fractal and fractional

Article

Forward Starting Option Pricing under Double Fractional Stochastic Volatilities and Jumps

Sumei Zhang [1,*], Haiyang Xiao [2] and Hongquan Yong [2]

[1] School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710121, China
[2] School of Computer Science and Technology, Xi'an University of Posts and Telecommunications, Xi'an 710121, China
* Correspondence: zhangsumei@xupt.edu.cn

Abstract: This paper aims to provide an effective method for pricing forward starting options under the double fractional stochastic volatilities mixed-exponential jump-diffusion model. The value of a forward starting option is expressed in terms of the expectation of the forward characteristic function of log return. To obtain the forward characteristic function, we approximate the pricing model with a semimartingale by introducing two small perturbed parameters. Then, we rewrite the forward characteristic function as a conditional expectation of the proportion characteristic function which is expressed in terms of the solution to a classic PDE. With the affine structure of the approximate model, we obtain the solution to the PDE. Based on the derived forward characteristic function and the Fourier transform technique, we develop a pricing algorithm for forward starting options. For comparison, we also develop a simulation scheme for evaluating forward starting options. The numerical results demonstrate that the proposed pricing algorithm is effective. Exhaustive comparative experiments on eight models show that the effects of fractional Brownian motion, mixed-exponential jump, and the second volatility component on forward starting option prices are significant, and especially, the second fractional volatility is necessary to price accurately forward starting options under the framework of fractional Brownian motion.

Keywords: forward starting option; fractional stochastic volatility; jump; forward characteristic function; pricing

Citation: Zhang, S.; Xiao, H.; Yong, H. Forward Starting Option Pricing under Double Fractional Stochastic Volatilities and Jumps. *Fractal Fract.* **2024**, *8*, 283. https://doi.org/10.3390/fractalfract8050283

Academic Editor: Leung Lung Chan

Received: 6 March 2024
Revised: 27 April 2024
Accepted: 4 May 2024
Published: 8 May 2024

1. Introduction

Forward starting options belong to European-style exotic options which start at the determination time of the strike and expirate further in the future. They are the building blocks of cliquets which are popular in the world of equity derivatives because of their retaining of the upside potential with protection against downside risk. However, the valuation of forward starting options is not easy, because these options not only depend on the terminal value of the underlying price, but also on the asset price at the determination time of the strike.

Rubinstein [1] first evaluated forward starting options based on the assumption of geometric Brownian motion. Via the changing of numeraire, Kruse and Nögel [2] considered the counterpart by deriving a semi-analytical pricing formula with two integrals which have to be evaluated under Heston stochastic volatility. With Feynman-Kac theorem, Ahlip and Rutkowski [3] and Lin and He [4] extend the method in [2] to stochastic volatility and stochastic interest rate, and regime-switching stochastic volatility models, respectively. Using a similar approach, Guo and Hung [5] and Ahlip et al. [6] considered forward starting options pricing under stochastic volatility, jumps, and stochastic interest rates, respectively. Lucié [7] and Haastrecht and Pelsser [8] reduced the formula in [2] to one in the form of a single integral under the Heston model and a stochastic volatility and stochastic interest rate model, respectively, and improved the efficiency of pricing using the fast Fourier transform technique (FFT). Given that the accuracy of FFT depends on a dampening factor, Zhang and Geng [9] and Zhang and Sun [10] expressed forward starting options prices by using a Fourier cosine series expansion under a stochastic volatility with jump model and a stochastic volatilities, stochastic interest rate, and double jumps model, respectively. Recently, Hata et al. [11] proposed a Taylor expansion approach for evaluating forward starting options under the Hull-White stochastic volatility model.

All of the above studies are conducted under the models driven by standard Brownian motion which are Markovian or memoryless. However, many studies show that asset price fluctuations exhibit "long memory" [12–18] or "short memory" [19–22] which can be captured by stochastic volatility models driven by fractional Brownian motion with the Hurst index $H \in (1/2, 1)$ or $H \in (0, 1/2)$, respectively. In addition, jumps in the asset price were observed by Coqueret and Tavin [23], Jin and Hong [24], Bates [25], and Wang and Xia [26]. Recently, by introducing two fractional volatilities into a mixed-exponential jump-diffusion model [27], Zhang et al. [28] propose a double fractional Heston mixed-exponential jump-diffusion model and demonstrate that the model fits the market better than the double Heston mixed-exponential jump-diffusion model and the double Heston model driven by standard Brownian motion.

However, because the model is not Markovian, all pricing methods based on the Feynman-Kac theorem are not easily applied, which poses a huge challenge for forward starting options pricing. Zhang et al. [28] proposed a model approximation method by approximating the pricing model with a semimartingale under which the Feynman-Kac theorem can be activated. Motivated by the performance of the model and the pricing method in [28], this paper extends the model and pricing method in [28] to the case of forward starting option. Recently, Wang and Guo [29] evaluated variance and volatility swaps under a similar model. Different from Wang and Guo [29] in which only one fractional volatility was used, we consider two fractional volatilities and combine model approximation and the Fourier transform technique to evaluate forward starting options.

The main goal of this paper is to provide a fast and efficient method for evaluating forward starting options under two fractional volatilities and jumps. The main contributions of this paper are twofold. Firstly, this paper derives the forward characteristic function of log return proposed in [28]. Secondly, this paper proposes a pricing method for forward starting options under double fractional volatilities and jumps. Compared to existing option pricing methods for fractional Brownian motion setting, our method can be used for both cases of $H \in (1/2, 1)$ and $H \in (0, 1/2)$. The rest of the paper is organized as follows. Section 2 presents the model and pricing problem. Section 3 provides an approximate model and the derivation of the forward characteristic function. Section 4 dilates the pricing method for forward starting options. Section 5 presents some numerical experiments. Section 6 concludes.

2. The Model and Pricing Problem

2.1. The Model

Assume that $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P\}$ is a complete probability space with the right continuous filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ and the risk-neutral probability measure P. In the double fractional Heston mixed-exponential jump-diffusion model, the dynamics of the asset price S_t are given by the following system:

$$\begin{cases} \frac{dS_t}{S_{t-}} = (r - \lambda\delta)dt + \sqrt{v_{1t}}dW_{1t}^S + \sqrt{v_{2t}}dW_{2t}^S + d\left(\sum_{k=1}^{N_t}(\zeta_k - 1)\right), \\ dv_{1t} = \kappa_1(\theta_1 - v_{1t})dt + \sigma_1\sqrt{v_{1t}}dB_{1t}^{H_1}, \\ dv_{2t} = \kappa_2(\theta_2 - v_{2t})dt + \sigma_2\sqrt{v_{2t}}dB_{2t}^{H_2}, \end{cases} \tag{1}$$

where W_{1t}^S and W_{2t}^S are both standard Brownian motion, $B_{1t}^{H_1}$ and $B_{2t}^{H_2}$ are both fractional Brownian motion with a Hurst index H_1 and H_2, respectively, and N_t is a Poisson process with rate λ. Let $\zeta = (\zeta_k)_{k\ge1}$ be a sequence of independent identically distributed nonnegative random variables, such that $Y = \ln \zeta$ has a mixed-exponential distribution with the density as follows:

$$f_Y(y) = p\sum_{k=1}^{m} p_k \eta_k e^{-\eta_k y} I_{y \geq 0} + q\sum_{j=1}^{n} q_j \hat{\theta}_j e^{\hat{\theta}_j y} I_{y<0}, \tag{2}$$

where $\eta_k > 1$, $\hat{\theta}_j > 0$, $p \geq 0$, $q = 1 - p \geq 0$, p_k, $q_j \in (-\infty, \infty)$, $\sum_{k=1}^{m} p_k = 1$, $\sum_{j=1}^{n} q_j = 1$. To make $f_Y(y)$ a density, a simple sufficient condition is $\sum_{k=1}^{M} p_k \eta_k \geq 0$ $(M = 1, \ldots, m)$, $\sum_{j=1}^{L} q_j \hat{\theta}_j \geq 0$ $(L = 1, \ldots, n)$ and a necessary condition is $p_1 > 0$, $q_1 > 0$, $\sum_{k=1}^{m} p_k \eta_k \geq 0$, $\sum_{j=1}^{n} q_j \hat{\theta}_j \geq 0$. Assume that $\delta = E[\zeta - 1]$. With the density (2), basic calculus gives the following:

$$\delta = p\sum_{k=1}^{m} p_k \eta_k \frac{1}{\eta_k - 1} + q\sum_{j=1}^{n} q_j \hat{\theta}_j \frac{1}{\hat{\theta}_j + 1} - 1.$$

