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Spectral and Oscillation Theory for an Unconventional Fractional Sturm–Liouville Problem

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Abstract: Here, we investigate the spectral and oscillation theory for a class of fractional differential equations subject to specific boundary conditions. By transforming the problem into a modified version with a classical structure, we establish the orthogonality properties of eigenfunctions and some major comparison theorems for solutions. We also derive a new type of integration by using parts of formulas for modified fractional integrals and derivatives. Furthermore, we analyze the variational characterization of the first eigenvalue, revealing its non-zero first eigenfunction within the interior. Our findings demonstrate the potential for novel definitions of fractional derivatives to mirror the classical Sturm–Liouville theory through simple isospectral transformations.

Keywords: Riemann–Liouville; Caputo; Sturm–Liouville; eigenvalues; fractional; spectral theory; oscillations; variational

MSC: 34B24; 34L10; 34C10

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1. Introduction

Lately, there has been considerable interest in the realm of fractional differential equations formulated using combinations of left and/or right Riemann and/or Caputo differential operators. Recent interest in fractional differential equations using combinations of left and/or right Riemann and/or Caputo operators motivates our exploration. This paper delves into the fundamental questions concerning the spectral and oscillation theory for equations of the form:

$$\mathbf{D}_b^{\alpha}(pD_a^{\alpha}y)(x) + q(x)y(x) = \lambda w(x)y(x), \quad x \in [a,b] \tag{1}$$

subject to the boundary conditions

$$I_a^{1-\alpha}y(a) = 0, \qquad I_a^{1-\alpha}y(b) = 0 \tag{2}$$

where $0 < \alpha < 1$ and p, q, and w are real or complex-valued and continuous (though these conditions can be substantially relaxed, as elaborated upon below and in, for instance, [1]). The operator $\mathbf{D}_b^{\alpha}$ represents a right-Caputo differential operator, and D_a^{α} symbolizes a left-Riemann–Liouville differential operator (see Section 2). This interest stems from the prospect that, when these operators are appropriately defined, they could serve as a comprehensive analogue to the Sturm–Liouville theory. The formulation's advantage lies in encompassing the following classical Sturm–Liouville problem upon taking the limit as $\alpha \to 1$:

$$-(p(x)y')' + q(x)y = \lambda y(x), \quad x \in [a,b]$$

with boundary conditions:

$$y(a) = 0, \qquad y(b) = 0.$$

The importance of the Sturm–Liouville equation in the preceding display cannot be underestimated. It appears in the most basic applications of the method for the separation of variables in both the one-dimensional vibrating string and heat equations and is at the centre of applied mathematics [1]. An attempt to extend the framework of the Sturm–Liouville equation to derivatives with fractional orders such as Equation (1) has been considered in various papers by the authors and others. See for example [2–6]. Numerical techniques for solving fractional equations can be found in the papers [7,8], among others. The potential applications of the field are widespread and we cite [9–12] as examples.

The existence and uniqueness of solutions to the initial value problems associated with

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + q(x)y(x) = 0, \quad x \in [a,b], \tag{3}$$

has already been considered in [13]. As in the ordinary derivative case, that is, the case where $\alpha = 1$, a representation of the solutions to (1) is generally unavailable, except in the simplest of cases where $q(x) = 0$, see Corollary 3.

In this paper, by means of a change of variable, we initially transform the problems (1) and (2) into a modified version of a differential equation with a principal term structured in the classical form, see (16). Subsequently, we investigate the spectral theory of this hybrid problem by establishing the orthogonality property of its eigenfunctions, establishing a major comparison theorem for its solutions, and formulating new integration by parts formulae for modified fractional integrals and derivatives with interior points in the limits of the integrals in question. Furthermore, we delve into the variational characterization of the first eigenvalue within this context and show that its first eigenfunction must be non-zero in the interior. We show that apparently new definitions of fractional derivatives can lead to results completely analogous to classical Sturm–Liouville theory by means of simple transformations.

2. Preliminaries

NOTATION:

We use the notation in [13]. Thus, in this paper, the Caputo (resp. the Riemann–Liouville) derivatives will be denoted by boldface (resp. upper case) letters, while ordinary derivatives have only superscripts that are an integer. As in [13] we will omit the obvious $\pm$ subscripts in expressions such $I_{a^+}^{1-\alpha}y(x)$, $D_{b^-}^\alpha y(x)$, and $D_b^{\alpha^+}y(x)$. Hence, they will be written as $I_a^{1-\alpha}y(x)$, and $D_b^\alpha y(x)$, etc., including expressions involving Caputo derivatives. The following abbreviations will also be used from time to time: $(pD_a^\alpha y)(x)$ for $p(x)D_a^\alpha y(x)$; $I_b^\alpha(qy)(x)$ for $I_b^\alpha(q(x)y(x))$. Moreover, the Caputo derivatives will be written with a bold face $\mathbf{D}$, so that $\mathbf{D}_a^\alpha, \mathbf{D}_b^\alpha$ will denote left- and right-Caputo derivatives, respectively, while D_a^α, D_b^α will refer to left- and right-Riemann–Liouville derivatives. As usual, D^n, D^j, etc., will be ordinary derivatives, as will expressions with a prime superscript, as usual.

We refer the reader to texts such as [14] for definitions and detailed proofs of the following propositions from fractional calculus. In the following definitions the subscripts a, b refer to the left and right endpoints of a given interval $[a,b]$.

Definition 1 ([14], p. 69)**.** *The left- and the right-Riemann–Liouville fractional integrals I_a^α and I_b^α of order $\alpha \in \mathbb{R}^+$ are defined by*

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a,b], \tag{4}$$

and

$$I_b^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t \in [a,b), \tag{5}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and $I_a^0(f) = f$, $I_a^{-n}(f) = f^{(n)}$ is the ordinary nth derivative of f [15].

Let $x \in [a, b]$ where x may be an interior point. We introduce the following definitions (which include the case where a, b are end points as per the previous definition).

Definition 2. *The left- and the right-Riemann–Liouville fractional integrals I^{α}_{x+} and I^{α}_{x-} of order $\alpha \in \mathbb{R}^+$ are defined by*

$$I^{\alpha}_{x+} f(t) := \frac{1}{\Gamma(\alpha)} \int_x^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a, b], \quad x \in [a, b), \tag{6}$$

and

$$I^{\alpha}_{x-} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^x \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t \in [a, b), \quad x \in (a, b], \tag{7}$$

respectively.

Definition 3 ([14], p. 92)**.** *The left- and right-Caputo fractional derivatives $\mathbf{D}^{\alpha}_a$ and $\mathbf{D}^{\alpha}_b$ are defined by*

$$\mathbf{D}^{\alpha}_a f(t) := I^{1-\alpha}_a \circ D f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \quad t > a, \tag{8}$$

and

$$\mathbf{D}^{\alpha}_b f(t) := -I^{1-\alpha}_b \circ D f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(s)}{(s-t)^{\alpha}} ds, \quad t < b, \tag{9}$$

where f is assumed to be differentiable and the integrals exist. Additionally, for $0 < x < 1$, we define the new symbols

$$\mathbf{D}^{\alpha}_{x+} f(t) := I^{1-\alpha}_{x^+} \circ D f(t) = \frac{1}{\Gamma(1-\alpha)} \int_x^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \quad t > a, \quad x \in [a, b) \tag{10}$$

and

$$\mathbf{D}^{\alpha}_{x-} f(t) := -I^{1-\alpha}_{x^-} \circ D f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^x \frac{f'(s)}{(s-t)^{\alpha}} ds, \quad t < b, \quad x \in (a, b] \tag{11}$$

Definition 4 ([14], pp. 70–71)**.** *The left- and the right-Riemann–Liouville fractional derivatives D^{α}_a and D^{α}_b are defined by*

$$D^{\alpha}_a f(t) := D \circ I^{1-\alpha}_a f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad t > a, \tag{12}$$

and

$$D^{\alpha}_b f(t) := -D \circ I^{1-\alpha}_b f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}} ds, \quad t < b, \tag{13}$$

where f is assumed to be differentiable and the integrals exist. Additionally, for $0 < x < 1$, we define the new symbols

$$D^{\alpha}_{x+} f(t) := D \circ I^{1-\alpha}_{x^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_x^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad t > a, \quad x \in [a, b) \tag{14}$$

and

$$D^{\alpha}_{x-} f(t) := -D \circ I^{1-\alpha}_{x^-} f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^x \frac{f(s)}{(s-t)^{\alpha}} ds, \quad t < b, \quad x \in (a, b] \tag{15}$$

