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# Mild Solutions for $w$-Weighted, Φ-Hilfer, Non-Instantaneous, Impulsive, $w$-Weighted, Fractional, Semilinear Differential Inclusions of Order $\mu \in (1,2)$ in Banach Spaces

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**Abstract:** The aim of this work is to obtain novel and interesting results for mild solutions to a semilinear differential inclusion involving a $w$-weighted, Φ-Hilfer, fractional derivative of order $\mu \in (1,2)$ with non-instantaneous impulses in Banach spaces with infinite dimensions when the linear term is the infinitesimal generator of a strongly continuous cosine family and the nonlinear term is a multi-valued function. First, we determine the formula of the mild solution function for the considered semilinear differential inclusion. Then, we give sufficient conditions to ensure that the mild solution set is not empty or compact. The desired results are achieved by using the properties of both the $w$-weighted Φ-Laplace transform, $w$-weighted $\psi$-convolution and the measure of non-compactness. Since the operator, the $w$-weighted Φ-Hilfer, includes well-known types of fractional differential operators, our results generalize several recent results in the literature. Moreover, our results are novel because no one has previously studied these types of semilinear differential inclusions. Finally, we give an illustrative example that supports our theoretical results.

**Keywords:** fractional differential inclusions; $w$-weighted Φ-Hilfer fractional derivative; infinitesimal generator of cosine family; $w$-weighted Φ-Laplace transform; measure of non-compactness; non-instantaneous impulses; multivaluted functions

**MSC:** Primary 26A33,34A08; Secondary 34A60

## 1. Introduction

Fractional differential inclusions and equations have many applications in our life [1–4]. Impulsive differential equations and impulsive differential inclusions are suitable models for studying the dynamics of actions in which a sudden change in state occurs. If this change occurs instantaneously, it is called an instantaneous impulse [5,6], but if this change continues for a period of time, it is called a non-instantaneous impulse [7–9].

There are many definitions for the fractional differential operator, and some of them are particular cases of others. Therefore, it is useful to consider fractional differential equations and fractional differential inclusions that contain a fractional differential operator which includes a large number of other fractional differential operators. This is our goal in this work. Indeed, in this paper, we consider a semilinear differential inclusion involving the $w$-weighted Φ-Hilfer fractional derivative, $D_{0,\sigma}^{\mu,v,\Phi,w}$ (Definition 3, below), which generalizes the concepts of fractional differential operators that were presented by Riemann–Liouville ($w(\sigma) = 1; \sigma \geq 0$, $\Phi(\sigma) = \sigma$, $v = 0$), Caputo ($w(\sigma) = 1; \sigma \geq 0$, $\Phi(\sigma) = \sigma$, $v = 1$), Hadamard ($w(\sigma) = 1; \sigma \geq 0$, $\Phi(\sigma) = \log \sigma$, $\sigma > 0, v = 1$), Φ- Riemann–Liouville ($w(\sigma) = 1; \sigma \geq 0$, $v = 0$), Φ- Caputo ($w(\sigma) = 1; \sigma \geq 0$, $v = 1$), Katugampola ($w(\sigma) = 1; \sigma \geq$

0, $\Phi(\sigma) = \sigma^\rho, \rho \geq 1, v = 1$), Hilfer–Hadamard, ($w(\sigma) = 1; \sigma \geq 0$, $\Phi(\sigma) = \log \sigma, \sigma > 0$), Hilfer ($w(\sigma) = 1; \sigma \geq 0$, $\Phi(\sigma) = \sigma$), Hilfer–Katugampola ($w(\sigma) = 1; \sigma \geq 0$, $\Phi(\sigma) = \sigma^\rho, \rho \geq 1$) and $\Phi$-Hilfer derivatives ($w(\sigma) = 1; \sigma \geq 0$).

Since the mild solution of a differential equation is not required to be continuously differentiable, like the classical solution, the study of the existence of mild solutions to differential equations or differential inclusions has been of interest for decades, especially for semi-linear differential equations and semi-linear differential inclusions. More than thirty years ago, the study of the existence of a mild solution to semi-linear differential Equations and semi-linear differential inclusions containing a fractional differential operator became of interest. Some of these equations contained the Caputo fractional derivative [10–12], some involved the Riemann–Liouville fractional differential operator [13,14], some contained the Caputo-Hadamard fractional differential operator [15,16], some included the Hilfer fractional differential operator of order $\alpha \in (0,1)$ in [17–26], some contained the Katugampola fractional differential operator [27], some contained the Hilfer–Katugampola fractional differential operator of order $\alpha \in (0,1)$ [28–32] and others involved the Hilfer fractional differential operator of order $\lambda \in (1,2)$ [33].

In this article, we will prove the existence of a mild solution to a semi-linear differential inclusion involving the $w$-weighted $\Phi$-Hilfer fractional differential operator. Because the fractional differential operators introduced by Caputo, Riemann–Liouville, Caputo-Hadamard, Hilfer and Hilfer–Katugampola are special cases of $w$-weighted $\Phi$-Hilfer fractional differential operators, our work generalizes many of the abovementioned results by replacing the fractional differential operator considered in these papers with the $w$-weighted $\Phi$-Hilfer fractional differential operator.

In order to formulate the problem, we mention some symbols that will be used during this paper.

- $\Im = [0, b]$.
- $E$ is a Banach space.
- $\mu \in (1,2)$, $v \in [0,1]$ and $\gamma = \mu + 2v - \mu v$.
- $w : \Im \to ]0, \infty[$ and $w^{-1}(\sigma) = \frac{1}{w(\sigma)}$.
- $\Phi : \Im \to \mathbb{R}$ is a strictly increasing continuously differentiable function with $\Phi'(\vartheta) \neq 0$ for any $\vartheta \in \Im$, and $\Phi^{-1}$ is its inverse.
- $m$ is a natural, $N_0 = \{0, 1, 2, .., m\}$, $N_1 = \{1, 2, .., m\}$ and $N_2 = \{2, 3, 4, ..., m\}$.
- $\vartheta_0 = 0 < \sigma_1 \leq \vartheta_1 < \sigma_2 \leq \vartheta_2 < ...... < \sigma_m \leq \vartheta_m < \sigma_{m+1} = b$.
- $\Im_k = (\vartheta_k, \sigma_{k+1}]; k \in N_0$, $T_i = (\vartheta_i, \sigma_{i+1}]; i \in N_1$.
- $D_{\vartheta_i, \sigma}^{\mu, v, \Phi, w}$ is the $w$-weighted $\Phi$-Hilfer derivative operator of order $\mu$ and of type $v$ and with a lower limit at $\vartheta_i$
- $I_{\vartheta_i, \sigma}^{2-\mu, \Phi, w}$ is the $w$-weighted $\Phi$-integral operator of order $2 - \mu$ and with a lower limit at $\vartheta_i$.
- $A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(\sigma)\}_{\sigma \in \mathbb{R}}$, where $C(\sigma)$ maps $E$ into itself.
- $F : \Im \times E \to P_{ck}(E)$ (the family of non-empty, convex and compact subsets of $E$ )
- $g_i, g_i^* : [\sigma_i, \vartheta_i] \times E \to E; i \in N_1$ are continuous functions, and $x_0, x_1 \in E$ are fixed points.
- $AC(\Im, E)$ is the Banach space of absolutely continuous functions from $\Im$ to $E$.

In this paper, and by using the properties of $w$-weighted $\Phi$-Laplace transform, we derive at first the formula of a mild solution to the following differential inclusion containing the $w$-weighted $\Phi$-Hilfer fractional derivative order $\mu$ and of type $v$ with the existence of non-instantaneous impulses in Banach spaces with infinite dimensions:

$$\begin{cases} D_{\vartheta_i,\sigma}^{\mu,v,\Phi,w} x(\sigma) \in Ax(\sigma) + F(\sigma, x(\sigma)), \text{ a.e., } \sigma \in (\vartheta_i, \sigma_{i+1}], i \in N_0, \\ x(\sigma_i^+) = g_i(\sigma_i, x(\sigma_i^-)), i \in N_1 \\ x(\sigma) = g_i(\sigma, x(\sigma_i^-)), \sigma \in (\sigma_i, \vartheta_i], i \in N_1, \\ \lim_{\sigma \to 0^+} w(\sigma) I_{0,\sigma}^{2-\gamma,\Phi,w} x(\sigma) = x_0, \lim_{\sigma \to 0^+} \frac{1}{\Phi'(\sigma)} \frac{d}{d\sigma} (w(\sigma) I_{0,\sigma}^{2-\gamma,\Phi,w} x(\sigma)) = x_1, \\ \lim_{\sigma \to \vartheta_i^-} (w(\sigma) I_{0,\sigma}^{2-\gamma} x(\sigma)) = g_i(\vartheta_i, x(\vartheta_i^-)), \forall i \in N_1 \\ \lim_{\sigma \to \vartheta_i^-} \frac{1}{\Phi'(\sigma)} \frac{d}{d\sigma} (w(\sigma) I_{0,\sigma}^{2-\gamma} x(\sigma)) = g_i^*(\vartheta_i, x(\vartheta_i^-)), \forall i \in N_1, \end{cases} \quad (1)$$

Then, without assuming the compactness of $C(\sigma); \sigma > 0$, we find the sufficient conditions that ensure that the mild solution set of Problem (1) is not empty or compact in the Banach space $PC_{2-\gamma,\Phi,w}(\Im, E)$, which will be defined latter.

To further explain our arguments that clarify our motivation for studying Problem (1) addressed in this manuscript, as well as the importance of our purpose, we state the following: In a very recently published paper, Alsheekhhussain et al. [34] considered Problem (1) when the operator $A$ is the zero operator. Zhou et al. [10] and He et al. [11] investigated the existence of mild solutions to Problem (1) when $\vartheta_i = \sigma_i = b, \forall i \in N_1$, $w(\sigma) = 1$, $\Phi(\sigma) = \sigma; \sigma \in \Im$ and $v = 1$. Wang et al. [12] considered Problem (1) when $w(\sigma) = 1$, $\Phi(\sigma) = \sigma; \sigma \in \Im$ and $v = 1$. Thongsalee et al. [13] proved that solutions for Problem (1) exist; when $A$ is the zero operator, $F$ is a single-valued function, $v = 0$, $\vartheta_i = \sigma_i = b, \forall i \in N_1$, $w(\sigma) = 1$, and $\Phi(\sigma) = \sigma \in \Im$, Shu et al. [14] studied Problem (1) in the particular cases $v = 0$, $w(\sigma) = 1$ and $\Phi(\sigma) = \sigma, \forall \sigma \in \Im$, Gu et al. [17] was the first to consider Problem (1) when $\mu \in (0,1)$, $\vartheta_i = \sigma_i = b, \forall i \in N_1$, $w(\sigma) = 1$ and $\Phi(\sigma) = \sigma; \sigma \in \Im$.

Working with the Laplace transform and density function, Gu et al. [17] was the first to define the mild solution for the semilinear differential equation:

$$\begin{cases} D_{0,\sigma}^{\delta,\eta} x(\sigma) = \Xi x(\sigma) + f(\sigma, x(\sigma)), \text{a.e., } \sigma \in (0, b], \\ \lim_{\sigma \to 0^+} I_{0,\sigma}^{1-\rho} x(\sigma) = x_0, \end{cases} \quad (2)$$

where $D_{0,\sigma}^{\delta,\eta}$ is the Hilfer fractional derivative of order $\delta, \eta \in (0,1), \rho = \delta + \eta - \delta\eta$, $\Xi$ is the infinitesimal generator of $C_0$-semigroup of linear bounded operators, $f : [0, b] \times E \to E$ and $x_0 \in E$ is a fixed point. Jaiwal et al. [18] presented the definition of a mild solution for (2) when $\Xi$ is an almost sectorial operator, and then they found the sufficient conditions that guarantee that the solution exists.

Yang et al. [19] proved the existence of mild solutions for the non-local semilinear differential equation:

$$\begin{cases} D_{0,\sigma}^{\mu,v} x(\sigma) = \Xi x(\sigma) + f(\sigma, x(\sigma)), \text{a.e., } \sigma \in (0, b], \\ \lim_{\sigma \to 0^+} I_{0,\sigma}^{1-\rho} x(\sigma) = x_0 - g(x), \end{cases} \quad (3)$$

where $\mu \in (0,1)$, $v \in [0,1]$, $\rho = \mu + v - \mu v$, and $\Xi$ generates an analytic semigroup of uniformly bounded linear operators. Wang et al. [20] showed solutions for (3) with the existence of non-instantaneous impulses and where $f$ is a multi-valued function and studied the controllability of the problem. Very recently, Elbukhari et al. [23] proved the existence of a mild solution for Problem (3), when $\Xi$ is the infinitesimal generator of a compact $C_0$-semigroup and $g$ does not satisfy any assumption such as compactness or Lipschitz continuity, making their findings interesting.

