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On Solutions of Two Post-Quantum Fractional Generalized Sequential Navier Problems: An Application on the Elastic Beam

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Abstract: Fractional calculus provides some fractional operators for us to model different real-world phenomena mathematically. One of these important study fields is the mathematical model of the elastic beam changes. More precisely, in this paper, based on the behavior patterns of an elastic beam, we consider the generalized sequential boundary value problems of the Navier difference equations by using the post-quantum fractional derivatives of the Caputo-like type. We discuss on the existence theory for solutions of the mentioned $(\mathrm{p};\mathrm{q})$-difference Navier problems in two single-valued and set-valued versions. We use the main properties of the $(\mathrm{p};\mathrm{q})$-operators in this regard. Application of the fixed points of the ρ-θ-contractions along with the endpoints of the multi-valued functions play a fundamental role to prove the existence results. Finally in two examples, we validate our models and theoretical results by giving numerical models of the generalized sequential $(\mathrm{p};\mathrm{q})$-difference Navier problems.

Keywords: endpoint; existence theorem; fixed point; navier problem; post-quantum derivative; set-valued function

MSC: 05A30; 26A33; 26E25; 34A12; 39A13

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Citation: Etemad, S.; Ntouyas, S.K.; Stamova, I.; Tariboon, J. On Solutions of Two Post-Quantum Fractional Generalized Sequential Navier Problems: An Application on the Elastic Beam. *Fractal Fract.* **2024**, *8*, 236. https://doi.org/10.3390/fractalfract8040236

Academic Editor: Rodica Luca

Received: 17 February 2024
Revised: 13 April 2024
Accepted: 15 April 2024
Published: 17 April 2024

1. Introduction

An important part of mathematics, which focuses on operators of arbitrary real orders and is known as fractional calculus, has brought significant theoretical and practical achievements and results in various fields of engineering and modeling. The main reason for emphasizing the importance of the results of this field can be found in various studies that have been conducted in recent years on topics such as the solutions of fractional differential equations (continuous and discrete) and their solution techniques and related algorithms. For example, fractional order techniques and models are abundantly seen in mathematical structures defined in economics, medical simulations, physics, image processing, clinical disciplines, etc., [1–3]. If we want to discuss examples, we can point out to the exact results of fractional algorithms and methods that clearly show their power in finding approximate numerical solutions and even the existence of analytical solutions of fractional equations. That is why fractional order modeling replaces integer order modeling and simulate solutions with the least error. We recommend some sources in this regard for finding more information [4–10].

In 1910, a mathematician named Jackson [11,12] began a systematic and classified study of quantum calculus, which was later abbreviated as q-calculus. In this new calculus, most of the basic operators are defined without using the concept of limit, an idea that opens a new path in mathematics. In the first step, q-calculus established its place in physics. For example, Fock [13] defined a kind of quantum difference equations and proved that hydrogen atoms are symmetric. Mathematicians then investigated various properties of q-series and q-operators, which led to the development of new concepts in the theory of quantum mechanics, orthogonal polynomials and hypergeometric functions and combinatorics [14,15]. Moreover, see other papers in this regard such as [16–20].

As the quantum calculus continued to develop and expand, a new calculus emerged that depends on two parameters $0 < \mathtt{q} < \mathtt{p} \leq 1$ and is known as post-quantum calculus or $(\mathtt{p};\mathtt{q})$-calculus. Of course, this new calculus is not a real generalization of q-calculus and it cannot be obtained by placing $\frac{\mathtt{q}}{\mathtt{p}}$ instead of q, but its rules and concepts are defined in such a way that assuming $\mathtt{p} = 1$, we can achieve the concepts of q-calculus.

The early and fundamental studies of $(\mathtt{p};\mathtt{q})$-calculus began with an article by Chakrabarti and Jagannathan [21]. Hypergeometric series [22], approximation methods [23,24], Bézier surfaces and curves [25], physical models [26], Lie groups are some of those theories in which $(\mathtt{p};\mathtt{q})$-operators play an important role. In the following, Sadjang [27] proved the fundamental theorems of $(\mathtt{p};\mathtt{q})$-calculus and introduced the Taylor's $(\mathtt{p};\mathtt{q})$-formula. In 2018 and 2019, Cheng et al. [28] and Milovanovic et al. [29] extended the $(\mathtt{p};\mathtt{q})$-Gamma and $(\mathtt{p};\mathtt{q})$-Beta functions, respectively. Soontharanon et al. combined $(\mathtt{p};\mathtt{q})$-calculus with fractional calculus and established fractional $(\mathtt{p};\mathtt{q})$-operators and their properties [30]. In the last three or four years, mathematicians have expanded their studies to the areas of existence theory, and with the help of the important tool of fixed point, they have proved the uniqueness of the solution for various types of post-quantum boundary value problems (BVPs) in the form of fractional $(\mathtt{p};\mathtt{q})$-difference equations.

Soontharanon et al. [31], in 2020, continued their studies on solutions of an r-th order $(\mathtt{p};\mathtt{q})$-integro-difference problem of the Riemann-Liouville-like type with the $(\mathtt{p};\mathtt{q})$-Robin conditions given by

$$\begin{cases} {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_0}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t}) = E\big(\mathtt{t}, \mathtt{y}(\mathtt{t}), {}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}_1}_{(\mathtt{p};\mathtt{q})}(f\mathtt{y})(\mathtt{t}), {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_2}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\big), \quad \mathtt{r}_1, \mathtt{r}_2 \in (0,1],\ \mathtt{t} \in \mathcal{I}^T_{(\mathtt{p};\mathtt{q})},\ \mathtt{r}_0 \in (1,2], \\ \alpha_1 \mathtt{y}(a) + \alpha_2 {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_3}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(a) = h_1(\mathtt{y}(\mathtt{t})), \quad \mathtt{r}_3 \in (0,1],\ \alpha_1, \alpha_2 \in \mathbb{R}^+, \\ \beta_1 \mathtt{y}(\frac{T}{\mathtt{p}}) + \beta_2 {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_3}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\frac{T}{\mathtt{p}}) = h_2(\mathtt{y}(\mathtt{t})), \quad \beta_1, \beta_2 \in \mathbb{R}^+,\ 0 < \mathtt{q} < \mathtt{p} \leq 1, \end{cases}$$

with $a \in \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} - \left\{0, \dfrac{T}{\mathtt{p}}\right\}$ so that $\mathcal{I}^T_{(\mathtt{p};\mathtt{q})} := \left\{\dfrac{\mathtt{q}^j}{\mathtt{p}^{j+1}}T : j \in \mathbb{N}_0\right\} \cup \{0\}$. Also, by assuming $\mathcal{A} = \mathbb{R}^3$, they consider the nonlinear continuous function $E : \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} \times \mathcal{A} \to \mathbb{R}$ and continuous functions $h_1, h_2 : \mathcal{C}(\mathcal{I}^T_{(\mathtt{p};\mathtt{q})}, \mathbb{R}) \to \mathbb{R}$ and $f : \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} \times \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} \to [0, \infty)$ so that

$${}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}_1}_{(\mathtt{p};\mathtt{q})}(f\mathtt{y})(\mathtt{t}) = \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1)\mathtt{p}^{\binom{\mathtt{r}_1}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{v})^{(\mathtt{r}_1 - 1)}_{(\mathtt{p};\mathtt{q})} f(\mathtt{t}, \mathtt{v}) \mathtt{y}\Big(\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1 - 1}}\Big) \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}.$$

Moreover, ${}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}^*}_{(\mathtt{p};\mathtt{q})}$ is the $\mathtt{r}^*$-th order $(\mathtt{p};\mathtt{q})$-derivative of the Riemann-Liouville-like type with $\mathtt{r}^* = \mathtt{r}_i$ $(i = 0, 2, 3)$.

Neang et al. [32] conducted another analysis on the existence results for a nonlinear r-th order $(\mathtt{p};\mathtt{q})$-difference problem of the Caputo-like type with the separated boundary conditions, formulated by

$$\begin{cases} {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t}) = E\big(\mathtt{t}, \mathtt{y}(\mathtt{p}^{\mathtt{r}}\mathtt{t})\big), \quad \mathtt{r} \in (1,2],\ 0 \leq \mathtt{t} \leq \dfrac{T}{\mathtt{p}^{\mathtt{r}}}, \\ \alpha_1 \mathtt{y}(0) + \beta_1 \mathbb{D}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(0) = \gamma_1 \mathtt{y}(\mathtt{w}_1), \quad \alpha_1, \beta_1, \gamma_1 \in \mathbb{R}, \\ \alpha_2 \mathtt{y}(T) + \beta_2 \mathbb{D}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\frac{T}{\mathtt{p}}) = \gamma_2 \mathtt{y}(\mathtt{w}_2), \quad \alpha_2, \beta_2, \gamma_2 \in \mathbb{R},\ 0 < \mathtt{q} < \mathtt{p} \leq 1, \end{cases}$$

so that $\mathbb{D}_{(\mathtt{p};\mathtt{q})}$ and ${}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}$ are the 1st order and $\mathtt{r}$-th order $(\mathtt{p};\mathtt{q})$-difference and $(\mathtt{p};\mathtt{q})$-derivative of the Caputo-like type, respectively, and $E \in \mathcal{C}([0, \frac{T}{\mathtt{p}^{\mathtt{r}}}] \times \mathbb{R}, \mathbb{R})$.

Once again in 2022, the same authors [33] defined a new function $E \in \mathcal{C}([0, T] \times \mathbb{R}, \mathbb{R})$ to simplify the domain of it, and to study the existence theorems, modeled the $\mathtt{r}$-th order $(\mathtt{p};\mathtt{q})$-difference problem of the Caputo-like type as

$$\begin{cases} {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t}) = E\big(\mathtt{p}^{\mathtt{r}}\mathtt{t}, \mathtt{y}(\mathtt{p}^{\mathtt{r}}\mathtt{t})\big), \ \ \mathtt{r} \in (1,2], \ \ 0 \le \mathtt{t} \le T, \\ \alpha_1 \mathtt{y}(0) + \alpha_2 \mathbb{D}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(0) = \alpha_3, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}, \\ \alpha_4 \mathtt{y}(T) + \alpha_5 \mathbb{D}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{p}T) = \alpha_6, \quad \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}, \ 0 < \mathtt{q} < \mathtt{p} \le 1, \end{cases}$$

with the above notations and $(\mathtt{p};\mathtt{q})$-operators. For more information on newly conducted studies in the context of $(\mathtt{p};\mathtt{q})$-calculus, refer to [34–39].

In order to fully understand the nature and dynamics of phenomena in the real world, modern science needs mathematical models of these phenomena and processes. Based on such exact mathematical models that are defined by operators and other mathematical concepts, we can examine the dynamical and behavioral structure of various engineering, mechanical and physical systems in laboratory environments and generalize the results to real dimensions in the surrounding world [40–42].

One of the important mathematical models that has attracted the attention of engineers in recent years and is used for technical simulations in advanced and complex structures is the mathematical dynamical model based on elastic beam. In huge engineering structures such as building structures, towers, bridges, aviation industry, giant ocean-going ships and spaceships, the application of elastic beam technology is considered one of the basic necessities. Based on this, from a mathematical point of view, the boundary value problem including a fourth-order Navier differential equation and two-point boundary conditions was modeled by Reiss et al. [43] in 1976, who studied the dynamic behavior of elastic beam changes, and its form is as follows

$$\begin{cases} \mathtt{y}^{(4)}(\mathtt{t}) = E\big(\mathtt{t}, \mathtt{y}(\mathtt{t}), \mathtt{y}''(\mathtt{t})\big), \quad \mathtt{t} \in \mathcal{I} := [0,1], \\ \mathtt{y}(0) = \mathtt{y}(1) = 0 = \mathtt{y}''(0) = \mathtt{y}''(1), \end{cases} \tag{1}$$

provided that the source function $E : \mathcal{I} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. In 1986, Aftabizadeh [44] transformed (1) into a 2-nd order integro–differential equation by assuming E to be a bounded nonlinear function, and proved the existence theorems under the Schauder's fixed point theorem. In 1997, Ma et al. [45] studied the Navier problem (1) and analyzed it based on the lower and upper solutions. Bai et al. [46], in 2004, adopted a monotone technique for the lower and upper solutions of the given beam elastic problem (1). Bachar et al. [47] extended the fractional version of the beam elastic problem (1) of the Riemann-Liouville type and completed their theoretical analysis on the existence and positivity of the unique solutions of the fractional Navier system given by

$$\begin{cases} {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_1}\big({}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_2}\mathtt{y}\big)(\mathtt{t}) = E\big(\mathtt{t}, \mathtt{y}(\mathtt{t}), {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t})\big), \\ \mathtt{y}(0) = \mathtt{y}(1) = 0 = {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_2}\mathtt{y}(0) = {}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_2}\mathtt{y}(1), \end{cases} \tag{2}$$

provided $\mathtt{r}_1, \mathtt{r}_2 \in (1,2]$, ${}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_1}$ and ${}^{\mathtt{R}}\mathbb{D}^{\mathtt{r}_2}$ specify the fractional derivatives of the Riemann-Liouville type and $E \in \mathcal{C}(\mathcal{I} \times \mathbb{R}^2, \mathbb{R})$. If $\mathtt{r}_1 = \mathtt{r}_2 = 2$, then (2) reduces to integer order problem (1).

In 2021, Etemad et al. [48] discussd a new sequential generalized fractional q-Navier problem given by

$$\begin{cases} {}^{c}\mathbb{D}^{r_1}_{(q)}\big({}^{c}\mathbb{D}^{r_2}_{(q)}y\big)(t) = E\big(t, y(t), {}^{c}\mathbb{D}^{r_2}_{(q)}y(t)\big), & t \in \mathcal{I} := [0,1],\ q \in (0,1), \\ \gamma y(0) = \delta y(1) = \lambda {}^{c}\mathbb{D}^{r_2}_{(q)}y(0) = \beta {}^{c}\mathbb{D}^{r_2}_{(q)}y(1) = 0, \end{cases} \tag{3}$$

so that $r_1 \in (1,2]$, $r_2 \in (1,2]$ and $\gamma, \delta, \lambda, \beta \in \mathbb{R}^+$. Moreover, here ${}^{c}\mathbb{D}^{(\cdot)}_{(q)}$ is the fractional q-derivative of the Caputo-like type.

Based on the integer order Navier Equation (1) and other aforementioned models, we inspired to study a new sequential $(p;q)$-model of the elastic beam with $(p;q)$-Navier difference equation of the Caputo-like type

$$\begin{cases} {}^{c}\mathbb{D}^{r_1}_{(p;q)}\big({}^{c}\mathbb{D}^{r_2}_{(p;q)}y\big)(t) = E\big(t, y(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big), & (t \in \mathcal{I}^{T}_{(p;q)} = [0, \frac{T}{p}],\ 0 < q < p \leq 1), \\ \beta y(t)\big|_{t=0} = \lambda y(t)\big|_{t=\frac{T}{p}} = \delta {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=0} = \gamma {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=\frac{T}{p}} = 0, \end{cases} \tag{4}$$

and the $(p;q)$-Navier difference inclusion of the Caputo-like type

$$\begin{cases} {}^{c}\mathbb{D}^{r_1}_{(p;q)}\big({}^{c}\mathbb{D}^{r_2}_{(p;q)}y\big)(t) \in \mathcal{E}\big(t, y(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big), & (t \in \mathcal{I}^{T}_{(p;q)} = [0, \frac{T}{p}],\ 0 < q < p \leq 1), \\ \beta y(t)\big|_{t=0} = \lambda y(t)\big|_{t=\frac{T}{p}} = \delta {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=0} = \gamma {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=\frac{T}{p}} = 0, \end{cases} \tag{5}$$

where $T > 0$, $r_1, r_2 \in (1,2]$, $\beta, \lambda, \delta, \gamma \in \mathbb{R}^+$, and the $(p;q)$-derivative of the Caputo-like type is denoted by ${}^{c}\mathbb{D}^{(\cdot)}_{(p;q)}$. We consider $E : [0, \frac{T}{p}] \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{E} : [0, \frac{T}{p}] \times \mathbb{R}^2 \to \mathbb{P}(\mathbb{R})$ as the continuous single-valued arbitrary function and multivalued arbitrary function, respectively, with some conditions.