Proposition 1 ([14], pp. 74–75)**.** *For $y(t) \in L^1(a, b)$ and $I^{1-\alpha}_a y, I^{1-\alpha}_b y \in AC[a, b]$, we have*

$$I_a^\alpha D_a^\alpha y(t) = y(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} I_a^{1-\alpha} y(a),$$
$$I_b^\alpha D_b^\alpha y(t) = y(t) - \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} I_b^{1-\alpha} y(b).$$

Proposition 2 ([14], p. 71)**.**

$$D_a^\alpha\left((x-a)^\beta\right) = \begin{cases} 0, & \text{if } \alpha-\beta-1 \in \mathbf{N} = \{0,1,\ldots\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}, & \text{otherwise.} \end{cases}$$

Proposition 3 ([14], p. 91)**.** *Whenever $0 < \alpha < 1$ then,*

$$D_a^\alpha f(t) = \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \mathbf{D}_a^\alpha f(t).$$

So, $D_a^\alpha f(t) = \mathbf{D}_a^\alpha f(t)$ if $f(a) = 0$.

Proposition 4 ([16], p. 44, [14], p. 77)**.** *For $0 < \alpha < 1$ and $f \in L^1(a,b)$ we have*

$$D_a^\alpha I_a^\alpha f(t) = f(t) \text{ and } D_b^\alpha I_b^\alpha f(t) = f(t).$$

Proposition 5 ([14], pp. 93)**.** *A semigroup property holds, i.e., for any $\alpha > 0, \beta$*

$$I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t), \qquad D(I^{\alpha+1} f)(t) = I^\alpha f(t),$$

if all the integrals exist.

Proposition 6 ([14], p. 71, proposition 2.1)**.** *For $\alpha, \beta > 0$ we have,*

$$I_a^\alpha((t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}.$$

Proposition 7 ([5], pp. 406)**.** *If all the integrals exist, we have two integration by parts formulae given by*

$$\int_a^b f(x) D_b^\alpha g(x) dx = \int_a^b g(x) \mathbf{D}_a^\alpha f(x) dx - f(x) I_b^{1-\alpha} g(x)|_{x=a}^b$$

and

$$\int_a^b f(x) D_a^\alpha g(x) dx = \int_a^b g(x) \mathbf{D}_b^\alpha f(x) dx + f(x) I_a^{1-\alpha} g(x)|_{x=a}^b$$

3. Reduction to a Hybrid Fractional Equation

We refer to [13] for fundamental existence and uniqueness questions related to (3). We will, therefore, tacitly assume that, besides the hypotheses presented in the sequel, additional assumptions are to be specified to allow for the existence and uniqueness of either absolutely continuous or $L^2(a,b)$ solutions on $[a,b]$ (see [13] for detailed assumptions in this case).

Although the results in [13] are formulated for the measurable coefficients $pandq$ in (3), in the sequel we will assume that p, q are continuous on $[a,b]$ and $p(x) > 0$ throughout. Unless otherwise specified we will always assume that $0 < \alpha < 1$.

In this section we will transform (3), which involves a right-Caputo derivative, to a more convenient form whose principal part is a classical Sturm–Liouville operator. Indeed, we will show the following:

Lemma 1. *The change of variable $z = I_a^{1-\alpha} y$ transforms (3) into an equation of the form*

$$-(p(x)z')' + D_b^{1-\alpha}(q(x)\, D_a^{1-\alpha} z(x)) = 0. \tag{16}$$

In addition, the boundary conditions

$$c_1 I_a^{1-\alpha}\, y(a) + c_2\, (pD_a^{\alpha} y)(a) = 0, \tag{17}$$
$$d_1 I_a^{1-\alpha}\, y(b) + d_2\, (pD_a^{\alpha} y)(b) = 0, \tag{18}$$

are transformed into the classical looking ones

$$c_1 z(a) + c_2\, (pz')(a) = 0, \tag{19}$$
$$d_1 z(b) + d_2\, (pz')(b) = 0. \tag{20}$$

Finally, $z \in C^1(a,b)$.

Proof. Using definitions (12) and (9), we obtain

$$-I_b^{1-\alpha} D\left(pDI_a^{1-\alpha} y\right)(x) + q(x)\, y(x) = 0,$$

or, since $z = I_a^{1-\alpha} y$ implies both $z' = DI_a^{1-\alpha} y$ and $y = D_a^{1-\alpha} z$ (by Proposition 4), we have

$$-I_b^{1-\alpha}(p(x)z'(x))' + q(x)\, D_a^{1-\alpha} z(x) = 0,$$

Applying $D_b^{1-\alpha}$ to the previous equation and using Proposition 4, we obtain (16). The form of the boundary conditions (19) and (18) transformed into (19)–(20) is clear from the definitions of z and its Riemann–Liouville derivative.

It was shown in [13] that, if y is a solution of (3) then

$$\begin{aligned} y(x) = K_1 \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + K_2\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) + I_b^{\alpha}(qy)(a)\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) \\ - I_a^{\alpha}\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x). \end{aligned} \tag{21}$$

Thus, whenever $z(a) = 0$ there holds

$$\begin{aligned} y(x) = K_2\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) + I_b^{\alpha}(qy)(a)\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) \\ - I_a^{\alpha}\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x). \end{aligned} \tag{22}$$

Applying the operator $I_a^{1-\alpha}$ to both sides of (22) and using the semigroup property, Proposition 5, we obtain

$$z(x) = K_2\, I_a^1\left(\frac{1}{p}\right)(x) + I_b^{\alpha}(qy)(a)\, I_a^1\left(\frac{1}{p}\right)(x) - I_a^1\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x)$$

Since I_a^1 is an ordinary integral and all integrands are continuous functions according to either a hypothesis or the existence theorem, it follows that each integral itself is at least absolutely continuous.

If $z(a) \neq 0$, using (21) along with Proposition 6 with $\alpha = \beta$ therein, gives

$$z(x) = \frac{K_1}{\Gamma(\alpha)} I_a^{1-\alpha}((t-a)^{\alpha-1})(x) + K_2\, I_a^1\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^1\left(\frac{1}{p}\right)(x)$$
$$-I_a^1\left(\frac{1}{p} I_b^\alpha(qy)\right)(x),$$
$$= K_1 + K_2\, I_a^1\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^1\left(\frac{1}{p}\right)(x) - I_a^1\left(\frac{1}{p} I_b^\alpha(qy)\right)(x),$$

a quantity that is always $L^2(a,b)$.

Finally, by definition, $z = I_a^{1-\alpha} y$ is necessarily absolutely continuous since y is and the integrand in $I_a^{1-\alpha}$ is $L^1(a,b)$. On the other hand, pz' is also necessarily absolutely continuous as it is the indefinite integral of an L^1 function. Hence both $z and pz'$ are absolutely continuous. Since p is continuous everywhere and is positive, this implies that $z \in C^1(a,b)$. Hence all solutions of (16) are at least $C^1(a,b)$. □

4. Spectral Theory

4.1. Fractional Boundary Conditions

In this part we develop a theory, analogous to the Sturm–Liouville theory, for equations in the form (26)–(28) or, equivalently, (23)–(25). We recall that the classical Sturm–Liouville theory is recovered by letting $\alpha \to 1^-$.

The basic idea behind Sturm's theory shines in the formulation of its oscillation theorem for the eigenfunctions arising from the eigenvalue problem. This is usually referred to as *Sturm's oscillation theorem*

$$(p(x)z')' + (\lambda\, w(x) - q(x))\, z = 0.$$

Thus, when $p(x) > 0$, the convexity is well understood. Things are not so clear if $p(x)$ and $w(x)$ have sign changes, but there are studies in this direction. Normally, we will assume that $w(x) > 0$ and is continuous in $[a,b]$ but the first few results here are of a general nature and do not necessarily involve the positivity of $w(x)$.