For $j = 1, 2$, κ_j, θ_j, σ_j satisfying $2\kappa_j\theta_j \geq \sigma_j^2$ denote the mean reversion rate, long-run mean, and volatilities of variance v_{jt}, respectively. According to Thao [30], fractional Brownian motion $B_{jt}^{H_j}$ has the following form:

$$B_{jt}^{H_j} = \frac{1}{\Gamma(H_j + 1/2)} \int_0^t (t-s)^{H_j - 1/2} dW_{js}^v (j = 1, 2), \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function, and W_{js}^v are both standard Brownian motion. Suppose processes W_{jt}^S and W_{jt}^v are independent of ζ and N_t, respectively, while W_{jt}^S and W_{jt}^v are correlated by setting $Cov(dW_{1t}^S, dW_{1t}^v) = \rho_1 dt$, $Cov(dW_{2t}^S, dW_{2t}^v) = \rho_2 dt$ to mimic the asset-volatility leverage effect. Suppose $S_0 = S$, $v_{10} = v_1$, $v_{20} = v_2$. According to Zhang et al. [28], model (1) can be written as the following stochastic partial integro-differential system:

$$\begin{cases} \frac{dS_t}{S_{t-}} = (r - \lambda\delta)dt + \sum_{j=1}^{2} \sqrt{v_{jt}} dW_{jt}^S + d\left(\sum_{k=1}^{N_t} (\zeta_k - 1)\right), \\ dv_{1t} = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \kappa_1(\theta_1 - v_{1s}) ds + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \sigma_1 \sqrt{v_{1s}} dW_{1s}^v, \\ dv_{2t} = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \kappa_2(\theta_2 - v_{2s}) ds + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \sigma_2 \sqrt{v_{2s}} dW_{2s}^v. \end{cases} \tag{4}$$

2.2. The Pricing Problem

Under the risk-neutral measure P, the value of a forward starting put option can be expressed on the forward log return as follows:

$$V_F(t_0, T) = e^{-rT} E^p \left[\left(e^{\ln K} - e^{\ln \frac{S_T}{S_{t_0}}} \right)^+ | \mathcal{F}_0 \right] \tag{5}$$

with determination time t_0, maturity T, constant interest rate r, and strike price K. According to [28], if the characteristic function of $\ln \frac{S_T}{S_{t_0}}$ is given, one can evaluate $V_F(t_0, T)$ effectively by using the Fourier transform technology.

Let $\varphi_F(u, \tau) = E^p \left[\exp\left(iu \ln \frac{S_T}{S_{t_0}} \right) | \mathcal{F}_{t_0} \right]$ $(\tau = T - t_0)$ be the characteristic function of $\ln \frac{S_T}{S_{t_0}}$, which we call the forward characteristic function. Because $B_{jt}^{H_j}$ are not semimartingales, $\varphi_F(u, \tau)$ is not a semimartingale. Therefore, we cannot obtain $\varphi_F(u, \tau)$ by solving a PDE like in the standard Brownian motion environment. However, according to El Euch and Gatheral [31], the forward characteristic function $\varphi_F(u, \tau)$ has a similar structure to that of the classic Heston model with jumps as follows:

$$\varphi_F(u, \tau) = E\left[\exp\left(-iu\lambda\tau\delta + iu\sum_{j=1}^{N_\tau} Y_j \right) \right] \exp\left(\sum_{j=1}^{2} h_j(u, \tau) + g_j(u, \tau) v_j \right), \tag{6}$$

where $h_j(u,\tau) = \kappa_j\theta_j\int_0^\tau g_j(u,t)dt$, $g_j(u,\tau) = \frac{1}{\Gamma(H_j+1/2)}\int_0^\tau (\tau-t)^{H_j-1/2}L_j(u,t)dt$, $L_j(u,\tau)$ are the solutions to the following fractional Riccati equations:

$$\begin{cases} D^{H_1+1/2}L_1(u,\tau) = 1/2(-u^2-iu) + (iu\rho_1\sigma_1-\kappa_1)L_1(u,\tau) + 1/2\sigma_1{}^2L_1{}^2(u,\tau), \\ D^{H_2+1/2}L_2(u,\tau) = 1/2(-u^2-iu) + (iu\rho_2\sigma_2-\kappa_2)L_2(u,\tau) + 1/2\sigma_2{}^2L_2{}^2(u,\tau), \\ I^{1/2-H_1}L_1(u,0) = I^{1/2-H_2}L_2(u,0) = 0, \end{cases} \tag{7}$$

where

$$D^{H_j+1/2}L_j(u,\tau) = \frac{1}{\Gamma(1/2-H_j)}\frac{d}{d\tau}\int_0^\tau (\tau-t)^{-H_j-1/2}L_j(u,t)dt,$$

$$I^{1/2-H_j}L_j(u,\tau) = \frac{1}{\Gamma(1/2-H_j)}\int_0^\tau (\tau-t)^{-H_j-1/2}L_j(u,t)dt.$$

3. The Forward Characteristic Function

3.1. Model Approximation

By introducing two small perturbed parameters $\varepsilon_1,\ \varepsilon_2\ (0<\varepsilon_1,\ \varepsilon_2\le 1)$, the processes $B_{jt}^{H_j}\ (j=1,\ 2)$ in Formula (3) can be approximated by the following semimartingales:

$$B_{jt}^{\varepsilon_j,H_j} = \int_0^t (t-s+\varepsilon_j)^{H_j-1/2}dW_{js}^v \tag{8}$$

in $L^2(\Omega)$ as $\varepsilon_1\to 0,\ \varepsilon_2\to 0$. Differentiating Formula (8), one has the following:

$$dB_{jt}^{\varepsilon_j,H_j} = (H_j-1/2)\psi_{jt}dt + \varepsilon_j^{H_j-1/2}dW_{jt}^v, \tag{9}$$

where $\psi_{jt} = \int_0^t (t-s+\varepsilon_j)^{H_j-3/2}dW_{js}^v$ are semimartingales. Putting Formula (9) into model (1), one can approximate model (1) by applying the following classic stochastic partial integro-differential system:

$$\begin{cases} \frac{dS_t^{\varepsilon_1,\varepsilon_2}}{S_{t-}{}^{\varepsilon_1,\varepsilon_2}} = (r-\lambda\delta)dt + +\sqrt{v_{1t}}dW_{1t}^S + \sqrt{v_{2t}}dW_{2t}^S + d\left(\sum_{k=1}^{N_t}(\zeta_k-1)\right), \\ dv_{1t}^{\varepsilon_1} = \left[(H_1-1/2)\psi_{1t}\sigma_1\sqrt{v_{1t}^{\varepsilon_1}} + \kappa_1\left(\theta_1-v_{1t}^{\varepsilon_1}\right)\right]dt + \varepsilon_1^{H_1-1/2}\sigma_1\sqrt{v_{1t}^{\varepsilon_1}}dW_{1t}^v, \\ dv_{2t}^{\varepsilon_2} = \left[(H_2-1/2)\psi_{2t}\sigma_2\sqrt{v_{2t}^{\varepsilon_2}} + \kappa_2\left(\theta_2-v_{2t}^{\varepsilon_2}\right)\right]dt + \varepsilon_2^{H_2-1/2}\sigma_2\sqrt{v_{2t}^{\varepsilon_2}}dW_{2t}^v. \end{cases} \tag{10}$$