The main contribution of this paper can be expressed as follows: here, we have extended the standard model of the elastic beam equation to two fractional structures of Navier $(p;q)$-difference equation and $(p;q)$-difference inclusion for the first time, and aslo, we have presented our numerical and theoretical results to study the existence of solutions for new sequential $(p;q)$-models (4) and (5) of the elastic beam. It is clear that if $r_1 = r_2 \to 2$, $p = 1$, $q \to 1$ and $\beta = \lambda = \delta = \gamma = 1$, we obtain the standard Navier model (1) of the elastic beam. Unlike many similar papers in the field of existence theory, we use a set of non-decreasing mappings and special contractions and endpoints to prove the existence of the solutions of the above two systems. The presented model of elastic beam based on the $(p;q)$-operators helps us to study some behaviors of fractional difference systems without the limit notion, and this can be an important advantage for the fast and easy simulation of the real phenomena in the context of discrete-type $(p;q)$-calculus.

We have prepared the next sections of the paper as follows: Section 2 aims to recall some basic notions of fractional q-calculus. Section 3 begins with a proposition for computing some $(p;q)$-integrals, and then, continues with a lemma which formulates the solution of the generalized sequential $(p;q)$-Navier problems (4) and (5) in the form of an $(p;q)$-integral equation. In the following, this section uses the well known fixed point theorem attributed to Krasnoselskii [49] and a special set of the operators proposed by Samet et al. [50] to study the existence theory for the solutions of single-valued operators. Section 4 considers the generalized sequential $(p;q)$-difference Navier inclusion problem (5) and examines the same existence theory for the solutions of (5), but this time, by applying the methods proposed by Mohammadi et al. [51] and also, by the approximate endpoint property. In Section 5, we assign our focus to validate the theorems proved in the previous Sections 3 and 4. The last section concludes our study by giving some remarks and future ideas.

2. Preliminaries

In the next three subsections, we are going to state some definitions, lemmas and theorems on the context of q-calculus, $(p;q)$-calculus, and fixed point theory as a reminder. Throughout this paper, let $q \in (0,1)$ and $0 < q < p \leq 1$.

2.1. *q-Calculus*

The q-power function is the q-analoge of $(s_1 - s_2)^m$ which is defined as

$$(s_1 - s_2)^{(0)}_{(q)} = 1, \quad (s_1 - s_2)^{(m)}_{(q)} = \prod_{a=0}^{m-1}(s_1 - s_2 q^a), \quad (s_1, s_2 \in \mathbb{R}, m \in \mathbb{N}_0),$$

Ref. [52]. If $m = r \in \mathbb{R}$, then generally we have

$$(s_1 - s_2)^{(r)}_{(q)} = s_1^r \prod_{a=0}^{\infty} \frac{1 - (\frac{s_2}{s_1})q^a}{1 - (\frac{s_2}{s_1})q^{r+a}}, \tag{6}$$

for $s_1 \neq 0$. In a special case, $(s_1)^{(r)}_{(q)} = s_1^r$ if $s_2 = 0$ [52]. The q-number $[s]_{(q)}$ and q-Gamma function $\Gamma_{(q)}(\cdot)$, for $s \in \mathbb{R}$ and $r \in \mathbb{R}\backslash\mathbb{Z}^{\leq 0}$, are given by

$$[0]_{(q)} = 0, \quad [s]_{(q)} = q^{s-1} + \cdots + q + 1 = \frac{1 - q^s}{1 - q} \ (s \neq 0), \quad \Gamma_{(q)}(r) = \frac{(1-q)^{(r-1)}_{(q)}}{(1-q)^{r-1}}. \tag{7}$$

Moreover, $\Gamma_{(q)}(r+1) = [r]_{(q)}\Gamma_{(q)}(r)$ [52].

Definition 1 ([53])**.** *The* q*-derivative of the function* y *is defined as*

$$\mathbb{D}_{(q)}y(t) = \left[\frac{d}{dt}\right]_{(q)} y(t) = \frac{y(t) - y(qt)}{(1-q)t}. \tag{8}$$

For the highr orders, we define $\mathbb{D}^m_{(q)}y(t) = \mathbb{D}_{(q)}(\mathbb{D}^{m-1}_{(q)}y(t))$ for each $m \in \mathbb{N}$ and also, define $\mathbb{D}^0_{(q)}y(t) = y(t)$ [53].

Definition 2 ([54,55])**.** *The* r*-th order* q*-integral of the Riemann-Liouville-like type for the function* $y \in \mathcal{C}([0,+\infty),\mathbb{R})$ *is given by*

$$^{R}\mathbb{I}^r_{(q)}y(t) = \begin{cases} \dfrac{1}{\Gamma_{(q)}(r)} \displaystyle\int_0^t (t - qv)^{(r-1)}_{(q)} y(v)\, d_{(q)}v, & r > 0, \\ y(t), & r = 0, \end{cases} \tag{9}$$

provided the integral converges.

Definition 3 ([54,55])**.** *Let* $\lambda = [r] + 1$*. The* r*-th order* q*-derivative of the Caputo-like type for* $y \in \mathcal{C}^{(\lambda)}([0,+\infty),\mathbb{R})$ *is defined by*

$$^{c}\mathbb{D}^r_{(q)}y(t) = \frac{1}{\Gamma_{(q)}(\lambda - r)} \int_0^t (t - qv)^{(\lambda - r - 1)}_{(q)} \mathbb{D}^{\lambda}_{(q)}y(v)\, d_{(q)}v, \tag{10}$$

provided the integral converges.

2.2. *(p; q)-Calculus*

All definitions of this subsection can be transformed into the aforementioned definitions in the previous subsections if $p = 1$.

The $(p;q)$-power function is the $(p;q)$-analogue of $(s_1 - s_2)^m$ which is given by

$$(s_1 - s_2)^{(0)}_{(p;q)} = 1, \quad (s_1 - s_2)^{(m)}_{(p;q)} = \prod_{a=0}^{m-1}(s_1 p^a - s_2 q^a), \quad (s_1, s_2 \in \mathbb{R}, m \in \mathbb{N}_0),$$

Ref. [30]. If $m = r \in \mathbb{R}$, then generally, we have

$$(s_1 - s_2)^{(r)}_{(p;q)} = s_1^r \prod_{a=0}^{\infty} \frac{1}{p^r}\left(\frac{1 - (\frac{s_2}{s_1})(\frac{q}{p})^a}{1 - (\frac{s_2}{s_1})(\frac{q}{p})^{r+a}}\right), \tag{11}$$

for $s_1 \neq 0$. Also, if $s_2 = 0$, then $(s_1)^{(r)}_{(p;q)} = \frac{1}{p^r}s_1^r$ [30]. The $(p;q)$-number $[s]_{(p;q)}$ and $(p;q)$-Gamma function $\Gamma_{(p;q)}(\cdot)$, for $s \in \mathbb{R}$ and $r \in \mathbb{R}\backslash\mathbb{Z}^{\leq 0}$, are given by

$$[0]_{(p;q)} = 0, \quad [s]_{(p;q)} = p^{s-1}[s]_{\frac{q}{p}} = \frac{p^s - q^s}{p - q}, \quad \Gamma_{(p;q)}(r) = \frac{(p-q)^{(r-1)}_{(p;q)}}{(p-q)^{r-1}}, \tag{12}$$

and also, $\Gamma_{(p;q)}(r+1) = [r]_{(p;q)}\Gamma_{(p;q)}(r)$ [30]. Furthermore, the $(p;q)$-Beta function $\mathbb{B}_{(p;q)}(\cdot,\cdot)$ is defined as

$$\mathbb{B}_{(p;q)}(r,\tilde{r}) = \int_0^1 v^{r-1}(1-qv)^{(\tilde{r}-1)}_{(p;q)} d_{(p;q)}v = \frac{\Gamma_{(p;q)}(r)\Gamma_{(p;q)}(\tilde{r})}{\Gamma_{(p;q)}(r+\tilde{r})} p^{\frac{1}{2}(\tilde{r}-1)(2r+\tilde{r}-2)}, \quad r, \tilde{r} > 0. \tag{13}$$

Definition 4 ([30])**.** *The $(p;q)$-derivative of the function y is given by*

$$\mathbb{D}_{(p;q)}y(t) = \frac{y(pt) - y(qt)}{(p-q)t}. \tag{14}$$

If $p = 1$, then $\mathbb{D}_{(1;q)}y(t) = \mathbb{D}_{(q)}y(t)$, and if $q \to 1$, then $\mathbb{D}_{(1;q\to 1)}y(t) = y'(t)$.

Definition 5 ([30])**.** *The $(p;q)$-integral of $y \in \mathcal{C}([0,T],\mathbb{R})$ is defined by*

$$\mathbb{I}_{(p;q)}y(t) = \int_0^t y(v)d_{(p;q)}v = (p-q)t\sum_{a=0}^{\infty}\frac{q^a}{p^{a+1}}y\left[\frac{q^a}{p^{a+1}}t\right].$$

Definition 6 ([30])**.** *The r-th order $(p;q)$-integral of the Riemann-Liouville-like type for the function $y \in \mathcal{C}([0,T],\mathbb{R})$ is defined by*

$$^{R}\mathbb{I}^{r}_{(p;q)}y(t) = \begin{cases} \dfrac{1}{\Gamma_{(p;q)}(r)p^{\binom{r}{2}}}\displaystyle\int_0^t (t-qv)^{(r-1)}_{(p;q)}y\left[\frac{v}{p^{r-1}}\right]d_{(p;q)}v, & r > 0, \\ y(t), & r = 0, \end{cases} \tag{15}$$

if the integral converges.

If $p = 1$, then $^{R}\mathbb{I}^{r}_{(1;q)}y(t) = {}^{R}\mathbb{I}^{r}_{(q)}y(t)$ which has been defined in Definition 2.

Definition 7 ([30])**.** *Let $\lambda = [r] + 1$. The r-th order $(p;q)$-derivative of the Caputo-like type for $y \in \mathcal{C}^{(\lambda)}([0,T],\mathbb{R})$ is defined by*

$$^{c}\mathbb{D}^{r}_{(p;q)}y(t) = {}^{R}\mathbb{I}^{\lambda-r}_{(p;q)}\mathbb{D}^{\lambda}_{(p;q)}y(t) = \frac{1}{\Gamma_{(p;q)}(\lambda-r)p^{\binom{\lambda-r}{2}}}\int_0^t (t-qv)^{(\lambda-r-1)}_{(p;q)}\mathbb{D}^{\lambda}_{(p;q)}y\left[\frac{v}{p^{\lambda-r-1}}\right]d_{(p;q)}v, \tag{16}$$

if the integral converges.

If $p = 1$, then $^{c}\mathbb{D}^{r}_{(1;q)}y(t) = {}^{c}\mathbb{D}^{r}_{(q)}y(t)$ which has been defined in Definition 3.

In the following, some important properties are recalled taken from [30].

Lemma 1 ([30]). *Let $\mathtt{r}, \tilde{\mathtt{r}} > 0$. Then*
$(A_{(\mathtt{p};\mathtt{q})})$ ${}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\left[{}^{\mathtt{R}}\mathbb{I}^{\tilde{\mathtt{r}}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\right] = {}^{\mathtt{R}}\mathbb{I}^{\tilde{\mathtt{r}}}_{(\mathtt{p};\mathtt{q})}\left[{}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\right] = {}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}+\tilde{\mathtt{r}}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})$.
$(B_{(\mathtt{p};\mathtt{q})})$ ${}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\left[{}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\right] = \mathtt{y}(\mathtt{t})$.

Lemma 2 ([30]). *Let $\mathtt{r}, \tilde{\mathtt{r}} > 0$ and $\mathtt{y}(\mathtt{t}) = \mathtt{t}^{\tilde{\mathtt{r}}}$. Then*
$(C_{(\mathtt{p};\mathtt{q})})$ ${}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t}) = \dfrac{\Gamma_{(\mathtt{p};\mathtt{q})}(\tilde{\mathtt{r}}+1)}{\Gamma_{(\mathtt{p};\mathtt{q})}(\tilde{\mathtt{r}}+\mathtt{r}+1)}\mathtt{t}^{\tilde{\mathtt{r}}+\mathtt{r}}$.
$(D_{(\mathtt{p};\mathtt{q})})$ ${}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t}) = \mathtt{p}^{\mathtt{r}}\dfrac{\Gamma_{(\mathtt{p};\mathtt{q})}(\tilde{\mathtt{r}}+1)}{\Gamma_{(\mathtt{p};\mathtt{q})}(\tilde{\mathtt{r}}-\mathtt{r}+1)}\mathtt{t}^{\tilde{\mathtt{r}}-\mathtt{r}}$, $\tilde{\mathtt{r}} > \mathtt{r}$.
$(E_{(\mathtt{p};\mathtt{q})})$ $\displaystyle\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{q}\mathtt{v})^{(\mathtt{r}-1)}_{(\mathtt{p};\mathtt{q})}\mathtt{v}^{\tilde{\mathtt{r}}}\mathtt{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} = \mathbb{B}_{(\mathtt{p};\mathtt{q})}(\tilde{\mathtt{r}}+1,\mathtt{r})\mathtt{t}^{\mathtt{r}+\tilde{\mathtt{r}}}$.

Theorem 1 ([30]). *Let $\lambda = [\mathtt{r}] + 1$. Then*

$$ {}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\left[{}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\right] = \mathtt{y}(\mathtt{t}) - \sum_{\mathtt{a}=0}^{\lambda-1}\frac{\mathbb{D}^{\mathtt{a}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(0)}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{a}+1)\mathtt{p}^{\binom{\mathtt{r}}{2}}}\mathtt{t}^{\mathtt{a}}. $$

In the simplified form, we have

$$ {}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\left[{}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\right] = \mathtt{y}(\mathtt{t}) + \mathtt{k}^*_0 + \mathtt{k}^*_1\mathtt{t} + \cdots + \mathtt{k}^*_{\lambda-1}\mathtt{t}^{\lambda-1}, $$

where $\mathtt{k}^*_{\mathtt{a}} \in \mathbb{R}$; $\mathtt{a} = 0, 1, \ldots, \lambda - 1$.

2.3. Fixed Point Theory

Now, we continue our preliminaries based on fixed point theory. We first begin by introducing some needed collections of sets.

Remark 1. *Let $(\mathcal{X}_*, \|\cdot\|_{\mathcal{X}_*})$ be a normed space. The collections $\mathrm{P}_{\mathbb{B}}(\mathcal{X}_*), \mathrm{P}_{\mathbb{CL}}(\mathcal{X}_*), \mathrm{P}_{\mathbb{CM}}(\mathcal{X}_*)$ and $\mathrm{P}_{\mathbb{CX}}(\mathcal{X}_*)$ contain all bounded, closed, compact and convex sets in $\mathcal{X}_*$, respectively.*

By Θ, we consider a subcollection of all operators $\theta : [0, \infty) \to [0, \infty)$ (which are non-decreasing) with

$$ \sum_{a=1}^{\infty}\theta^a(\mathtt{t}) < \infty, \ \theta(\mathtt{t}) < \mathtt{t}, \ \text{for all}\, \mathtt{t} > 0. $$

Definition 8 ([50]). *Let $E : \mathcal{X}_* \to \mathcal{X}_*$ and $\rho : \mathcal{X}_*^2 \to \mathbb{R}_{\geq 0}$. Then*
(a) *E is ρ-θ-contraction if*

$$ \rho(\mathtt{y}_1, \mathtt{y}_2)d(E\mathtt{y}_1, E\mathtt{y}_2) \leq \theta(d(\mathtt{y}_1, \mathtt{y}_2)), \quad \forall\, \mathtt{y}_1, \mathtt{y}_2 \in \mathcal{X}_*. $$

(b) *E is ρ-admissible if $\rho(E\mathtt{y}_1, E\mathtt{y}_2) \geq 1$ whenever $\rho(\mathtt{y}_1, \mathtt{y}_2) \geq 1$.*

Definition 9 ([56]). *Let $\mathcal{E} : \mathcal{X}_* \to \mathrm{P}(\mathcal{X}_*)$ be a multi-valued function.*
(c) *$\mathtt{y} \in \mathcal{X}_*$ is an endpoint of $\mathcal{E}$ if $\mathcal{E}(\mathtt{y}) = \{\mathtt{y}\}$.*
(d) *$\mathcal{E}$ has an approximate endpoint property if*

$$ \inf_{\mathtt{y}_1 \in \mathcal{X}_*}\Big[\sup_{\mathtt{y}_2 \in \mathcal{E}(\mathtt{y}_1)} d(\mathtt{y}_1, \mathtt{y}_2)\Big] = 0. $$

Definition 10 ([51]). *Assume $\mathcal{E} : \mathcal{X}_\star \to \mathrm{P}_{\mathbb{CL},\mathbb{B}}(\mathcal{X}_\star)$, $\rho : \mathcal{X}_\star^2 \to [0, +\infty)$ and $\theta \in \Theta$. Let $\mathbb{H}_d$ be the Pompeiu-Hausdorff metric. Then*

(e) $\mathcal{E}$ *is* ρ*-admissible if for every* $y_1 \in \mathcal{X}_\star$ *and* $y_2 \in \mathcal{E}y_1$,

$$\rho(y_1, y_2) \geq 1 \;\Rightarrow\; \rho(y_2, y_3) \geq 1, \;\textit{for all}\; y_3 \in \mathcal{E}y_2.$$

(f) $\mathcal{E}$ *is an* ρ*-*θ*-contraction if*

$$\rho(y_1, y_2)\mathbb{H}_d(\mathcal{E}y_1, \mathcal{E}y_2) \leq \theta(d(y_1, y_2)), \quad \textit{for all}\; y_1, y_2 \in \mathcal{X}_\star.$$

In the following, we recall needed fixed point and endpoint theorems as a reminder.