Now, we consider the fractional eigenvalue problem

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + (q(x) - \lambda w(x))y(x) = 0, \tag{23}$$

subject to the boundary conditions

$$c_1 I_a^{1-\alpha}\, y(a) + c_2\, (pD_a^\alpha y)(a) = 0, \tag{24}$$
$$d_1 I_a^{1-\alpha}\, y(b) + d_2\, (pD_a^\alpha y)(b) = 0, \tag{25}$$

in which c_1, c_2, d_1, d_2 are constants and we are looking for values of λ such that there is a non-trivial solution satisfying the boundary conditions (24) and (25). Such a solution, when it exists, will be called an eigenfunction corresponding to the eigenvalue λ.

Remark 1. *We note that* (16) *has a leading part that is a classical Sturm–Liouville operator (i.e., with ordinary derivatives) while the remaining part consists of fractional derivatives. Thus, whenever y is a solution to* (3)*, the functions $I_a^{1-\alpha} y$ themselves satisfy a Sturm-type differential equation. We shall see below that this transformation to the hybrid form* (16) *has definite advantages for spectral theory.*

Replacing q with $q - \lambda w$ in (3) and applying Lemma 1, the eigenvalue problem (23), subject to both (24) and (25) is now converted to a hybrid eigenvalue problem of the form,

$$-(p(x)z')' + D_b^{1-\alpha}((q(x) - \lambda\, w(x))\, D_a^{1-\alpha} z) = 0, \tag{26}$$

but now, as per Lemma 1, it is subject to the transformed boundary conditions,

$$c_1 z(a) + c_2(pz')(a) = 0, \tag{27}$$
$$d_1 z(b) + d_2(pz')(b) = 0. \tag{28}$$

Remark 2. *The existence and uniqueness theorems for initial value problems proved in [13] for (3) now carry over to the Equation (26) for each λ. So, in particular, the initial value problem for an equation of the form*

$$-(p(x)z')' + D_b^{1-\alpha}(Q(x)\, D_a^{1-\alpha} z) = 0, \quad z(a) = 0, (pz')(a) = K_2$$

has a unique C^1-solution in (a, b).

Remark 3. *Once we have two solutions, z_1, z_2 of (16) such that $z_1(a)z_2'(a) - z_2(a)z_1'(a) \neq 0$, (so they must be linearly independent) then any other solution φ can be written as a linear combination of these two, as all we need are its initial values and these are given at the outset to ensure its existence and uniqueness.*

Remark 4. *Note that, on account of the earlier existence results, solutions z of (26) are at least absolutely continuous functions with pz' which is also absolutely continuous and (26) is satisfied a.e. In addition, note that the transformation in Lemma 1 is isospectral in the sense that the spectrum is preserved when passing from (23) to (26).*

We start with the basic theory that will be necessary in the sequel. Once again, as this is an inaugural paper in a new area, we will consider the homogeneous Dirichlet problem for (26).

Theorem 1. *Let $w(x)$ not be identically zero. Then, any solution z (real or complex) to the proble, (26) with $z(a) = z(b) = 0$ satisfies*

$$\int_a^b \left(p(x)|z'(x)|^2 + (q(x) - \lambda w(x))|D_a^{1-\alpha} z(x)|^2 \right) dx = 0. \tag{29}$$

Proof. By multiplying (26) by $\bar{z}(x)$ and integrating we obtain, after simplification,

$$\int_a^b p(x)|z'(x)|^2\, dx + \int_a^b \bar{z}(x) D_b^{1-\alpha}\Big((q(x) - \lambda w(x)) D_a^{1-\alpha} z(x) \Big) dx = 0.$$

By applying Proposition 7 to the second integral, we find that

$$\int_a^b p(x)|z'(x)|^2\, dx + \int_a^b (q(x) - \lambda w(x)) D_a^{1-\alpha} z(x) \mathbf{D}_a^{1-\alpha} \bar{z}(x)\, dx$$
$$- \bar{z}(x)\, I_b^{\alpha}((q(x) - \lambda w(x)) D_a^{\alpha} z(x))|_{x=a}^{x=b} = 0.$$

Since $z(a) = z(b) = 0$, Proposition 3 and its conjugation shows us that $\mathbf{D}_a^{1-\alpha} z(x) = D_a^{1-\alpha} z(x)$ with a similar relation to $\bar{z}$, i.e.,

$$D_a^{1-\alpha} z(x) \mathbf{D}_a^{1-\alpha} \bar{z}(x) = D_a^{1-\alpha} z(x) D_a^{1-\alpha} \bar{z}(x) = D_a^{1-\alpha} z(x) \overline{D_a^{1-\alpha} z(x)} = |D_a^{1-\alpha} z(x)|^2.$$

from which there follows (29). □

The next result is analogous to the ordinary Sturm–Liouville case and deals with weight functions w that may possibly change sign in (a, b).

Theorem 2. *Assume that p, q, w are continuous in $[a, b]$. Let $w(x) \neq 0$ somewhere in (a, b), $q(x) \geq 0$ in (a, b) and $p(x) > 0$ in $[a, b]$. Then, any eigenvalue of the Dirichlet problem associated with (26) must be real.*

Proof. From Theorem 1 we know that such a solution z must satisfy,

$$\int_a^b \left(p(x)|z'(x)|^2 + (q(x) - \lambda w(x))|D_a^{1-\alpha} z(x)|^2 \right) dx = 0.$$

Let $\lambda \in \mathbf{C}$, $\operatorname{Im} \lambda \neq 0$ be a possibly complex eigenvalue and z a corresponding complex eigenfunction. Then,

$$\int_a^b \left(p(x)|z'(x)|^2 + q(x)\,|D_a^{1-\alpha} z(x)|^2 \right) dx = \lambda \int_a^b w(x)|D_a^{1-\alpha} z(x)|^2\, dx. \tag{30}$$

Since p, q, w are real-valued we can take the imaginary part of both sides to find that

$$\operatorname{Im} \lambda \int_a^b w(x)|D_a^{1-\alpha} z(x)|^2\, dx = 0.$$

However, since $\operatorname{Im} \lambda \neq 0$, we must have

$$\int_a^b w(x)|D_a^{1-\alpha} z(x)|^2\, dx = 0.$$

Thus, the left hand side of (30) must be zero. The positivity of $p(x)$ now implies that $|z'(x)| = 0$ identically on $[a, b]$. Since z is absolutely continuous, this implies that $z(x) = z(a)$ is a constant, i.e., $z(x) = 0$, contrary to the hypothesis that z is an eigenfunction. Thus, $\operatorname{Im} \lambda = 0$ and all eigenvalues are real. □

Remark 5. *The proof above is actually valid for the measurable coefficients p, q, w subject to the usual conditions for the existence and uniqueness of solutions to the initial value problems, see [13].*

Next, we exhibit an orthogonality relationship between the eigenfunctions corresponding to distinct eigenvalues.

Theorem 3. *Let w be continuous and non-zero somewhere in (a, b), q be continuous, and p be continuous and positive (although this too can be relaxed). Then eigenfunctions z_n and z_k of (26), corresponding to distinct eigenvalues $\lambda_n \neq \lambda_k$ from the Dirichlet problem, are orthogonal in the sense that*

$$\int_a^b D_a^{1-\alpha} z_n(x)\, D_a^{1-\alpha} z_k(x)\, w(x)\, dx = 0.$$

Proof. Let λ_n and λ_k ($\lambda_n \neq \lambda_k$) be the eigenvalue of the eigenfunctions $z_n(x)$ and $z_k(x)$, respectively. Then,

$$-(p(x)z_n'(x))' + D_b^{1-\alpha}\Big((q(x) - \lambda_n w(x)) D_a^{1-\alpha} z_n(x) \Big) = 0, \tag{31}$$

with a similar equation for $z_k(x)$ in the form

$$-(p(x)z_k'(x))' + D_b^{1-\alpha}\Big((q(x) - \lambda_n w(x)) D_a^{1-\alpha} z_k(x) \Big) = 0, \tag{32}$$

Multiplying (31) by $z_k(x)$ and (32) by $z_n(x)$, and then integrating the first term by parts and the second term by parts using Proposition 7, we obtain, after some simplification,

$$\int_a^b \Big(p(x)z_n'(x)z_k'(x) + z_k(x)\,(q(x) - \lambda_n w(x)) D_a^{1-\alpha} z_n(x) D_a^{1-\alpha} z_k(x) \Big) dx = 0.$$

and

$$\int_a^b \Big(p(x)z_k'(x)z_n'(x) + z_n(x)\,(q(x) - \lambda_k w(x))\, D_a^{1-\alpha} z_k(x)\, D_a^{1-\alpha} z_n(x)\Big)dx = 0.$$

Subtracting the last two equations we obtain

$$(\lambda_n - \lambda_k)\int_a^b w(x)\, D_a^{1-\alpha} z_n(x)\, D_a^{1-\alpha} z_k(x) dx = 0, \tag{33}$$

and the result follows. □

The next result will be useful later on and deals with the uniqueness of solutions to certain initial value problems associated with (26).