Because $\psi_{jt} = \int_0^t (t-s+\varepsilon_j)^{H_j-3/2}dW_{js}^v$ are semimartingales, the forward characteristic function $\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau) = E^p\left[\exp\left(iu\ln\frac{S_T^{\varepsilon_1,\varepsilon_2}}{S_{t_0}^{\varepsilon_1,\varepsilon_2}}\right)|\mathcal{F}_{t_0}\right]$ is a semimartingale. Based on the tower law of conditional expectations, one can write $\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau)$ as follows:

$$\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau) = E^P\left[E^P(\exp\left(iu\ln\frac{S_T^{\varepsilon_1,\varepsilon_2}}{K} - iu\ln\frac{S_{t_0}^{\varepsilon_1,\varepsilon_2}}{K}\right)|\mathcal{F}_{t_0})|\mathcal{F}_0\right] = E^P\left[\varphi(u,\tau)\exp\left(-iu\ln\frac{S_{t_0}^{\varepsilon_1,\varepsilon_2}}{K}\right)|\mathcal{F}_0\right], \tag{11}$$

where $\varphi(u,\tau)$ is the characteristic function of $\ln(S_T^{\varepsilon_1,\varepsilon_2}/K)$ conditional on the filtration $\mathcal{F}_{t_0}$. If we consider $\exp\left(-iu\ln\frac{S_{t_0}^{\varepsilon_1,\varepsilon_2}}{K}\right)$ as the proportion of $\varphi(u,\tau)$, we only need to calculate the expectation of the proportion characteristic function to obtain the forward characteristic function.

3.2. Calculation of the Expectation of the Proportion Characteristic Function

Lemma 1. Under the risk-neutral measure P, the characteristic function of the log asset price $\ln(S_T^{\varepsilon_1,\varepsilon_2}/K)$ conditional on the filtration $\mathcal{F}_{t_0}$, $\varphi(u,\tau)$ is given by the following:

$$\varphi(u,\tau) = \exp\left(iu\ln\frac{S_{t_0}^{\varepsilon_1,\varepsilon_2}}{K} + M(u,\tau) + \sum_{j=1}^{2} N_j(u,\tau)v_{jt_0}\right), \tag{12}$$

where

$$\begin{aligned}
&M(u,\tau) = iur\tau + \lambda\tau\Lambda + \sum_{j=1}^{2}\tfrac{\kappa_j\theta_j}{\Delta_j^2}\left[-(D_j+\gamma_j)\tau - 2\ln\tfrac{1-G_je^{-D_j\tau}}{1-G_j}\right],\\
&N_j(u,\tau) = \sum_{j=1}^{2}\tfrac{1}{\Delta_j^2}\left[-(D_j+\gamma_j)\left(\tfrac{1-e^{-D_j\tau}}{1-G_je^{-D_j\tau}}\right)\right], \tau = T - t_0,\\
&D_j = \sqrt{\gamma_j^2 - 4a_jb_j},\ \gamma_j = iu\rho_j\Delta_j - \kappa_j,\ G_j = \tfrac{\gamma_j+D_j}{\gamma_j-D_j},\\
&a_j = \tfrac{1}{2}\Delta_j^2,\ b_j = -\tfrac{1}{2}u(i+u), \Delta_j = \varepsilon_j^{H_j-1/2}\sigma_j,\\
&\Lambda = p\sum_{i=1}^{m} p_i\eta_i\tfrac{1}{\eta_i - iu} + q\sum_{j=1}^{n} q_j\hat{\theta}_j\tfrac{1}{\hat{\theta}_j+iu} - 1 - iu\delta.
\end{aligned}$$

Proof. See Zhang et al. [28]. □

Lemma 2. Under the risk-neutral measure P,

$$E^P\left[\sum_{j=1}^{2} N_j(u,\tau)v_{jt_0}(t)|\mathcal{F}_0\right] = \exp\left[\overline{M}(u,\tau) + \sum_{j=1}^{2}\widetilde{N}_j(u,t)v_j\right],$$

where for $j = 1,\ 2$,

$$\begin{aligned}
&\overline{M}(u,\tau) = -\sum_{j=1}^{2}\tfrac{2\kappa_j\theta_j}{\Delta_j^2}\ln\left(1 - \tfrac{N_j(u,\tau)}{l_{t_j}}\right),\\
&\widetilde{N}_j(u,\tau) = \tfrac{N_j(u,\tau)e^{-\kappa t_0}}{1-\frac{N_j(u,\tau)}{l_{t_j}}}, l_{t_j} = \tfrac{2\kappa_j}{\Delta_j^2(1-e^{-\kappa_j t_0})}.
\end{aligned}$$

Proof. Set $\Pi = E^P\left[\sum_{j=1}^{2} N_j(u,\tau)v_{jt_0}(t)|\mathcal{F}_0\right]$. Π As a semimartingale; the Feynman-Kac PDE for Π is followed as follows:

$$\begin{cases}
\frac{\partial\Pi}{\partial t} = \sum\limits_{j=1}^{2}\left[\kappa_j(\theta_j - v_{jt}\varepsilon_j) + \left(H_j - \frac{1}{2}\right)\psi_{jt}\sigma_j\sqrt{v_{jt}^{\varepsilon_j}}\right]\frac{\partial\Pi}{\partial v_{jt}^{\varepsilon_j}} + \frac{1}{2}\Delta_j^2 v_{jt}^{\varepsilon_j}\frac{\partial^2\Pi}{\partial v_{jt}^{\varepsilon_j 2}},\\
\Pi\Big|_{t=0} = \exp\left[\sum\limits_{j=1}^{2} N_j(u,\tau)v_j\right].
\end{cases} \tag{13}$$

Since ψ_{jt} $(j = 1,2)$ are semimartingales, one has $\psi_{j0} = E(\psi_{jt}) = 0$. In the light of the affine structure of model (10), we conjuncture that Π has the following form:

$$\Pi = \exp\left[\overline{M}(u,t) + \sum_{j=1}^{2}\widetilde{N}_j(u,t)v_j\right]. \tag{14}$$

Putting Formula (14) into Formula (13), one has the following ODEs:

$$\begin{cases} \frac{\partial \overline{M}}{\partial t} = \kappa_1\theta_1\widetilde{N}_1 + k_2\theta_2\widetilde{N}_2, \\ \frac{\partial \widetilde{N}_1}{\partial t} = -\kappa_1\widetilde{N}_1 + 1/2\Delta_1^2\widetilde{N}_1^2, \\ \frac{\partial \widetilde{N}_2}{\partial t} = -\kappa_2\widetilde{N}_2 + 1/2\Delta_2^2\widetilde{N}_2^2 \end{cases} \tag{15}$$

with $\overline{M}(u,0) = 0$, $\widetilde{N}_1(u,0) = N_1(u,\tau)$, $\widetilde{N}_2(u,0) = N_2(u,\tau)$. Solving the ODEs (15), Lemma 2 follows. □

Combining Lemma 1 with Lemma 2, one can obtain the expectation of the proportion characteristic function; thus, one has the forward characteristic function $\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau)$.

Theorem 1. Assuming that the dynamics of asset price $S_t^{\varepsilon_1,\varepsilon_2}$ are given by model (10), then the forward characteristic function of $\ln\frac{S_T^{\varepsilon_1,\varepsilon_2}}{S_{t_0}^{\varepsilon_1,\varepsilon_2}}$, $\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau)$ is given by the following:

$$\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau) = \exp\left[\widetilde{M}(u,\tau) + \sum_{j=1}^{2}\widetilde{N}_j(u,\tau)v_{jt_0}\right], \tag{16}$$

where

$$\begin{aligned} \widetilde{M}(u,\tau) &= M(u,\tau) - \sum_{j=1}^{2}\frac{2\kappa_j\theta_j}{\Delta_j^2}\ln\Big(1 - N_j(u,\tau)/l_{t_j}\Big), \\ \widetilde{N}_j(u,\tau) &= \frac{N_j(u,\tau)e^{-\kappa t_0}}{1-\frac{N_j(u,\tau)}{l_{t_j}}}, l_{t_j} = \frac{2\kappa_j}{\Delta_j^2(1-e^{-\kappa_j t_0})}. \end{aligned}$$

Based on Theorem 1, differential calculation gives the following outcome.