Theorem 2 ([50])**.** *Let a metric space* $(\mathcal{X}_*, \mathrm{d})$ *be complete,* $\rho : \mathcal{X}_* \times \mathcal{X}_* \to \mathbb{R}$*, and* $\theta \in \Theta$*. Let* $E : \mathcal{X}_* \to \mathcal{X}_*$ *be* ρ*-*θ*-contraction. Moreover,*

(1) E *is* ρ*-admissible on* $\mathcal{X}_*$*;*
(2) $\exists\, y_0 \in \mathcal{X}_*$ *s.t.* $\rho(y_0, Ey_0) \geq 1$*;*
(3) *For every sequence* $\{y_i\}$ *in* $\mathcal{X}_*$ *with* $y_i \to y$*, if* $\rho(y_i, y_{i+1}) \geq 1$ *for all* $i \geq 1$*, then* $\rho(y_i, y) \geq 1$ *for each* $i \geq 1$*.*

Then E has a fixed point.

Theorem 3 ([49])**.** *Let* $Y \subseteq \mathcal{X}_*$ *be a non-empty bounded, closed, convex set, and* E_1 *and* E_2 *be defined on* Y *so that*

(1) $E_1 y_1 + E_2 y_2 \in Y$*, for all* $y_1, y_2 \in Y$*;*
(2) *The continuous function* E_1 *is compact;*
(3) E_2 *is contraction.*

Then, $\exists\, y \in Y$ *so that* $y = E_1 y + E_2 y$ *(Krasnoselskii's fixed point theorem).*

Theorem 4 ([51])**.** *Let a metric space* $(\mathcal{X}_*, \mathrm{d})$ *be complete,* $\theta \in \Theta$*, and* $\rho : \mathcal{X}_* \times \mathfrak{A}_* \to [0, \infty)$ *be strictly increasing. Let* $\mathcal{E} : \mathcal{X}_* \to \mathrm{P}_{\mathbb{CL},\mathbb{B}}(\mathcal{X}_*)$ *be an* ρ*-*θ*-contraction. Moreover,*

1. $\mathcal{E}$ *is* ρ*-admissible;*
2. $\rho(y_0, y_1) \geq 1$ *for some* $y_0 \in \mathfrak{A}_*$ *and* $y_1 \in \mathcal{E}y_0$*;*
3. *For every sequence* $\{y_i\}$ *in* $\mathcal{X}_*$ *with* $\rho(y_i, y_{i+1}) \geq 1$ *and* $y_i \to y$ *for all* $i \in \mathbb{N}$*, there is a subsequence* $\{y_{i_r}\}$ *of* $\{y_i\}$ *so that* $\rho(y_{i_r}, y) \geq 1$ *for each* $r \in \mathbb{N}$*.*

Then, $\mathcal{E}$ *has a fixed point.*

Theorem 5 ([56])**.** *Let a metric space* $(\mathcal{X}_*, \mathrm{d})$ *be complete. Moreover,*

1. *The upper semi-continuous function* $\theta : [0, \infty) \to [0, \infty)$ *is so that* $\liminf_{t \to \infty}(t - \theta(t)) > 0$ *and* $\theta(t) < t$ *for all* $t > 0$*;*
2. $\mathcal{E} : \mathcal{X}_* \to \mathrm{P}_{\mathbb{CL},\mathbb{B}}(\mathcal{X}_*)$ *is so that* $\mathbb{H}_d(\mathcal{E}y_1, \mathcal{E}y_2) \leq \theta(d(y_1, y_2))$ *for each* $y_1, y_2 \in \mathcal{X}_*$*.*

Then $\mathcal{E}$ *has a unique endpoint if and only if it has an approximate endpoint property.*

3. On the Generalized $(p;q)$-Difference Navier Problem (4)

To conduct an analysis on the solutions of the generalized sequential $(p;q)$-difference Navier problem (4), we introduce $\mathcal{X}_* = \{y(t) : y(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)} y(t) \in \mathcal{C}_{\mathbb{R}}(\mathcal{I}^T_{(p;q)})\}$ as a spece containing all real-valued functions on $\mathcal{I}^T_{(p;q)}$, which is a Banach space with the norm

$$\|y\|_{\mathcal{X}_*} = \sup_{t \in \mathcal{I}^T_{(p;q)}} |y(t)| + \sup_{t \in \mathcal{I}^T_{(p;q)}} \left|{}^{c}\mathbb{D}^{r_2}_{(p;q)} y(t)\right|,$$

for each $y \in \mathcal{X}_*$.

In the following, we provide a lemma under which the structure of the solutions related to the proposed generalized sequential $(p;q)$-difference Navier problem (4) is determined in the context of an integral equation.

Lemma 3. *Let* $g \in \mathcal{C}_{\mathbb{R}}(\mathcal{I}^{T}_{(p;q)})$, $r_1 \in (1,2]$, $r_2 \in (1,2]$ *and* $\beta, \lambda, \delta, \gamma \in \mathbb{R}^+$. *Then* y^* *is a solution to the generalized* $(p;q)$*-difference Navier problem*

$$\begin{cases} {}^{c}\mathbb{D}^{r_1}_{(p;q)}\big({}^{c}\mathbb{D}^{r_2}_{(p;q)}y\big)(t) = g(t), \qquad (t \in [0, \frac{T}{p}],\ 0 < q < p \leq 1,\ T > 0), \\ \beta y(t)\big|_{t=0} = \lambda y(t)\big|_{t=\frac{T}{p}} = \delta\, {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=0} = \gamma\, {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=\frac{T}{p}} = 0, \end{cases} \tag{17}$$

if and only if it is satisfied the $(p;q)$*-integral equation*

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^t (t-qv)^{(r_1+r_2-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v \\ & - \frac{pt}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)^{(r_1+r_2-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v \\ & + \frac{\big[p^2(\frac{T}{p})^{r_2+1}t - pTt^{r_2+1}\big]}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v. \end{aligned} \tag{18}$$

Proof. Let y^* be a solution of the generalized $(p;q)$-Navier problem (17). Simply, we see that

$$ {}^{c}\mathbb{D}^{r_1}_{(p;q)}\big({}^{c}\mathbb{D}^{r_2}_{(p;q)}y^*\big)(t) = g(t).$$

Since $r_1 \in (1,2]$, by applying the r_1-th order $(p;q)$-integral of the Riemann-Liouville-like type, we get

$$ {}^{c}\mathbb{D}^{r_2}_{(p;q)}y^*(t) = \frac{1}{\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^t (t-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v + c_0^* + c_1^* t,$$

for each the constants $c_0^*, c_1^* \in \mathbb{R}$. The third condition, i.e., $\delta\, {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=0} = 0$, gives $c_0^* = 0$ immediately. Thus, we have

$$ {}^{c}\mathbb{D}^{r_2}_{(p;q)}y^*(t) = \frac{1}{\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^t (t-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v + c_1^* t. \tag{19}$$

Now, in view of (19) and by the fourth condition, i.e., $\gamma\, {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big|_{t=\frac{T}{p}} = 0$, we obtain

$$\frac{\gamma}{\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v + \gamma c_1^* \frac{T}{p} = 0,$$

and accordingly,

$$c_1^* = -\frac{p}{T\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v. \tag{20}$$

In the following, by (20), the Equation (19) becomes

$$\begin{aligned} {}^{c}\mathbb{D}^{r_2}_{(p;q)}y^*(t) = {} & \frac{1}{\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^t (t-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v \\ & - \frac{pt}{T\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)^{(r_1-1)}_{(p;q)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v. \end{aligned} \tag{21}$$

Similarly, since $r_2 \in (1,2]$, by taking the r_2-th order $(p;q)$-integral of the Riemann-Liouville-like type on (21), we may write

$$y^*(t) = \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^t (t-qv)_{(p;q)}^{(r_1+r_2-1)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$
$$- \frac{pt^{1+r_2}}{T\Gamma_{(p;q)}(2+r_2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v + c_0^{**} + c_1^{**}t,$$

for each the constants $c_0^{**}, c_1^{**} \in \mathbb{R}$. The first condition, i.e., $\beta y(t)\big|_{t=0} = 0$, yields $c_0^{**} = 0$ immediately. Hence,

$$y^*(t) = \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^t (t-qv)_{(p;q)}^{(r_1+r_2-1)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$
$$- \frac{pt^{1+r_2}}{T\Gamma_{(p;q)}(2+r_2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v + c_1^{**}t. \tag{22}$$

Next, the second condition, i.e., $\lambda y(t)\big|_{t=\frac{T}{p}} = 0$, gives

$$c_1^{**} = -\frac{p}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1+r_2-1)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$
$$+ \frac{p^2\big(\frac{T}{p}\big)^{1+r_2}}{T^2\Gamma_{(p;q)}(2+r_2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v.$$

Put c_1^{**} into (22). Therefore,

$$y^*(t) = \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^t (t-qv)_{(p;q)}^{(r_1+r_2-1)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$
$$- \frac{pt}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1+r_2-1)} g\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$
$$+ \frac{\big[p^2(\frac{T}{p})^{r_2+1}t - pTt^{r_2+1}\big]}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)} g\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v,$$

and this shows that y^* is satisfied $(p;q)$-integral Equation (18). We can prove the converse by direct computation and this completes the proof. □

Based on the previous lemma, a new operator $G : \mathcal{X}_* \to \mathcal{X}_*$ is defined by

$$(Gy)(t) = \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^t (t-qv)_{(p;q)}^{(r_1+r_2-1)} E\Big[\frac{v}{p^{r_1+r_2-1}}, y\big(\frac{v}{p^{r_1+r_2-1}}\big), {}^c\mathbb{D}_{(p;q)}^{r_2} y\big(\frac{v}{p^{r_1+r_2-1}}\big)\Big]\, d_{(p;q)}v$$
$$- \frac{pt}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^{\frac{T}{p}} (\frac{T}{p}-qv)_{(p;q)}^{(r_1+r_2-1)} E\Big[\frac{v}{p^{r_1+r_2-1}}, y\big(\frac{v}{p^{r_1+r_2-1}}\big), {}^c\mathbb{D}_{(p;q)}^{r_2} y\big(\frac{v}{p^{r_1+r_2-1}}\big)\Big]\, d_{(p;q)}v$$

$$+\frac{p^2(\frac{T}{p})^{r_2+1}t-pTt^{r_2+1}}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}}\int_0^{\frac{T}{p}}(\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)}E\Big[\frac{v}{p^{r_1-1}},y(\frac{v}{p^{r_1-1}}),{}^{c}\mathbb{D}^{r_2}_{(p;q)}y(\frac{v}{p^{r_1-1}})\Big]\,d_{(p;q)}v.$$

The function y^* as a solution of the generalized $(p;q)$-difference Navier problem (4) is a fixed point of the newly defined operator G.

Before proving the main results, we recall a double $(p;q)$-integral in the following proposition which is used later.

Proposition 1 ([30]). *Let $r_1, r_2 \in \mathbb{R}$. Then*

$$I=\int_0^t\int_0^{\frac{v}{p^{1-r_2}}}(t-qv)_{(p;q)}^{(1-r_2)}(\frac{v}{p^{(1-r_2)}}-qw)_{(p;q)}^{(r_1+r_2-5)}d_{(p;q)}w\,d_{(p;q)}v$$

$$=p^{\binom{2-r_2}{2}+\binom{r_1+r_2-4}{2}}\frac{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_1+r_2-4)}{\Gamma_{(p;q)}(r_1-1)}t^{r_1-2}.$$

Also, keep in mind the following notations for simplicity:

$$L_1=\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+r_2+1)}+\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+1)\Gamma_{(p;q)}(r_2+2)},\tag{23}$$

$$L_2=\frac{(\frac{T}{p})^{r_1-2}}{\Gamma_{(p;q)}(r_1-1)}+\frac{p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_1+r_2+1)}+\frac{[1+\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)]p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1+1)},$$

and

$$L_3=\frac{(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+r_2+1)}+\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+1)\Gamma_{(p;q)}(r_2+2)},$$

$$L_4=\frac{p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_1+r_2+1)}+\frac{[1+\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)]p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1+1)}.\tag{24}$$

We start our first existence theorem as follows, with the help of the ρ-θ-contractions and ρ-admissible functions.

Theorem 6. *Let $\Lambda:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$, $E\in\mathcal{C}([0,\frac{T}{p}]\times\mathcal{X}_*^2,\mathcal{X}_*)$ and $\theta\in\Theta$. Moreover,*

($\mathfrak{P}1$) *For each $y_1, y_2, x_1, x_2 \in \mathcal{X}_*$ and $t\in[0,\frac{T}{p}]$, we have*

$$|E(t,y_1,x_1)-E(t,y_2,x_2)|\le\tilde{L}\,\theta(|y_1-y_2|+|x_1-x_2|),$$

and

$$\Lambda\big((y_1(t),x_1(t)),(y_2(t),x_2(t))\big)\ge0,$$

so that $\tilde{L}=\dfrac{1}{L_1+L_2}$;

($\mathfrak{P}2$) *Some $y_0\in\mathcal{X}_*$ exists s.t. $\forall\,t\in[0,\frac{T}{p}]$,*

$$\Lambda\big((y_0(t),{}^{c}\mathbb{D}^{r_2}_{(p;q)}y_0(t)),(Gy_0(t),{}^{c}\mathbb{D}^{r_2}_{(p;q)}(Gy_0(t)))\big)\ge0,$$

and also, we have

$$\Lambda\big((y_1(t),{}^{c}\mathbb{D}^{r_2}_{(p;q)}y_1(t)),(y_2(t),{}^{c}\mathbb{D}^{r_2}_{(p;q)}y_2(t))\big)\ge0,$$

which gives

$$\Lambda\Big(\big(Gy_1(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}\big(Gy_1(t)\big)\big), \big(Gy_2(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}\big(Gy_2(t)\big)\big)\Big) \geq 0,$$

for each $y_1, y_2 \in \mathcal{X}_*$ *and all* $t \in [0, \frac{T}{p}]$;

($\mathfrak{P}3$) *For every sequence* $\{y_i\}_{i\geq 1} \subseteq \mathcal{X}_*$ *converging to* y, *the inequality*

$$\Lambda\Big(\big(y_i(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_i(t)\big), \big(y_{i+1}(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_{i+1}(t)\big)\Big) \geq 0,$$

gives

$$\Lambda\Big(\big(y_i(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_i(t)\big), \big(y(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y(t)\big)\Big) \geq 0,$$

for each i *and* $t \in [0, \frac{T}{p}]$.

Then, the generalized sequential $(p;q)$*-difference Navier problem* (4) *has a solution on* $[0, \frac{T}{p}]$.