Suechoei et al. [35] derived the formula of a mild solution for Problem (1) in the particular cases $\mu \in (0,1)$, $\vartheta_i = \sigma_i = b, \forall i \in N_1$ $w(\sigma) = 1$, $\sigma \in \Im$ and $v = 1$.

Later on, Asawasamrit et al. [36] studied non-instantaneous, impulsive differential equations involving the $\Phi$-Caputo fractional derivative of order $\alpha \in (0,1)$ with Riemann–Stieltjes fractional integral boundary conditions.

Sousa et al. [37] introduced the concept of the $\Phi$-Hilfer fractional derivative of order $\mu \in (0,1)$ and obtained important results. Kucche et al. [38] showed that solutions for the following non-linear differential equation involving the $\Phi$-Hilfer fractional derivative exist:

$$\begin{cases} D_{a^+,\sigma}^{\mu,v,\Phi} x(\sigma) = f(\sigma, x(\sigma)), a.e., \sigma \in (a,b] - \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \\ \lim_{\epsilon \to 0^+} I_{a^+,\sigma}^{1-\rho} x(\sigma_k + \epsilon) - \lim_{\epsilon \to 0^+} I_{a^+,\sigma}^{1-\rho} x(\sigma_k - \epsilon) = \xi_k \in \mathbb{R}, k = 1,..,m, \\ \lim_{\epsilon \to a^+} I_{a^+,\sigma}^{1-\rho} x(\sigma) = x_0 \in \mathbb{R}, \end{cases}$$

where $\mu \in (0,1)$, $v \in [0,1]$, $D_{a^+,\sigma}^{\mu,v,\Phi}$ is the $\Phi$-Hilfer fractional derivative, $\rho = \mu + v - \mu v$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$.

In [39–46], there are studies on the existence of mild solutions of differential equations and inclusions involving the $w$ -weighted $\Phi$-Hilfer fractional derivative of order $\mu \in (0,1)$ and of type $v \in [0,1]$ in the special case $w(\sigma) = 1; \forall \sigma \in \Im$.

Very recently, Benial et al. [47] considered $w$-weighted $\Phi$-Riemann–Liouville differential equation of order $\sigma(\xi)$, where $\sigma(\xi) : [0,b] \to (1,2]$.

For other contributions on weighted fractional boundary value problems, we refer to [48–50].

**Remark 1.** *Our work is novel and interesting because:*

*1- To date, none of the researchers in the field have considered studying semilinear differential equations or semilinear differential inclusions containing the $w$-weighted $\Phi$-Hilfer fractional derivative of order $\mu \in (1,2)$ and of type $v \in [0,1]$.*

*2- Our studied problem is considered with the existence of non-instantaneous impulses and in infinite-dimensional Banach spaces.*

*3- Our problem contains the $w$-weighted $\Phi$-Hilfer fractional derivative, which interpolates many fractional differential operators, and hence, it includes the majority of problems cited above.*

*4- Li et al. [33] derived the representation of mild solutions to Problem (1) in the particular situations when $w(\sigma) = 1$ and $\Phi(\sigma) = \sigma, \forall \sigma \in \Im$.*

The following summarizes the focal contributions of our work.

- A new class of differential inclusions is formulated, involving the $w$-weighted $\Phi$-Hilfer differential operator, $D_{0,\sigma}^{\mu,v,\Phi,w}$, of order $\mu \in (1,2)$ and of type $v$ in Branch spaces with finite dimension, when the linear term is the infinitesimal generator of a strongly continuous cosine family, and the nonlinear term is a multi-valued function

- By utilizing both the $w$-weighted $\Phi$-Laplace transform and $w$- weighted $\Phi$-convolution, the representation of mild solutions for Problem (1) is derived (Lemma 10 and Definition 12).

- Our obtained formula for mild solutions coincides with the formula that was obtained by Li et al. [33] in the special case $w(\sigma) = 1$ and $\Phi(\sigma) = \sigma, \forall \sigma \in \Im$ (Corollary 1).

- The conditions that ensure that the mild solution set for Problem (1) is not empty or compact are obtained (Theorem 1).

- This work is a generalization of what was achieved in [17,19,33–35].

- An example is given to show the possibility of applying our results (Example 1).

- Our method helps interested researchers to generalize the majority of the aforementioned works to the case where the non-linear term is a multifunction and the space is infinite-dimensional.

- Since a large class of fractional differential operators can be obtained from $D_{0,\sigma}^{\mu,v,\Phi,w}$, the works in many results mentioned above can be generalized by replacing the considered fractional differential operator with $D_{0,\sigma}^{\mu,v,\Phi,w}$ and making the dimension of the space infinite, and this is considered as a suggestion for future research work as a result of our work.

- One can obtain a broad class of fractional differential equations and inclusions as a particular case of Problem (1) (see Remark 1).

We organize our work as follows: in Section 2, we present definitions and results from previous work that we will need to obtain our results. In the third section, we obtain the relation between Problem (1) and the correlating fractional integral equation and the representation of mild solutions. Moreover, we prove that the mild solution set for Problem (1) is not empty or compact. Finally, an example is presented to clarify the possibility of the application of our results.

## 2. Preliminaries and Notations

We commence this section by recalling some symbols that will be used later.

For any $p \geq 1$, denote by $L_w^{p,\Phi}((0,b), E)$ the Banach space of all Lebesgue measurable functions $f$ such that $fw(\Phi')^{\frac{1}{p}} \in L^p((a,b), E)$ when $p \in [1, \infty)$, and $fw \in L^\infty((0,b), E)$ when $p = \infty$, where

$$||f||_{L_w^{p,\Phi}((0,b),E)} := \begin{cases} (\int_0^b ||f(x)w(x)||^P \Phi'(x)dx)^{\frac{1}{p}}, \text{ if } p \in [1,\infty), \\ ||fw||_{L^\infty((0,b),E)}, \text{ if } p = \infty. \end{cases}$$

For any function $f : [0,b] \to E$, define [50]

$$D^{1,\Phi,w}f(\sigma) := w^{-1}(\sigma)\frac{1}{\Phi'(\sigma)}\frac{d}{d\sigma}(f(\sigma)w(\sigma)),$$

$$D^{k,\Phi,w}f(\sigma) := D^{1,\Phi,w}D^{k-1,\Phi,w}f(\sigma) = w^{-1}(\sigma)[\frac{1}{\Phi'(\sigma)}\frac{d}{d\sigma}]^k(f(\sigma)w(\sigma)), k \in N_2,$$

and

$$f_{k,\Phi,w}(\sigma) := w(\sigma)D^{k,\Phi,w}f(\sigma) = [\frac{1}{\Phi'(\sigma)}\frac{d}{d\sigma}]^k(f(\sigma)w(\sigma)), k \in N_2.$$

Let us consider the Banach spaces:

$-C_w([a,b], E) := \{x \in C(\Im, E) : wx \in C([a,b], E)\}$, where $||x|| = \max_{s \in \Im} ||w(s)x(s)||$.

$-C_{2-\gamma,\Phi,w}(\Im, E) := \{x \in C((0,b], E) : (\Phi(.) - \Phi(0))^{2-\gamma}x(.) \in C_w(\Im, E)\}$, where

$$||x||_{C_{2-\gamma,\Phi,w}(\Im,E)} := \sup_{\varrho \in \Im} || (\Phi(\varrho) - \Phi(0))^{2-\gamma}w(\varrho)x(\varrho)||.$$

$-PC_{2-\gamma,\Phi,w}(\Im, E) := \{x : [0,b] \to E, (\Phi(.) - \Phi(\vartheta_k))^{2-\gamma}x(.) \in C_w(\Im_k, E), x_{|\sigma_i} \in C(\sigma_i, E),$
$\lim_{\sigma \to \vartheta_k^+}(\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}x(\sigma)$ and $\lim_{\sigma \to \sigma_i^+} x(\sigma)$ exists, $k \in N_0, i \in N_1\}$, where

$$||x||_{PC_{2-\gamma,\Phi,w}(\Im,E)} := \max\{\sup_{\substack{\sigma \in \overline{\Im_k} \\ k \in N_0}} (\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}||w(\sigma)x(\sigma)||_E, \sup_{\substack{\sigma \in \overline{T_k} \\ k \in N_1}} ||x(\sigma)||_E\},$$

$-AC^{1,w}(\Im, E) := \{x : [a,b] \to E, xw \in AC(\Im, E)\}$, where $||x||_{AC^{1,w}(\Im,E)} = ||xw||_{AC(\Im,E)}.$

$-C_{2-\mu,\Phi,w}^1(\Im, E) := \{x \in C_{2-\mu,\Phi,w}(\Im, E) : D^{1,\Phi,w}x \in C_{2-\mu,\Phi,w}(\Im, E)\}.$

$-C_{2-\gamma,\Phi,w}^n(\Im, E) := \{x \in C_{2-\gamma,\Phi,w}^{n-1}(\Im, E) : D^{n,\Phi,w}x \in C_{2-\gamma,\Phi,w}(\Im, E)\}, n \in \mathbb{N} - \{1\}$, where

$$||x||_{C_{2-\mu,\Phi,w}^n(\Im,E)} := \sum_{k=1}^{k=n-1} ||D^{k,\Phi,w}x||_{C_{2-\mu,\Phi,w}^k(\Im,E)} + ||D^{n,\Phi,w}x||_{C_{2-\mu,\Phi}(\Im,E)}.$$

$-AC^{n,\Phi,w}(\Im, E) := \{x : \Im \to E, D^{n-1,\Phi,w}x \in AC^{1,w}(\Im, E), D^{n,\Phi,w}x \in L_w^{1,\Phi}((0,b), E)\};$
$n \in \mathbb{N} - \{1\}$, where $||x||_{AC^{n,\Phi,w}(\Im,E)} := ||x_{n-1,\Phi,w}||_{AC(\Im,E)}.$

The function $\chi_{PC_{2-\gamma,\Phi,w}(\Im,E)} : P_b(PC_{2-\gamma,\Phi,w}(\Im, E)) \to [0, \infty)$, which is given by :

$$\chi_{PC_{2-\gamma,\Phi,w}(\Im,E)}(D) := \max\{\max_{k \in N_0} \chi_{C(\overline{\Im_k},E)}(D_{|\overline{\Im_k}}), \max_{i \in N_1} \chi_{C(\overline{T_i},E)}(D_{|\overline{T_i}})\},$$

is a measure of noncompactness on $PC_{2-\gamma,\Phi,w}(\Im, E)$, where

$$D_{|\overline{\Im_k}} := \{h^* \in C(\overline{\Im_k}, E) : h^*(\sigma) = (\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}w(\sigma)h(\sigma), \sigma \in \Im_k,$$
$$h^*(\vartheta_k) = \lim_{\sigma \to \vartheta_k^+} h^*(\sigma), h \in D\},$$

and

$$D_{|\overline{T_i}} := \{h^* \in C(\overline{T_i}, E) : h^*(\sigma) = h(t), t \in T_i, h^*(t_i) = h(t_i^+), h \in D\}.$$

**Definition 1** ([50])**.** *Let $\alpha > 0$. The w-weighted Riemann–Liouville fractional integral of order $\alpha$ where the lower limit at a of a function $f \in L_w^{p,\Phi}([a,b], E)$ in regard to $\Phi$ is given by:*

$$(I_{a,t}^{\alpha,\Phi,w}f)(\sigma) := \frac{w^{-1}(\sigma)}{\Gamma(\alpha)} \int_a^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\alpha-1}\Phi'(\vartheta)f(\vartheta)w(\vartheta)d\vartheta.$$

**Lemma 1** ([50], Theorem 2.4)**.** *Assume $f \in L_w^{p,\Phi}((a,b], E), 1 \le p \le \infty, \alpha > 0$ and $\beta > 0$, then $I_{a,\sigma}^{\alpha,\Phi,w}I_{a,\sigma}^{\beta,\Phi,w}f = I_{a,\sigma}^{\alpha+\beta,\Phi,w}f$.*

**Definition 2** ([50])**.** *Let $n \in \mathbb{N}$ and $\alpha \in ]n-1, n[$. The w-weighted Riemann–Liouville fractional derivative whose order $\alpha$ where the lower limit at a of a function $f : [a,b] \to E$ in regard to $\Phi$ is given by:*

$$(D_{a,\sigma}^{\alpha,\Phi,w}f)(\sigma) := D_\sigma^{n,\Phi,w}(I_{a,\sigma}^{n-\alpha,\Phi,w}f)(\sigma)$$
$$= \frac{1}{w(\sigma)\Gamma(n-\alpha)}D_\sigma^{n,\Phi,w}(\int_a^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{n-\alpha-1}\Phi'(\vartheta)f(\vartheta)w(\vartheta)d\vartheta,$$

*assuming that the right-hand side is well defined.*

**Lemma 2** ([50], Proposition 1.3)**.**
*i- If $\alpha > 0$ and $\beta > 0$, then*

$$I_{a,\sigma}^{\alpha,\Phi,w}\left(\frac{(\Phi(\sigma) - \Phi(a))^{\beta-1}}{w(\sigma)}\right)$$
$$= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \frac{(\Phi(\sigma) - \Phi(a))^{\beta+\alpha-1}}{w(\sigma)}; \forall \sigma \in \Im. \tag{4}$$

*ii- If $\beta > 0$ and $\alpha < \beta$, then*

$$D_{a,\sigma}^{\alpha,\Phi,w}\left(\frac{(\Phi(\sigma) - \Phi(a))^{\beta-1}}{w(\sigma)}\right)$$
$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \frac{(\Phi(\sigma) - \Phi(a))^{\beta-\alpha-1}}{w(\sigma)}; \forall \sigma \in \Im.$$

One can use similar arguments to the ones used in $E = \mathbb{R}$ ([50], Theorems 3.3–3.6) to prove the next lemma.