Theorem 4 ([13], Theorem 3.4). *Let p, q be measurable complex-valued functions satisfying*

$$c_1 \equiv \sup_{x\in[a,b]} I_a^\alpha\left(\frac{1}{|p|}\right)(x) < \infty, \tag{34}$$

and

$$c_2 \equiv \sup_{x\in[a,b]} I_b^\alpha(|q|)(x) < \infty. \tag{35}$$

In addition, let $x_0 \in (a,b]$,

$$I_a^\alpha\left(\frac{1}{p}\right)(x_0) \neq 0.$$

and assume that $2c_1\, c_2 < 1$. Then, the only solution to the initial value problem (3) that satisfies

$$I_a^{1-\alpha} y(x_0) = 0 \quad (pD_a^\alpha y)(x_0) = 0 \tag{36}$$

and is continuous on $[a,b]$ is the trivial solution.

Now, as is well known in the ordinary Sturm–Liouville theory, we show that any (non-trivial) solution to (26) can only have a finite number of zeros in $[a,b]$, if any at all.

Lemma 2. *Let $p, q \in C[a,b]$, $p(x) > 0$ there. Consider the solution to the problem*

$$-(p(x)z')' - D_b^{1-\alpha}(q(x)\, D_a^{1-\alpha} z) = 0, \tag{37}$$

where $z(a) = 0$, $(pz')(a) \neq 0$. Then, z can only have a finite number of zeros in (a,b).

Proof. The solution is clearly non-trivial. We assume, on the contrary, that this solution has an infinite number of zeros, say t_n; then, they must accumulate somewhere at t_0, say $t_0 \in (a,b)$. Since $z \in C[a,b]$ by existence and $t_n \to t_0$ (as $n \to \infty$), it follows that $z(t_0) = 0$. On the other hand, since $z \in C^1$, $z'(t_0) = \lim_{t_n \to t_0}(z(t_n) - z(t_0))/(t_n - t_0) = 0$. Since z and pz' are both zero at $t = t_0$, this violates Theorem 4 as it implies that z is the trivial solution. The case where $t_0 = b$ is similar and so is omitted. □

Corollary 1. *For a fixed λ, every non-trivial solution y to (23) satisfying $I_a^{1-\alpha} y(a) = 0$, $(pD_a^\alpha y)(a) \neq 0$, has the property that $I_a^{1-\alpha} y$ has a finite number of zeros.*

Proof. Replace q in (37) by $q - \lambda w$. If, on the contrary, there was a solution y with an $I_a^{1-\alpha} y$ with an infinite number of zeros, then, according to Rolle's theorem, so would its derivative, i.e., $D_a^\alpha y$. But this would contradict Lemma 2 in our equations. □

Remark 6. *The relationship between the number of zeros of a function and the number of zeros of its left-Riemann–Liouville integral is an old one and goes back to at least Steinig [17]. There* *the author shows that, generally speaking, there exists functions f having a finite number of zeros whose left-Riemann–Liouville integral has an infinite number of zeros. (Here, we showed that, if f satisfies some specific differential equation, then this is impossible, i.e., the left-Riemann–Liouville integral must have a finite number of zeros.)*

In order to prove the existence of eigenvalues for the boundary problem (26) that satisfy $z(a) = z(b) = 0$, we can consider the solutions $z(x)$ as a function of λ, denoted by $z(x,\lambda)$, and show that, in fact, $z(x,\lambda)$ is, for each $x \in [a,b]$ an entire function of $\lambda \in \mathbf{C}$. This will show that the zeros of $z(b,\lambda)$ are at most countable and we will prove that there is at least one of them (thereby proving the existence of eigenvalues). We will also show that the zeros of $z(b,\lambda)$ (each one of which gives an eigenvalue) move to the left as λ increases so that there can be finite limit as to the number of zeros, such as $\lambda \to \infty$.

Lemma 3. *Let y, g, h be non-negative continuous functions on $[a,b]$ satisfying*

$$y(x) \le c + \int_x^b g(s)\,y(s)\,ds + \int_a^x h(s)\,y(s)\,ds, \quad x \in [a,b]. \tag{38}$$

where $c \ge 0$ is a constant. If

$$M = \sup_{x\in[a,b]} \left(\int_x^b g(s)\,ds + \int_a^x h(s)\,ds \right) < 1, \tag{39}$$

then

$$||y||_\infty \le \frac{c}{1-M}.$$

Remark 7. *At this point one may conjecture that a general a priori-type inequality similar to the Gronwall inequality is valid for (38); for example, something like the following:*

Conjecture 5. *Let y, g, h be non-negative continuous functions on $[a,b]$ satisfying (38); then, $c \ge 0$ is a constant. Then,*

$$y(x) \le C\,e^{-\int_x^b g(s)\,ds + \int_a^x h(s)\,ds}, \quad x \in [a,b]. \tag{40}$$

where $C = c\,e^{\int_a^b g(s)\,ds}$.

Since the right-hand side of (40) is a solution to (38) with equality for the value of c chosen therein, it appears to be a maximal solution. It is not, however, a maximal solution because the initial condition fails, i.e., the initial condition $y(a)$ depends on y itself

$$y(a) = \int_a^b g(s)y(s)\,ds.$$

A counterexample to this general conjecture is given by setting $c = 0$, $a = 0, b = 1$, $g(x) \equiv 1, h(x) \equiv 1$, for all $x \in [0,1]$. It is easy to verify that (38) holds for $y(x) \equiv 1$, but since $c = 0$, (40) fails.

In this last set of lemmata we assume that all integrals and derivatives exist and provide a more general integration by parts formula when interior points are involved in the limits of integration.

Lemma 4 (Another integration by parts formula)**.** *For any $x \in [a,b]$ there holds*

$$\int_a^x f(t)\, D_b^\alpha g(t)\, dt = -f(t)\, I_b^{1-\alpha} g(t)\Big|_a^x + \int_a^x g(t)\, \mathbf{D}_a^\alpha f(t)\, dt \\ + \frac{1}{\Gamma(1-\alpha)} \int_x^b \int_a^x \frac{f'(t)\, g(s)}{(s-t)^\alpha}\, dt\, ds. \tag{41}$$

Proof. Integrating the left hand-side by parts and applying the definitions of the various fractional terms along with Fubini's theorem, we find

$$\begin{aligned} \int_a^x f(t)\, D_b^\alpha g(t)\, dt &= -\frac{1}{\Gamma(1-\alpha)} \int_a^x f(t)\, \frac{d}{dt} \int_t^b \frac{g(s)\, ds}{(s-t)^\alpha}\, dt \\ &= -f(t)\, I_b^{1-\alpha} g(t)\Big|_a^x + \frac{1}{\Gamma(1-\alpha)} \int_a^x \int_t^b \frac{f'(t)\, g(s)}{(s-t)^\alpha}\, ds\, dt \\ &= -f(t)\, I_b^{1-\alpha} g(t)\Big|_a^x + \frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\int_a^s \frac{f'(t)}{(s-t)^\alpha}\, dt \right) g(s)\, ds \\ &+ \frac{1}{\Gamma(1-\alpha)} \int_x^b \int_a^x \frac{f'(t)\, g(s)}{(s-t)^\alpha}\, dt\, ds \\ &= -f(t)\, I_b^{1-\alpha} g(t)\Big|_a^x + \int_a^x g(s)\, \mathbf{D}_a^\alpha f(s)\, ds \\ &+ \frac{1}{\Gamma(1-\alpha)} \int_x^b \int_a^x \frac{f'(t)\, g(s)}{(s-t)^\alpha}\, dt\, ds \end{aligned}$$

as required. □

Remark 8. *We observe that, by setting $x = b$ in Lemma 4, the first part of Proposition 7 holds.*

This idea can be utilized to generate analogous formulae for other fractional integrals, as demonstrated below:

Lemma 5. *For any $x \in [a,b]$ there holds,*

$$\int_a^x f(t)\, D_a^\alpha g(t)\, dt = f(t)\, I_b^{1-\alpha} g(t)\Big|_a^x + \int_a^x g(s)\, \mathbf{D}_{x-}^\alpha f(s)\, ds. \tag{42}$$

Proof.

$$\begin{aligned} \int_a^x f(t)\, D_a^\alpha g(t)\, dt &= \frac{1}{\Gamma(1-\alpha)} \int_a^x f(t)\, \frac{d}{dt} \int_a^t \frac{g(s)\, ds}{(t-s)^\alpha}\, dt \\ &= f(t)\, I_a^{1-\alpha} g(t)\Big|_a^x - \frac{1}{\Gamma(1-\alpha)} \int_a^x \int_a^t \frac{f'(t)\, g(s)}{(t-s)^\alpha}\, ds\, dt \\ &= f(t)\, I_a^{1-\alpha} g(t)\Big|_a^x - \frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\int_s^x \frac{f'(t)}{(t-s)^\alpha}\, dt \right) g(s)\, ds \\ &= f(t)\, I_a^{1-\alpha} g(t)\Big|_a^x + \int_a^x g(s)\, \mathbf{D}_{x-}^\alpha f(s)\, ds. \end{aligned}$$

as required. □

Lemma 6. *For any $x \in [a,b]$, there holds*

$$\int_x^b f(t)\, \mathbf{D}_b^\alpha g(t)\, dt = -g(t)\, I_{x+}^{1-\alpha} f(t)\Big|_x^b + \int_x^b g(t)\, D_{x+}^\alpha f(t)\, dt. \tag{43}$$

Proof.

$$\begin{aligned}\int_x^b f(t)\,\mathbf{D}_b^\alpha g(t)\,dt &= -\frac{1}{\Gamma(1-\alpha)}\int_x^b f(t)\int_t^b \frac{g'(s)ds}{(s-t)^\alpha}dt \\ &= -\frac{1}{\Gamma(1-\alpha)}\int_x^b g'(s)\int_x^s \frac{f(t)}{(s-t)^\alpha}dt\,ds \\ &= -g(s)I_{x^+}^{1-\alpha}f(s)\Big|_x^b + \int_x^b g(s)D_{x^+}^\alpha f(s)\,ds.\end{aligned}$$

and the result follows. □

Lemma 7. *For any $x \in [a,b]$, there holds*

$$\begin{aligned}\int_x^b f(t)\,\mathbf{D}_a^\alpha g(t)\,dt &= g(t)\,I_b^{1-\alpha}f(t)\Big|_x^b + \int_x^b g(t)\,D_b^\alpha f(t)\,dt \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_a^x \int_x^b \frac{f(t)\,g'(s)}{(t-s)^\alpha}\,dt\,ds.\end{aligned} \tag{44}$$

Proof.

$$\begin{aligned}\int_x^b f(t)\,\mathbf{D}_a^\alpha g(t)\,dt &= \frac{1}{\Gamma(1-\alpha)}\int_x^b f(t)\int_a^t \frac{g'(s)ds}{(t-s)^\alpha}dt \\ &= \frac{1}{\Gamma(1-\alpha)}\int_x^b g'(s)\int_s^b \frac{f(t)dt}{(t-s)^\alpha}ds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_a^x \int_x^b \frac{f(t)g'(s)}{(t-s)^\alpha}dtds \\ &= g(s)\,I_b^{1-\alpha}f(s)\Big|_x^b + \int_x^b g(s)\,D_b^\alpha f(s)\,ds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_a^x \int_x^b \frac{f(t)\,g'(s)}{(t-s)^\alpha}\,dt\,ds.\end{aligned}$$

□

Lemma 8. *For any $x \in [a,b]$, there holds*

$$\begin{aligned}\int_a^x f(t)\,\mathbf{D}_b^\alpha g(t)\,dt &= -g(t)\,I_a^{1-\alpha}f(t)\Big|_a^x + \int_a^x g(t)\,D_a^\alpha f(t)\,dt \\ &- \frac{1}{\Gamma(1-\alpha)}\int_x^b \int_a^x \frac{f(t)\,g'(s)}{(s-t)^\alpha}\,dt\,ds.\end{aligned} \tag{45}$$

Proof.

$$\begin{aligned}\int_a^x f(t)\,\mathbf{D}_b^\alpha g(t)\,dt &= -\frac{1}{\Gamma(1-\alpha)}\int_a^x f(t)\int_t^b \frac{g'(s)\,ds}{(s-t)^\alpha}\,dt \\ &= -\frac{1}{\Gamma(1-\alpha)}\int_a^x \int_a^s \frac{f(t)\,g'(s)}{(s-t)^\alpha}\,dt\,ds \\ &- \frac{1}{\Gamma(1-\alpha)}\int_x^b \int_a^x \frac{f(t)\,g'(s)}{(s-t)^\alpha}\,dt\,ds \\ &= -g(t)\,I_a^{1-\alpha}f(t)\Big|_a^x + \int_a^x g(s)\,D_a^\alpha f(s)\,ds \\ &- \frac{1}{\Gamma(1-\alpha)}\int_x^b \int_a^x \frac{f(t)\,g'(s)}{(s-t)^\alpha}\,dt\,ds\end{aligned}$$

□

Remark 9. *We observe that, by setting $x = b$ in Lemma 8, the second part of Proposition 7 holds.*

4.2. Classical Boundary Conditions

Lemma 9 (A Sturm comparison theorem)**.** *Let q_1, q_2 be real-valued continuous functions over $[a,b]$ such that $q_1(x) \geq q_2(x)$ for all $x \in [a,b]$ with strict inequality for at least one point in (a,b). Let $y = y_1$ be a nontrivial solution to*

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + q_1(x)y(x) = 0, \tag{46}$$

such that $y_1(a) = y_1(b) = 0$, $y_1(x) \neq 0$ in (a,b). Then, every real solution, y_2, to

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + q_2(x)y(x) = 0, \tag{47}$$

has at least one zero in (a,b).

Proof. Without a loss of generality, we may take it that $y_1(x) > 0$ on (a,b). The proof is by contradiction, as usual. We assume, on the contrary, that there is a solution $y_2(x) > 0$ on (a,b). By multiplying (46) with $y = y_1$ by $y_2(x)$ and (52) with $y = y_2$ by $y_1(x)$ and subtracting we obtain

$$\mathbf{D}_b^\alpha(pD_a^\alpha y_1)(x)y_2(x) - \mathbf{D}_b^\alpha(pD_a^\alpha y_2)(x)y_1(x) = (q_2(x) - q_1(x))\, y_1(x)y_2(x),$$

for all $x \in (a,b)$. On the other hand, integrating one of these integrals by parts, say the first, and using the second of Proposition 7 we obtain (suppressing the variables of integration for simplicity of form)

$$\int_a^b \mathbf{D}_b^\alpha(pD_a^\alpha y_1)\, y_2\, dx = \int_a^b (pD_a^\alpha y_1)(D_a^\alpha y_2)\, dx - \Big[(pD_a^\alpha\, y_1)\,(I_a^{1-\alpha} y_2)\Big]\Big|_a^b.$$

Similarly,

$$\int_a^b \mathbf{D}_b^\alpha(pD_a^\alpha y_2)\, y_1\, dx = \int_a^b (pD_a^\alpha y_2)(D_a^\alpha y_1)\, dx - \Big[(pD_a^\alpha\, y_2)\,(I_a^{1-\alpha} y_1)\Big]\Big|_a^b.$$

By subtracting the previous two equations and simplifying we obtain

$$\Big[(pD_a^\alpha\, y_2)\,(I_a^{1-\alpha} y_1) - (pD_a^\alpha\, y_1)\,(I_a^{1-\alpha} y_2)\Big]\Big|_a^b = \int_a^b (q_2 - q_1)\, y_1\, y_2\, dx. \tag{48}$$

Using the transformed variables z defined in Lemma 1 and the definitions of the various fractional derivatives, we rewrite the left hand side of (48) so as to obtain

$$(pz_2')(b)\, z_1(b) - (pz_2')(a)\, z_1(a) - (pz_1')(b)\, z_2(b) + (pz_1')(a)\, z_2(a),$$

which, when combined with (48), yields the identity,

$$\begin{aligned}&(pz_2')(b)\, z_1(b) - (pz_2')(a)\, z_1(a) - (pz_1')(b)\, z_2(b) + (pz_1')(a)\, z_2(a)\\ &= \int_a^b (q_2 - q_1)\, y_1\, y_2\, dx.\end{aligned} \tag{49}$$