Theorem 2. Assuming that the dynamics of asset price $S_t^{\varepsilon_1,\varepsilon_2}$ are given by model (10), and $c_n = \left.\frac{\partial^n \ln\varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau)}{\partial u^n}\right|_{u=0}$ denotes the n-th cumulant of $\ln\frac{S_T^{\varepsilon_1,\varepsilon_2}}{S_{t_0}^{\varepsilon_1,\varepsilon_2}}$, one has the following:

$$\begin{aligned} c_1 = &\sum_{j=1}^{2}\left[\frac{\theta_j(1-e^{-\kappa_j\tau}-\kappa_j\tau)}{2\kappa_j} + \frac{e^{-\kappa_j\tau}-1}{2k_j}\left(\frac{2\kappa_j\theta_j}{\Delta_j^2 h_{t_j}} + e^{-\kappa_j t}v_j\right)\right] \\ &+\lambda\tau\left[p\sum_{k=1}^{m}\frac{p_k}{\eta_k} - q\sum_{l=1}^{n}\frac{q_l}{\hat{\theta}_l} - \left(p\sum_{k=1}^{m}\frac{p_k\eta_k}{\eta_k-1} + q\sum_{l=1}^{n}\frac{q_l\hat{\theta}_l}{\hat{\theta}_l+1} - 1\right)\right], \\ c_2 = &\sum_{j=1}^{2}\left[\frac{\kappa_j\theta_j\gamma_j''\tau}{\Delta_j^2} - \frac{5\theta_j\Delta_j^2}{8\kappa_j^3} + \frac{2\theta_j\rho_j\Delta_j}{\kappa_j^2} - \frac{\theta_j}{\kappa_j}\right. \\ &+e^{-\kappa_j\tau}\left(\frac{\theta_j\Delta_j^2+\theta_j\kappa_j\tau\Delta_j^2}{2\kappa_j^3} + \frac{\theta_j-\theta_j\rho_j\tau\Delta_j}{\kappa_j} - \frac{2\theta_j\rho_j\Delta_j}{\kappa_j^2}\right) \\ &\left.+\frac{\theta_j\Delta_j^2}{8\kappa_j^3}e^{-2\kappa_j\tau} - \frac{2\kappa_j\theta_j}{\Delta_j^2}\frac{B_j''h_{t_j}+(B_j')^2}{{h_{t_j}}^2} - e^{-\kappa_j t}\frac{B_j''ht_j+2B_j'}{h_{t_j}}v_j\right] \\ &+\lambda\tau\left(p\sum_{k=1}^{m}\frac{2p_k}{\eta_k^2} + q\sum_{l=1}^{n}\frac{2q_l}{\hat{\theta}_l^2}\right), \\ c_4 = &\ 0, \end{aligned}$$

where

$$\begin{aligned} \gamma_j'' &= \frac{\Delta_j^2}{\kappa_j} - \frac{\rho_j\Delta_j^3}{\kappa_j^2} + \frac{\Delta_j^4}{4\kappa_j^3}, B_j' = -\frac{i\left(1-e^{-\kappa_j\tau}\right)}{2\kappa_j}, \\ B_j'' &= \frac{\Delta_j^2}{4\kappa_j^3}e^{-2\kappa_j\tau} + e^{-\kappa_j\tau}\left(\frac{\Delta_j^2\tau}{2\kappa_j^2} + \frac{1-\Delta_j\rho_j\tau}{\kappa_j} - \frac{\Delta_j\rho_j}{\kappa_j^2}\right) - \frac{1}{\kappa_j} - \frac{\Delta_j^2}{4\kappa_j^3} + \frac{\Delta_j\rho_j}{\kappa_j^2}. \end{aligned}$$

4. The Pricing Algorithm

Let $\xi = \ln \frac{S_T^{\varepsilon_1,\varepsilon_2}}{S_{t_0}^{\varepsilon_1,\varepsilon_2}}$; one can express (5) as follows:

$$V_F(t_0,T) = e^{-rT}K\int_{-\infty}^{\infty} -(e^{\xi}-1)^{+} f_F^{\varepsilon_1,\varepsilon_2}(\xi|x)d\xi, \tag{17}$$

where $f_F^{\varepsilon_1,\varepsilon_2}(\xi|x)$ is the density of ξ conditional on $x = \ln(S/K)$.

Since $f_F^{\varepsilon_1,\varepsilon_2}(\xi|x)$ decays to zero as $\xi \to \pm\infty$, $f_F^{\varepsilon_1,\varepsilon_2}(\xi|x)$ can be approximated by applying a truncated Fourier cosine series expansion (COS) on the domain $[a,b]$ as follows:

$$f_F^{\varepsilon_1,\varepsilon_2}(\xi|x) \approx \frac{2}{b-a}\sum_{k=0}^{N-1} \Re\left[\varphi_F^{\varepsilon_1,\varepsilon_2}\left(\frac{k\pi}{b-a},\tau\right)e^{-i\frac{k\pi}{b-a}(\ln K+a)}\right]\cos\left[\frac{k\pi}{b-a}(\xi-a)\right], \tag{18}$$

where $\Re[.]$ denotes taking the real part of the argument, and N denotes the number of terms in series expansion. Inserting (18) into (17) and interchanging summation and integration, one can approximate forward starting option value by the following:

$$V_F(t_0,T) \approx \frac{2}{b-a}Ke^{-rT}\sum_{k=0}^{N-1} \Re\left[\varphi_F^{\varepsilon_1,\varepsilon_2}\left(\frac{k\pi}{b-a},\tau\right)e^{-i\frac{k\pi a}{b-a}}\right]U_K \tag{19}$$

where $U_k = \int_a^b -(e^{\xi}-1)^{+}\cos(\frac{\xi-a}{b-a}k\pi)d\xi$ denotes the cosine series coefficients. By using a basic calculus calculation, one has the analytical solutions of U_k as follows:

$$U_k = \begin{cases} \frac{1}{1+\left(\frac{k\pi}{(b-a)}\right)^2}\left[\frac{k\pi}{b-a}\sin\left(\frac{ak\pi}{b-a}\right) - \cos\left(\frac{ak\pi}{b-a}\right) + e^a\right] + \frac{a-b}{k\pi}\sin\left(\frac{ak\pi}{b-a}\right) & k \neq 0, \\ e^a - a - 1 & k = 0 \end{cases} \tag{20}$$

.

According to Fang and Oosterrlee [32], for some constant L, the integration domain $[a,\ b]$ is chosen as follows:

$$[a,\ b] = \left[c_1 - L\sqrt{|c_2|+|c_4|},\ c_1 + L\sqrt{|c_2|+|c_4|}\right] \tag{21}$$

where $c_j\ (j = 1,\ 2,\ 4)$ can be obtained by the following:

$$c_j = \frac{1}{i^j}\frac{\partial^j(\ln \varphi_F^{\varepsilon_1,\varepsilon_2}(u,\tau))}{\partial u^j}|u = 0\ (j = 1,\ 2,\ 4).$$

We present the following Algorithm 1 for approximating a forward starting put option price.