**Lemma 3.** *i- If $\alpha \in (n-1, n)$ and $f \in AC^{n,\Phi,w}([a,b], E)$, then $D_{a,\sigma}^{\alpha,\Phi,w}f$ exits almost everywhere and*

$$(D_{a,\sigma}^{\alpha,\Phi,w}f)(\sigma) := \frac{w^{-1}(\sigma)}{\Gamma(n-\alpha)}(\int_a^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{n-\alpha-1}\Phi'(\vartheta)D_\vartheta^{n,\Phi,w}(w(\vartheta)f(\vartheta))d\vartheta$$
$$- \sum_{k=0}^{k=n-1} \frac{(\Phi(\sigma) - \Phi(a))^k}{k!}f_{k,\Phi,w}(a), a.e.,$$

*where $f_{0,\Phi,w}(a) = f(a)w(a)$.*

　*ii- If $f \in L_w^{p,\Phi}((a,b], E)$ and $\alpha > \beta > 0$, then $D_{a,\sigma}^{\beta,\Phi,w} I_{a,\sigma}^{\alpha,\Phi,w} f(\sigma) = I_{a,\sigma}^{\alpha-\beta,\Phi,w} f(\sigma), \forall \sigma \in \Im$.*

　*iii- If $f \in L_w^{p,\Phi}((a,b], E)$, then $D_{a,\sigma}^{\alpha,\Phi,w} I_{a,\sigma}^{\alpha,\Phi,w} f(\sigma) = f(\sigma), \forall \sigma \in \Im$.*

　*iv- If $\alpha \in (n-1,n)$ and $f \in L_w^{p,\Phi}((a,b], E)$ such that $I_{a,\sigma}^{n-\alpha,\Phi,w} f \in AC^{n,\Phi,w}([a,b], E)$, then for any $\sigma \in [a,b]$*

$$I_{a,\sigma}^{\alpha,\Phi,w}(D_{a,\sigma}^{\alpha,\Phi,w} f)(\sigma)$$

$$= f(\sigma) - w^{-1}(\sigma) \sum_{k=1}^{n} \frac{(\Phi(\sigma) - \Phi(a))^{\alpha-k}}{\Gamma(\alpha - k + 1)} \lim_{\sigma \to a} \left(\frac{1}{\Phi'(\sigma)} \frac{d}{d\sigma}\right)^{n-k} (w(\sigma) I_{a,\sigma}^{n-\alpha} f(\sigma)). \tag{5}$$

**Definition 3** ([50]). *Let $n \in \mathbb{N}$ and $\alpha \in ]n-1,n[$. The w-weighted $\Phi-$Caputo fractional derivative of order $\alpha$ where the lower limit at $a$ of a function $f$ ; $[a,b] \to E$ in regard to to $\Phi$ is given by:*

$${}^c D_{a,\sigma}^{\alpha,\Phi,w} f)(\sigma) := D_{\sigma}^{\alpha,\Phi,w} (f(\sigma) - \sum_{k=0}^{k=n-1} \frac{(\Phi(\sigma) - \Phi(a))^k}{k!} f_{k,\Phi,w}(a)),$$

*assuming that the right-hand side is well defined.*

In the following, we recall some properties for $I_{a,\sigma}^{\alpha,\Phi,w}$, $D_{a,\sigma}^{\alpha,\Phi,w}$ and ${}^c D_{a,\sigma}^{\alpha,\Phi,w}$.

**Lemma 4** ([50], Theorems 4.2–4.5). *i- If $\alpha \in (n-1,n)$ and $f \in AC^{n,\Phi,w}([a,b], E)$, then for a.e.*

$${}^c D_{a,\sigma}^{\alpha,\Phi,w} f)(\sigma) = \frac{w^{-1}(\sigma)}{\Gamma(1-\alpha)} \left(\int_a^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{n-\alpha-1} [\frac{1}{\Phi'(\sigma)} \frac{d}{d\sigma}]^n (f(\vartheta) w(\vartheta)) d\vartheta.$$

*ii- If $\alpha \in (0,1)$ and $f \in AC^{n,\Phi,w}([a,b], E)$, then for a.e.*

$${}^c D_{a,\sigma}^{\alpha,\Phi,w} f)(\sigma) = \frac{w^{-1}(\sigma)}{\Gamma(1-\alpha)} \left(\int_a^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{n-\alpha-1} \frac{d}{d\vartheta} (f(\vartheta) w(\vartheta)) d\vartheta.$$

*iii- ${}^c D_{a,\sigma}^{\alpha,\Phi,w} (w^{-1}(\sigma)(\Phi(\sigma) - \Phi(a))^k(\sigma) = 0; \forall k = 0, 1, \ldots, n-1$, where $n-1 < \alpha < n$.*

As a result of definitions (3) and (1), we give in the following definition the concept of the $w$-weighted $\Phi$-Hilfer derivative operator.

**Definition 4.** *The w-weighted $\Phi$-Hilfer derivative of order $\mu$ and of type $v$ where the lower limit at $a$ for a function $f : [a,b] \to E$ is given by*

$$D_{a,\sigma}^{\mu,v,\Phi,w} f(\sigma) := I_{a,\sigma}^{v(2-\mu)} D_{\sigma}^{2,\Phi,w} I_{a,\sigma}^{(2-\mu)(1-v),\Phi,w} f(\sigma)$$

$$= I_{a,\sigma}^{v(2-\mu)} D_{\sigma}^{2,\Phi,w} I_{a,\sigma}^{2-\gamma,\Phi,w} f(\sigma)$$

$$= I_{a,\sigma}^{v(2-\mu)} D_{a,\sigma}^{\gamma,\Phi,w} f(\sigma), \tag{6}$$

*where $\gamma = \mu + 2vs. - \mu vs.$, assuming that the right-hand side is well defined.*

**Remark 2.** *1- If $f \in C^{2,\Phi,w}([a.b], E)$, then $D_{a,\sigma}^{\gamma,\Phi,w} f(\sigma)$ exists a.e., and consequently $D_{a,\sigma}^{\mu,v,\Phi,w} f(\sigma)$ exists for $\sigma \in (a,b]$.*

　*2- If $I_{a,\sigma}^{2-\gamma,\Phi,w} f(\sigma) \in C_{2-\mu,\Phi,w}^\gamma([a,b], E)$, then $D_{a,\sigma}^{\gamma,\Phi,w} f(\sigma) \in C_{2-\mu,\Phi,w}([a,b], E)$, and consequently $D_{a,\sigma}^{\mu,v,\Phi,w} f(\sigma)$ exists for $\sigma \in (a,b]$.*

　*3- Let $x \in L_w^{p,\Phi}((a,b], E)$ be such that $I_{a,\sigma}^{2-\gamma,\Phi,w} x \in C_{2-\gamma,\Phi,w}^2([a,b], E)$. Since $C_{2-\gamma,\Phi,w}^2([a,b], E) \subseteq C^{2,\Phi,w}([a,b], E)$, then, $I_{a,\sigma}^{2-\gamma,\Phi,w} x \in C^{2,\Phi,w}([a,b], E)$. Therefore, by Lemma 1, (5) and (6), we obtain for $\sigma \in (a,b]$,*

$$I_{a,\sigma}^{\mu,\Phi,w}(D_{a,\sigma}^{\mu,v,\Phi,w}x)(\sigma) = I_{a,\sigma}^{\mu,\Phi,w}I_{a,\sigma}^{v(2-\mu),\Phi,w}D_{a,\sigma}^{\gamma,\Phi,w}x(\sigma)$$

$$= I_{a,\sigma}^{\gamma,\Phi,w}D_{a,\sigma}^{\gamma,\Phi,w}x(\sigma)$$

$$= x(\sigma) - \frac{(\Phi(\sigma)-\Phi(a))^{\gamma-1}}{w(\sigma)\Gamma(\gamma)}\lim_{\sigma\to a^+}\left(\frac{1}{\Phi'(\sigma)}\frac{d}{d\sigma}\right)(w(\sigma)I^{2-\gamma,\Phi,w}x(\sigma))$$

$$- \frac{(\Phi(\sigma)-\Phi(a))^{\gamma-2}}{w(\sigma)\Gamma(\gamma-1)}\lim_{\sigma\to a^+}(w(\sigma)I^{2-\gamma,\Phi,w}x(\sigma)). \tag{7}$$

**Definition 5** ([51]). *We call a one-parameter family of bounded linear operators $\{C(\sigma)\}_{\sigma\in\mathbb{R}}$ which maps the Banach space E into itself a strongly cosine family if and only if*

*(i)* $C(0) = I$,
*(ii)* $C(\vartheta+\sigma) + C(\vartheta-\sigma) = 2C(\vartheta)C(\sigma)$ *for all* $\vartheta,\sigma\in\mathbb{R}$,
*(iii)* *The map* $\sigma\longmapsto C(\sigma)x$ *is continuous for each* $x\in E$.

**Definition 6** ([51]). *Let* $\{C(\sigma)\}_{\sigma\in\mathbb{R}}$ *be a strongly cosine family, Then, we call the family* $\{S(\sigma)\}_{\sigma\in\mathbb{R}}$, *where*

$$S(\sigma)x = \int_0^\sigma C(\vartheta)xd\vartheta, \tag{8}$$

*a strongly continuous sine family correlated with* $\{C(\sigma)\}_{\sigma\in\mathbb{R}}$.

**Lemma 5** ([51]). *Let* $\{C(\sigma)\}_{\sigma\in\mathbb{R}}$ *be a strongly cosine family on E. Then, the following are true.*
    1- $S(0) = 0, C(\sigma) = C(-\sigma), S(\sigma) = -S(-\sigma)$ *for all* $\sigma\in\mathbb{R}$;
    2- $C(\sigma), C(\vartheta), S(\sigma)$ *and* $S(\vartheta)$ *are commute for all* $\sigma,\vartheta\in\mathbb{R}$;
    3- *For any* $x\in E, \sigma\longmapsto C(\sigma)x$ *is continuous;*
    4- $S(\sigma+\vartheta) + S(\sigma-\vartheta) = 2S(\sigma)C(\vartheta)$ *for all* $\sigma,\vartheta\in\mathbb{R}$;
    5- $S(\sigma+\vartheta) = S(\sigma)C(\vartheta) + C(\sigma)S(\vartheta)$ *for all* $\sigma,\vartheta\in\mathbb{R}$;
    6- *There are positive constants* $M_0$ *and* $\varsigma$ *such that* $||C(\sigma)|| \le M_0 e^{\varsigma|\sigma|}$ *for all* $\sigma\in\mathbb{R}$ *and*

$$||S(\sigma)x - S(\vartheta)x|| \le M_0 |\int_\vartheta^\sigma e^{\varsigma|\tau|}xd\tau|, \text{ for all } \sigma,\vartheta\in\mathbb{R}.$$

**Definition 7** ([51]). *The infinitesimal generator of a strongly cosine family* $\{C(\sigma)\}_{\sigma\in\mathbb{R}}$ *is given by*

$$Ax = \frac{d^2}{d\sigma^2}C(\sigma)x|_{x=0},$$

*where* $D(A) = \{x\in E : C(\sigma)x \text{ is twice continuously differentiable of } \sigma\}$.