Proof. Let $y_1, y_2 \in \mathcal{X}_*$ be arbitrary so that

$$\Lambda\Big(\big(y_1(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_1(t)\big), \big(y_2(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_2(t)\big)\Big) \geq 0,$$

for all $t \in [0, \frac{T}{p}]$. For simplicity in the computations, we put

$$\begin{cases} E^{r_1,r_2}_{t,p}(y) = E\Big[\dfrac{t}{p^{r_1+r_2-1}}, y\big(\dfrac{t}{p^{r_1+r_2-1}}\big), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y\big(\dfrac{t}{p^{r_1+r_2-1}}\big)\Big], \quad (t \in [0, \dfrac{T}{p}]), \\ E^{r_1}_{t,p}(y) = E\Big[\dfrac{t}{p^{r_1-1}}, y\big(\dfrac{t}{p^{r_1-1}}\big), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y\big(\dfrac{t}{p^{r_1-1}}\big)\Big], \quad (t \in [0, \dfrac{T}{p}]). \end{cases} \tag{25}$$

By the hypotheses, we clearly have

$$\begin{aligned} \big|E^{r_1,r_2}_{t,p}(y_1) - E^{r_1,r_2}_{t,p}(y_2)\big| &= \bigg|E\Big[\frac{t}{p^{r_1+r_2-1}}, y_1\big(\frac{t}{p^{r_1+r_2-1}}\big), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_1\big(\frac{t}{p^{r_1+r_2-1}}\big)\Big] \\ &\quad - E\Big[\frac{t}{p^{r_1+r_2-1}}, y_2\big(\frac{t}{p^{r_1+r_2-1}}\big), {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_2\big(\frac{t}{p^{r_1+r_2-1}}\big)\Big]\bigg| \\ &\leq \tilde{L}\,\theta\bigg(\Big|y_1\big(\frac{t}{p^{r_1+r_2-1}}\big) - y_2\big(\frac{t}{p^{r_1+r_2-1}}\big)\Big| + \Big|{}^{c}\mathbb{D}^{r_2}_{(p;q)}y_1\big(\frac{t}{p^{r_1+r_2-1}}\big) - {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_2\big(\frac{t}{p^{r_1+r_2-1}}\big)\Big|\bigg) \\ &\leq \tilde{L}\,\theta(\|y_1 - y_2\|_{\mathcal{X}_*}). \end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned} \big|E^{r_1}_{t,p}(y_1) - E^{r_1}_{t,p}(y_2)\big| &\leq \tilde{L}\,\theta\bigg(\Big|y_1\big(\frac{t}{p^{r_1-1}}\big) - y_2\big(\frac{t}{p^{r_1-1}}\big)\Big| + \Big|{}^{c}\mathbb{D}^{r_2}_{(p;q)}y_1\big(\frac{t}{p^{r_1-1}}\big) - {}^{c}\mathbb{D}^{r_2}_{(p;q)}y_2\big(\frac{t}{p^{r_1-1}}\big)\Big|\bigg) \\ &\leq \tilde{L}\,\theta(\|y_1 - y_2\|_{\mathcal{X}_*}). \end{aligned}$$

Now, by (26), we may write

$$\begin{aligned} &\big|Gy_1(t) - Gy_2(t)\big| \\ &\leq \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^t (t-qv)^{(r_1+r_2-1)}_{(p;q)}\big|E^{r_1,r_2}_{v,p}(y_1) - E^{r_1,r_2}_{v,p}(y_2)\big|\, d_{(p;q)}v \\ &\quad + \frac{pt}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^{\frac{T}{p}} \big(\frac{T}{p} - qv\big)^{(r_1+r_2-1)}_{(p;q)}\big|E^{r_1,r_2}_{v,p}(y_1) - E^{r_1,r_2}_{v,p}(y_2)\big|\, d_{(p;q)}v \end{aligned}$$

$$+\frac{\left|p^2(\frac{T}{p})^{r_2+1}t\right|+\left|pTt^{r_2+1}\right|}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}}\int_0^{\frac{T}{p}}(\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)}\left|E_{v,p}^{r_1}(y_1)-E_{v,p}^{r_1}(y_2)\right|d_{(p;q)}v$$

$$\leq\frac{\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^t(t-qv)_{(p;q)}^{(r_1+r_2-1)}d_{(p;q)}v$$

$$+\frac{pt\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^{\frac{T}{p}}(\frac{T}{p}-qv)_{(p;q)}^{(r_1+r_2-1)}d_{(p;q)}v$$

$$+\frac{\left(\left|p^2(\frac{T}{p})^{r_2+1}t\right|+\left|pTt^{r_2+1}\right|\right)\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}}\int_0^{\frac{T}{p}}(\frac{T}{p}-qv)_{(p;q)}^{(r_1-1)}d_{(p;q)}v$$

$$\leq\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+r_2+1)}\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})+\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+1)\Gamma_{(p;q)}(r_2+2)}\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})$$

$$=\tilde{L}\,L_1\theta(\|y_1-y_2\|_{\mathcal{X}_*}).$$

Also, we have

$$\left|({}^{c}\mathbb{D}_{(p;q)}^{r_2}Gy_1)(t)-({}^{c}\mathbb{D}_{(p;q)}^{r_2}Gy_2)(t)\right|$$

$$\leq\frac{(\frac{T}{p})^{r_1-2}}{\Gamma_{(p;q)}(r_1-1)}\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})+\frac{p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_1+r_2+1)}\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})$$

$$+\frac{[1+\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)]p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1+1)}\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})$$

$$=\tilde{L}\,L_2\theta(\|y_1-y_2\|_{\mathcal{X}_*}).$$

These give $\|Gy_1-Gy_2\|_{\mathcal{X}_*}\leq(L_1+L_2)\tilde{L}\,\theta(\|y_1-y_2\|_{\mathcal{X}_*})=\theta(\|y_1-y_2\|_{\mathcal{X}_*})$. Now, define $\rho:\mathcal{X}_*\times\mathcal{X}_*\to[0,\infty)$ by

$$\rho(y_1,y_2)=\begin{cases}1, & \text{if }\ \Lambda\Big(\big(y_1(t),{}^{c}\mathbb{D}_{(p;q)}^{r_2}y_1(t)\big),\big(y_2(t),{}^{c}\mathbb{D}_{(p;q)}^{r_2}y_2(t)\big)\Big)\geq0,\\ 0, & \quad\text{otherwise,}\end{cases}$$

for each $y_1,y_2\in\mathcal{X}_*$. Then, by using these arbitrary members $y_1,y_2\in\mathcal{X}_*$, we get

$$\rho(y_1,y_2)d(Gy_1,Gy_2)\leq\theta(d(y_1,y_2)).$$

This inequality confirms that G is an ρ-θ-contraction. Moreover, G can be proved easily to be ρ-admissible and $\rho(y_0,Gy_0)\geq1$. Finally, consider $\{y_i\}_{i\geq1}\subseteq\mathcal{X}_*$ converging to y and let $\rho(y_i,y_{i+1})\geq1$ for all i. Based on the definition of the non-negative function ρ, one can write

$$\Lambda\Big(\big(y_i(t),{}^{c}\mathbb{D}_{(p;q)}^{r_2}y_i(t)\big),\big(y_{i+1}(t),{}^{c}\mathbb{D}_{(p;q)}^{r_2}y_{i+1}(t)\big)\Big)\geq0.$$

So the hypothesis of theorem implies that

$$\Lambda\Big(\big(y_i(t),{}^{c}\mathbb{D}_{(p;q)}^{r_2}y_i(t)\big),\big(y(t),{}^{c}\mathbb{D}_{(p;q)}^{r_2}y(t)\big)\Big)\geq0.$$

Therefore, $\rho(y_i,y)\geq1$ for all i. Since all hypotheses of Theorem 2 are fulfilled, so $G(y^{**})=y^{**}\in\mathcal{X}_*$. That means that y^{**} is a solution of the generalized sequential $(p;q)$-difference Navier problem (4). $\square$

The next existence theorem gets help from the standard contractions along with the compact operators used in the Krasnoselskii's fixed point theorem.

Theorem 7. *Let* $E \in \mathcal{C}([0, \frac{T}{\mathrm{p}}] \times \mathcal{X}_*^2, \mathcal{X}_*)$*. Moreover,*

($\mathfrak{P}4$) $\exists\, \Bbbk \in C([0, \frac{T}{\mathrm{p}}], \mathbb{R})$ *s.t.* $\forall\, \mathrm{t} \in [0, \frac{T}{\mathrm{p}}]$ *and* $\mathrm{y}_1, \mathrm{y}_2, \mathrm{x}_1, \mathrm{x}_2 \in \mathcal{X}_*$*,*

$$|E(\mathrm{t}, \mathrm{y}_1, \mathrm{x}_1) - E(\mathrm{t}, \mathrm{y}_2, \mathrm{x}_2)| \leq \Bbbk(\mathrm{t})(|\mathrm{y}_1 - \mathrm{y}_2| + |\mathrm{x}_1 - \mathrm{x}_2|);$$

($\mathfrak{P}5$) $\exists\, \mathfrak{f} \in \mathcal{C}([0, \frac{T}{\mathrm{p}}], \mathbb{R}^+)$ *and there is a non-decreasing function* $\theta \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ *s.t. for all* $\mathrm{t} \in [0, \frac{T}{\mathrm{p}}]$ *and each* $\mathrm{y}_1, \mathrm{y}_2 \in \mathcal{X}_*$*,*

$$|E(\mathrm{t}, \mathrm{y}_1, \mathrm{y}_2)| \leq \mathfrak{f}(\mathrm{t})\theta(|\mathrm{y}_1| + |\mathrm{y}_2|).$$

Then, the generalized sequential $(\mathrm{p};\mathrm{q})$*-difference Navier problem* (4) *has at least one solution if*

$$\mathbb{L} = \|\Bbbk\|(L_3 + L_4) < 1, \tag{27}$$

where $\|\Bbbk\| = \sup_{\mathrm{t} \in [0, \frac{T}{\mathrm{p}}]} |\Bbbk(\mathrm{t})|$ *and* L_3, L_4 *are supposed in* (24)*.*

Proof. Set $\|\mathfrak{f}\| = \sup_{\mathrm{t} \in [0, \frac{T}{\mathrm{p}}]} |\mathfrak{f}(\mathrm{t})|$ and choose an approximate value $\varepsilon > 0$ so that

$$\varepsilon \geq \theta(\|\mathrm{y}\|_{\mathcal{X}_*})\|\mathfrak{f}\|(L_1 + L_2), \tag{28}$$

where L_1 and L_2 are assumed in (23), and also, define $\mathbb{Y}_\varepsilon = \{\mathrm{y} \in \mathcal{X}_* : \|\mathrm{y}\|_{\mathcal{X}_*} \leq \varepsilon\}$. This non-empty set is bounded, closed and convex in $\mathcal{X}_*$. Two operators G_1 and G_2 can be defined on $\mathbb{Y}_\varepsilon$ as

$$(G_1\mathrm{y})(\mathrm{t}) = \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1 + \mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\mathrm{t}} (\mathrm{t} - \mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-1)} E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y})\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v},$$

and

$$\begin{aligned}(G_2\mathrm{y})(\mathrm{t}) = &-\frac{\mathrm{p}\mathrm{t}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1 + \mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}} - \mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-1)} E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y})\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\ &+ \frac{\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}\mathrm{t} - \mathrm{p}T\mathrm{t}^{\mathrm{r}_2+1}}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2 + 2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}} - \mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1-1)} E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1}(\mathrm{y})\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v},\end{aligned}$$

for all $\mathrm{t} \in [0, \frac{T}{\mathrm{p}}]$. Put $\hat{\theta} = \sup_{\mathrm{y} \in \mathcal{X}_*} \theta(\|\mathrm{y}\|_{\mathcal{X}_*})$. For each $\mathrm{y}_1, \mathrm{y}_2 \in \mathbb{Y}_\varepsilon$, the following inequalities are satisfied as follow

$$\begin{aligned}&\big|(G_1\mathrm{y}_1 + G_2\mathrm{y}_2)(\mathrm{t})\big| \\ &\quad\leq \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1 + \mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\mathrm{t}} (\mathrm{t} - \mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-1)} \big|E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y}_1)\big|\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\ &\quad+ \frac{\mathrm{p}\mathrm{t}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1 + \mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}} - \mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-1)} \big|E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y}_2)\big|\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\ &\quad+ \frac{|\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}\mathrm{t}| + |\mathrm{p}T\mathrm{t}^{\mathrm{r}_2+1}|}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2 + 2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}} - \mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1-1)} \big|E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1}(\mathrm{y}_2)\big|\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^t (t-qv)_{(p;q)}^{(r_1+r_2-1)}\mathfrak{f}(v)\theta\big(|y_1(v)|+|{}^c\mathbb{D}_{(p;q)}^{r_2}y_1(v)|\big)\,d_{(p;q)}v\\
&+\frac{pt}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^{\frac{T}{p}}\big(\frac{T}{p}-qv\big)_{(p;q)}^{(r_1+r_2-1)}\mathfrak{f}(v)\theta\big(|y_2(v)|+|{}^c\mathbb{D}_{(p;q)}^{r_2}y_2(v)|\big)\,d_{(p;q)}v\\
&+\frac{\big|p^2(\frac{T}{p})^{r_2+1}t\big|+\big|pTt^{r_2+1}\big|}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}}\int_0^{\frac{T}{p}}\big(\frac{T}{p}-qv\big)_{(p;q)}^{(r_1-1)}\mathfrak{f}(v)\theta\big(|y_2(v)|+|{}^c\mathbb{D}_{(p;q)}^{r_2}y_2(v)|\big)\,d_{(p;q)}v\\
&\leq \hat{\theta}\|\mathfrak{f}\|\Big[\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+r_2+1)}+\frac{2(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+1)\Gamma_{(p;q)}(r_2+2)}\Big]=\hat{\theta}\|\mathfrak{f}\|L_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\big|({}^c\mathbb{D}_{(p;q)}^{r_2}G_1y_1+{}^c\mathbb{D}_{(p;q)}^{r_2}G_2y_2)(t)\big| &\leq \hat{\theta}\|\mathfrak{f}\|\Big[\frac{(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(r_1-1)}+\frac{p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_1+r_2+1)}\\
&+\frac{\big[1+\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\big]p^{r_2}(\frac{T}{p})^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1+1)}\Big]\\
&=\hat{\theta}\|\mathfrak{f}\|L_2.
\end{aligned}
$$

Therefore, $\|G_1y_1+G_2y_2\|_{\mathcal{X}_*}\leq\hat{\theta}\|\mathfrak{f}\|(L_1+L_2)\leq\varepsilon$. That is,

$$
(G_1y_1+G_2y_2)\in\mathbb{Y}_\varepsilon.
$$

The continuity property for the single-valued function E follows that G_1 is continuous too. So, on the set $\mathbb{Y}_\varepsilon$ and for each member y in it, we estimate

$$
\begin{aligned}
\big|(G_1y)(t)\big| &\leq \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}}\int_0^t (t-qv)_{(p;q)}^{(r_1+r_2-1)}\\
&\times\Big|E\Big[\frac{v}{p^{r_1+r_2-1}},y\big(\frac{v}{p^{r_1+r_2-1}}\big),{}^c\mathbb{D}_{(p;q)}^{r_2}y\big(\frac{v}{p^{r_1+r_2-1}}\big)\Big]\Big|\,d_{(p;q)}v\\
&\leq \frac{(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+r_2+1)}\|\mathfrak{f}\|\theta(\|y\|_{\mathcal{X}_*}),
\end{aligned}
$$

and

$$
\begin{aligned}
\big|({}^c\mathbb{D}_{(p;q)}^{r_2}G_1y)(t)\big| &\leq \frac{1}{\Gamma_{(p;q)}(r_1+r_2-4)\Gamma_{(p;q)}(2-r_2)p^{\binom{r_1+r_2-4}{2}+\binom{2-r_2}{2}}}\\
&\times\int_0^t\int_0^{\frac{v}{p^{1-r_2}}}(t-qv)_{(p;q)}^{(1-r_2)}\big(\frac{v}{p^{(1-r_2)}}-qw\big)_{(p;q)}^{(r_1+r_2-5)}\\
&\times\Big|E\Big[\frac{w}{p^{r_1+r_2-5}},y\big(\frac{w}{p^{r_1+r_2-5}}\big),{}^c\mathbb{D}_{(p;q)}^{r_2}y\big(\frac{w}{p^{r_1+r_2-5}}\big)\Big]\Big|\,d_{(p;q)}w\,d_{(p;q)}v\\
&\leq \frac{(\frac{T}{p})^{r_1-2}}{\Gamma_{(p;q)}(r_1-1)}\|\mathfrak{f}\|\theta(\|y\|_{\mathcal{X}_*}).
\end{aligned}
$$

Thus,

$$
\|G_1y\|_{\mathcal{X}_*}\leq\Big[\frac{(\frac{T}{p})^{r_1+r_2}}{\Gamma_{(p;q)}(r_1+r_2+1)}+\frac{(\frac{T}{p})^{r_1-2}}{\Gamma_{(p;q)}(r_1-1)}\Big]\|\mathfrak{f}\|\theta(\varepsilon).
$$