Algorithm 1. The COS-based algorithm for pricing a forward starting put option

Step 1: Initialization $K,\ S,\ t_0, T,\ r, \kappa_1,\ \theta_1,\ \sigma_1, v_1,\ \rho_1,\ \kappa_2,\ \theta_2, \sigma_2,\ v_2,\ \rho_2,\ \lambda,\ m, n,\ p,\ p_1,$ $q_1,\ \hat{\theta}_1,\ \eta_1,\ \hat{\theta}_2,\ \eta_2,\ H_1,\ H_2,\ L,\ N$
Step 2: Choose $\varepsilon_1,\ \varepsilon_2$ to approximate the pricing model (1) by applying model (10)
Step 3: Compute the forward characteristic function by applying Formula (16)
Step 4: Compute the cumulant $c_1,\ c_2,\ c_4$ by applying Theorem 2
Step 5: Compute cosine series coefficients U_k by applying Formula (20)
Step 6: Compute the truncated domain $[a,\ b]$ by applying Formula (21)
Step 7: Approximate the value of a forward starting put option by applying Formula (19)

5. Numerical Experiments

Under the approximate model (10), we used the COS-based algorithm to evaluate forward starting put options. According to [28], the parameters selected were as follows:

$\kappa_1 = 12$, $\theta_1 = 0.05$, $\sigma_1 = 0.9$, $\rho_1 = -0.5$, $v_1 = 0.05$, $\kappa_2 = 16$, $\theta_2 = 0.03$, $\sigma_2 = 0.9$, $\rho_2 = -0.5$, $v_2 = 0.02$, $p = 0.4$, $p_1 = 1.3$, $q_1 = 1.2$, $\hat{\theta}_1 = 20$, $\hat{\theta}_2 = 20$, $\eta_1 = 50$, $\eta_2 = 50$, $H_1 = 0.8$, $H_2 = 0.7$, $\lambda = 1$, $m = 2$, $n = 2$, $r = 0.0165$, $S = 100$, $K = 100$, $t_0 = 1$, $T = 5$. To test the convergence of the pricing algorithm, we set four groups $(\varepsilon_1,\ \varepsilon_2)$: (0.01, 0.01), (0.001, 0.001), (0.0001, 0.0001), (0.00001, 0.00001), specifying $L = 5$, $L = 10$, $L = 15$, and specifying $N = 16$, $N = 32$, $N = 64$ for each L.

To test the effectiveness of the pricing algorithm, we also evaluated forward starting put options by using a Euler-Maruyama scheme for the approximate model (10) as follows:

$$\begin{cases} S_t^{\varepsilon_1,\varepsilon_2} = S_{t-}{}^{\varepsilon_1,\varepsilon_2} e(r - \lambda\delta - \frac{1}{2}\sum\limits_{j=1}^{2} v_{jt-1}^{\varepsilon_j})\Delta t + \sum_{j=1}^{2} \sqrt{v_{jt-1}^{\varepsilon_j}}\sqrt{\Delta t} W_{jt}^S + d\left(\sum_{k=1}^{N_t} (\zeta_k - 1)\right) \prod\limits_{j=N_{t-1}}^{N_t} e^{Y_j}, \\ v_{1t}^{\varepsilon_1} = v_{1t-1}^{\varepsilon_1} + \left[(H_1 - 1/2)\psi_{1t-1}\sigma_1\sqrt{v_{1t-1}^{\varepsilon_1}} + \kappa_1\left(\theta_1 - v_{1t-1}^{\varepsilon_1}\right)\right]\Delta t \\ \quad +\varepsilon_1^{H_1-1/2}\sigma_1\sqrt{v_{1t-1}^{\varepsilon_1}}\sqrt{\Delta t}\left(\rho_1 W_{1t}{}^S + \sqrt{1-\rho_1{}^2} Z_{1t}\right), \\ v_{2t}^{\varepsilon_2} = v_{2t-1}^{\varepsilon_2} + \left[(H_2 - 1/2)\psi_{2t-1}\sigma_2\sqrt{v_{2t-1}^{\varepsilon_2}} + \kappa_2\left(\theta_2 - v_{2t-1}^{\varepsilon_2}\right)\right]\Delta t \\ \quad +\varepsilon_2^{H_2-1/2}\sigma_2\sqrt{v_{2t-1}^{\varepsilon_2}}\sqrt{\Delta t}\left(\rho_2 W_{2t}{}^S + \sqrt{1-\rho_2{}^2} Z_{2t}\right). \end{cases} \tag{22}$$

where Y_j is a random number generated by two double exponential distributions, $W_{1t}{}^S, W_{2t}{}^S$, Z_{1t}, Z_{2t} are independent standard normal random variables, $\Delta t = T/N_2$ denotes the time step. $\psi_{jt} \approx \sqrt{\frac{t}{N_1}}\sum\limits_{k=0}^{N_1-1}\left[t\left(1 - \frac{k}{N_1}\right) + \varepsilon_j\right]^{H_j - \frac{3}{2}} z_{jk} (j = 1, 2)$, where $\left\{z_{jk}\right\}$ denotes the sequence of independent standard normal random variables and N_1 is the number of subintervals in $[0,\ t]$. For Monte Carlo simulation, we used the number of subintervals $N_1 = 100$, the number of simulations $M = 100{,}000$, and the number of time steps $N_2 = 1000$. Table 1 examines the convergence and accuracy of the pricing algorithm for evaluating forward starting puts under approximate model (10).

Table 1. Convergence and accuracy of the COS-based algorithm for evaluating forward starting put options under the approximate model.

ε_1	ε_2	L	COS(16)	COS(32)	COS(64)	MC ± Std
		5	18.5682	18.5682	18.5682	
0.01	0.01	10	18.6135	18.5682	18.5682	18.5686 ± 0.0159
		15	19.1058	18.5700	18.5682	
		5	18.5838	18.5838	18.5838	
0.001	0.001	10	18.6274	18.5838	18.5838	18.5842 ± 0.0162
		15	19.1109	18.5855	18.5838	
		5	18.5918	18.5918	18.5918	
0.0001	0.0001	10	18.6344	18.5918	18.5918	18.5920 ± 0.0150
		15	19.1133	18.5934	18.5918	
		5	18.5960	18.5960	18.5960	
0.00001	0.00001	10	18.6381	18.5960	18.5960	18.5960 ± 0.0146
		15	19.1145	18.5975	18.5960	

From Table 1, we can see that the accuracy of the COS-based algorithm depends on the values of $(\varepsilon_1,\ \varepsilon_2)$, L, and the number of terms N in series expansion. As the value of $(\varepsilon_1,\ \varepsilon_2)$ decreases, option prices tend to stabilize. The effect of L on option prices depends on the number of terms N. When $N = 64$, the effect of L on option prices is almost negligible. Table 1 shows that when $(\varepsilon_1,\ \varepsilon_2) = (0.00001,\ 0.00001)$ and $N = 64$, the option values calculated by the two methods are the very closest.

Furthermore, we examined the effects of the fractional Brownian motion, jump, and the second volatility component on forward starting put option prices. To this end, we set $(\varepsilon_1,\ \varepsilon_2) = (0.00001,\ 0.00001)$, $N = 64$, $L = 10$ and specified two determination time t_0 =1/4, 1, and two maturities T =1/2, 5 years. We evaluated forward starting put options under model (10) (FDHestonMEM) applying the model parameters and option parameters used in Table 1. For comparison, we also evaluated forward starting put options under the FDHeston mode by setting $\lambda = 0$, the DHeston model by setting $\lambda = 0, H_1 = H_2 = 0.5$, the FHestonMEM model by setting $\kappa_2 = \theta_2 = \sigma_2 = v_2 = \rho_2 = 0$, the FHeston model by setting $\kappa_2 = \theta_2 = \sigma_2 = v_2 = \rho_2 = \lambda = 0$, the DHestonMEM model by setting $H_1 = H_2 = 0.5$, the HestonMEM model by setting $\kappa_2 = \theta_2 = \sigma_2 = v_2 = \rho_2 = 0,\ H_1 = 0.5$, and the Heston model by setting $\kappa_2 = \theta_2 = \sigma_2 = v_2 = \rho_2 = \lambda = 0,\ H_1 = 0.5$. Table 2 compares forward starting put option prices generated from the above eight models under $t_0 = 1/4,\ T = 1/2$ and $t_0 = 1,\ T = 5$, respectively.