**Lemma 6.** *Let A be the infinitesimal generator of a strongly continuous cosine family* $\{C(\sigma)\}_{\sigma\in\mathbb{R}}$. *Then, for* $\lambda$ *with* $\text{Rel }\lambda > \varsigma$, $\lambda^2$ *belongs to the resolvent set of A,* $\lambda(\lambda^2-A)^{-1}u = \int_0^\infty e^{-\lambda\sigma}C(\sigma)u\,d\sigma$ *and* $(\lambda^2-A)^{-1}u = \int_0^\infty e^{-\lambda\sigma}S(\sigma)u\,d\sigma$ ; $u\in E$, *where* $\varsigma$ *is defined in the sixth item 6 of Lemma 5.*

**Definition 8** ([50]). *The* $w-$ *weighted* $\Phi-$*Laplace transform for a function* $f : [a,\infty) \to E$ *is given by*

$$\{L_{\Phi,w}f(.)\}(\lambda) := \int_a^\infty e^{-\lambda(\Phi(\sigma)-\Phi(0))}w(\sigma)f(\sigma)\Phi'(\sigma)d\sigma, \lambda > 0. \tag{9}$$

**Definition 9** ([50], Definition 5.9). *The* $w-$ *weighted* $\Phi-$*convolution of functions* $f : [a,\infty) \to E$ *and* $h : [a,\infty) \to \mathbb{R}$ *is given by*

$$(h *_{\Phi,w} f)\sigma$$
$$= w^{-1}(\sigma)\int_a^\sigma w(\Phi^{-1}(\Phi(\sigma)-\Phi(a)-\Phi(\vartheta)))h(\Phi^{-1}(\Phi(\sigma)-\Phi(a)-\Phi(\vartheta))w(\vartheta)f(\vartheta)\Phi'(\vartheta)d\vartheta.$$

where $\Phi^{-1}$ is the inverse function of $\Phi$; that is, $\sigma = \Phi^{-1}(\Phi(\sigma) - \Phi(a) - \Phi(\vartheta)) \Longleftrightarrow \Phi(\sigma) = \Phi(\sigma) - \Phi(a) - \Phi(\vartheta)$.

**Definition 10** ([50]). *A function $f : [0, \infty) \to E$ is considered to be a w-weighted $\Phi$-exponential function if there are positive constants $L, C, K$ such that $||w(x)f(x)|| \leq Le^{C\Phi(x)}, \forall x > K$.*

**Lemma 7** ([50], Proposition 5.2, Remark 5.8, Theorem 5.9 and Corollary 5.11). *Let $a \geq 0$.*
1- $f *_{\Phi,w} h = h *_{\Phi,w} f$.
2-

$$\{L_{\Phi,w}w^{-1}(.)((\Phi(.) - \Phi(a))^{\beta-1}\}(\lambda) = \frac{\Gamma(\beta)}{\lambda^\beta}; \beta > -1. \tag{10}$$

3- *If $\alpha > 0$ and $x : [a, \infty] \to E$ is a piecewise continuous function on each interval $[0, \sigma]$ and $w-$weighted $\Phi-$exponential, then*

$$\{L_{\Phi,w}(I_{a,\sigma}^{\alpha,\Phi,w}x)\}(\lambda) := \frac{\{L_{\Phi,w}x\}(\lambda)}{\lambda^\alpha}; \lambda > 0. \tag{11}$$

4- *If the $w-$weighted $\Phi-$Laplace transform of $f$ and $h$ exist for $\lambda > 0$, then*

$$\{L_{\Phi,w}(f *_{\Phi,w} h)\}(\lambda) = \{L_{\Phi,w}f\}(\lambda) \times \{L_{\Phi,w}h\}(\lambda). \tag{12}$$

5- *Let $\alpha \in (n-1, n)$. If $(D_{a,\sigma}^{\alpha,\Phi,w}f)(\sigma)$ is well defined for almost $a > 0$, then*

$$\{L_{\Phi,w}(D_{a,\sigma}^{\alpha,\Phi,w}f(\sigma))\}(\lambda) = \lambda^\alpha\{L_{\Phi,w}(f(\sigma)\}(\lambda) - \lim_{\sigma \to 0}\sum_{k=0}^{n-1}(\frac{d}{\Phi'(x)dx})^k w(x)I_{a,\sigma}^{n-\alpha,\Phi,w}f(a^+). \tag{13}$$

**Lemma 8** ([52], Corollary 3.3.1.). *Assume that $U$ is a not empty, closed and convex subset of $E$ and $Y : U \to P_{ck}(E)$ is $\chi-$condensing and with a closed graph, where $\chi$ is a non-singular measure of noncompactness defined on subsets of $U$. Then, $Y$ has a fixed point.*

**Lemma 9** ([52], Propostion 3.5.1.). *Assume that $U$ is as in Lemma 8 and $Y : U \to P_{ck}(E)$ is $\chi-$condensing on all bounded subsets of $U$, where $\chi$ is a monotone measure of noncompactness defined on $E$. If $Y$ has a closed graph, and the fixed-points set for $Y$, $Fix(Y)$, is a bounded subset of $E$, then it is compact.*

### 3. Non-Emptiness and Compactness of the Mild Solution Set for (1)

Firstly, by using the properties of the $w-$weighted $\Phi-$Laplace transform, we derive the formula of a mild solution function for Problem (1). For this purpose, we consider the following $w$-weighted semilinear differential equation of order $\mu$ and of type $v$ :

$$\begin{cases} D_{0,\sigma}^{\mu,v,\Phi,w}x(\sigma) = Ax(\sigma) + f(\sigma, x(\sigma)), \sigma \in (0, b], \\ \lim_{\sigma \to 0^+} I_{0,\sigma}^{2-\gamma,\Phi,w}w(\sigma)x(\sigma) = x_0, \\ \lim_{\sigma \to 0^+} \frac{1}{\Phi'(\sigma)}\frac{d}{d\sigma}(I_{0,\sigma}^{2-\gamma,\Phi,w}w(\sigma)x(\sigma) = x_1, \end{cases} \tag{14}$$

where $f : \Im \times E \to E$. By applying the operator $I_{0,\sigma}^{\mu,\Phi,w}$ on both sides of Equation (14) and using (7), we have for $\sigma \in (0, b]$

$$x(\sigma) = x_0 w^{-1}(\sigma)\frac{(\Phi(\sigma) - \Phi(0))^{\gamma-2}}{\Gamma(\gamma-1)} + x_1 w^{-1}(\sigma)\frac{(\Phi(\sigma) - \Phi(0))^{\gamma-1}}{\Gamma(\gamma)}$$
$$+ I_{0,\sigma}^{\mu,\Phi,w}Ax(\sigma) + I_{0,\sigma}^{\mu,\Phi,w}f(\sigma, x(\sigma)). \tag{15}$$

Equation (15) is called the corresponding integral equation for Problem (14). In the next lemma, we obtain the formula of a mild solution function for (14).

**Lemma 10.** *Assume that $A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(\sigma)\}_{\sigma \in \mathbb{R}}$ and $x \in L_w^{p,\Phi}(\Im, E)$ such that $I_{0,\sigma}^{2-\gamma,\Phi,w}x \in C_{2-\gamma,\Phi,w}^2([0,b], E)$. If $x$ satisfies (15), then, for $\sigma \in (0, b]$,*

$$x(\sigma) = D_{0,\sigma}^{2\delta-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma)$$

$$+ \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f(\vartheta, x(\vartheta))w(\vartheta)\Phi'(\vartheta)d\vartheta, \tag{16}$$

*where* $\sigma \in (0,b], \delta = \frac{\mu}{2}$, $Z_1(\sigma) = \frac{w^{-1}(\sigma)(\Phi(\sigma)-\Phi(0))^{\gamma-2\delta-1}}{\Gamma(\gamma-2\delta)}$, $Z_2(\sigma) = (\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\Phi,\delta,w}(\sigma,0)x_1$, $Z_3(\sigma) = (\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\Phi,\delta,w}(\sigma,0)x_0$,

$$K^{\delta,\Phi,w}(\sigma,\vartheta)u = \frac{1}{w(\sigma)} \int_0^\infty \delta\theta\xi_\delta(\theta)\vartheta((\Phi(\sigma) - \Phi(\vartheta))^\delta\theta)ud\theta; \sigma \in (0,\infty), 0 \le \vartheta \le \sigma, \tag{17}$$

$\xi_\delta(\theta) = \frac{1}{\delta}\theta^{-1-\frac{1}{\delta}}\rho_\delta(\theta^{-\frac{1}{\delta}})$ *and* $\rho_\delta(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\delta n-1}\frac{\Gamma(n\,\delta+1)}{n!}\sin(n\pi\delta), \sigma \in (0,\infty)$.

**Proof.** Let $\lambda > 0$ be such that $Rel\,\lambda^\delta > \varsigma$, where $\varsigma$ is given as in the sixth item of Lemma 5. Then, using Lemma 6, $(\lambda^{2\delta} - A)^{-1}$ is well defined and

$$(\lambda^{2\delta} - A)^{-1}u = \int_0^\infty e^{-\lambda^\delta\sigma}\vartheta(\sigma)u\,d\sigma\,; u \in E. \tag{18}$$

Applying the generalized $w-$weighted $\Phi-$Laplace transform, defined by (9), on both sides of Equation (15) and using (10) and (11), it follows that

$$\{L_{\Phi,w}x(.)\}(\lambda)$$

$$= \{L_{\Phi,w}x_0w^{-1}(.)\frac{(\Phi(.) - \Phi(0))^{\gamma-2}}{\Gamma(\gamma-1)}\}(\lambda)$$

$$+ \{L_{\Phi,w}x_1w^{-1}(.)\frac{(\Phi(.) - \Phi(0))^{\gamma-1}}{\Gamma(\gamma)}\}(\lambda)$$

$$+ \{L_{\Phi,w}I_{0,\sigma}^{\mu,\Phi,w}Ax(.)\}(\lambda) + \{L_{\Phi,w}I_{0,\sigma}^{\mu,\Phi,w}f(\sigma, x(.))\}(\lambda). \tag{19}$$

Set

$$X(\lambda) = \{L_{\Phi,w}^0 x\}(\lambda),$$

and

$$F(\lambda) = \{L_{\Phi,w}^0 f(., x(.))\}(\lambda).$$

Then, (19) becomes

$$X(\lambda) = \frac{x_0}{\lambda^{\gamma-1}} + \frac{x_1}{\lambda^\gamma} + \frac{AX(\lambda)}{\lambda^\mu} + \frac{F(\lambda)}{\lambda^\mu}$$

$$= \frac{x_0}{\lambda^{\gamma-1}} + \frac{x_1}{\lambda^\gamma} + \frac{AX(\lambda)}{\lambda^{2\delta}} + \frac{F(\lambda)}{\lambda^{2\delta}}. \tag{20}$$

From (18), it yields

$$X(\lambda) = \lambda^{2\delta-\gamma+1}(\lambda^{2\delta} - A)^{-1}x_0 + \lambda^{2\delta-\gamma}(\lambda^{2\delta} - A)^{-1}x_1$$

$$+ (\lambda^{2\delta} - A)^{-1}F(\lambda)$$

$$= \lambda^{2\delta-\gamma+1}\int_0^\infty e^{-\lambda^\delta\sigma}S(\sigma)x_0\,d\sigma + \lambda^{2\delta-\gamma}\int_0^\infty e^{-\lambda^\delta\sigma}S(\sigma)x_1\,d\sigma$$

$$+ \int_0^\infty e^{-\lambda^\delta\sigma}S(\sigma)\,F(\lambda)d\sigma$$

$$= I_1 + I_2 + I_3. \tag{21}$$

In $I_2$, replacing $\sigma$ with $(\Phi(\vartheta) - \Phi(0))^\delta$, we obtain

$$I_2 = \lambda^{2\delta - \gamma} \int_0^\infty \delta(\Phi(\vartheta) - \Phi(0))^{\delta - 1} e^{-(\lambda(\Phi(\vartheta) - \Phi(0))^\delta} S((\Phi(\vartheta) - \Phi(0))^\delta) \Phi'(\vartheta) x_1 d\vartheta. \quad (22)$$

Because $\delta \in (\frac{1}{2}, 1)$, then [53]

$$e^{-(\lambda(\Phi(\vartheta) - \Phi(0))^\delta} = \int_0^\infty e^{-\lambda\theta(\Phi(\vartheta) - \Phi(0))} \rho_\delta(\theta) d\theta. \quad (23)$$

Equations (22) and (23) imply that

$$\begin{aligned}
I_2 &= \lambda^{2\delta - \gamma} \int_0^\infty \int_0^\infty \delta(\Phi(\vartheta) - \Phi(0))^{\delta - 1} e^{-\lambda\theta(\Phi(\vartheta) - \Phi(0))} \times \\
&\quad \rho_\delta(\theta) S((\Phi(\vartheta) - \Phi(0))^\delta) \Phi'(\vartheta) x_1 d\theta d\vartheta.
\end{aligned}$$