Immediately, and from the above inequality, one can find the uniform boundedness of G_1 on $\mathbb{Y}_\varepsilon$. To survey the compactness of G_1 on $\mathbb{Y}_\varepsilon$, let $\mathtt{t}_1, \mathtt{t}_2 \in [0, \frac{T}{\mathtt{p}}]$ be chosen arbitrarily so that $\mathtt{t}_1 < \mathtt{t}_2$. Then, by the notation (25), estimate

$$\begin{aligned}
&\big|(G_1\mathtt{y})(\mathtt{t}_2) - (G_1\mathtt{y})(\mathtt{t}_1)\big| \\
&= \Bigg|\frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}}\int_0^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}E_{\mathtt{v},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
&- \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_1}{2}}}\int_0^{\mathtt{t}_1}(\mathtt{t}_1-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}E_{\mathtt{v},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}\Bigg| \\
&\le \Bigg|\frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}}\int_0^{\mathtt{t}_1}\Big[(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}-(\mathtt{t}_1-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}\Big]E_{\mathtt{v},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}\Bigg| \\
&+ \Bigg|\frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_1}{2}}}\int_{\mathtt{t}_1}^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}E_{\mathtt{v},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}\Bigg| \\
&\le \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}}\int_0^{\mathtt{t}_1}\Big[(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}-(\mathtt{t}_1-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}\Big]\Big|E_{\mathtt{v},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\Big|\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
&+ \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_1}{2}}}\int_{\mathtt{t}_1}^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}\Big|E_{\mathtt{v},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\Big|\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
&\le \|\mathfrak{f}\|\theta(\varepsilon)\Bigg[\frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}}\int_0^{\mathtt{t}_1}\Big[(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}-(\mathtt{t}_1-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}\Big]\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
&+ \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_1}{2}}}\int_{\mathtt{t}_1}^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}\Bigg].
\end{aligned}$$

The last inequality (its right-hand side) is independent of $\mathtt{y}$. So $\big|(G_1\mathtt{y})(\mathtt{t}_2) - (G_1\mathtt{y})(\mathtt{t}_1)\big|$ tends to zero as $\mathtt{t}_1$ tends to $\mathtt{t}_2$. Also,

$$\begin{aligned}
&\big|({}^{c}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}G_1\mathtt{y})(\mathtt{t}_2) - ({}^{c}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}G_1\mathtt{y})(\mathtt{t}_1)\big| \le \Bigg|\frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2-4)\Gamma_{(\mathtt{p};\mathtt{q})}(2-\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2-4}{2}+\binom{2-\mathtt{r}_2}{2}}} \\
&\times\int_0^{\mathtt{t}_2}\int_0^{\frac{\mathtt{v}}{\mathtt{p}^{1-\mathtt{r}_2}}}(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(1-\mathtt{r}_2)}\Big(\frac{\mathtt{v}}{\mathtt{p}^{(1-\mathtt{r}_2)}}-\mathtt{q}\mathtt{w}\Big)_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-5)}E_{\mathtt{w},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{w}\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
&- \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2-4)\Gamma_{(\mathtt{p};\mathtt{q})}(2-\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2-4}{2}+\binom{2-\mathtt{r}_2}{2}}} \\
&\times\int_0^{\mathtt{t}_1}\int_0^{\frac{\mathtt{v}}{\mathtt{p}^{1-\mathtt{r}_2}}}(\mathtt{t}_1-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(1-\mathtt{r}_2)}\Big(\frac{\mathtt{v}}{\mathtt{p}^{(1-\mathtt{r}_2)}}-\mathtt{q}\mathtt{w}\Big)_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-5)}E_{\mathtt{w},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{w}\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}\Bigg| \\
&\le \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2-4)\Gamma_{(\mathtt{p};\mathtt{q})}(2-\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2-4}{2}+\binom{2-\mathtt{r}_2}{2}}} \\
&\times\int_0^{\mathtt{t}_1}\int_0^{\frac{\mathtt{v}}{\mathtt{p}^{1-\mathtt{r}_2}}}\Big[(\mathtt{t}_2-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(1-\mathtt{r}_2)}-(\mathtt{t}_1-\mathtt{q}\mathtt{v})_{(\mathtt{p};\mathtt{q})}^{(1-\mathtt{r}_2)}\Big] \\
&\times\Big(\frac{\mathtt{v}}{\mathtt{p}^{(1-\mathtt{r}_2)}}-\mathtt{q}\mathtt{w}\Big)_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-5)}\Big|E_{\mathtt{w},\mathtt{p}}^{\mathtt{r}_1,\mathtt{r}_2}(\mathtt{y})\Big|\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{w}\,\mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}
\end{aligned}$$

$$+\frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2-4)\Gamma_{(\mathrm{p};\mathrm{q})}(2-\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2-4}{2}+\binom{2-\mathrm{r}_2}{2}}}$$

$$\times\int_{\mathrm{t}_1}^{\mathrm{t}_2}\int_0^{\frac{\mathrm{v}}{\mathrm{p}^{1-\mathrm{r}_2}}}(\mathrm{t}_2-\mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(1-\mathrm{r}_2)}\Big(\frac{\mathrm{v}}{\mathrm{p}^{(1-\mathrm{r}_2)}}-\mathrm{q}\mathrm{w}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-5)}\Big|E_{\mathrm{w},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y})\Big|\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{w}\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$\leq\|\mathfrak{f}\|\theta(\varepsilon)\Bigg[\frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2-4)\Gamma_{(\mathrm{p};\mathrm{q})}(2-\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2-4}{2}+\binom{2-\mathrm{r}_2}{2}}}$$

$$\times\int_0^{\mathrm{t}_1}\int_0^{\frac{\mathrm{v}}{\mathrm{p}^{1-\mathrm{r}_2}}}\Big[(\mathrm{t}_2-\mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(1-\mathrm{r}_2)}-(\mathrm{t}_1-\mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(1-\mathrm{r}_2)}\Big]\Big(\frac{\mathrm{v}}{\mathrm{p}^{(1-\mathrm{r}_2)}}-\mathrm{q}\mathrm{w}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-5)}\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{w}\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$+\frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2-4)\Gamma_{(\mathrm{p};\mathrm{q})}(2-\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2-4}{2}+\binom{2-\mathrm{r}_2}{2}}}$$

$$\times\int_{\mathrm{t}_1}^{\mathrm{t}_2}\int_0^{\frac{\mathrm{v}}{\mathrm{p}^{1-\mathrm{r}_2}}}(\mathrm{t}_2-\mathrm{q}\mathrm{v})_{(\mathrm{p};\mathrm{q})}^{(1-\mathrm{r}_2)}\Big(\frac{\mathrm{v}}{\mathrm{p}^{(1-\mathrm{r}_2)}}-\mathrm{q}\mathrm{w}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-5)}\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{w}\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}\Bigg].$$

Once again, the last inequality (its right-hand side) is independent of y and so,

$$\big|\big({}^{c}\mathbb{D}_{(\mathrm{p};\mathrm{q})}^{\mathrm{r}_2}G_1\mathrm{y}\big)(\mathrm{t}_2)-\big({}^{c}\mathbb{D}_{(\mathrm{p};\mathrm{q})}^{\mathrm{r}_2}G_1\mathrm{y}\big)(\mathrm{t}_1)\big|\to 0,$$

as $\mathrm{t}_1\to\mathrm{t}_2$. Therefore,

$$\big\|\big(G_1\mathrm{y}\big)(\mathrm{t}_2)-\big(G_1\mathrm{y}\big)(\mathrm{t}_1)\big\|_{\mathcal{X}_*}\to 0,$$

as $\mathrm{t}_1\to\mathrm{t}_2$. Hence, the equi-continuity of the operator G_1 is proved. By the Arzelá-Ascoli theorem, the compactness of G_1 is to be held on $\mathbb{Y}_\varepsilon$. Lastly, it is showed that G_2 is a contraction. For each $\mathrm{y}_1,\mathrm{y}_2\in\mathbb{Y}_\varepsilon$, we have

$$\big|\big(G_2\mathrm{y}_1\big)(\mathrm{t})-\big(G_2\mathrm{y}_2\big)(\mathrm{t})\big|$$

$$\leq\frac{\mathrm{p}|\mathrm{t}|}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}}\int_0^{\frac{T}{\mathrm{p}}}\Big(\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-1)}\big|E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y}_1)-E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1,\mathrm{r}_2}(\mathrm{y}_2)\big|\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$+\frac{\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}|\mathrm{t}|+\mathrm{p}T|\mathrm{t}^{\mathrm{r}_2+1}|}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}}\int_0^{\frac{T}{\mathrm{p}}}\Big(\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1-1)}\big|E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1}(\mathrm{y}_1)-E_{\mathrm{v},\mathrm{p}}^{\mathrm{r}_1}(\mathrm{y}_2)\big|\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$\leq\frac{\mathrm{p}|\mathrm{t}|}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}}\int_0^{\frac{T}{\mathrm{p}}}\Big(\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1+\mathrm{r}_2-1)}$$

$$\times\,\mathbb{k}(\mathrm{v})\theta\big(|\mathrm{y}_1(\mathrm{v})-\mathrm{y}_2(\mathrm{v})|+|{}^{c}\mathbb{D}_{(\mathrm{p};\mathrm{q})}^{\mathrm{r}_2}\mathrm{y}_1(\mathrm{v})-{}^{c}\mathbb{D}_{(\mathrm{p};\mathrm{q})}^{\mathrm{r}_2}\mathrm{y}_2(\mathrm{v})|\big)\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$+\frac{\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}|\mathrm{t}|+\mathrm{p}T|\mathrm{t}^{\mathrm{r}_2+1}|}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}}\int_0^{\frac{T}{\mathrm{p}}}\Big(\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v}\Big)_{(\mathrm{p};\mathrm{q})}^{(\mathrm{r}_1-1)}$$

$$\times\,\mathbb{k}(\mathrm{v})\theta\big(|\mathrm{y}_1(\mathrm{v})-\mathrm{y}_2(\mathrm{v})|+|{}^{c}\mathbb{D}_{(\mathrm{p};\mathrm{q})}^{\mathrm{r}_2}\mathrm{y}_1(\mathrm{v})-{}^{c}\mathbb{D}_{(\mathrm{p};\mathrm{q})}^{\mathrm{r}_2}\mathrm{y}_2(\mathrm{v})|\big)\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$\leq\frac{(\frac{T}{\mathrm{p}})^{\mathrm{r}_1+\mathrm{r}_2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2+1)}\|\mathbb{k}\|\|\mathrm{y}_1-\mathrm{y}_2\|_{\mathcal{X}_*}+\frac{2(\frac{T}{\mathrm{p}})^{\mathrm{r}_1+\mathrm{r}_2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+1)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)}\|\mathbb{k}\|\|\mathrm{y}_1-\mathrm{y}_2\|_{\mathcal{X}_*}$$

$$= \|\mathbb{k}\| L_3 \|y_1 - y_2\|_{\mathcal{X}_*},$$

and

$$\big|\big({}^{c}\mathbb{D}^{r_2}_{(p;q)} G_2 y_1\big)(t) - \big({}^{c}\mathbb{D}^{r_2}_{(p;q)} G_2 y_2\big)(t)\big|$$

$$\le \frac{p^{1+r_2}|t^{1-r_2}|}{T\Gamma_{(p;q)}(r_1+r_2)\Gamma_{(p;q)}(2-r_2)p^{\binom{r_1+r_2}{2}}} \int_0^{\frac{T}{p}} \big(\frac{T}{p} - qv\big)^{(r_1+r_2-1)}_{(p;q)}$$

$$\times\, \mathbb{k}(v)\theta\big(|y_1(v) - y_2(v)| + |{}^{c}\mathbb{D}^{r_2}_{(p;q)} y_1(v) - {}^{c}\mathbb{D}^{r_2}_{(p;q)} y_2(v)|\big)\, d_{(p;q)}v$$

$$+ \frac{\big[p^{2+r_2}\big(\frac{T}{p}\big)^{r_2+1}|t^{1-r_2}| + p^{1+r_2}T\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(2-r_2)|t|\big]}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} \big(\frac{T}{p} - qv\big)^{(r_1-1)}_{(p;q)}$$

$$\times\, \mathbb{k}(v)\theta\big(|y_1(v) - y_2(v)| + |{}^{c}\mathbb{D}^{r_2}_{(p;q)} y_1(v) - {}^{c}\mathbb{D}^{r_2}_{(p;q)} y_2(v)|\big)\, d_{(p;q)}v$$

$$\le \frac{p^{r_2}\big(\frac{T}{p}\big)^{r_1}}{\Gamma_{(p;q)}(r_1+r_2+1)\Gamma_{(p;q)}(2-r_2)} \|\mathbb{k}\| \|y_1 - y_2\|_{\mathcal{X}_*}$$

$$+ \frac{\big[1 + \Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\big]p^{r_2}\big(\frac{T}{p}\big)^{r_1}}{\Gamma_{(p;q)}(2-r_2)\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1+1)} \|\mathbb{k}\| \|y_1 - y_2\|_{\mathcal{X}_*}$$

$$= \|\mathbb{k}\| L_4 \|y_1 - y_2\|_{\mathcal{X}_*}.$$

Thus,

$$\big\|G_2 y_1 - G_2 y_2\big\|_{\mathcal{X}_*} \le \|\mathbb{k}\|(L_3 + L_4)\|y_1 - y_2\|_{\mathcal{X}_*} = \mathbb{L}\|y_1 - y_2\|_{\mathcal{X}_*},$$

where $\mathbb{L} < 1$, and under this Lipschitz constant, G_2 is a contraction on $\mathbb{Y}_\varepsilon$. The conclusion of Theorem 3 implies the existence of solution for the generalized sequential $(p;q)$-difference Navier problem (4). □

4. On the Generalized $(p;q)$-Difference Navier Problem (5)

The existence theorems for the generalized sequential $(p;q)$-difference Navier inclusion problem (5) are established in this section.

For the generalized sequential $(p;q)$-difference Navier inclusion problem (5), we call $y \in \mathcal{C}_{\mathcal{X}_*}(\mathcal{I}^T_{(p;q)}, \mathcal{X}_*)$ as a solution if the given boundary conditions are satisfied for y and also, there is some $\bar{F} \in \mathcal{L}^1(\mathcal{I}^T_{(p;q)})$ so that $\bar{F}(t) \in \mathcal{E}\big(t, y(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)} y(t)\big)$ for almost all $t \in \mathcal{I}^T_{(p;q)}$ and

$$y(t) = \frac{1}{\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^t (t - qv)^{(r_1+r_2-1)}_{(p;q)} \bar{F}\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$

$$- \frac{pt}{T\Gamma_{(p;q)}(r_1+r_2)p^{\binom{r_1+r_2}{2}}} \int_0^{\frac{T}{p}} \big(\frac{T}{p} - qv\big)^{(r_1+r_2-1)}_{(p;q)} \bar{F}\Big[\frac{v}{p^{r_1+r_2-1}}\Big]\, d_{(p;q)}v$$

$$+ \frac{\big[p^2\big(\frac{T}{p}\big)^{r_2+1}t - pTt^{r_2+1}\big]}{T^2\Gamma_{(p;q)}(r_2+2)\Gamma_{(p;q)}(r_1)p^{\binom{r_1}{2}}} \int_0^{\frac{T}{p}} \big(\frac{T}{p} - qv\big)^{(r_1-1)}_{(p;q)} \bar{F}\Big[\frac{v}{p^{r_1-1}}\Big]\, d_{(p;q)}v,$$

for all $t \in \mathcal{I}^T_{(p;q)}$. The set of all selections of the multi-valued function $\mathcal{E}$ is given by

$$(\mathbb{S})_{\mathcal{E},y} = \big\{\bar{F} \in \mathcal{L}^1(\mathcal{I}^T_{(p;q)}) : \bar{F}(t) \in \mathcal{E}\big(t, y(t), {}^{c}\mathbb{D}^{r_2}_{(p;q)} y(t)\big) \text{ for all } t \in \mathcal{I}^T_{(p;q)}\big\},$$

for each $\mathrm{y} \in \mathcal{X}_*$. Moreover, the operator $\mathcal{F} : \mathcal{X}_* \to \mathrm{P}(\mathcal{X}_*)$ is considered as

$$\mathcal{F}(\mathrm{y}) = \{\hbar \in \mathcal{X}_* : \text{ there exists } \bar{F} \in (\mathbb{S})_{\mathcal{E},\mathrm{y}} : \hbar(\mathrm{t}) = \breve{g}(\mathrm{t}) \text{ for all } \mathrm{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}\}, \tag{29}$$

where

$$\begin{aligned}\breve{g}(\mathrm{t}) = {}& \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\ & - \frac{\mathrm{p}\mathrm{t}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\ & + \frac{\big[\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}\mathrm{t} - \mathrm{p}T\mathrm{t}^{\mathrm{r}_2+1}\big]}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1-1}}\Big]\, \mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}.\end{aligned}$$

For simplicity, put

$$\alpha_1 = \|\alpha\| L_1 \text{ and } \alpha_2 = \|\alpha\| L_2. \tag{30}$$

In this step, we are going to use the special multi-valued functions of the ρ-θ-contractive type and ρ-admissible type for proving the existence result related to solutions of the generalized sequential $(\mathrm{p};\mathrm{q})$-difference Navier inclusion problem (5).