Table 2. Comparison of forward starting put options prices under the FDHestonMEM, FDHeston, FDHestonMEM, FDHeston, DHestonMEM, DHeston, HestonMEM, and Heston model.

t_0	T	K	FDHestonMEM	FDHeston	FHestonMEM	Heston	DHestonMEM	DHeston	HestonMEM	FHeston
		80	0.3377	0.2963	0.1152	0.2338	0.5163	0.4806	0.2633	0.0808
		85	0.8357	0.7710	0.3789	0.5292	1.0338	0.9768	0.5838	0.3135
		90	1.7720	1.6860	1.0201	1.1199	1.9213	1.8407	1.2084	0.9212
		95	3.2961	3.1972	2.2817	2.2118	3.3242	3.2239	2.3342	2.1609
1/4	1/2	100	5.4970	5.3967	4.3501	4.0521	5.3689	5.2599	4.1908	4.2280
		105	8.3794	8.2885	7.2696	6.8269	8.1202	8.0172	6.9499	7.1656
		110	11.8723	11.7976	10.9349	10.5137	11.5519	11.4676	10.5975	10.8588
		115	15.8585	15.8022	15.1550	14.8625	15.5523	15.4920	14.9075	15.1063
		120	20.2087	20.1695	19.7321	19.5711	19.9646	19.9265	19.5916	19.7044
		80	9.4981	9.1760	6.2624	6.0294	9.5588	9.2443	6.4006	5.8740
		85	11.5033	11.1574	7.9640	7.6361	11.5188	11.1782	8.0497	7.5372
		90	13.6941	13.3281	9.8907	9.4638	13.6606	13.2971	9.9150	9.4315
		95	16.0616	15.6788	12.0357	11.5096	15.9769	15.5937	11.9928	11.5507
1	5	100	18.5960	18.1999	14.3897	13.7678	18.4591	18.0596	14.2768	13.8855
		105	21.2872	20.8811	16.9421	16.2303	21.0985	20.6860	16.7588	16.4249
		110	24.1252	23.7121	19.6808	18.8873	23.8861	23.4638	19.4290	19.1565
		115	27.1001	26.6827	22.5932	21.7280	26.8128	26.3836	22.2769	22.0673
		120	30.2023	29.7832	25.6666	24.7405	29.8695	29.4363	25.2909	25.1438

From Table 2, we can see that for the above eight models, the mixed-exponential jump and the second volatility component both increase forward starting options prices for all maturities and determination time. The effects of fractional Brownian motion on forward starting option prices depend on the determination time, maturity, and the value of K/S. To comprehensively examine the effects of fractional Brownian motion, jump, and the second volatility component on forward starting put option prices, we computed the relative error between forward starting put options prices under the above models with the same parameters as in Table 2. Figures 1–3 report the main outcomes.

To examine the effect of mixed-exponential jump on forward starting put options prices, we investigated four cases including the single volatility model and two volatilities model driven by fractional Brownian motion and standard Brownian motion, respectively. Figure 1 reports the effect of the mixed-exponential jump on the values of short-term ($t_0 = 1/4,\ T = 1/2$) and long-term ($t_0 = 1,\ T = 5$) forward starting put options.

From Figure 1, we can see that for the short-term options, when $K/S < 1.05$, the effect of jump on option prices with single volatility is more significant than that with two volatilities. Moreover, the effects of jump on option prices under the framework of fractional Brownian motion are more significant than those in the standard Brownian motion case. When $K/S \geq 1.05$, the effect of jump on option prices with single volatility and with two volatilities are almost the same. For the long-term options, the effects of the mixed-exponential jump on the options prices with single volatility are both more significant than the counterpart with two volatilities under the two frameworks of Brownian motion. However, these effects of jump on the options prices are significantly weaker than those in the short-term case.

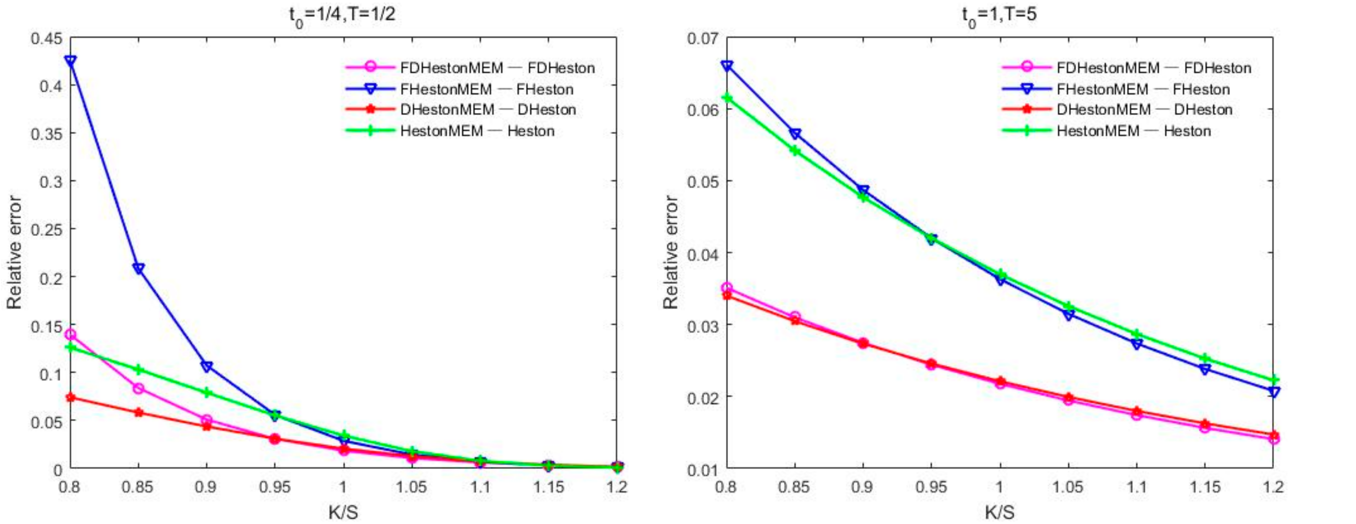


Figure 1. The effect of the mixed-exponential jump on the values of short-term ($t_0 = 1/4,\ T = 1/2$) and long-term ($t_0 = 1,\ T = 5$) forward starting put options.

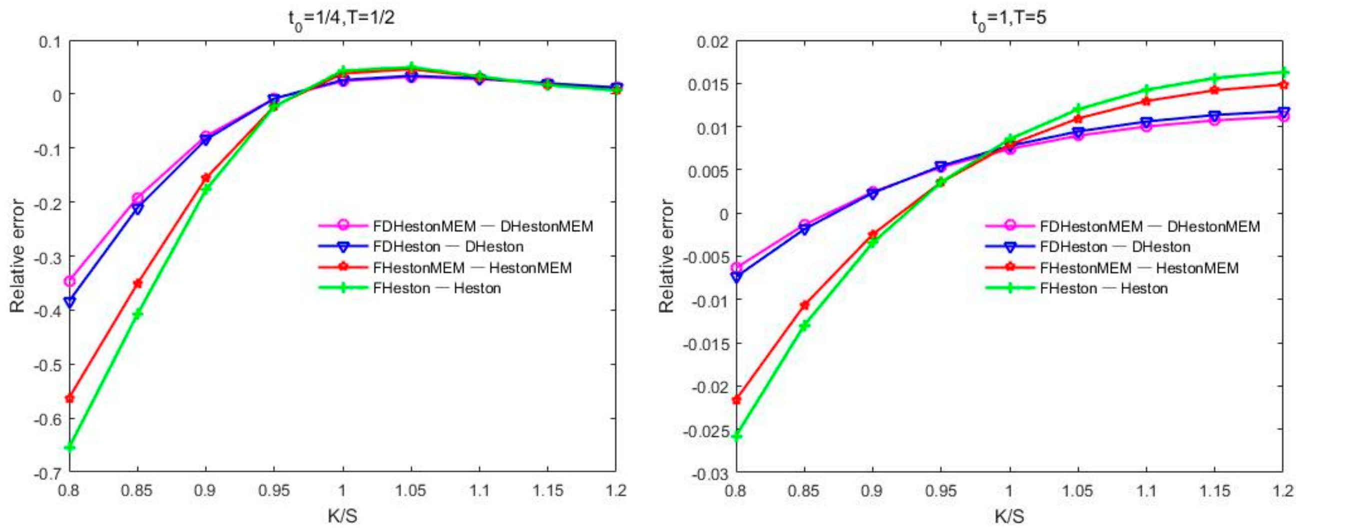


Figure 2. The effects of fractional Brownian motion on the values of short-term ($t_0 = 1/4,\ T = 1/2$) and long-term ($t_0 = 1,\ T = 5$) forward starting put options.