Replacing $\theta(\Phi(\vartheta) - \Phi(0))$ with $\Phi(\tau) - \Phi(0)$,

$$\begin{aligned}
I_2 &= \lambda^{2\delta - \gamma} \int_0^\infty \int_0^\infty \delta \frac{(\Phi(\tau) - \Phi(0))^{\delta - 1}}{\theta^\delta} e^{-\lambda(\Phi(\tau) - \Phi(0))} \rho_\delta(\theta) \times \\
&\quad S\left(\frac{(\Phi(\tau) - \Phi(0))^\delta}{\theta^\delta}\right) \Phi'(\tau) x_1 d\theta d\tau.
\end{aligned}$$

Replacing $\theta$ with $\sigma^{\frac{-1}{\delta}}$ and using (11), (12) and (17), we obtain

$$\begin{aligned}
I_2 &= \lambda^{2\delta - \gamma} \int_0^\infty e^{-\lambda(\Phi(\tau) - \Phi(0))} (\Phi(\tau) - \Phi(0))^{\delta - 1} \times \\
&\quad \int_0^\infty \sigma(\sigma^{\frac{-1}{\delta} - 1} \rho_\delta(\sigma^{\frac{-1}{\delta}})) S((\Phi(\tau) - \Phi(0))^\delta \sigma) \Phi'(\tau) x_1 d\sigma d\tau \\
&= \lambda^{2\delta - \gamma} \int_0^\infty e^{-\lambda(\Phi(\tau) - \Phi(0))} (\Phi(\tau) - \Phi(0))^{\delta - 1} \Phi'(\tau) \times \\
&\quad \left(\int_0^\infty \delta\sigma\xi_\delta(\sigma) S((\Phi(\tau) - \Phi(0))^\delta \sigma) d\sigma\right) x_1 d\tau \\
&= \lambda^{2\delta - \gamma} \int_0^\infty e^{-\lambda(\Phi(\tau) - \Phi(0))} (\Phi(\tau) - \Phi(0))^{\delta - 1} \Phi'(\tau) w(\tau) K^{\delta, \Phi, w}(\tau, 0) x_1 d\tau.
\end{aligned}$$

Since, from (10),

$$L_{\Phi, w}(w^{-1}(\sigma)(\Phi(\sigma) - \Phi(0))^{\gamma - 2\delta - 1} = \lambda^{2\delta - \gamma} \Gamma(\gamma - 2\delta),$$

it yields,

$$\begin{aligned}
I_2 &= L_{\Phi, w}(Z_1)(\lambda) \times \{L_{\Phi, w}(Z_2)(\lambda) \\
&= L_{\Phi, w}(Z_1 *_{\Phi, w} Z_2).
\end{aligned} \quad (24)$$

Similarly,

$$\begin{aligned}
I_1 &= \lambda^{2\delta - \gamma + 1} \int_0^\infty e^{-\lambda(\Phi(\tau) - \Phi(0))} (\Phi(\tau) - \Phi(0))^{\delta - 1} \Phi'(\tau) w(\tau) K^{\Phi, \delta}(\tau, 0) x_0 d\tau \\
&= \lambda^{2\delta - \gamma + 1} \{L_{\Phi, w}((\Phi(\tau) - \Phi(0))^{\delta - 1}\}(\lambda).
\end{aligned}$$

Since $\lim_{\sigma\to 0} I^{1-(2\delta-\gamma+1),\Phi,w}[(\Phi(\sigma)-\Phi(0))^{\delta-1}K^{\Phi,\delta,w}(\sigma,0)x_0] = 0$ and $1-(2\delta-\gamma+1) \in (0,1)$, by (13), we have

$$I_1 = \lambda^{2\delta-\gamma+1}\{L_{\Phi,w}(Z_3)\}(\lambda) = \{L_{\Phi,w}\,(D_{0,\sigma}^{2\delta-\gamma+1,\Phi,w}Z_3\}(\lambda). \tag{25}$$

For $I_3$, by arguing as in (21)–(23), we can arrive at

$$I_3 = \int_0^\infty \int_0^\infty \delta \frac{(\Phi(\tau)-\Phi(0))^{\delta-1}}{\theta^\delta} e^{-\lambda(\Phi(\tau)-\Phi(0))} \rho_\delta(\theta) \times$$

$$S(\frac{(\Phi(\tau)-\Phi(0))^\delta}{\theta^\delta})\Phi'(\tau)F(\lambda)d\theta d\tau$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \delta \frac{(\Phi(\tau)-\Phi(0))^{\delta-1}}{\theta^\delta} e^{-\lambda(\Phi(\tau)-\Phi(0))} \rho_\delta(\theta) \times$$

$$S(\frac{(\Phi(\tau)-\Phi(0))^\delta}{\theta^\delta})e^{-\lambda(\Phi(\kappa)-\Phi(0))}f(\kappa,x(\kappa))w(\kappa)\Phi'(\kappa)\Phi'(\tau)d\kappa d\theta d\tau$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \delta \frac{(\Phi(\tau)-\Phi(0))^{\delta-1}}{\theta^\delta} e^{-\lambda(\Phi(\tau)+\Phi(\kappa)-\Phi(0))} \rho_\delta(\theta) \times$$

$$S(\frac{(\Phi(\tau)-\Phi(0))^\delta}{\theta^\delta})f(\kappa,x(\kappa))w(\kappa)\Phi'(\kappa)\Phi'(\tau)d\kappa d\theta d\tau. \tag{26}$$

Replacing, in (26), $\Phi(\kappa)$ with $\Phi(\sigma)-\Phi(\tau)+\Phi(0)$, one can obtain

$$I_3 = \int_0^\infty \int_\tau^\infty \int_0^\infty \delta \frac{(\Phi(\tau)-\Phi(0))^{\delta-1}}{\theta^\gamma} e^{-(\lambda(\Phi(\sigma)-\Phi(0))} \rho_\delta(\theta) \times$$

$$S((\frac{\Phi(\tau)-\Phi(0)}{\theta})^\delta)\Phi'(\tau) \times$$

$$f(\Phi^{-1}(\Phi(\sigma)-\Phi(\tau)+\Phi(0)), x(\Phi^{-1}(\Phi(\sigma)-\Phi(\tau)+\Phi(0)))) \times$$

$$w(\Phi^{-1}(\Phi(\sigma)-\Phi(\tau)+\Phi(0)))\Phi'(\sigma)d\theta d\sigma d\tau$$

$$= \int_0^\infty e^{-(\lambda(\Phi(\sigma)-\Phi(0))} \int_0^\sigma \int_0^\infty \delta \frac{(\Phi(\tau)-\Phi(0))^{\delta-1}}{\theta^\gamma} \rho_\delta(\theta) \times$$

$$S((\frac{\Phi(\tau)-\Phi(0)}{\theta})^\delta)\Phi'(\sigma)\Phi'(\tau) \times$$

$$f(\Phi^{-1}(\Phi(\sigma)-\Phi(\tau)+\Phi(0)), x(\Phi^{-1}(\Phi(\sigma)-\Phi(\tau)+\Phi(0)))) \times$$

$$w(\Phi^{-1}(\Phi(\sigma)-\Phi(\tau)+\Phi(0)))d\theta d\tau d\sigma. \tag{27}$$

Replacing $\Phi(\tau)$, in (27), with $\Phi(\sigma)-\Phi(\vartheta)+\Phi(0)$, we obtain:

$$I_3 = \int_0^\infty e^{-(\lambda(\Phi(\sigma)-\Phi(0))} \int_0^\sigma \int_0^\infty \gamma \frac{(\Phi(\sigma)-\Phi(\vartheta))^{\delta-1}}{\theta^\gamma} \rho_\delta(\theta) \times$$

$$S((\frac{\Phi(\sigma)-\Phi(\vartheta)}{\theta})^\delta)f(\vartheta,x(\vartheta))w(\vartheta)\Phi'(\vartheta)\Phi'(\sigma)d\theta d\vartheta d\sigma.$$

Replacing $\theta$ with $\theta^{\frac{-1}{\delta}}$ and using (17), we get

$$
\begin{aligned}
I_3 &= \int_0^\infty e^{-(\lambda(\Phi(\sigma)-\Phi(0)))} \int_0^\sigma \int_0^\infty \theta(\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} e^{-\lambda\Phi(\sigma)} \rho_\delta(\theta^{\frac{-1}{\delta}})\theta^{\frac{-1}{\delta}-1} \times \\
&\quad S((\Phi(\sigma)-\Phi(\vartheta))^\delta\theta) f(\vartheta,x(\vartheta))w(\vartheta)\Phi'(\vartheta)\Phi'(\sigma)d\theta d\vartheta d\sigma \\
&= \int_0^\infty e^{-(\lambda(\Phi(\sigma)-\Phi(0)))} \int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} [\int_0^\infty \delta\theta\xi_\delta(\theta) \times \\
&\quad S((\Phi(\sigma)-\Phi(\vartheta))^\delta\theta) f(\vartheta,x(\vartheta))d\theta]w(\vartheta)\Phi'(\vartheta)\Phi'(\sigma)d\vartheta d\sigma \\
&= \int_0^\infty e^{-(\lambda(\Phi(\sigma)-\Phi(0)))} w(\sigma) [\int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta) \times \\
&\quad f(\vartheta,x(\vartheta))w(\vartheta)\Phi'(\vartheta)d\vartheta]\Phi'(\sigma)d\sigma \\
&= L_{\Phi,w}\{Z_4\}(\lambda),
\end{aligned}
\tag{28}
$$

where $Z_4(\sigma) = \int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} w(\vartheta) K^{\delta,\Phi,w}(\sigma,\vartheta) f(\vartheta,x(\vartheta))\Phi'(\vartheta)d\vartheta$.

Equations (20), (25), (26) and (28) give us

$$
\begin{aligned}
\{L_{\Phi,w}x\}(\lambda) &= \{L_{\Phi,w} (D_{0,t}^{2\delta-\gamma+1,\Phi,w} Z_2\}(\lambda) + L_{\Phi,w}(Z_2 *_{\Phi,w} Z_1\}(\lambda) \\
&\quad + +L_{\Phi,w}\{Z_4\}(\lambda).
\end{aligned}
$$

By utilizing the $w-$weighted $\Phi-$Laplace transform, we obtain, for any $\sigma \in (0,b]$,

$$
\begin{aligned}
x(\sigma) &= D_{0,\sigma}^{2\delta-\gamma+1,\Phi,w} Z_3(\sigma) + (Z_1 *_{\Phi,w} Z_2)(\sigma) + Z_4(\sigma) \\
&= D_{0,\sigma}^{2\delta-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\sigma,0)x_0) + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\
&\quad + \int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta) f(\vartheta,x(\vartheta))w(\vartheta)\Phi'(\vartheta)d\vartheta.
\end{aligned}
$$

So, (16) is satisfied. $\square$

As a consequence of Lemma 10, we obtain the next definitions:

**Definition 11.** *A function $x \in C_{2-\gamma,\Phi,w}(\Im,E)$ is called a mild solution for Problem (14) when it satisfies the next fractional integral equation:*

$$
\begin{aligned}
x(\sigma) &= D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)x_0) + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\
&\quad + \int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta) f(\vartheta,x(\vartheta))\Phi'(\vartheta)w(\vartheta)d\vartheta.
\end{aligned}
\tag{29}
$$

**Definition 12.** *A function $x \in PC_{2-\gamma,\Phi,w}(\Im,E)$ is called a mild solution for Problem (1) if it satisfies the following fractional integral equation:*

$$
x(\sigma) = \begin{cases}
D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\
\quad + \int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta) f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_0, \\
g_i(\sigma,x(\sigma_i^-)), \sigma \in \sigma_i, i \in N_1, \\
D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(\vartheta_i))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta_i))g_i(\vartheta_i,x(\vartheta_i^-)) \\
\quad + (Z_{5,i} *_{\Phi,w} Z_{6,i}^x)(\sigma) \\
\quad + \int_{\vartheta_i}^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta) f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_i, i \in N_2,
\end{cases}
\tag{30}
$$

where $\Im_i = (\vartheta_i,\sigma_{i+1}]; i \in N_0$, $T_i = (\sigma_i,\vartheta_i]; i \in N_1$, $Z_1(\sigma) = \frac{w^{-1}(\sigma)(\Phi(\sigma)-\Phi(0))^{\gamma-2\delta-1}}{\Gamma(\gamma-2\delta)}$, $Z_2(\sigma) = (\Phi(\sigma)-\Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\sigma,0)x_1$, $Z_{5,i}(\sigma) = \frac{w^{-1}(\sigma)(\Phi(\sigma)-\Phi(\vartheta_i))^{\gamma-2\delta-1}}{\Gamma(\gamma-2\delta)}$, $Z_{6,i}^x(\sigma) = (\Phi(\sigma)-\Phi(\vartheta_i))^{\delta-1} K^{\Phi,\delta,w}(\sigma,\vartheta_i)g_i^*(\vartheta_i,x(\vartheta_i^-))$, $\delta = \frac{\mu}{2}$ and $f \in L^1((0,b),E)$ with $f(\vartheta) \in F(\vartheta,x(\vartheta)), a.e.$