Theorem 8. *Let $\mathcal{E} : \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})} \times \mathcal{X}_*^2 \to \mathrm{P}_{\mathbb{CM}}(\mathcal{X}_*)$. Moreover,*

($\mathfrak{P}6$) *$\mathcal{E}$ is bounded and integrable so that $\mathcal{E}(\cdot, \mathrm{y}_1, \mathrm{y}_2) : \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})} \to \mathrm{P}_{\mathbb{CM}}$ is measurable for each $\mathrm{y}_1, \mathrm{y}_2 \in \mathcal{X}_*$;*

($\mathfrak{P}7$) *There are $\alpha \in \mathcal{C}(\mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}, [0, \infty))$ and $\theta \in \Theta$ so that*

$$\mathbb{H}_{\mathrm{d}}\big(\mathcal{E}(\mathrm{t}, \mathrm{y}_1, \mathrm{y}_2), \mathcal{E}(\mathrm{t}, \tilde{\mathrm{y}}_1, \tilde{\mathrm{y}}_2)\big) \leq \alpha(\mathrm{t})\bigg(\frac{\tilde{L}}{\|\alpha\|}\bigg)\theta(|\mathrm{y}_1 - \tilde{\mathrm{y}}_1| + |\mathrm{y}_2 - \tilde{\mathrm{y}}_2|), \tag{31}$$

for all $\mathrm{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$ and each $\mathrm{y}_1, \mathrm{y}_2, \tilde{\mathrm{y}}_1, \tilde{\mathrm{y}}_2 \in \mathcal{X}_*$, where $\sup_{\mathrm{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}} |\alpha(\mathrm{t})| = \|\alpha\|$, $\tilde{L} = \frac{1}{L_1+L_2}$ and L_1, L_2 are the constants assumed in (23);

($\mathfrak{P}8$) *There is $\Lambda : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ so that $\Lambda((\mathrm{y}_1, \mathrm{y}_2), (\tilde{\mathrm{y}}_1, \tilde{\mathrm{y}}_2)) \geq 0$ for each $\mathrm{y}_1, \mathrm{y}_2, \tilde{\mathrm{y}}_1, \tilde{\mathrm{y}}_2 \in \mathcal{X}_*$;*

($\mathfrak{P}9$) *There is a sequence $\{\mathrm{y}_i\}_{i\geq 1} \subseteq \mathcal{X}_*$ converging to y so that the inequality*

$$\Lambda\big((\mathrm{y}_i(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}_i(\mathrm{t})), (\mathrm{y}_{i+1}(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}_{i+1}(\mathrm{t}))\big) \geq 0, \quad \forall\, \mathrm{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})},\ i \geq 1,$$

implies the existence of a subsequence $\{\mathrm{y}_{i_r}\}_{r\geq 1}$ of $\{\mathrm{y}_i\}$ with

$$\Lambda\big((\mathrm{y}_{i_r}(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}_{i_r}(\mathrm{t})), (\mathrm{y}(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathrm{t}))\big) \geq 0,$$

for all $\mathrm{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$ and $r \geq 1$;

($\mathfrak{P}10$) *There are $\mathrm{y}_0 \in \mathcal{X}_*$ and $\hbar \in \mathcal{F}(\mathrm{y}_0)$ so that*

$$\Lambda\big((\mathrm{y}_0(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}_0(\mathrm{t})), (\hbar(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar(\mathrm{t}))\big) \geq 0,$$

for all $\mathrm{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$, where $\mathcal{F} : \mathcal{X}_* \to \mathrm{P}(\mathcal{X}_*)$ is defined by (29);

($\mathfrak{P}11$) *For each $\mathrm{y} \in \mathcal{X}_*$ and $\hbar \in \mathcal{F}(\mathrm{y})$ such that*

$$\Lambda\big((\mathrm{y}(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathrm{t})), (\hbar(\mathrm{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar(\mathrm{t}))\big) \geq 0,$$

there is some $\breve{g} \in \mathcal{F}(\mathrm{y})$ *so that*

$$\Lambda\big((\hbar(\mathtt{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar(\mathtt{t})), (\breve{g}(\mathtt{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t}))\big) \geq 0,$$

for all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$.

Then, the generalized sequential $(\mathrm{p};\mathrm{q})$*-difference Navier inclusion problem* (5) *has a solution.*

Proof. It is known that a fixed point of $\mathcal{F} : \mathcal{X}_* \to \mathrm{P}(\mathcal{X}_*)$ is the solution of the generalized sequential $(\mathrm{p};\mathrm{q})$-difference Navier inclusion problem (5). The closed-valued multi-valued function $\mathtt{t} \to \mathcal{E}\big(\mathtt{t}, \mathrm{y}(\mathtt{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})\big)$ is measurable for each $\mathrm{y} \in \mathcal{X}_*$. So, $\mathcal{E}$ has also a measurable selection; that is, $(\mathbb{S})_{\mathcal{E},\mathrm{y}} \neq \emptyset$.

In the next step, we establish that $\mathcal{F}(\mathrm{y}) \subseteq \mathcal{X}_*$ is closed for each $\mathrm{y} \in \mathcal{X}_*$. Let $\{\mathrm{y}_i\}_{i\geq 1}$ be a sequence in $\mathcal{F}(\mathrm{y})$ so that $\mathrm{y}_i \to \mathrm{y}$. For every i, $\exists \bar{F}_i \in (\mathbb{S})_{\mathcal{E},\mathrm{y}}$ so that

$$\begin{aligned}
\mathrm{y}_i(\mathtt{t}) = & \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\mathtt{t}} (\mathtt{t}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}_i\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\
& - \frac{\mathrm{p}\mathtt{t}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}_i\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\
& + \frac{\big[\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}\mathtt{t} - \mathrm{p}T\mathtt{t}^{\mathrm{r}_2+1}\big]}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}_i\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v},
\end{aligned}$$

for almost all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. Note that $\mathcal{E}$ is compact-valued. Hence, a subsequence $\{\bar{F}_i\}_{i\geq 1}$ exists so that converges to some $\bar{F} \in \mathcal{L}^1(\mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})})$. Immediately, we find that $\bar{F} \in (\mathbb{S})_{\mathcal{E},\mathrm{y}}$ and

$$\begin{aligned}
& \mathrm{y}_i(\mathtt{t}) \to \mathrm{y}(\mathtt{t}) \\
& = \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\mathtt{t}} (\mathtt{t}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\
& - \frac{\mathrm{p}\mathtt{t}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v} \\
& + \frac{\big[\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}\mathtt{t} - \mathrm{p}T\mathtt{t}^{\mathrm{r}_2+1}\big]}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}} \int_0^{\frac{T}{\mathrm{p}}} (\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v},
\end{aligned}$$

for all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. Consequently, $\mathrm{y} \in \mathcal{F}(\mathrm{y})$ and $\mathcal{F}$ possesses the closed values. Again, since $\mathcal{E}$ has the compact values, clearly one can show that $\mathcal{F}(\mathrm{y})$ is bounded for every $\mathrm{y} \in \mathcal{X}_*$.

In this step, we establish that $\mathcal{F}$ is an ρ-θ-contraction. First, define a non-negative function ρ on $\mathcal{X}_* \times \mathcal{X}_*$ as

$$\rho(\mathrm{y}, \tilde{\mathrm{y}}) = \begin{cases} 1 & \text{if } \Lambda\big((\mathrm{y}(\mathtt{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})), (\tilde{\mathrm{y}}(\mathtt{t}), {}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\tilde{\mathrm{y}}(\mathtt{t}))\big) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for each $\mathrm{y}, \tilde{\mathrm{y}} \in \mathcal{X}_*$. Let $\mathrm{y}, \tilde{\mathrm{y}} \in \mathcal{X}_*$ and $\hbar_1 \in \mathcal{F}(\tilde{\mathrm{y}})$. Choose $\bar{F}_1 \in (\mathbb{S})_{\mathcal{E},\tilde{\mathrm{y}}}$ so that

$$\hbar_1(\mathtt{t}) = \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}} \int_0^{\mathtt{t}} (\mathtt{t}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})} \bar{F}_1\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}$$

$$- \frac{\mathtt{pt}}{T\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}} \int_0^{\frac{T}{\mathtt{p}}} (\frac{T}{\mathtt{p}} - \mathtt{qv})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)} \bar{F}_1\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big]\, \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}$$

$$+ \frac{\big[\mathtt{p}^2(\frac{T}{\mathtt{p}})^{\mathtt{r}_2+1}\mathtt{t} - \mathtt{p}T\mathtt{t}^{\mathtt{r}_2+1}\big]}{T^2\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_2+2)\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1)\mathtt{p}^{\binom{\mathtt{r}_1}{2}}} \int_0^{\frac{T}{\mathtt{p}}} (\frac{T}{\mathtt{p}} - \mathtt{qv})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1-1)} \bar{F}_1\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1-1}}\Big]\, \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v},$$

for all $\mathtt{t} \in \mathcal{I}_{(\mathtt{p};\mathtt{q})}^{T}$. From (31), one may write

$$\mathbb{H}_{\mathrm{d}}\big(\mathcal{E}\big(\mathtt{t},\mathtt{y},{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}\big),\mathcal{E}\big(\mathtt{t},\tilde{\mathtt{y}},{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}\big)\big) \leq \alpha(\mathtt{t})\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta\big(|\mathtt{y}-\tilde{\mathtt{y}}| + \big|{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y} - {}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}\big|\big),$$

for every $\mathtt{y},\tilde{\mathtt{y}} \in \mathcal{X}_*$ with

$$\Lambda\big(\big(\mathtt{y}(\mathtt{t}),{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t})\big),\big(\tilde{\mathtt{y}}(\mathtt{t}),{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}(\mathtt{t})\big)\big) \geq 0,$$

for almost all $\mathtt{t} \in \mathcal{I}_{(\mathtt{p};\mathtt{q})}^{T}$. There is some $\breve{g} \in \mathcal{E}\big(\mathtt{t},\mathtt{y}(\mathtt{t}),{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t})\big)$ so that

$$|\bar{F}_1(\mathtt{t}) - \breve{g}| \leq \alpha(\mathtt{t})\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta\big(|\mathtt{y}(\mathtt{t})-\tilde{\mathtt{y}}(\mathtt{t})| + \big|{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t}) - {}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}(\mathtt{t})\big|\big).$$

Define $\mathfrak{F} : \mathcal{I}_{(\mathtt{p};\mathtt{q})}^{T} \to \mathrm{P}(\mathcal{X}_*)$ by

$$\mathfrak{F}(\mathtt{t}) = \left\{\breve{g} \in \mathcal{X}_* : |\bar{F}_1(\mathtt{t}) - \breve{g}| \leq \alpha(\mathtt{t})\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta\big(|\mathtt{y}(\mathtt{t})-\tilde{\mathtt{y}}(\mathtt{t})| + \big|{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t}) - {}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}(\mathtt{t})\big|\big)\right\},$$

for all $\mathtt{t} \in \mathcal{I}_{(\mathtt{p};\mathtt{q})}^{T}$. Note that $\bar{F}_1$ and $\omega = \alpha\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta\big(|\mathtt{y}-\tilde{\mathtt{y}}| + \big|{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y} - {}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}\big|\big)$ are measurable. Therefore, $\mathfrak{F}(\cdot) \cap \mathcal{E}\big(\cdot,\mathtt{y}(\cdot),{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\cdot)\big)$ is measurable. This time, we choose $\bar{F}_2 \in \mathcal{E}\big(\mathtt{t},\mathtt{y}(\mathtt{t}),{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t})\big)$ so that for all $\mathtt{t} \in \mathcal{I}_{(\mathtt{p};\mathtt{q})}^{T}$,

$$|\bar{F}_1(\mathtt{t}) - \bar{F}_2(\mathtt{t})| \leq \alpha(\mathtt{t})\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta\big(|\mathtt{y}(\mathtt{t})-\tilde{\mathtt{y}}(\mathtt{t})| + \big|{}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\mathtt{y}(\mathtt{t}) - {}^{\mathtt{c}}\mathbb{D}_{(\mathtt{p};\mathtt{q})}^{\mathtt{r}_2}\tilde{\mathtt{y}}(\mathtt{t})\big|\big).$$

Consider $\hbar_2 \in \mathcal{F}(\mathtt{y})$ as

$$\hbar_2(\mathtt{t}) = \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{qv})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)} \bar{F}_2\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big]\, \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}$$

$$- \frac{\mathtt{pt}}{T\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}} \int_0^{\frac{T}{\mathtt{p}}} (\frac{T}{\mathtt{p}} - \mathtt{qv})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)} \bar{F}_2\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big]\, \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}$$

$$+ \frac{\big[\mathtt{p}^2(\frac{T}{\mathtt{p}})^{\mathtt{r}_2+1}\mathtt{t} - \mathtt{p}T\mathtt{t}^{\mathtt{r}_2+1}\big]}{T^2\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_2+2)\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1)\mathtt{p}^{\binom{\mathtt{r}_1}{2}}} \int_0^{\frac{T}{\mathtt{p}}} (\frac{T}{\mathtt{p}} - \mathtt{qv})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1-1)} \bar{F}_2\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1-1}}\Big]\, \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v},$$

for all $\mathtt{t} \in \mathcal{I}_{(\mathtt{p};\mathtt{q})}^{T}$. Then,

$$|\hbar_1(\mathtt{t}) - \hbar_2(\mathtt{t})|$$

$$\leq \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{qv})_{(\mathtt{p};\mathtt{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)} \Big|\bar{F}_1\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big] - \bar{F}_2\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big]\Big|\, \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}$$

$$+\frac{\mathtt{pt}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+\mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}}\int_0^{\frac{T}{\mathrm{p}}}(\frac{T}{\mathrm{p}}-\mathtt{qv})_{(\mathrm{p};\mathrm{q})}^{(\mathtt{r}_1+\mathtt{r}_2-1)}\Big|\bar{F}_1\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big]-\bar{F}_2\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\Big]\Big|\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathtt{v}$$

$$+\frac{|\mathtt{p}^2(\frac{T}{\mathrm{p}})^{\mathtt{r}_2+1}\mathtt{t}|+|\mathtt{p}T\mathtt{t}^{\mathtt{r}_2+1}|}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1)\mathtt{p}^{\binom{\mathtt{r}_1}{2}}}\int_0^{\frac{T}{\mathrm{p}}}(\frac{T}{\mathrm{p}}-\mathtt{qv})_{(\mathrm{p};\mathrm{q})}^{(\mathtt{r}_1-1)}\Big|\bar{F}_1\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1-1}}\Big]-\bar{F}_2\Big[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1-1}}\Big]\Big|\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathtt{v}$$

$$\leq\frac{2(\frac{T}{\mathrm{p}})^{\mathtt{r}_1+\mathtt{r}_2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+\mathtt{r}_2+1)}\|\alpha\|\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*})$$

$$+\frac{2(\frac{T}{\mathrm{p}})^{\mathtt{r}_1+\mathtt{r}_2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+1)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_2+2)}\|\alpha\|\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*})$$

$$\leq\left[\frac{2(\frac{T}{\mathrm{p}})^{\mathtt{r}_1+\mathtt{r}_2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+\mathtt{r}_2+1)}+\frac{2(\frac{T}{\mathrm{p}})^{\mathtt{r}_1+\mathtt{r}_2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+1)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_2+2)}\right]\|\alpha\|\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*})$$

$$=\tilde{L}\,L_1\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*}),$$

and

$$\begin{aligned}\left|{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar_1(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar_2(\mathtt{t})\right|\leq&\left[\frac{(\frac{T}{\mathrm{p}})^{\mathtt{r}_1-2}}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1-1)}+\frac{\mathtt{p}^{\mathtt{r}_2}(\frac{T}{\mathrm{p}})^{\mathtt{r}_1}}{\Gamma_{(\mathrm{p};\mathrm{q})}(2-\mathtt{r}_2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+\mathtt{r}_2+1)}\right.\\&\left.+\frac{\left[1+\Gamma_{(\mathrm{p};\mathrm{q})}(2-\mathtt{r}_2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_2+2)\right]\mathtt{p}^{\mathtt{r}_2}(\frac{T}{\mathrm{p}})^{\mathtt{r}_1}}{\Gamma_{(\mathrm{p};\mathrm{q})}(2-\mathtt{r}_2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathtt{r}_1+1)}\right]\\&\times\|\alpha\|\left(\frac{\tilde{L}}{\|\alpha\|}\right)\theta\big(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*}\big)\\&=\tilde{L}\,L_2\theta\big(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*}\big),\end{aligned}$$

for all $\mathtt{t}\in\mathcal{I}^T_{(\mathrm{p};\mathrm{q})}$. Thus,

$$\begin{aligned}\|\hbar_1-\hbar_2\|_{\mathcal{X}_*}=&\sup_{\mathtt{t}\in\mathcal{I}^T_{(\mathrm{p};\mathrm{q})}}|\hbar_1(\mathtt{t})-\hbar_2(\mathtt{t})|+\sup_{\mathtt{t}\in\mathcal{I}^T_{(\mathrm{p};\mathrm{q})}}\left|{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar_1(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar_2(\mathtt{t})\right|\\&\leq(L_1+L_2)\tilde{L}\,\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*})=\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*}).\end{aligned}$$

Hence,

$$\rho(\mathtt{y},\tilde{\mathtt{y}})\mathbb{H}_{\mathrm{d}}\big(\mathcal{F}(\mathtt{y})-\mathcal{F}(\tilde{\mathtt{y}})\big)\leq\theta(\|\mathtt{y}-\tilde{\mathtt{y}}\|_{\mathcal{X}_*}),$$

is fulfilled for each $\mathtt{y},\tilde{\mathtt{y}}\in\mathcal{X}_*$. This means that $\mathcal{F}$ is an ρ-θ-contraction.