However, since both $y_1(x) > 0, y_2(x) > 0$ and $q_1(x) \geq q_2(x)$ on (a,b), the right-hand side of (49) must be strictly negative. On the other hand, since $y_1(x) > 0$ we also have $I_a^{1-\alpha} y_1(x) > 0$, i.e., $z_1(x) > 0$ on (a,b). Similarly, $z_2(x) > 0$ on (a,b). But, the positivity of p and the fact that z_1, z_2 are differentiable in (a,b) implies that $(pz_2')(b)\, z_1(b) \leq 0$. Similarly, $-(pz_2')(a)\, z_1(a) \leq 0$. Similar arguments show that, finally, $-(pz_1')(b)\, z_2(b) \geq 0$ and that

$(pz_1')(a)\, z_2(a) \geq 0$. Thus, the left side of (49) is positive or zero while the right-side is strictly negative. This contradiction proves the lemma. □

Remark 10. *Note that when $\alpha = 1$ this becomes Sturm's comparison theorem as the Caputo derivatives introduce a negative sign before the leading coefficient.*

Corollary 2. *Let q_1, q_2 be real-valued continuous functions over $[a, b]$ such that $q_1(x) > q_2(x)$ for all $x \in (a, b)$. Let $z = z_1$ be a nontrivial solution to*

$$-(p(x)z_1')' + D_b^{1-\alpha}(q_1(x)\, D_a^{1-\alpha} z_1) = 0, \tag{50}$$

such that $D_a^{1-\alpha} z_1(a) = D_a^{1-\alpha} z_1(b) = 0$. Then, every real solution to

$$-(p(x)z_2')' + D_b^{1-\alpha}(q_2(x)\, D_a^{1-\alpha} z_2) = 0, \tag{51}$$

has the property that $D_a^{1-\alpha} z_2(x)$ has at least one zero in (a, b).

Proof. This is an immediate consequence of Lemmas 1, 9, and Proposition 4. □

Corollary 3. *Let $p(t) > 0$ and q both be continuous in $[a, b]$ and $q(x) \geq 0$ with strict inequality in at least one point of $[a, b]$. Then every solution to the equation*

$$\mathbf{D}_b^{\alpha}(pD_a^{\alpha} y)(x) + q(x)\, y(x) = 0, \tag{52}$$

has **at most** *one zero in $[a, b]$.*

Proof. The proof is by contradiction. Simply use Lemma 9 with $q_1 = q$, $q_2 = 0$. Note that $\mathbf{D}_b^{\alpha}(pD_a^{\alpha} y)(x) = 0$ has the particular solution

$$y(t) = (pD_a^{\alpha} y)(b)\, I_a^{\alpha}(1/p)(t)$$

whose only zero is at $t = a$ (since $p(t) > 0$). Consequently, no nontrivial solution of (52) can exist and have two zeros in $[a, b]$. □

Lemma 9, can now be used to guarantee the existence of oscillations in (a, b).

Theorem 6. *Consider the equations*

$$\mathbf{D}_b^{\alpha}(pD_a^{\alpha} y_1)(x) + (q(x) - \lambda_1 w(x))\, y_1(x) = 0, \tag{53}$$

and

$$\mathbf{D}_b^{\alpha}(pD_a^{\alpha} y_2)(x) + (q(x) - \lambda_2 w(x))\, y_2(x) = 0, \tag{54}$$

where $y_1(x)$ is a (nontrivial) solution of (53) satisfying $y_1(a) = y_1(b) = 0$ and $y_1(x) \neq 0$ in (a, b). If $\lambda_1 < \lambda_2$, then every solution to (54) has at least one zero in (a, b).

Proof. The proof is a straightforward consequence of Lemma 9 as the condition $q_1(x) > q_2(x)$ is equivalent to $\lambda_1 < \lambda_2$. □

Remark 11. *It will be shown that there exists a value of $\lambda = \lambda_0$ and a corresponding solution to (23) satisfying the classical Dirichlet boundary conditions at a, b, i.e., $y(a) = y(b) = 0$, and that it is indeed positive in (a, b). Of course, this means that λ_0 is an eigenvalue of (23) with $y = y_1$ as an eigenfunction. Theorem 6 then guarantees that any other eigenvalues must have eigenfunctions with zeros in (a, b).*

Conjecture 7. *The number of zeros in any non-trivial solution to (3) satisfying $y(a) = 0$ on $[a, b]$ is necessarily finite.*

Discussion: This is clear if $q(x) \geq 0$ according to Corollary 3. If there is a nontrivial solution, y, with $y(t_n) = 0$, then the t_n must accumulate somewhere at t_0, say $t_0 \in [a,b]$. Since $y \in C[a,b]$ by existence, it follows that $y(t_0) = 0$. There are two then cases: either $t_0 = b$ or $t_0 < b$.

Let $t_0 = b$. Now, (3) and definition (9) force

$$\int_{t_n}^{b} \frac{D((pD_a^\alpha y)(s))}{(s-t_n)^\alpha}\,ds = 0, \quad n = 1,2,\ldots, \tag{55}$$

which, in turn, implies that $D((pD_a^\alpha y)(t))$ must change sign an infinite number of times in (a,b). However, it is conceivable that $(pD_a^\alpha y)(t)$ remains of one sign throughout a closed interval near b. We will show that this is impossible. Assume for the moment that $(pD_a^\alpha y)(t) > 0$ for all $t \in [t_m, b)$ and for some m (and therefore for infinitely many subsequent m). Since $p(t) > 0$, we obtain $D_a^\alpha y(t) > 0$ for all $t \in [t_m, b)$. On the other hand, $y(a) = 0$ shows us that $y \in C[a,b]$ and is therefore bounded by M, say. Thus,

$$|I_a^{1-\alpha} y(a)| \leq \frac{M}{\Gamma(1-\alpha)} \lim_{t \to a^+} \int_a^t (t-s)^{\alpha-1}\,ds = 0. \tag{56}$$

Of course, boundedness near $x = a$ is sufficient in the preceding argument. A case where $t_0 < b$ is unclear and open.

5. Variational Characterization of the First Eigenvalue

In this section we show that whenever the minimum λ_0 of the functional

$$\frac{\int_a^b p\,|z'|^2 + q\,|D_a^{1-\alpha} z|^2}{\int_a^b w |D_a^{1-\alpha} z|^2} \tag{57}$$

exists and is attained for a function z in some appropriate space, then z must be an eigenfunction of (26) with λ_0 as an eigenvalue and $z(x)$ must be of one sign. The existence of the minimum of (57) was essentially shown in [5] but under the additional boundary condition that $I_a^{1-\alpha} y(b) = 0$, (i.e., $z(b) = 0$), a condition that is generally *independent* of $y(b) = 0$. (Note that $y(a) = 0$ implies $I_a^{1-\alpha} y(a) = 0$ as a result of (56)). We therefore assume in this section that the infinitum of (57) is attained for all z in an appropriate function space defined below without assuming that $y(b) = 0$ but by requiring that $z(b) = 0$.

We begin with some necessary conditions. Using (26) we obtain

$$-z(pz')' + zD_b^{1-\alpha}((q-\lambda w)D_a^{1-\alpha} z) = 0,$$

which, after an integration and use of the homogeneous Dirichlet boundary conditions

$$z(a) = z(b) = 0,$$

in the first boundary term and Proposition 7, gives

$$\int_a^b p|z'|^2\,dx + \int_a^b (q-\lambda w)\left|D_a^{1-\alpha} z\right|^2 dx - zI_b^\alpha (q-\lambda w) D_b^{1-\alpha} z\Big|_a^b = 0,$$

so that

$$\int_a^b p|z'|^2 + (q-\lambda w)\left|D_a^{1-\alpha} z\right|^2 dx = 0.$$

Now consider the quadratic functional,

$$L(x, z', D_a^{1-\alpha} z) = p(x)|z'|^2 + (q(x) - \lambda w(x))\left|D_a^{1-\alpha} z\right|^2$$

and its integral,

$$J(z) = \int_{x_0}^{x_1} L(x, z'(x), D_a^{1-\alpha} z(x))\, dx.$$

where $x_0 < x_1$ and $x_0, x_1 \in [a, b]$. Let $h \in C_0^\infty(a, b)$ satisfy $h(x_0) = h(x_1) = 0$. For $\varepsilon \geq 0$, define the function φ by

$$\varphi(\epsilon) = J(z + \epsilon h) = \int_{x_0}^{x_1} L(x, (z + \epsilon h)', D_a^{1-\alpha}(z + \epsilon h))\, dx.$$

where we suppressed the variables of integration for clarity. Using the chain rule, in order for φ to have a minimum at ε, we must have

$$\begin{aligned} \varphi'(\epsilon) = \int_{x_0}^{x_1} & (L_{z'}(x, z' + \epsilon h', D_a^{1-\alpha}(z + \epsilon h))\, h' \\ & + L_{D_a^{1-\alpha}}(x, z' + \epsilon h', D_a^{1-\alpha}(z + \epsilon h))\, D_a^{1-\alpha} h\,)\, dx \\ & = 0. \end{aligned} \tag{58}$$

(The subscripts on the L denote partial derivatives with respect to that variable.)