**Corollary 1.** *If $\sigma_i = \vartheta_i = b, \forall i \in N_1, w(\sigma) = 1$ and $\Phi(\sigma) = \sigma, \forall \sigma \in \Im$, then the mild solution function of the following problem:*

$$\begin{cases} D_{0,\sigma}^{\mu,v} x(\sigma) \in Ax(\sigma) + F(\sigma, x(\sigma)), a.e., \sigma \in (0, b] \\ \lim_{\sigma \to 0^+} I_{0,\sigma}^{2-\gamma} x(\sigma) = x_0, \ \lim_{\sigma \to 0^+} \frac{d}{d\sigma}(I_{0,\sigma}^{2-\gamma} x(\sigma)) = x_1 \end{cases} \quad (31)$$

*becomes*

$$x(\sigma) = \begin{cases} D_{0,\sigma}^{\mu-\gamma+1}(\sigma^{\delta-1} K^\delta(\sigma, 0)x_0) + (Z_1 * Z_2)(\sigma) \\ + \int_0^\sigma (\sigma - \vartheta)^{\delta-1} K^\delta(\sigma, \vartheta)f(\vartheta)d\vartheta, \sigma \in (0, b], \end{cases} \quad (32)$$

*where $\delta = \frac{\mu}{2}$,*

$$K^\delta(\sigma, \vartheta)u = \int_0^\infty \delta\theta \xi_\delta(\theta)\vartheta(\sigma - \vartheta)^\delta\theta)ud\theta, \quad (33)$$

$K^\delta(\sigma, \vartheta)u = \int_0^\infty \delta\theta \xi_\delta(\theta)S(\sigma - \vartheta)^\delta\theta)ud\theta$ *and*

$$(Z_1 * Z_2)(\sigma) = \frac{1}{\Gamma(\gamma - 2\delta)} \int_0^\sigma (\sigma - \vartheta)^{\gamma-2\delta-1}\vartheta^{\delta-1} K^\delta(\vartheta, 0)x_1 = I_{0,\sigma}^{\gamma-2\delta}\sigma^{\delta-1} K^\delta(\sigma, 0)x_1, \quad (34)$$

*and this coincides with Definition (8) in [33].*

The next lemma illustrates some properties of $K^\delta(.,.)$.

**Lemma 11.** *Suppose that the operator $A$ satisfies the next condition:*
*(A) $A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(\sigma)\}_{\sigma \in \mathbb{R}}$, and there is $M > 0$ such that $\sup_{\sigma \geq 0}||C(\sigma)|| \leq M$.*
*Then,*
*1- For every $\sigma > 0$, every $\vartheta \in (0, \sigma)$ and every $u \in E$,*

$$w(\sigma)||K^{\delta,\Phi,w}(\sigma, \vartheta)u|| \leq \frac{M(\Phi(\sigma) - \Phi(\vartheta))^\delta||u||}{\Gamma(2\delta)}. \quad (35)$$

*2- For any $\sigma_2 \geq \sigma_1 \geq \vartheta$,*

$$\lim_{\sigma_2 \to \sigma_1} ||w(\sigma_2)K^{\delta,\Phi,w}(\sigma_2, \vartheta) - w(\sigma_1)K^{\delta,\Phi,w}(\sigma_1, \vartheta)|| = 0. \quad (36)$$

**Proof.** 1- Since $\sup_{\sigma \geq 0}||C(\sigma)|| \leq M$, it yields

$$w(\sigma)||K^{\delta,\Phi,w}(\sigma, \vartheta)u|| \leq \int_0^\infty \delta\theta \xi_\delta(\theta)||S((\Phi(\sigma) - \Phi(\vartheta))^\delta\theta))||ud\theta$$

$$\leq \int_0^\infty \delta\theta^2 \xi_\delta(\theta)M(\Phi(\sigma) - \Phi(\vartheta))^\delta||u||d\theta$$

$$= M(\Phi(\sigma) - \Phi(\vartheta))^\delta||u||\delta \int_0^\infty \theta^2 \xi_\delta(\theta)d\theta.$$

Since $\int_0^\infty \theta^2 \xi_\delta(\theta)d\theta = \frac{\Gamma(3)}{\Gamma(1+2\delta)}$, it follows that

$$w(\sigma)||K^{\delta,\Phi,w}(\sigma, \vartheta)u|| \leq M(\Phi(\sigma) - \Phi(\vartheta))^\delta||u||\delta\frac{\Gamma(3)}{\Gamma(1 + 2\delta)}$$

$$= \frac{M(\Phi(\sigma) - \Phi(\vartheta))^\delta||u||}{\Gamma(2\delta)}.$$

2- For any $u \in E$ with $||u|| \leq 1$, we have

$$\lim_{\sigma_2 \to \sigma_1} ||w(\sigma_2)(K^{\delta,\Phi,w}(\sigma_2,\vartheta)u - w(\sigma_1)(K^{\delta,\Phi,w}(\sigma_1,\vartheta)u||$$

$$\leq \lim_{\sigma_2 \to \sigma_1} \int_0^\infty \delta\theta\xi_\delta(\theta)||S((\Phi(\sigma_2) - \Phi(\vartheta))^\delta\theta) - S((\Phi(\sigma_1) - \Phi(\vartheta))^\delta\theta)||ud\theta$$

$$= \lim_{\sigma_2 \to \sigma_1} M|(\Phi(\sigma_2) - \Phi(\vartheta))^\delta\theta) - S((\Phi(\sigma_1) - \Phi(\vartheta))^\delta\theta)|||u||\delta\int_0^\infty \theta^2\xi_\delta(\theta)d\theta$$

$$\leq \lim_{\sigma_2 \to \sigma_1} \frac{M|(\Phi(\sigma_2) - \Phi(\vartheta))^\delta\theta) - S((\Phi(\sigma_1) - \Phi(\vartheta))^\delta\theta)|}{\Gamma(2\delta)}$$

$$= 0.$$

$\square$

**Lemma 12.** *For every $u \in E$ and every $\sigma \in (\vartheta_i, \sigma_{i+1}], i \in N_0$*

$$\frac{d}{d\sigma}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta_i)u)$$
$$= (\delta - 1)(\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-2}\Phi(\sigma)w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta_i)u$$
$$+ \delta^2(\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)\int_0^\infty \theta^2\xi_\delta(\theta)C((\Phi(\sigma) - \Phi(\vartheta_i))^\delta\theta)\xi_\delta(\theta)ud\theta, \quad (37)$$

*and*

$$||\frac{d}{d\sigma}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta_i)u)||$$
$$\leq (\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)\frac{M||u||}{\Gamma(2\delta)}. \quad (38)$$

**Proof.**

$$\frac{d}{d\sigma}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta_i)u)$$
$$= (\delta - 1)(\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-2}\Phi'w(\sigma)K^{\delta,\Phi,w}(\sigma,\Phi)u$$
$$+ (\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}\frac{d}{d\sigma}(\int_0^\infty \delta\theta\xi_\delta(\theta)S((\Phi(\sigma) - \Phi(\vartheta_i))^\delta\theta)ud\theta. \quad (39)$$

However,

$$\frac{d}{d\sigma}(\int_0^\infty \delta\theta\xi_\delta(\theta)S((\Phi(\sigma) - \Phi(\vartheta_i))^\delta\theta)ud\theta$$
$$= \int_0^\infty \delta\theta\xi_\delta(\theta)\frac{d}{d\sigma}[S((\Phi(\sigma) - \Phi(\vartheta_i))^\delta\theta]ud\theta$$
$$= \int_0^\infty \delta\theta\xi_\delta(\theta)\delta(\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}\theta\Phi(\sigma)C((\Phi(\sigma) - \Phi(\vartheta_i))^\delta\theta)ud\theta$$
$$= \delta^2(\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}\Phi(\sigma)\int_0^\infty C((\Phi(\sigma) - \Phi(0)^\delta\theta)\theta^2\xi_\delta(\theta)ud\theta. \quad (40)$$

Then, Equation (37) is yielded from (39) and (41). Now, by (35), (37) and assumption (A), we obtain

$$||\frac{d}{d\sigma}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta_i)u)||$$

$$\leq ||(\delta-1)(\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-2}\Phi(\sigma)w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta_i)u||$$

$$+ ||(\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)\int_0^\infty \delta^2\theta^2\xi_\delta(\theta)C((\Phi(\sigma) - \Phi(\vartheta_i))^\delta\theta)ud\theta||$$

$$\leq (1-\delta)(\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-2}\Phi(\sigma)\frac{M(\Phi(\sigma) - \Phi(\vartheta_i))^\delta||u||}{\Gamma(2\delta)}$$

$$+ (\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)M\,||u||\int_0^\infty \delta^2\theta^2\xi_\delta(\theta)d\theta$$

$$= (1-\delta)(\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)\frac{M||u||}{\Gamma(2\delta)}$$

$$+ \frac{\delta((\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)M||u||}{\Gamma(2\delta)}$$

$$= (\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2}\Phi(\sigma)\frac{M||u||}{\Gamma(2\delta)}.$$

So, (38) holds. □

**Lemma 13.** *If $\sigma \in \sigma \in (\vartheta_i, \sigma_{i+1}], i \in N_0$, then*

$$w(\sigma)((\Phi(\sigma) - \Phi(\vartheta_i))^{2-\gamma}||D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta_i))x_0)||$$

$$\leq \frac{M||x_0||}{(2\delta-1)\Gamma(\gamma-1)}. \tag{41}$$

**Proof.** In view of ($i$) of Lemma 3, (ii) of Lemma 4 and (38), we have

$$||D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta_i))x_0)||$$

$$= ||\frac{w^{-1}(\sigma)}{\Gamma(\gamma-\mu)}\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\gamma-\mu-1}\frac{d}{d\vartheta}[(\Phi(\vartheta) - \Phi(\vartheta_i))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\vartheta,\vartheta_i)x_0]d\vartheta||$$

$$\leq \frac{w^{-1}(\sigma)}{\Gamma(\gamma-\mu)}\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\gamma-\mu-1}||\frac{d}{d\vartheta}[(\Phi(\vartheta) - \Phi(\vartheta_i))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\vartheta,0)x_0]||d\vartheta$$

$$\leq \frac{w^{-1}(\sigma)}{\Gamma(\gamma-\mu)}\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\gamma-\mu-1}\Phi(\vartheta)\frac{(\Phi(\vartheta) - \Phi(\vartheta_i))^{2\delta-2}M||x_0||}{\Gamma(2\delta)}d\vartheta$$

$$= \frac{M||x_0||w^{-1}(\sigma)}{\Gamma(\gamma-\mu)\Gamma(2\delta)}\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\gamma-\mu-1}\Phi(\vartheta)(\Phi(\vartheta) - \Phi(\vartheta_i))^{2\delta-2}d\vartheta$$

$$= \frac{M||x_0||}{\Gamma(2\delta)}I_{0,\sigma}^{\gamma-\mu,\Phi,w}(w^{-1}(\sigma)(\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2})$$

$$= \frac{M||x_0||\Gamma(2\delta-1)}{\Gamma(2\delta)\Gamma(2\delta-1+\gamma-\mu)}w^{-1}(\sigma)(\Phi(\sigma) - \Phi(\vartheta_i))^{2\delta-2+\gamma-\mu}$$

$$= \frac{M||x_0||}{(2\delta-1)\Gamma(\gamma-1)}w^{-1}(\sigma)(\Phi(\sigma) - \Phi(\vartheta_i))^{\gamma-2}.$$

So, (41) is true. □

In the next theorem, we demonstrate that the mild solution set for Problem (4) is not empty or compact.

In addition to condition ($A$), assume $F : \Im \times E \to P_{ck}(E)$ to be such that:

($F_1$) For any $u \in E$, $F(.,u)$ is measurable and for almost $\sigma \in \cup_{k \in N_1}\Im_k$, $F(\sigma,.)$ upper semicontinuous.