Now, let $\mathtt{y}\in\mathcal{X}_*$ and $\tilde{\mathtt{y}}\in\mathcal{F}(\mathtt{y})$ so that $\rho(\mathtt{y},\tilde{\mathtt{y}})\geq 1$ and

$$\Lambda\Big(\big(\mathtt{y}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\mathtt{y}(\mathtt{t})\big),\big(\tilde{\mathtt{y}}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\tilde{\mathtt{y}}(\mathtt{t})\big)\Big)\geq 0.$$

There is some $\breve{g}\in\mathcal{F}(\tilde{\mathtt{y}})$ so that

$$\Lambda\Big(\big(\tilde{\mathtt{y}}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\tilde{\mathtt{y}}(\mathtt{t})\big),\big(\breve{g}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t})\big)\Big)\geq 0.$$

This implies that $\rho(\tilde{\mathtt{y}},\breve{g})\geq 1$ establishing the fact that $\mathcal{F}$ is ρ-admissible. Next, consider $\mathtt{y}_0\in\mathcal{X}_*$ and $\tilde{\mathtt{y}}\in\mathcal{F}(\mathtt{y}_0)$ so that $\forall\,\mathtt{t}\in\mathcal{I}^T_{(\mathrm{p};\mathrm{q})}$,

$$\Lambda\Big(\big(\mathtt{y}_0(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\mathtt{y}_0(\mathtt{t})\big),\big(\tilde{\mathtt{y}}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathrm{p};\mathrm{q})}\tilde{\mathtt{y}}(\mathtt{t})\big)\Big)\geq 0.$$

Clearly, $\rho(\mathtt{y}_0, \tilde{\mathtt{y}}) \geq 1$. Let $\{\mathtt{y}_i\}_{i\geq 1} \subseteq \mathcal{X}_*$ be a sequence converging to $\mathtt{y}$ and let, for all i, $\rho(\mathtt{y}_i, \mathtt{y}_{i+1}) \geq 1$. Then,

$$\Lambda\big(\big(\mathtt{y}_i(\mathtt{t}), {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathtt{p};\mathtt{q})}\mathtt{y}_i(\mathtt{t})\big), \big(\mathtt{y}_{i+1}(\mathtt{t}), {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathtt{p};\mathtt{q})}\mathtt{y}_{i+1}(\mathtt{t})\big)\big) \geq 0.$$

By $(\mathfrak{P}9)$, there is a subsequence $\{\mathtt{y}_{i_r}\}_{r\geq 1}$ of $\{\mathtt{y}_i\}$ so that

$$\Lambda\big(\big(\mathtt{y}_{i_r}(\mathtt{t}), {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathtt{p};\mathtt{q})}\mathtt{y}_{i_r}(\mathtt{t})\big), \big(\mathtt{y}(\mathtt{t}), {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\big)\big) \geq 0$$

for all $\mathtt{t} \in \mathcal{I}^T_{(\mathtt{p};\mathtt{q})}$. We find that $\rho(\mathtt{y}_{i_r}, \mathtt{y}) \geq 1$ for all r. By considering the above statements, all hypotheses of Theorem 4 are fulfilled. The existence of some fixed point for $\mathcal{F}$ is proved, and so, the generalized sequential $(\mathtt{p};\mathtt{q})$-difference Navier inclusion problem (5) involves a solution. $\square$

The last existence theorem is related to the existence of an endpoint for the given multi-valued function by using the approximate endpoint property.

Theorem 9. *Let $\mathcal{E} : \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} \times \mathcal{X}^2_* \to \mathrm{P}_{\mathbb{CM}}(\mathcal{X}_*)$. Moreover,*

$(\mathfrak{P}12)$ *There is $\theta : [0,\infty) \to [0,\infty)$ so that $\liminf_{\mathtt{t}\to\infty}(\mathtt{t} - \theta(\mathtt{t})) \geq 0$ and $\theta(\mathtt{t}) \leq \mathtt{t}$, $\forall\, \mathtt{t} > 0$. Here, θ is non-decreasing and upper semi-continuous;*

$(\mathfrak{P}13)$ *$\mathcal{E} : \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} \times \mathcal{X}^2_* \to \mathrm{P}_{\mathbb{CM}}(\mathcal{X}_*)$ is bounded and integrable such that $\mathcal{E}(\cdot, \mathtt{y}_1, \mathtt{y}_2) : \mathcal{I}^T_{(\mathtt{p};\mathtt{q})} \to \mathrm{P}_{\mathbb{CM}}(\mathcal{X}_*)$ is measurable for each $\mathtt{y}_1, \mathtt{y}_2 \in \mathcal{X}_*$;*

$(\mathfrak{P}14)$ *There is some $\alpha \in \mathcal{C}(\mathcal{I}^T_{(\mathtt{p};\mathtt{q})}, [0,\infty))$ so that*

$$\mathbb{H}_{\mathtt{d}}\big(\mathcal{E}(\mathtt{t}, \mathtt{y}_1, \mathtt{y}_2), \mathcal{E}(\mathtt{t}, \tilde{\mathtt{y}_1}, \tilde{\mathtt{y}_2})\big) \leq \alpha(\mathtt{t})\tilde{L}_\star\theta(|\mathtt{y}_1 - \tilde{\mathtt{y}_1}| + |\mathtt{y}_2 - \tilde{\mathtt{y}_2}|) \tag{32}$$

holds for each $\mathtt{t} \in \mathcal{I}^T_{(\mathtt{p};\mathtt{q})}$ and $\mathtt{y}_1, \mathtt{y}_2, \tilde{\mathtt{y}_1}, \tilde{\mathtt{y}_2} \in \mathcal{X}_*$, where $\tilde{L}_\star = \frac{1}{\alpha_1+\alpha_2}$ and α_1, α_2 are assumed in (30);

$(\mathfrak{P}15)$ *$\mathcal{F}$, given by (29), has the approximate endpoint property.*

Then, the generalized sequential $(\mathtt{p};\mathtt{q})$-difference Navier inclusion problem (5) has a solution.

Proof. Here, we shall prove that $\mathcal{F} : \mathcal{X}_* \to \mathrm{P}(\mathcal{X}_*)$ has an endpoint. As the first step, we establish that $\mathcal{F}(\mathtt{y}) \subseteq \mathcal{X}_*$ is closed for each $\mathtt{y} \in \mathcal{X}_*$. Note that the multi-valued function $\mathtt{t} \to \mathcal{E}\big(\mathtt{t}, \mathtt{y}(\mathtt{t}), {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_2}_{(\mathtt{p};\mathtt{q})}\mathtt{y}(\mathtt{t})\big)$ possesses the closed values and is measurable for every $\mathtt{y} \in \mathcal{X}_*$. Hence, there is a measurable selection for $\mathcal{E}$, and accordingly, $(\mathbb{S})_{\mathcal{E},\mathtt{y}} \neq \emptyset$. Immediately, similar to the proof of Theorem 8, one can easily prove that $\mathcal{F}(\mathtt{y})$ admits the closed values. Also, $\mathcal{F}(\mathtt{y})$ is bounded for every $\mathtt{y} \in \mathcal{X}_*$ because $\mathcal{E}$ has the compact values.

In the following, we focus on the establishment of the inequality

$$\mathbb{H}_{\mathtt{d}}\big(\mathcal{F}(\mathtt{y}), \mathcal{F}(\breve{\mathtt{y}})\big) \leq \theta(\|\mathtt{y} - \breve{\mathtt{y}}\|_{\mathcal{X}_*}).$$

For finding this purpose, let $\mathtt{y}, \breve{\mathtt{y}} \in \mathcal{X}_*$, $\hbar_1 \in \mathcal{F}(\breve{\mathtt{y}})$ and select $\bar{F}_1 \in (\mathbb{S})_{\mathcal{E},\breve{\mathtt{y}}}$ so that

$$\begin{aligned}
\hbar_1(\mathtt{t}) = {} & \frac{1}{\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1 + \mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{v})^{(\mathtt{r}_1+\mathtt{r}_2-1)}_{(\mathtt{p};\mathtt{q})} \bar{F}_1\left[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\right] \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
& - \frac{\mathtt{p}\mathtt{t}}{T\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1 + \mathtt{r}_2)\mathtt{p}^{\binom{\mathtt{r}_1+\mathtt{r}_2}{2}}} \int_0^{\frac{T}{\mathtt{p}}} (\frac{T}{\mathtt{p}} - \mathtt{q}\mathtt{v})^{(\mathtt{r}_1+\mathtt{r}_2-1)}_{(\mathtt{p};\mathtt{q})} \bar{F}_1\left[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1+\mathtt{r}_2-1}}\right] \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\
& + \frac{\big[\mathtt{p}^2(\frac{T}{\mathtt{p}})^{\mathtt{r}_2+1}\mathtt{t} - \mathtt{p}T\mathtt{t}^{\mathtt{r}_2+1}\big]}{T^2\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_2 + 2)\Gamma_{(\mathtt{p};\mathtt{q})}(\mathtt{r}_1)\mathtt{p}^{\binom{\mathtt{r}_1}{2}}} \int_0^{\frac{T}{\mathtt{p}}} (\frac{T}{\mathtt{p}} - \mathtt{q}\mathtt{v})^{(\mathtt{r}_1-1)}_{(\mathtt{p};\mathtt{q})} \bar{F}_1\left[\frac{\mathtt{v}}{\mathtt{p}^{\mathtt{r}_1-1}}\right] \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v},
\end{aligned}$$

for almost all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. From (32), we get

$$\mathbb{H}_{\mathrm{d}}\big(\mathcal{E}\big(\mathtt{t},\mathrm{y}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})\big),\mathcal{E}\big(\mathtt{t},\breve{g}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t})\big)\big) \leq \alpha(\mathtt{t})\tilde{L}_{\star}\theta\big(|\mathrm{y}(\mathtt{t})-\breve{g}(\mathtt{t})|+\big|{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t})\big|\big),$$

for all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. There is some $\bar{\mu} \in \mathcal{E}\big(\mathtt{t},\mathrm{y}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})\big)$ so that

$$|\bar{F}_1(\mathtt{t})-\bar{\mu}| \leq \alpha(\mathtt{t})\tilde{L}_{\star}\theta\big(|\mathrm{y}(\mathtt{t})-\breve{g}(\mathtt{t})|+\big|{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t})\big|\big).$$

Define $\sigma : \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})} \to \mathrm{P}(\mathcal{X}_*)$ by

$$\sigma(\mathtt{t}) = \big\{\bar{\mu} \in \mathcal{X}_* : |\bar{F}_1(\mathtt{t})-\bar{\mu}| \leq \alpha(\mathtt{t})\tilde{L}_{\star}\theta\big(|\mathrm{y}(\mathtt{t})-\breve{g}(\mathtt{t})|+\big|{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t})\big|\big)\big\},$$

for all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. Notice that $\bar{F}_1$ and $b_* = \alpha\tilde{L}_{\star}\theta\big(|\mathrm{y}-\breve{g}|+\big|{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}-{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}\big|\big)$ are measurable. Therefore, $\sigma(\cdot)\cap\mathcal{E}\big(\cdot,\mathrm{y}(\cdot),{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\cdot)\big)$ is measurable.

This time, take $\bar{F}_2 \in \mathcal{E}\big(\mathtt{t},\mathrm{y}(\mathtt{t}),{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})\big)$ so that

$$|\bar{F}_1(\mathtt{t})-\bar{F}_2(\mathtt{t})| \leq \alpha(\mathtt{t})\tilde{L}_{\star}\theta\big(|\mathrm{y}(\mathtt{t})-\breve{g}(\mathtt{t})|+\big|{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\mathrm{y}(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\breve{g}(\mathtt{t})\big|\big),$$

for all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. Also, take $\hbar_2 \in \mathcal{F}(\mathrm{y})$ so that

$$\begin{aligned}\hbar_2(\mathtt{t}) = {}& \frac{1}{\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}}\int_0^{\mathtt{t}}(\mathtt{t}-\mathrm{q}\mathrm{v})^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})}\bar{F}_2\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}\\ &-\frac{\mathrm{p}\mathtt{t}}{T\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1+\mathrm{r}_2)\mathrm{p}^{\binom{\mathrm{r}_1+\mathrm{r}_2}{2}}}\int_0^{\frac{T}{\mathrm{p}}}\big(\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v}\big)^{(\mathrm{r}_1+\mathrm{r}_2-1)}_{(\mathrm{p};\mathrm{q})}\bar{F}_2\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1+\mathrm{r}_2-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v}\\ &+\frac{\big[\mathrm{p}^2(\frac{T}{\mathrm{p}})^{\mathrm{r}_2+1}\mathtt{t}-\mathrm{p}T\mathtt{t}^{\mathrm{r}_2+1}\big]}{T^2\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_2+2)\Gamma_{(\mathrm{p};\mathrm{q})}(\mathrm{r}_1)\mathrm{p}^{\binom{\mathrm{r}_1}{2}}}\int_0^{\frac{T}{\mathrm{p}}}\big(\frac{T}{\mathrm{p}}-\mathrm{q}\mathrm{v}\big)^{(\mathrm{r}_1-1)}_{(\mathrm{p};\mathrm{q})}\bar{F}_2\Big[\frac{\mathrm{v}}{\mathrm{p}^{\mathrm{r}_1-1}}\Big]\,\mathrm{d}_{(\mathrm{p};\mathrm{q})}\mathrm{v},\end{aligned}$$

for all $\mathtt{t} \in \mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}$. If we continue the implemented steps in Theorem 8, we have

$$\begin{aligned}\|\hbar_1-\hbar_2\|_{\mathcal{X}_*} &= \sup_{\mathtt{t}\in\mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}}|\hbar_1(\mathtt{t})-\hbar_2(\mathtt{t})|+\sup_{\mathtt{t}\in\mathcal{I}^{T}_{(\mathrm{p};\mathrm{q})}}\big|{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar_1(\mathtt{t})-{}^{\mathrm{c}}\mathbb{D}^{\mathrm{r}_2}_{(\mathrm{p};\mathrm{q})}\hbar_2(\mathtt{t})\big|\\ &\leq (\alpha_1+\alpha_2)\tilde{L}_{\star}\theta(\|\mathrm{y}-\breve{g}\|_{\mathcal{X}_*}) = \theta(\|\mathrm{y}-\breve{g}\|_{\mathcal{X}_*}).\end{aligned}$$

Therefore,

$$\mathbb{H}_{\mathrm{d}}\big(\mathcal{F}(\mathrm{y}),\mathcal{F}(\breve{g})\big) \leq \theta(\|\mathrm{y}-\breve{g}\|_{\mathcal{X}_*}),$$

for every $\mathrm{y},\breve{g} \in \mathcal{X}_*$. On the other hand, $\mathcal{F}$ has the approximate endpoint property by the hypothesis ($\mathfrak{P}$15). Accordingly, the existence of $\mathrm{y}^{**} \in \mathcal{X}_*$ is established by Theorem 5 so that $\mathcal{F}(\mathrm{y}^{**}) = \{\mathrm{y}^{**}\}$; that is, as we expected, y^{**} is an endpoint for $\mathcal{F}$. Therefore, the generalized sequential $(\mathrm{p};\mathrm{q})$-difference Navier inclusion problem (5) has a solution y^{**}. □

5. Examples

Analysis of some numerical examples are conducted here, to validate the theoretical results in the previous sections.