For the special case where $x_0 = a, x_1 = b$ we find that $h(a) = 0 = h(b)$ for our test functions and, in addition, Proposition 4 provides us with $D_a^{1-\alpha} h(x) = \mathbf{D}_a^{1-\alpha} h(x)$.

With these identifications and $\epsilon = 0$, (58) can now be rewritten as

$$\begin{aligned} \varphi'(0) &= \int_a^b (L_{z'}(x, z', D_a^{1-\alpha} z)\, h' + L_{D_a^{1-\alpha}}(x, z', D_a^{1-\alpha} z)\, \mathbf{D}_a^{1-\alpha}\, h)\, dx \\ &= L_{z'}(x, z', D_a^{1-\alpha} z)\, h\big|_a^b - \int_a^b h\, \frac{d}{dx} L_{z'}(x, z', D_a^{1-\alpha} z)\, dx \\ &+ \int_a^b h\, D_b^{1-\alpha} \Big(L_{D_a^{1-\alpha}}(x, z', D_a^{1-\alpha} z) \Big) + h\, I_a^{1-\alpha} \Big(L_{D_a^{1-\alpha}}(x, z', D_a^{1-\alpha} z) \Big) \Big|_a^b \\ &= 0, \end{aligned} \tag{59}$$

where we used Proposition 7 in order to evaluate the second integral in (58). Simplifying (59) we find

$$\int_a^b h \left(D_b^{1-\alpha} \Big(L_{D_a^{1-\alpha}}(x, z', D_a^{1-\alpha} z) \Big) - \frac{d}{dx} (L_{z'}(x, z', D_a^{1-\alpha} z) \right) dx = 0,$$

for every h in a dense subset of $L^2(a, b)$. Consequently, for $x \in [a, b]$,

$$D_b^{1-\alpha} \Big(L_{D_a^{1-\alpha}}(x, z', D_a^{1-\alpha} z) \Big) - \frac{d}{dx} \Big(L_{z'}(x, z', D_a^{1-\alpha} z) \Big) = 0,$$

(a.e.) is the Euler–Lagrange equation of the functional,

$$L(x, z', D_a^{1-\alpha} z) = p|z'|^2 + (q - \lambda w) \big| D_a^{1-\alpha} z \big|^2.$$

Writing $u = z'$, $v = D_a^{1-\alpha} z$, we obtain $L_u = 2pu$ and $L_v = 2(q - \lambda w)v$. Therefore,

$$D_b^{1-\alpha}(2(q - \lambda w) D_a^{1-\alpha} z) - \frac{d}{dx}(2pz') = 0,$$

or, equivalently,

$$-(pz')' + D_b^{1-\alpha}((q - \lambda w) D_a^{1-\alpha} z) = 0,$$

and, as expected, this is (26).

Remark 12. *The previous argument is independent of the sign of the weight function, $w(x)$, so long as it appears explicitly in (26). For this to happen it suffices that w be continuous and non-zero somewhere in (a,b).*

Now, the proof of Theorem 2 implies that, for any eigenvalue/eigenfunction pair of (26) satisfying $z(a) = z(b) = 0$, we must have

$$\lambda \int_a^b w(x)|D_a^{1-\alpha} z(x)|^2\, dx > 0.$$

Assuming for the moment that $\lambda_0 > 0$ is the smallest eigenvalue whose eigenfunction z_0 satisfies $\lambda_0 \int_a^b w(x)|D_a^{1-\alpha} z_0(x)|^2\, dx > 0$, then

$$\lambda_0 = \frac{\int_a^b p\,|z|'^2 + q\,|D_a^{1-\alpha} z|^2}{\int_a^b w|D_a^{1-\alpha} z|^2}.$$

We define the space V as a collection of functions z such that $z(a) = z(b) = 0$, $z \in AC[a,b]$, $pz' \in AC[a,b]$ and $D_a^{1-\alpha} z \in C[a,b]$. We have already shown above that if

$$\lambda_0 = \min_{z \in V} \frac{\int_a^b p\,|z|'^2 + q\,|D_a^{1-\alpha} z|^2}{\int_a^b w|D_a^{1-\alpha} z|^2}$$

exists and is attained for some minimizer z ($z = z_0$ here) , then λ_0 must be an eigenvalue of (26) and z must be a corresponding eigenfunction. We now show that z must be of one sign. For $z \in V$, $|z(x)| \in V$, and $|z(x)|$ is also a minimizer. However, both z and $|z|$ must satisfy (26) with the same eigenvalue λ_0. If these eigenfunctions were independent then every solution to (26) would be a linear combination of these two. In particular, every solution would have to satisfy the boundary conditions at a, b, which is impossible according to the uniqueness results in [13]. Hence, z and $|z|$ must be linearly dependent, i.e., $|z(x)| = z(x)$ is necessary for all $x \in [a,b]$. So, $z(x) \geq 0$, i.e., $z(x)$ has one sign. If $z(x_0) = 0$ for some $x_0 \in (a,b)$, then $z'(x_0) = 0$ as $z(x)$ has one sign and z is continuous. So, according to (36) and Theorem 4, $z \equiv 0$. Thus, $z(x) \neq 0$ in (a,b) and z can only vanish at the end points. This proves,

Theorem 8. *If λ_0 is the smallest positive eigenvalue of the Dirichlet problem for (26), then $z_0(x) \neq 0$ for $x \in (a,b)$.*

Remark 13. *Theorem 8 is valid not only for $w(x) > 0$ and $q(x)$ arbitrarily, but for $w(x) > 0$ somewhere, and $q(x) \geq 0$.*

Example 1. *Consider (23) where the coefficients are defined by $q(x) = x(1-x)$, $w(x) \equiv 1$, $p(x) \equiv 1$, and $[a,b] = [0,1]$. We apply the finite difference method to compute the eigenvalues for the resulting Dirichlet fractional Sturm–Liouville problem, i.e.,*

$$\mathbf{D}_1^{\alpha}(D_0^{\alpha} y)(x) + (x(1-x) - \lambda) y(x) = 0, \tag{60}$$

$$I_0^{1-\alpha}\, y(0) = 0, \qquad I_0^{1-\alpha}\, y(1) = 0. \tag{61}$$

Our methods show that the preceding eigenvalue problem is equivalent to the eigenvalue problem for the hybrid equation

$$-z'' + D_1^{1-\alpha}((x(1-x) - \lambda)\, D_0^{1-\alpha} z) = 0, \tag{62}$$

subject to the boundary conditions

$$z(0) = 0, \qquad z(1) = 0,$$

via the isospectral transformation, $z(x) = I_0^{1-\alpha} y(x)$ *applied to* (60).

The table below gives the first five eigenvalues of the Dirichlet problem for (62) *(and so* (60)*) on* $[0,1]$*, where the eigenvalues corresponding to the classical Sturm–Liouville problem* ($\alpha = 1$) *were computed using the MATSLISE software [18]. The table indicates that the eigenvalues for our problem converge towards those of the classical Sturm–Liouville problem as* α *approaches 1.*

The Figure 1 illustrates the first eigenfunction of our problem for varying values of α *as in Table 1. The plot clearly demonstrates that as* α *approaches 1, the graph of the eigenfunction increasingly aligns with that of the classical eigenfunction.*

Table 1. The first five eigenvalues of (60) and (61) for varying α.