($F_2$) There is a function $\varphi \in L_{\Phi,w}^1(\Im, [0,\infty))$ such that for every $u \in E$ and any $k \in N_0$,

$$||F(\sigma,u)|| \leq \varphi(\sigma)(1 + ||u||), a.e. \text{ for } \sigma \in \cup_{k \in N_1}\Im_k.$$

$(F_3)$ There is $\varsigma \in L_w^{1,\Phi}(\Im, \mathbb{R}^+)$ such that for every bounded set $D \subseteq E$,

$$\chi(\cup_{u \in D} F(\sigma, u)) \leq (\Phi(\sigma) - \Phi(0))^{2-\gamma} \varsigma(\sigma) \chi(D), a.e. \text{or } \sigma \in \cup_{k \in N_1} \Im_k,$$

where $\chi$ is the Hausdorff measure of non-compactness on $E$.

$(H_1)$ For every $i \in N_1$, $g_i : [\sigma_i, \vartheta_i] \times E \to E$ such that for every $\sigma \in [\sigma_i, \vartheta_i]$, $g_i(\sigma, .)$ map, every bounded set to a relatively compact subset and for every bounded set $D, \subseteq E$,

$$\lim_{\tau_2 \to \tau_1} \sup_{x \in D} ||g_i(\tau_2, x) - g_i(\tau_1, x)|| = 0.$$

Moreover, there is $h_1 > 0$ with

$$||g_i(\sigma, u)|| \leq h_i ||u||, \ \forall \sigma \in T_i, \text{ and } \forall u \in E.$$

$(H_2)$ For any $i \in N_1$, $g_i^* : [\sigma_i, \vartheta_i] \times E \to E$ is defined such that it maps bounded sets to relatively compact sets, and there is $h_i^* > 0$ with

$$||g_i^*(\sigma, u))|| \leq h_i^* ||u||, \ \forall \sigma \in T_i, \text{ and } \forall u \in E.$$

Hence, Problem (1) has a mild solution assuming that the next inequalities are satisfied.

$$\frac{3hM\Phi(b)}{\Gamma(\gamma - 2\delta + 1)\Gamma(2\delta)} + \frac{M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varphi||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)} + h < 1. \tag{42}$$

and

$$\frac{2M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varsigma||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)} < 1, \tag{43}$$

where $h = \max_{i \in N_1}\{h_i, h_i^*\}$. Moreover, the set of mild solutions is compact in Banach space $C_{2-\gamma,\Phi,w}(\Im, E)$.

**Proof.** Let $u \in C_{2-\gamma,\Phi,w}(\Im, E)$. Assumptions $(F_1)$ and $(F_2)$ imply the existence of a measurable function $f \in L^1((0,b), E)$ with $f(\sigma) \in F(\sigma, u(\sigma)), a.e.$ [52]. We define a multi-valued function $\Re : PC_{2-\gamma,\Phi,w}(\Im, E) \to P(C_{2-\gamma,\Phi,w}(\Im, E))$ (the family of non-empty subsets of $C_{2-\gamma,\Phi,w}(\Im, E)$) in the following manner: $v \in \Re(u)$ means that

$$v(\sigma) = \begin{cases} D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\ \quad + \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_0, \\ g_i(\sigma, x(\sigma_i^-)), \sigma \in T_i, i \in N_1, \\ D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta_i))g_i(\vartheta_i, x(\vartheta_i^-)) \\ \quad + (Z_{5,i} *_{\Phi,w} Z_{6,i}^x)(\sigma) \\ \quad + \int_{\vartheta_i}^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_i, i \in N_1, \end{cases} \tag{44}$$

where $f \in L^1((0,b), E)$ with $f(\sigma) \in F(\sigma, u(\sigma)), a.e.$ Our aim in the following steps is to show that, by using Lemma 8, the $\Re$ has a fixed point, and it is clear that this point is a mild solution for Problem (4).

Step 1. For any $u \in C_{2-\gamma,\Phi,w}(\Im, E)$, $\Re(u)$ is convex. This is, clearly, achieved since the set of values of $F$ is convex.

Step 2. There exists $n_0 \in \mathbb{N}$ such that $\Re(\mho_{n_0}) \subseteq \mho_{n_0}$, where $\mho_{n_0} = \{u \in C_{2-\gamma,\Phi,w}(\Im, E) : ||u||_{C_{2-\gamma,\Phi,w}(\Im,E)} \leq n_0\}$. To clarify this, assume that there are $u_n, v_n; n \in \mathbb{N}$ with $v_n \in$

$\Re(u_n), ||u_n||_{C_{2-\gamma,\Phi,w}(\Im,E)} \leq n$ and $||u_n||_{C_{2-\gamma,\Phi,w}(\Im,E)} > n$. So, according to (44), there are $f_n \in L^1((0,b),E)$ ; $n \in \mathbb{N}$ with $f_n(\sigma) \in F(\sigma,u_n(\sigma))$, a.e., and

$$
v_n(\sigma) = \begin{cases}
D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\
+ \int_0^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f_n(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_0, \\
g_i(\sigma, v_n(\sigma_i^-)), \sigma \in T_i, i \in N_1, \\
D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(\vartheta_i))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta_i))g_i(\vartheta_i, v_n(\vartheta_i^-)) \\
+ (Z_{5,i} *_{\Phi,w} Z_{6,i}^{v_n})(\sigma) \\
+ \int_{\vartheta_i}^\sigma (\Phi(\sigma)-\Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f_n(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_i, i \in N_1.
\end{cases}
\tag{45}
$$

Let $\sigma \in (0,\sigma_1]$. By (41),

$$
(\Phi(\sigma)-\Phi(0))^{2-\gamma}w(\sigma)||D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma)-\Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0||
$$
$$
\leq \frac{M||x_0||}{(2\delta-1)\Gamma(\gamma-1)}.
\tag{46}
$$

Moreover, according to the definition of $*_{\Phi,w}$, we have

$$
(\Phi(\sigma)-\Phi(0))^{2-\gamma}w(\sigma)||(Z_1 *_{\Phi,w} Z_2)(\sigma)||
$$
$$
= (\Phi(\sigma)-\Phi(0))^{2-\gamma}||\int_0^\sigma w(\Phi^{-1}(\Phi(\sigma)-\Phi(0)-\Phi(\vartheta))w^{-1}(\Phi^{-1}(\Phi(\sigma)-\Phi(0)-\Phi(\vartheta))\times
$$
$$
\frac{(\Phi(\sigma)-2\Phi(0)-\Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma-2\delta)}w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta)-\Phi(0))^{\delta-1}K^{\Phi,\delta,w}(\vartheta,0)x_1 d\vartheta||
$$
$$
\leq \Phi(b)^{2-\gamma}||\int_0^\sigma \frac{(\Phi(\sigma)-2\Phi(0)-\Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma-2\delta)}(\Phi(\vartheta)-\Phi(0))^{\delta-1}\times
$$
$$
w(\vartheta)K^{\Phi,\delta,w}(\vartheta,0)x_1\Phi'(\vartheta)d\vartheta||.
$$

This inequality with (35) leads to

$$
(\Phi(\sigma)-\Phi(0))^{2-\gamma}w(\sigma)||(Z_1 *_{\Phi,w} Z_2)(\sigma)||
$$
$$
\leq (\Phi(\sigma)-\Phi(0))^{2-\gamma}\int_0^\sigma \frac{|(\Phi(\sigma)-2\Phi(0)-\Phi(\vartheta))^{\gamma-2\delta-1}|}{\Gamma(\gamma-2\delta)}\Phi'(\vartheta)\times
$$
$$
\frac{M(\Phi(\vartheta)-\Phi(0))^{2\delta-1}}{\Gamma(2\delta)}d\vartheta
$$
$$
\leq \frac{M(\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)\Gamma(\gamma-2\delta)}\int_0^\sigma |(\Phi(\sigma)-2\Phi(0)-\Phi(\vartheta))^{\gamma-2\delta-1}|\Phi'(\vartheta)d\vartheta
$$
$$
= \frac{M(\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)\Gamma(\gamma-2\delta)}\frac{(\Phi(\sigma)-3\Phi(0))^{\gamma-2\delta}+2\Phi(0)^{\gamma-2\delta}}{(\gamma-2\delta)}
$$
$$
\leq \frac{3M\Phi(b)}{\Gamma(\gamma-2\delta+1)\Gamma(2\delta)}.
\tag{47}
$$

Next, from assumption $(F_2)$ and (35), one obtains for almost $\vartheta \in (0,\sigma]$,

$$
||w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta)f_n(\vartheta)|| \leq \frac{M(1+n)}{\Gamma(2\delta)}(\Phi(\sigma)-\Phi(\vartheta))^\delta||f_n(\vartheta)||
$$
$$
\leq \frac{M(1+n)}{\Gamma(2\delta)}(\Phi(\sigma)-\Phi(\vartheta))^\delta\varphi(\vartheta)
$$
$$
\leq \frac{M(1+n)}{\Gamma(2\delta)}(\Phi(\sigma)-\Phi(\vartheta))^\delta\varphi(\vartheta).
\tag{48}
$$

This inequality leads to

$$(\Phi(\sigma) - \Phi(0))^{2-\gamma} w(\sigma) \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1} ||K^{\delta,\Phi,w}(\sigma,\vartheta) f_n(\vartheta) w(\vartheta)|| \Phi'(\vartheta) d\vartheta$$

$$= (\Phi(\sigma) - \Phi(0))^{2-\gamma} \int_0^\sigma |(\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}| \, ||w(\sigma) K^{\delta,\Phi,w}(\sigma,\vartheta) f_n(\vartheta)|| w(\vartheta) \Phi'(\vartheta) d\vartheta$$

$$\leq \frac{M(1+n)\Phi(b)^{2-\gamma}}{\Gamma(2\delta)} |\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{2\delta-1} w(\vartheta) \Phi'(\vartheta) \varphi(\vartheta) d\vartheta|$$

$$\leq \frac{M(1+n)\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)} \int_0^\sigma w(\vartheta) \Phi'(\vartheta) \varphi(\vartheta) d\vartheta$$

$$= \frac{M(1+n)\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varphi||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)}. \tag{49}$$

For $\sigma \in (\sigma_i, \vartheta_i]$, the assumption $(h_1)$ leads to

$$||v_n(\sigma)|| \leq hn. \tag{50}$$

Next, let $\sigma \in (\vartheta_i, \sigma_{i+1}]$. Set

$$L(\sigma) = ||D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta_i)) g_i(\vartheta_i, v(\vartheta_i^-))||.$$

It is yielded from (41) that

$$w(\sigma)((\Phi(\sigma) - \Phi(\vartheta_i))^{2-\gamma} L(\sigma)$$

$$\leq \frac{M||x_0||}{(2\delta-1)\Gamma(\gamma-1)}. \tag{51}$$

Moreover, from the definition of $*_{\Phi,w}$, (35) and $(H_2)$, we get,

$$||(\Phi(\sigma) - \Phi(\vartheta_i))^{2-\gamma} w(\sigma)(Z_{5,i} *_{\Phi,w} Z_{6,i}^x)(\sigma)||$$

$$\leq \frac{\Phi(b)^{2-\gamma}}{\Gamma(\gamma-2\delta)} \int_{\vartheta_i}^\sigma (\Phi(\sigma) - 2\Phi(\vartheta_i) - \Phi(\vartheta))^{\gamma-2\delta-1} (\Phi(\vartheta) - \Phi(\vartheta_i))^{\delta-1} \times$$

$$||w(\vartheta) K^{\Phi,\delta,w}(\vartheta,\vartheta_i) g_i^*(\vartheta_i, x(\vartheta_i^-))|| \Phi'(\vartheta) d\vartheta$$

$$\leq \frac{nMh \, \Phi(b)^{2-\gamma}}{\Gamma(2\delta)\Gamma(\gamma-2\delta)} \int_{\vartheta_i}^\sigma (\Phi(\sigma) - 2\Phi(\vartheta_i) - \Phi(\vartheta))^{\gamma-2\delta-1} (\Phi(\vartheta) - \Phi(\vartheta_i))^{2\delta-1} \Phi'(\vartheta) d\vartheta$$

$$\leq \frac{nMh\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)\Gamma(\gamma-2\delta)} \int_{\vartheta_i}^\sigma (\Phi(\sigma) - 2\Phi(\vartheta_i) - \Phi(\vartheta))^{\gamma-2\delta-1} \Phi'(\vartheta) d\vartheta$$

$$\leq \frac{3nMh\Phi(b)}{\Gamma(2\delta)\Gamma(\gamma-2\delta+1)}. \tag{52}$$

Next, by arguing as in (49), one has

$$||(\Phi(\sigma) - \Phi(\vartheta_i))^{2-\gamma} w(\sigma) \int_{\vartheta_i}^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1} ||K^{\delta,\Phi,w}(\sigma,\vartheta) f_n(\vartheta) w(\vartheta)|| \Phi'(\vartheta) d\vartheta$$

$$\leq \frac{M(1+n)\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varphi||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)}. \tag{53}$$

Relations (46), (47) and (49)–(53) give us

$$n < ||v_n||$$

$$\leq \frac{M||x_0||}{(2\delta-1)\Gamma(\gamma-1)} + \frac{3(1+nh)M\Phi(b)}{\Gamma(\gamma-2\delta+1)\Gamma(2\delta)} + \frac{M(1+n)\Phi(b)^{1-\gamma+2\delta}}{\delta\Gamma(2\delta)} ||\varphi||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)}$$

$$+ hn.$$

Dividing both sides of this inequality and then letting $n \to \infty$ yields

$$1 < \frac{3hM\Phi(b)}{\Gamma(\gamma - 2\delta + 1)\Gamma(2\delta)} + \frac{M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varphi||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)} + h,$$

but this inequality contradicts (42).