Example 1. *We model the following generalized sequential* $(\mathtt{p};\mathtt{q})$*-difference Navier problem of the elastic beam as*

$$\begin{cases} {}^{c}\mathbb{D}^{1.56}_{(0.5;0.45)}\left({}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}\right)(\mathtt{t}) = \dfrac{\mathtt{t}|\tan^{-1}(\mathtt{y}(\mathtt{t}))|}{40+40|\tan^{-1}(\mathtt{y}(\mathtt{t}))|} + 0.025\mathtt{t}\,{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(\mathtt{t}), \\ 3.5\mathtt{y}(0) = 8.2\mathtt{y}(0.5) = 12.24\,{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(0) = 2.2\,{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(0.5) = 0, \end{cases} \tag{33}$$

where $\mathtt{p} = 0.5$, $\mathtt{q} = 0.45$, $\mathtt{r}_1 = 1.56$, $\mathtt{r}_2 = 1.79$, $\beta = 3.5$, $\lambda = 8.2$, $\delta = 12.24$, $\gamma = 2.2$, *and* $T = 0.25$ *with* $\frac{T}{\mathtt{p}} = 0.5$ *and* $\mathtt{t} \in \mathcal{I}^{0.25}_{(0.5;0.45)} = [0, 0.5]$*. Moreover, the continuous nonlinear function* $E : [0, 0.5] \times \mathbb{R}^2 \to \mathbb{R}$ *is defined by*

$$E\big(\mathtt{t}, \mathtt{y}(\mathtt{t}), \mathtt{w}(\mathtt{t})\big) = \frac{\mathtt{t}|\tan^{-1}(\mathtt{y}(\mathtt{t}))|}{40+40|\tan^{-1}(\mathtt{y}(\mathtt{t}))|} + 0.025\mathtt{t}\mathtt{w}(\mathtt{t}).$$

For each $\mathtt{y}_1, \mathtt{y}_2, \mathtt{w}_1, \mathtt{w}_2 \in \mathbb{R}$*, we can estimate*

$$\begin{aligned} \big|E\big(\mathtt{t}, \mathtt{y}_1(\mathtt{t}), \mathtt{w}_1(\mathtt{t})\big) - E\big(\mathtt{t}, \mathtt{y}_2(\mathtt{t}), \mathtt{w}_2(\mathtt{t})\big)\big| &\le 0.025\mathtt{t}(|\tan^{-1}(\mathtt{y}_1(\mathtt{t})) - \tan^{-1}(\mathtt{y}_2(\mathtt{t}))| + |\mathtt{w}_1(\mathtt{t}) - \mathtt{w}_2(\mathtt{t})|) \\ &\le 0.025\mathtt{t}(|\mathtt{y}_1(\mathtt{t}) - \mathtt{y}_2(\mathtt{t})| + |\mathtt{w}_1(\mathtt{t}) - \mathtt{w}_2(\mathtt{t})|). \end{aligned}$$

Put $\mathbb{k}(\mathtt{t}) = 0.025\mathtt{t}$ *for all* $\mathtt{t}$*. Then,* $\|\mathbb{k}\| = \sup_{\mathtt{t}\in[0,0.5]} |0.025\mathtt{t}| = 0.0125$*. Now, the nondecreasing function* $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ *is defined by* $\theta(\varsigma) = \varsigma$ *for each* $\varsigma \in \mathbb{R}^+$*. Then,*

$$\begin{aligned} \big|E\big(\mathtt{t}, \mathtt{y}(\mathtt{t}), {}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(\mathtt{t})\big)\big| &\le 0.025\mathtt{t}\big(|\mathtt{y}(\mathtt{t})| + \big|{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(\mathtt{t})\big|\big) \\ &= 0.025\mathtt{t}\theta\big(|\mathtt{y}(\mathtt{t})| + \big|{}^{c}\mathbb{D}^{1.79}_{0.57(0.5;0.45)}\mathtt{y}(\mathtt{t})\big|\big). \end{aligned}$$

Obviously, $\mathfrak{f} : \mathcal{I}^{0.25}_{(0.5;0.45)} = [0, 0.5] \to \mathbb{R}^+$ *given by* $\mathfrak{f}(\mathtt{t}) = 0.025\mathtt{t}$ *is continuous. By* (24)*, we compute*

$$L_3 = \frac{0.098}{\Gamma_{(0.5;0.45)}(4.35)} + \frac{0.196}{\Gamma_{(0.5;0.45)}(2.56)\Gamma_{(0.5;0.45)}(3.79)} \approx 0.019204,$$

$$L_4 = \frac{0.098}{\Gamma_{(0.5;0.45)}(0.21)\Gamma_{(0.5;0.45)}(4.35)} + \frac{\big[1 + \Gamma_{(0.5;0.45)}(0.21)\Gamma_{(0.5;0.45)}(3.79)\big]0.098}{\Gamma_{(0.5;0.45)}(0.21)\Gamma_{(0.5;0.45)}(3.79)\Gamma_{(0.5;0.45)}(2.56)} \approx 0.058137.$$

By (27)*, we have* $\mathbb{L} \approx 0.0009667625 < 1$*. Hence, Theorem 7 concludes that the generalized sequential* $(\mathtt{p};\mathtt{q})$*-difference Navier problem* (33) *has at least one solution on* $\mathcal{I}^{0.25}_{(0.5;0.45)}$*.*

The next example deals with the inclusion version of the Navier $(\mathtt{p};\mathtt{q})$-difference equation.

Example 2. *By using the given parameters in the previous example, i.e.,* $\mathtt{p} = 0.5$, $\mathtt{q} = 0.45$, $\mathtt{r}_1 = 1.56$, $\mathtt{r}_2 = 1.79$, $\beta = 3.5$, $\lambda = 8.2$, $\delta = 12.24$, $\gamma = 2.2$, *and* $T = 0.25$ *with* $\frac{T}{\mathtt{p}} = 0.5$ *and* $\mathtt{t} \in \mathcal{I}^{0.25}_{(0.5;0.45)} = [0, 0.5]$*, we model the generalized sequential* $(\mathtt{p};\mathtt{q})$*-difference Navier inclusion problem of the elastic beam as*

$$\begin{cases} {}^{c}\mathbb{D}^{1.56}_{(0.5;0.45)}\left({}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}\right)(\mathtt{t}) = \left[0, \dfrac{\mathtt{t}|\sin(\mathtt{y}(\mathtt{t}))|}{48(1+\mathtt{t}^2)(1+|\sin(\mathtt{y}(\mathtt{t}))|)} + \dfrac{2\mathtt{t}|{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(\mathtt{t})|}{96(1+2\mathtt{t})(1+|{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(\mathtt{t})|)}\right], \\ 3.5\mathtt{y}(0) = 8.2\mathtt{y}(0.5) = 12.24\,{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(0) = 2.2\,{}^{c}\mathbb{D}^{1.79}_{(0.5;0.45)}\mathtt{y}(0.5) = 0, \end{cases} \tag{34}$$

Define $\mathcal{E} : \mathcal{I}^{0.25}_{(0.5;0.45)} = [0, 0.5] \times \mathbb{R}^2 \to \mathrm{P}(\mathbb{R})$ *by*

$$\mathcal{E}\big(\mathtt{t}, \mathtt{y}_1(\mathtt{t}), \mathtt{y}_2(\mathtt{t})\big) = \left[0, \frac{\mathtt{t}|\sin(\mathtt{y}_1(\mathtt{t}))|}{48(1+\mathtt{t}^2)(1+|\sin(\mathtt{y}_1(\mathtt{t}))|)} + \frac{2\mathtt{t}|\mathtt{y}_2(\mathtt{t})|}{96(1+2\mathtt{t})(1+|\mathtt{y}_2(\mathtt{t})|)}\right],$$

for each $\mathtt{t} \in [0, 0.5]$. *The function* $\alpha \in \mathcal{C}([0, 0.5], [0, \infty))$ *is chosen so that* $\alpha(\mathtt{t}) = \frac{\mathtt{t}}{8}$ *for each* $\mathtt{t} \in [0, 0.5]$. *So,* $\|\alpha\| = \sup_{\mathtt{t} \in [0,0.5]} |\frac{\mathtt{t}}{8}| = \frac{0.5}{8} = 0.0625$. *Also, choose the function* $\theta : [0, \infty) \to [0, \infty)$ *as* $\theta(\mathtt{t}) = \frac{\mathtt{t}}{6}$ *for almost all* $\mathtt{t} > 0$. *Note that* θ *is non-decreasing and upper semi-continuous. In this case,* $\liminf_{\mathtt{t} \to \infty}(\mathtt{t} - \theta(\mathtt{t})) > 0$ *and* $\theta(\mathtt{t}) < \mathtt{t}$ *for all* $\mathtt{t} > 0$. *Now,* (23) *and* (30) *give*

$$L_1 = \frac{0.196}{\Gamma_{(0.5;0.45)}(4.35)} + \frac{0.196}{\Gamma_{(0.5;0.45)}(2.56)\Gamma_{(0.5;0.45)}(3.79)} \approx 0.028804,$$

$$L_2 = \frac{1.3568}{\Gamma_{(0.5;0.45)}(0.56)} + \frac{0.098}{\Gamma_{(0.5;0.45)}(0.21)\Gamma_{(0.5;0.45)}(4.35)}$$
$$+ \frac{0.098\left[1 + \Gamma_{(0.5;0.45)}(0.21)\Gamma_{(0.5;0.45)}(3.79)\right]}{\Gamma_{(0.5;0.45)}(0.21)\Gamma_{(0.5;0.45)}(3.79)\Gamma_{(0.5;0.45)}(2.56)} \approx 1.898857,$$

and

$$\alpha_1 = \|\alpha\| L_1 \approx 0.00180025 \quad \textit{and} \quad \alpha_2 = \|\alpha\| L_2 \approx 0.1186785625.$$

For every $\mathtt{y}_1, \mathtt{y}_2, \tilde{\mathtt{y}_1}, \tilde{\mathtt{y}_2} \in \mathbb{R}$, *we get*

$$\begin{aligned} \mathbb{H}_{\mathtt{d}}\left(\mathcal{E}\left(\mathtt{t}, \mathtt{y}_1(\mathtt{t}), \mathtt{y}_2(\mathtt{t})\right), \mathcal{E}\left(\mathtt{t}, \tilde{\mathtt{y}_1}(\mathtt{t}), \tilde{\mathtt{y}_2}(\mathtt{t})\right)\right) &\le \frac{\mathtt{t}}{8}.\frac{1}{6}(|\mathtt{y}_1(\mathtt{t}) - \tilde{\mathtt{y}_1}(\mathtt{t})| + |\mathtt{y}_2(\mathtt{t}) - \tilde{\mathtt{y}_2}(\mathtt{t})|) \\ &= \frac{\mathtt{t}}{8}\theta(|\mathtt{y}_1(\mathtt{t}) - \tilde{\mathtt{y}_1}(\mathtt{t})| + |\mathtt{y}_2(\mathtt{t}) - \tilde{\mathtt{y}_2}(\mathtt{t})|) \\ &\le \alpha(\mathtt{t})\theta(|\mathtt{y}_1(\mathtt{t}) - \tilde{\mathtt{y}_1}(\mathtt{t})| + |\mathtt{y}_2(\mathtt{t}) - \tilde{\mathtt{y}_2}(\mathtt{t})|)\left[\frac{1}{\alpha_1 + \alpha_2}\right]. \end{aligned}$$

In the last step, a set-valued map $\mathcal{F} : \mathcal{X}_* \to \mathrm{P}(\mathcal{X}_*)$ *is defined as*

$$\mathcal{F}(\mathtt{y}) = \left\{\hbar \in \mathcal{X}_* : \ \exists \bar{F} \in (\mathbb{S})_{\mathcal{E},\mathtt{y}} \textit{ so that } \hbar(\mathtt{t}) = \breve{g}(\mathtt{t}) \ \forall\, \mathtt{t} \in \mathcal{I}^{0.25}_{(0.5;0.45)} = [0.0.5]\right\},$$

where

$$\begin{aligned} \breve{g}(\mathrm{t}) = &\frac{1}{\Gamma_{(0.5;0.45)}(3.35)0.5^{\binom{3.35}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - 0.45\mathtt{v})^{(2.35)}_{(0.5;0.45)} \bar{F}\left[\frac{\mathtt{v}}{0.5^{2.35}}\right] \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\ &- \frac{0.5\mathtt{t}}{0.25\Gamma_{(0.5;0.45)}(3.35)0.5^{\binom{3.35}{2}}} \int_0^{0.5} (0.5 - 0.45\mathtt{v})^{2.35}_{(0.5;0.45)} \bar{F}\left[\frac{\mathtt{v}}{0.5^{2.35}}\right] \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v} \\ &+ \frac{\left[0.5^2(0.5)^{2.79}\mathtt{t} - 0.125\mathtt{t}^{2.79}\right]}{0.0625\Gamma_{(0.5;0.45)}(3.79)\Gamma_{(0.5;0.45)}(1.56)0.5^{\binom{1.56}{2}}} \int_0^{0.5} (0.5 - 0.45\mathtt{v})^{0.56}_{(0.5;0.45)} \bar{F}\left[\frac{\mathtt{v}}{0.5^{0.56}}\right] \mathrm{d}_{(\mathtt{p};\mathtt{q})}\mathtt{v}. \end{aligned}$$

Finally, Theorem 9 implies that the generalized sequential $(\mathtt{p};\mathtt{q})$*-difference Navier inclusion problem* (34) *of the elastic beam has a solution.*

6. Conclusions

In this paper, we dealt with two different cases of an elastic beam modeling in the context of the notions of $(\mathtt{p};\mathtt{q})$-calculus. In fact, in the framework of the existing mathematical definitions related to the single-valued and multi-valued functions, we generalized the fourth-order differential equation of the elastic beam changes to the two $(\mathtt{p};\mathtt{q})$-difference Navier equation and $(\mathtt{p};\mathtt{q})$-difference Navier inclusion, separately. Of course, we emphasize that the main goal of this study is to investigate the existence results of the solutions for both systems, not to obtain a new method for finding numerical $(\mathtt{p};\mathtt{q})$-solutions. We recalled the main definitions about the ρ-θ-contractions and ρ-admissible functions, and then, by using the fixed point theorems and endpoint theorems, proved our desired theorems about the existence property of the solutions. Based on these results, one can extend the studies later for defining new $(\mathtt{p};\mathtt{q})$-integral transforms or other new $(\mathtt{p};\mathtt{q})$-algorithms to approximate the numerical solutions with the help of the real numerical data.

Author Contributions: Conceptualization: S.E. and S.K.N.; Formal Analysis: S.E., S.K.N., I.S. and J.T.; Investigation: S.E. and S.K.N.; Methodology: I.S. and J.T.; Software: S.E.; Writing—Original Draft: S.E. and S.K.N.; Writing—Review & Editing: I.S. and J.T. All authors have read and agreed to the last version of the manuscript.

Funding: This research budget was allocated by National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-67-B-01.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The first author would like to thank Azarbaijan Shahid Madani University.

Conflicts of Interest: The authors declare no conflict of interest.

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