λ_k	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.999$	$\alpha = 1$ [18]
λ_1	6.9295	8.0786	9.1256	9.6161	10.0840	10.0869
λ_2	14.6601	20.5433	28.6143	33.7137	39.5553	39.6577
λ_3	25.7724	39.1419	59.1001	72.5553	88.6937	88.9987
λ_4	36.8219	60.0181	97.5375	124.2440	157.4271	158.0835
λ_5	50.4162	85.8096	145.6650	189.7199	245.7714	246.9088

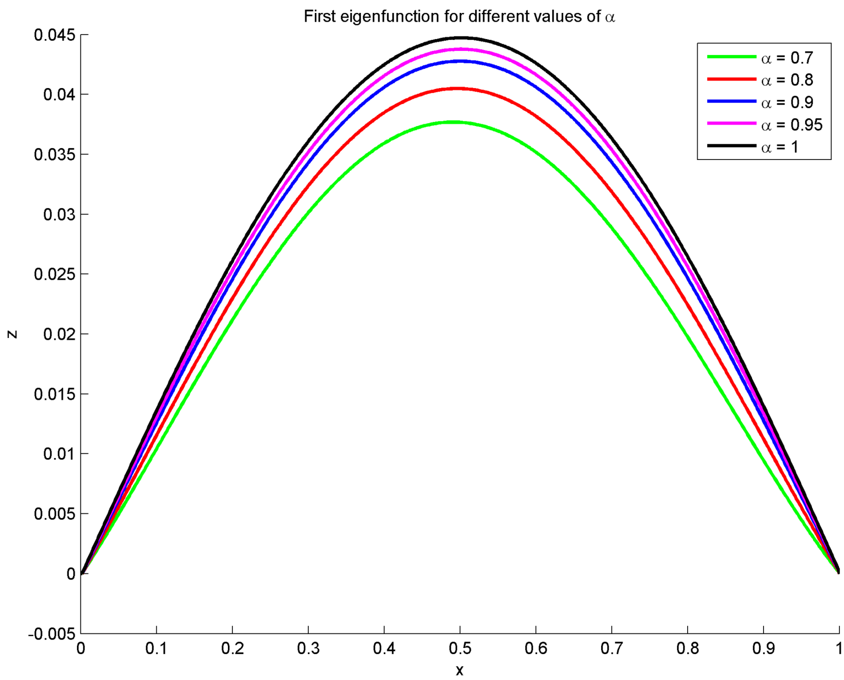

Figure 1. The first eigenfunction for varying values of α in (60) and (61).

Other Fractional Differential Operators

We note that fractional differential operators can be defined in such a way so as to generate a spectral theory completely analogous to Sturm–Liouville theory in every respect. For example, for $0 < \alpha < 1$, we must consider the fractional differential equation

$$D(p(x)\, D_a^{\alpha} y)(x) + (\lambda\, w(x) - q(x))\, I_a^{1-\alpha} y(x) = 0. \tag{63}$$

obtained by replacing the Caputo derivative in (3) with an ordinary derivative and the unknown variable with a left Riemann–Liouville integral of order $1-\alpha$. We associate the fractional boundary conditions (17) and (18) with this equation, i.e.,

$$c_1 I_a^{1-\alpha}\, y(a) + c_2\, (p D_a^{\alpha} y)(a) = 0, \tag{64}$$
$$d_1 I_a^{1-\alpha}\, y(b) + d_2\, (p D_a^{\alpha} y)(b) = 0. \tag{65}$$

Using the same idea as in Section 3 above, we note that the change in variable $z(x) = I_a^{1-\alpha} y(x)$ will produce, in this case, the classical Sturm–Liouville equation

$$(p(x)z')' + (\lambda\, w(x) - q(x))\, z(x) = 0, \tag{66}$$

as opposed to (26), and therefore a spectral theory can be deduced for the eigenvalues and eigenfunctions of (63) subject to (64) and (65) without much effort. The point of this transformation is that every solution to (63) has the property that its left-Riemann–Liouville integral of order $1-\alpha$ satisfies an ordinary Sturm–Liouville equation, (66), and consequently the ordinary Sturm–Liouville theory applies to those particular solutions.

This transformation, being isospectral, preserves the eigenvalues, λ_n, while the eigenfunctions of (63)–(65) are now in the form $y_n(x) = D_a^{1-\alpha} z_n(x)$ where the $z_n(x)$ are the eigenfunctions of a classical Sturm–Liouville problem satisfying boundary conditions of the form (27) and (28), i.e.,

$$c_1 z(a) + c_2 (pz')(a) = 0, \tag{67}$$
$$d_1 z(b) + d_2 (pz')(b) = 0. \tag{68}$$

Besides the existence of real eigenvalues, we also obtain their asymptotic distribution (in possibly very general settings (see [19])) and an oscillation theorem for the functions $I_a^{1-\alpha} y_n(x)$, i.e., the eigenfunctions $y_n(x)$ of (66)–(68) now have the property that, for each α, $0 < \alpha < 1$, $I_a^{1-\alpha} y_n(x)$ has exactly n zeros in (a, b). The variational characterization of the eigenvalues being routine in the case of (63), (67), and (68) (see [20]), now becomes new in the case of (63)–(65). The same is true of the expansion theorem for arbitrary functions in various spaces (see [20])).

As a sample of what is readily provable by known methods and under very general conditions on p, q, w, (see, e.g., ([1], Chapter 8)) and so whose proofs will be omitted, is the following:

Theorem 9. *Let $w(x) > 0$ in $[a, b]$. The eigenvalue problem (63)–(65) has a countable number of real eigenvalues*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

whose eigenfunctions y_n are orthogonal in the sense that

$$\int_a^b (I_a^{1-\alpha} y_n)(x)\, (I_a^{1-\alpha} y_k)(x)\, w(x)\, dx = 0,$$

for $k \neq n$ and they form a complete set in $L^2(a, b)$. Indeed, any function $\varphi \in C^2[a, b]$ satisfying the boundary conditions may be expanded into a uniformly convergent series of the eigenfunctions of our problem, i.e.,

$$\varphi(x) = \sum_{n=0}^{\infty} c_n (I_a^{1-\alpha} y_n)(x),$$

with the expansion holding uniformly on $[a, b]$, and

$$c_n = \int_a^b \varphi(x)\, (I_a^{1-\alpha} y_n)(x)\, w(x)\, dx.$$

In addition, the eigenfunctions y_n corresponding to λ_n have the property that $I_a^{1-\alpha} y_n(x)$ have exactly n zeros in (a, b). The eigenvalues admit the asymptotic distribution

$$\lambda_n \sim \frac{n^2 \pi^2}{\left(\int_a^b \sqrt{\frac{w(x)}{p(x)}}\, dx \right)^2},$$

as $n \to \infty$ [19].

6. Conclusions

In this paper we considered a Sturm–Liouville-type problem (23)–(25) generated by a composition of various fractional derivatives, namely one of Riemann–Liouville type and the other of Caputo type. Using a transformation of dependent variables we showed that it could be reduced to the study of a hybrid differential equation where the leading term is classical Sturm–Liouville type and the other terms involve fractional derivatives, see (26), and whose boundary conditions look like well-known classical, homogeneous, separated boundary conditions. We also formulated an analog of Sturm's comparison theorem, as in Lemma 9, and, in so doing, planted the seeds for a new qualitative theory for the study of the solutions to our equations (number and placement of zeros, etc.) in Section 4.2. We also extended the formulae for integration by parts by including cases where the limits of integration may be an interior point, see Lemmas 4–8.

Among the open questions and directions for future research we note the following.

1. Prove Conjecture 7, i.e., that under suitable hypotheses on the coefficients, the number of zeros of any non-trivial solution of (3) is always finite.
2. Prove an oscillation theorem of Sturm type for (23)–(25). All that is known so far is that the first eigenfunction, corresponding to the eigenvalue λ_o, is zero-free in (a,b), Theorem 8. Thus, for example, we expect that the next eigenvalue, λ_1, has an eigenfunction with exactly one zero in (a,b). More generally, we believe that (whenever $w(x) > 0$) all subsequent eigenfunctions λ_n, $n \geq 1$, have exactly n zeros in (a,b).
3. Study the possible presence of singularities in the solution as in [21,22]. The use of methods from the classical analysis of singularity formation (e.g., [23]) can be applied in this context under suitable conditions.

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