheswari, P.; Balachandran, K.; Annapoorani, N. Existence and stability results for Caputo fractional stochastic differential equations with Lévy noise. *Filomat* **2020**, *34*, 1739–1751. [CrossRef]
18. Ye, H.P.; Gao, J.M.; Ding, Y.S. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
19. Xu, Y.; Duan, J.Q.; Xu, W. An averaging principle for stochastic dynamical systems with Lévy noise. *Phys. D* **2011**, *240*, 1395–1401. [CrossRef]