To examine the effects of fractional Brownian motion on forward starting put options prices, we investigated four cases including the single volatility model and two volatilities model with jump and without jump, respectively. Figure 2 reports the effects of fractional Brownian motion on the values of short-term ($t_0 = 1/4,\ T = 1/2$) and long-term ($t_0 = 1$, $T = 5$) forward starting put options.

From Figure 2 we can see that for the short-term options, when $K/S \leq$0.95, fractional Brownian motion both decreases option prices with single volatility and with two volatilities, and the effects of fractional Brownian motion on option prices with single volatility are more significant than those with two volatilities. When $K/S >$0.95, fractional Brownian motion both increase the option prices with single volatility and with two volatilities, and the effects of fractional Brownian motion on option prices with single volatility and with two volatilities are almost the same. For the long-term options, when $K/S < 0.9$ or $K/S > 1$, the effects of fractional Brownian motion on the options prices with single volatility are both more significant than the counterpart with two volatilities under two frameworks with jump and without jump. When $0.95 \leq K/S < 1$, the effects of fractional Brownian motion on the options prices with two volatilities are slightly significant than the counterpart with single volatility. However, these effects of fractional Brownian motion on the options prices are significantly weaker than those in the short-term case.

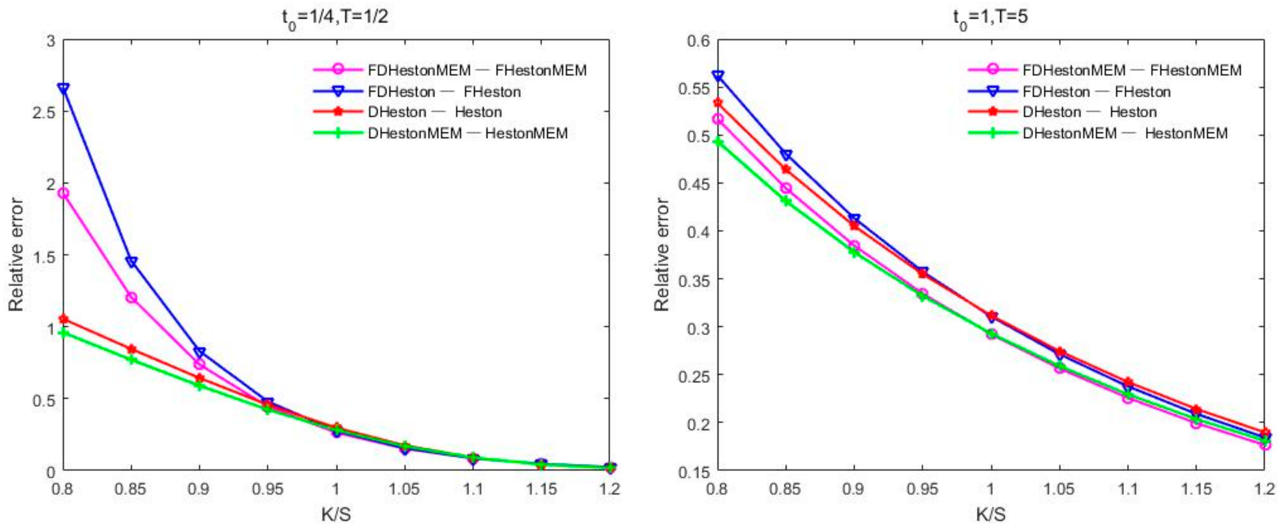


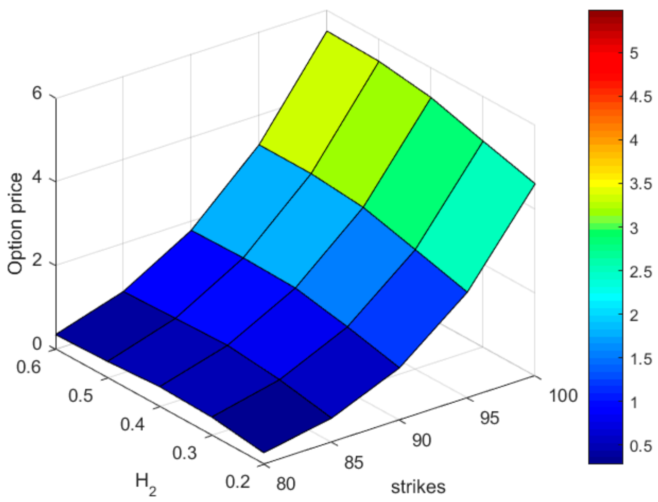
Figure 3. The effects of the second volatility on the values of short-term ($t_0 = 1/4,\ T = 1/2$) and long-term ($t_0 = 1,\ T = 5$) forward starting put options.

To examine the effects of the second volatility on forward starting put options prices, we investigated four cases including the jump-diffusion model and diffusion model under the framework of fractional Brownian motion and standard Brownian motion, respectively. Figure 3 reports the effects of the second volatility on the values of short-term ($t_0 = 1/4,\ T = 1/2$) and long-term ($t_0 = 1,\ T = 5$) forward starting put options.

From Figure 3 we can see that for the short-term options, when $K/S < 0.95$, the effects of the second volatility on option prices without jump are more significant than those with jump. As K/S increases, the intensity of the effect gradually weakens. When $K/S \geq 0.95$, the effects of the second volatility on option prices with jump and without jump are almost the same. For the long-term options, the effects of the second volatility on the options prices are significantly weaker than those in the short-term case. For all maturities and determination time, the effects of the second volatility on option prices in the framework of fractional Brownian motion are more significant than those in the standard Brownian motion case, which implies that the second volatility is necessary to price accurately forward starting options under the framework of fractional Brownian motion.

Furthermore, based on the model parameters and option parameters used in Table 2, we examined the effects of the main parameters of the second fractional volatility on the short-term ($t_0 = 1/4,\ T = 1/2$) forward starting put options prices. Due to limited space, Figure 4 only reports the effects of Hurst index H_2 and long-run mean θ_2 on the options prices.

From Figure 4 we can see that when $H_2 > 0.5$, increasing the value of H_2 rapidly raises the options prices and the intensity of this effect gradually tends to stabilize. When $H_2 < 0.5$, the effect of H_2 on the options prices is similar to that in the case of $H_2 > 0.5$, and this effect is more significant. Figure 4 displays this compared to the standard Brownian motion ($H_2 = 0.5$) case, which shows that increasing or decreasing the value of H_2 both significantly change forward starting options prices. Increasing θ_2 raises the options prices and the intensity of this effect gradually increases as the strike prices increase. Figure 4 shows that the effects of H_2 and θ_2 on forward starting options prices are both significant, which implies that the effect of the second fractional volatility on forward starting options prices is significant and the FDHestonMEM model may fit the forward implied volatility better.



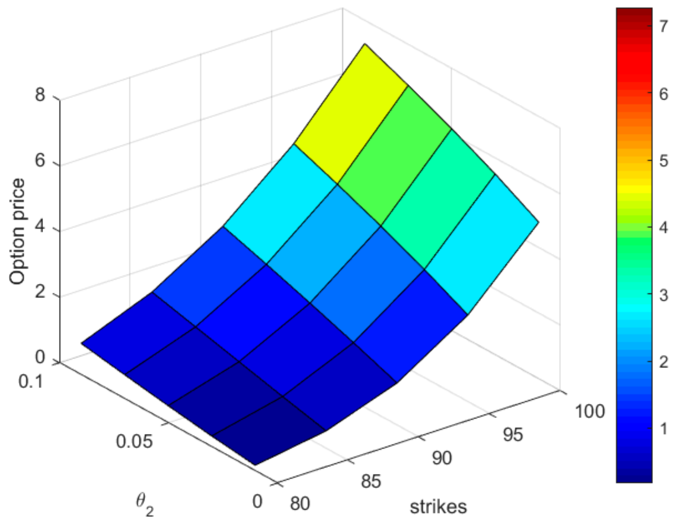


Figure 4. The effects of Hurst parameter H_2 and long-run mean θ_2 of the second volatility on the values of short-term ($t_0 = 1/4,\ T = 1/2$) forward starting put options.

6. Conclusions

This paper proposes an effective pricing algorithm for forward starting options under double fractional Heston stochastic volatilities and jumps. We express the value of a forward starting option in terms of the expectation of the forward characteristic function of log return. By introducing two small perturbed parameters, we approximate the pricing model using a semimartingale. Then, we rewrite the forward characteristic function as a conditional expectation of the proportion characteristic function which is expressed in terms of the solution to a classic PDE. With the affine structure of the approximate model, we obtained the solution to the PDE. Based on the obtained forward characteristic function and Fourier transform technique, we propose a pricing algorithm for forward starting options. For comparison, we also developed a simulation scheme for evaluating forward starting options. The numerical results show that the proposed pricing algorithm is effective. Via exhaustive comparative experiments on eight models, we found that the effects of fractional Brownian motion, mixed-exponential jump, and the second volatility component on forward starting option prices are all significant, and especially, the second fractional volatility is necessary to price accurately the forward starting options under the framework of fractional Brownian motion. The pricing algorithm for forward starting options proposed in the paper can be extended to the case of stochastic interest rates.

Author Contributions: Conceptualization, S.Z.; Formal analysis, S.Z. and H.X.; Methodology, S.Z. and H.Y.; Validation, H.Y. and H.X.; Writing—original draft, S.Z.; Writing—review & editing, S.Z. and H.X. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the National Natural Science Foundation (No. 11601420) and the Natural Science Foundation of Shaanxi Province (No. 2024JC-YBMS-008).

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Acknowledgments: The authors would like to thank the editors and anonymous reviewers for their detailed and valuable suggestions and comments, which improved the quality of the original manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Rubinstein, M. Pay now, choose later. *Risk* **1991**, *4*, 44–47.
2. Kruse, S.; Nögel, U. On the pricing of forward starting options in Heston's model on stochastic volatility. *Financ. Stoch.* **2005**, *9*, 233–250. [CrossRef]

3. Ahlip, R.; Rutkowski, M. Forward start options under stochastic volatility and stochastic interest rates. *Int. J. Theor. Appl. Financ.* **2011**, *12*, 209–225. [CrossRef]
4. Lin, S.; He, X.J. A closed-form pricing formula for forward start options under a regime-switching stochastic volatility model. *Chaos Solitons Fractals* **2021**, *144*, 110644. [CrossRef]
5. Guo, J.H.; Hung, M.W. A generalization of Rubinstein's "Pay now, choose later". *J. Futures Mark.* **2008**, *28*, 488–515. [CrossRef]
6. Ahlip, R.; Park, L.A.F.; Prodan, A.; Weissenhofer, S. Forward start options under Heston affine jump-diffusions and stochastic interest rate. *Int. J. Financ. Eng.* **2021**, *8*, 2150005. [CrossRef]
7. Lucié, V. Forward-start options in stochastic volatility models. *Wilmott Mag.* **2003**, *5*, 72–75. [CrossRef]
8. Haastrecht, A.V.; Pelsser, A.A.J. Accounting for stochastic interest rates, stochastic volatility and a general dependency structure in the valuation of forward starting options. *J. Futures Markets.* **2011**, *31*, 103–125. [CrossRef]
9. Zhang, S.M.; Geng, J.H. Fourier-cosine method for pricing forward starting options with stochastic volatility and jumps. *Commun. Stat.–Theory Methods.* **2017**, *46*, 9995–10004. [CrossRef]
10. Zhang, S.M.; Sun, Y.D. Forward starting options pricing with double stochastic volatility, stochastic interest rates and double jumps. *J. Comput. Appl. Math.* **2017**, *325*, 34–41. [CrossRef]
11. Hata, H.; Liu, N.L.; Yasuda, K. Expressions of forward starting option price in Hull–White stochastic volatility model. *Decis. Econ. Financ.* **2021**, *45*, 101–135. [CrossRef]
12. Comte, F.; Renault, E. Long memory in continuous-time stochastic volatility models. *Math. Financ.* **1998**, *8*, 291–323. [CrossRef]
13. Hassani, H.; Yarmohammadi, M.; Mashhad, L.M. Uncovering hidden insights with long-memory process detection: An in-depth overview. *Risks* **2023**, *11*, 113. [CrossRef]
14. Han, Y.; Li, Z.; Liu, C. Option pricing under the fractional stochastic volatility model. *Anziam J.* **2021**, *63*, 123–142.
15. Najafi, A.; Mehrdoust, F. Conditional expectation strategy under the long memory Heston stochastic volatility model. *Commun. Stat.—Simul. Comput.* **2023**. [CrossRef]
16. Alhagyan, M.; Yassen, M.F. Incorporating stochastic volatility and long memory into geometric Brownian motion model to forecast performance of standard and Poor's 500 index. *AIMS Math.* **2023**, *8*, 18581–18595. [CrossRef]
17. Alhagyan, M. The effects of incorporating memory and stochastic volatility into GBM to forecast exchange rates of Euro. *Alex. Eng. J.* **2022**, *61*, 9601–9608. [CrossRef]
18. Hassler, U.; Pohle, M.O. Forecasting under long memory. *J. Financ. Econom.* **2023**, *21*, 742–778. [CrossRef]
19. Gatheral, J.; Jaisson, T.; Rosenbaum, M. Volatility is rough. *Quant. Financ.* **2018**, *18*, 933–949. [CrossRef]
20. Livieri, G.; Mouti, S.; Pallavicini, A.; Rosenbaum, M. Rough volatility: Evidence from option prices. *IISE Trans.* **2018**, *50*, 767–776. [CrossRef]
21. Fukasawa, M. Volatility has to be rough. *Quant. Financ.* **2021**, *21*, 1–8. [CrossRef]
22. Brandi, G.; Di Matteo, T. Multiscaling and rough volatility: An empirical investigation. *Int. Rev. Financ. Anal.* **2022**, *84*, 102324. [CrossRef]
23. Coqueret, G.; Tavin, B. An investigation of model risk in a market with jumps and stochastic volatility. *Eur. J. Oper. Res.* **2016**, *253*, 648–658. [CrossRef]
24. Jin, X.; Hong, Y. Jump-diffusion volatility models for variance swaps: An empirical performance analysis. *Int. Rev. Financ. Anal.* **2023**, *87*, 102606. [CrossRef]
25. Bates, D.S. Empirical option pricing models. *Annu. Rev. Financ. Econ.* **2022**, *14*, 369–389. [CrossRef]
26. Wang, L.; Xia, W.X. Power-type derivatives for rough volatility with jumps. *J. Futures Mark.* **2022**, *42*, 1369–1406. [CrossRef]
27. Cai, N.; Kou, S.G. Option pricing under a mixed-exponential jump diffusion model. *Manag. Sci.* **2011**, *57*, 2067–2081. [CrossRef]
28. Zhang, S.; Yong, H.; Xiao, H. Option pricing with fractional stochastic volatilities and jumps. *Fractal Fract.* **2023**, *7*, 680. [CrossRef]
29. Wang, K.; Guo, X.X. Valuations of variance and volatility swaps under double Heston jump-difusion model with approximative fractional stochastic volatility. *Comput. Econ.* **2024**, *63*, 1543–1573. [CrossRef]
30. Thao, T.H. An approximate approach to fractional analysis for finance. *Nonlinear Anal.–Real World Appl.* **2006**, *7*, 124–132. [CrossRef]
31. Euch, O.E.; Rosenbaum, M. The characteristic function of rough Heston models. *Math. Financ.* **2019**, *29*, 3–38. [CrossRef]
32. Fang, F.; Oosterlee, C.W. A novel pricing method for European options based on Fourier cosine series expansions. *SIAM J. Sci. Comput.* **2008**, *31*, 826–848. [CrossRef]