Step 2. The graph of $\Re$ is closed on $\mho_{n_0}$. Let $v_n \in \Re(u_n); n \in \mathbb{N}$, with $u_n \epsilon \mho_{n_0}$, $u_n \to u$ and $v_n \to v$. Then, there are $f_n \in L^1((0,b), E); n \in \mathbb{N}$ with $f_n(\sigma) \in F(\sigma, u_n(\sigma))$, a.e. such that (45) is fulfilled. From (49), $(f_n)$ is uniformly bounded, and hence, it has a weakly convergent subsequence. We denote it, again, by $(f_n)$ to a function $f$ in $L^1((0,b), E)$. From Mazur's lemma, there exists a subsequence of $(f_n)$, $(z_n)$, which converges almost everywhere to $f$. For any $n \in \mathbb{N}$, let

$$v_n^*(\sigma) = \begin{cases} D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\ + \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)z_n(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_0, \\ g_i(\sigma, v_n(\sigma_i^-)), \sigma \in T_i, i \in N_1, \\ D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta_i))g_i(\vartheta_i, v_n(\vartheta_i^-)) \\ + (Z_{5,i} *_{\Phi,w} Z_{6,i}^{v_n})(\sigma) \\ + \int_{\vartheta_i}^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)z_n(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_i, i \in N_1. \end{cases}$$

Obviously, $(v_n^*)$ is a subsequence of $(v_n)$ and converges to the function

$$v_n^*(\sigma) = \begin{cases} D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\ + \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_0, \\ g_i(\sigma, v(\sigma_i^-)), \sigma \in T_i, i \in N_1, \\ D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta_i))g_i(\vartheta_i, v(\vartheta_i^-)) \\ + (Z_{5,i} *_{\Phi,w} Z_{6,i}^v)(\sigma) \\ + \int_{\vartheta_i}^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_i, i \in N_1. \end{cases}$$

Then, $v^* = v$; moreover, the upper semi-continuity of $F(\sigma,.)$ implies $f(\sigma) \in F(\sigma, u(\sigma))$, a.e., and so, $v \in \Re(u)$.

Step 3. Let $\pounds = \Re(\mho_{n_0})$. For every $k \in N_0$, and every $i \in N_1$, let

$$\pounds_{|\overline{\Im_k}} := \{z \in C(\overline{\Im_k}, E) : z(\sigma) = (\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}w(\sigma)v(\sigma), \sigma \in \Im_k,$$

$$z(\vartheta_k) = \lim_{\sigma \to \vartheta_k^+} z(\sigma), v \in \pounds\},$$

and

$$\pounds_{|\overline{T_i}} := \{z \in C(\overline{T_i}, E) : z(\sigma) = v(\sigma), \sigma \in T_i, z(\sigma_i) = z(\sigma_i^+), v \in \pounds\}.$$

In this step, our aim is to show that the sets $\pounds_{|\overline{\Im_k}}$ and $\pounds_{|\overline{T_i}}$ are equicontinuous in the Banach spaces $C(\overline{\Im_k}, E)$ and $C(\overline{T_i}, E)$.

Case 1. Suppose that $z \in \pounds_{|\overline{\Im_0}}$. Then, there is $v \in \Re(\mho_{n_0})$ with

$$z(\sigma) = \begin{cases} (\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}w(\sigma)v(\sigma), \sigma \in \Im_0 \\ z(\vartheta_k) = \lim_{\sigma \to 0^+} z(\sigma). \end{cases}$$

According to the definition of $\Re$, there exists $u \in \mho_{n_0}$ and $f(\sigma) \in L^1((0,b), E)$ with $f(\sigma) \in F(\sigma, u(\sigma))$, a.e., such that

$$v(\sigma) = \begin{cases} D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1}K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma) \\ + \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\sigma,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta, \sigma \in \Im_0, \end{cases}$$

Let $\tau_1, \tau_2 \in \mathfrak{S}_0$ and $\tau_1 < \tau_2$. We have

$$
\begin{aligned}
\Delta_1 &= ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} w(\tau_2) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_2) \\
&\quad - (\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_1) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_1)|| \\
&\leq |(\Phi(\tau_2) - \Phi(0))^{2-\gamma} - (\Phi(\tau_1) - \Phi(0))^{2-\gamma}| \times \\
&\quad ||w(\tau_2) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_2)|| \\
&\quad + (\Phi(\tau_1) - \Phi(0))^{2-\gamma} ||w(\tau_2) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_2) \\
&\quad - w(\tau_1) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_1)|| \\
&= \Delta_{11} + \Delta_{12}.
\end{aligned}
$$

Due to the continuity of $\Phi$ and (41), $\lim_{\sigma_1 \to \sigma_1} \Delta_{11} = 0$. Moreover, using (35), we obtain

$$
\begin{aligned}
&\lim_{\sigma \to 0} (\Phi(\sigma) - \Phi(0))^{2-\gamma} w(\sigma) || K^{\delta,\Phi,w}(\sigma,0) u || \\
&\leq \lim_{\sigma \to 0} \frac{M(\Phi(\sigma) - \Phi(0))^{2-\gamma+\delta} ||u||}{\Gamma(2\delta)} = 0.
\end{aligned}
$$

Then, by Definition 3,

$$
\begin{aligned}
&D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma_1,0)) x_0(\sigma) \\
&= {}^c D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma_1,0)) x_0(\sigma).
\end{aligned}
$$

Therefore, by (38),

$$
\begin{aligned}
&\lim_{\tau_2 \to \tau_1} \Delta_{12} \\
&= \lim_{\tau_2 \to \tau_1} ||w(\tau_2) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_2) \\
&\quad - w(\tau_1) D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0)) x_0(\tau_1)|| \\
&\leq \lim_{\tau_2 \to \tau_1} \frac{1}{\Gamma(\gamma-\mu)} || \int_0^{\tau_2} ((\Phi(\tau_2) - \Phi(\vartheta))^{\gamma-\mu-1} \\
&\quad \times \frac{d}{d\vartheta} (((\Phi(\vartheta) - \Phi(0))^{\delta-1} w(\vartheta) K^{\delta,\Phi,w}(\vartheta,0)) x_0) d\vartheta \\
&\quad - \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\gamma-\mu-1} \\
&\quad \times \frac{d}{d\vartheta} (((\Phi(\vartheta) - \Phi(0))^{\delta-1} w(\vartheta) K^{\delta,\Phi,w}(\vartheta,0)) x_0) d\vartheta || \\
&\leq \lim_{\tau_2 \to \tau_1} \frac{\Phi(b)^{2\delta-2} M ||x_0|}{\Gamma(2\delta)\Gamma(\gamma-\mu)} \int_{\tau_1}^{\tau_2} ((\Phi(\tau_2) - \Phi(\vartheta))^{\gamma-\mu-1} \Phi(\vartheta) d\vartheta \\
&\quad + \lim_{\tau_2 \to \tau_1} \frac{\Phi(b)^{2\delta-2} M ||x_0||}{\Gamma(2\delta)\Gamma(\gamma-\mu)} \int_0^{\tau_1} |((\Phi(\tau_1) - \Phi(\vartheta))^{\gamma-\mu-1} \\
&\quad - ((\Phi(\tau_2) - \Phi(\vartheta))^{\gamma-\mu-1}| \Phi(\vartheta) d\vartheta \\
&= 0.
\end{aligned}
$$

Next, from (47), we obtain

$$
\begin{aligned}
\Delta_2 &= ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} w(\tau_2)(Z_1 *_{\Phi,w} Z_2)(\tau_2) \\
&\quad - (\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_1)((Z_1 *_{\Phi,w} Z_2)(\tau_1)|| \\
&= ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} \int_0^{\tau_2} \frac{(\Phi(\tau_2) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta \\
&\quad - (\Phi(\tau_1) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} \frac{(\Phi(\tau_1) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta|| \\
&\leq ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} \int_{\tau_1}^{\tau_2} \frac{(\Phi(\tau_2) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta|| \\
&\quad + ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} \frac{(\Phi(\tau_2) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta \\
&\quad - (\Phi(\tau_2) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} \frac{(\Phi(\tau_1) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta|| \\
&\quad + ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} \frac{(\Phi(\tau_1) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta \\
&\quad - (\Phi(\tau_1) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} \frac{(\Phi(\tau_1) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}}{\Gamma(\gamma - 2\delta)} \times \\
&\quad w(\vartheta)\Phi'(\vartheta)(\Phi(\vartheta) - \Phi(0))^{\delta-1} K^{\Phi,\delta,w}(\vartheta,0) x_1 d\vartheta|| \\
&= \Delta_{21} + \Delta_{22} + \Delta_{23}.
\end{aligned}
$$

Relation (35) implies that

$$
\begin{aligned}
&\lim_{\tau_2 \to \tau_1} \Delta_{21} \\
&\leq \frac{M\Phi(b)^{1-\gamma+2\delta}||x_1||}{\Gamma(\gamma - 2\delta)\Gamma(2\delta)} \lim_{\tau_2 \to \tau_1} \int_{\tau_1}^{\tau_2} (\Phi(\tau_2) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1} \Phi'(\vartheta) d\vartheta \\
&= 0.
\end{aligned}
$$

Again, by (35), one has

$$
\begin{aligned}
&\lim_{\tau_2 \to \tau_1} \Delta_{22} \\
&\leq \frac{M\Phi(b)^{1-\gamma+2\delta}||x_1||}{\Gamma(\gamma - 2\delta)\Gamma(2\delta)} \times \\
&\lim_{\tau_2 \to \tau_1} ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} [(\Phi(\tau_2) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1} \\
&\quad - \Phi(\tau_2) - 2\Phi(0) - \Phi(\vartheta))^{\gamma-2\delta-1}] \Phi'(\vartheta) d\vartheta \\
&= 0,
\end{aligned}
$$

and

$$\lim_{\tau_2 \to \tau_1} \Delta_{23}$$
$$\leq \lim_{\tau_2 \to \tau_1} |(\Phi(\tau_2) - \Phi(0))^{2-\gamma} - (\Phi(\tau_2) - \Phi(0))^{2-\gamma}| \times$$
$$\frac{M\Phi(b)^\delta}{\Gamma(\gamma - 2\delta)\Gamma(2\delta)} \int_0^{\tau_1} (\Phi(\tau_1) - 2\Phi(0) - \Phi(\vartheta))^{\gamma - 2\delta - 1} \Phi'(\vartheta) d\vartheta$$
$$= 0.$$

Next,

$$||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} w(\tau_2) \int_0^{\tau_2} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta$$

$$- (\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_1) \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_1,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta||$$

$$\leq ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} w(\tau_2) \int_{\tau_1}^{\tau_2} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta||$$

$$+ ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} w(\tau_2) \int_0^{\tau_1} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta$$

$$- (\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_2) \int_0^{\tau_1} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta||$$

$$+ ||(\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_2) \int_0^{\tau_1} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta$$

$$- (\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_2) \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta||$$

$$+ ||(\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_2) \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta$$

$$- (\Phi(\tau_1) - \Phi(0))^{2-\gamma} w(\tau_1) \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_1,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta||$$

$$= \Delta_{31} + \Delta_{32} + \Delta_{33} + \Delta_{34}.$$

Note that assumption $(F_2)$ leads to

$$||f(\vartheta)|| \leq \varphi(\sigma)(1 + n_0), a.e.$$

Then, relations (35) and (50) tell us

$$||w(\tau_2) K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta)||$$
$$\leq \frac{M(1 + n_0)(\Phi(\tau_2) - \Phi(\vartheta))^\delta \varphi(\vartheta) w(\vartheta)}{\Gamma(2\delta)}$$
$$\leq \frac{M(1 + n_0)\Phi(b)^\delta \varphi(\vartheta) w(\vartheta)}{\Gamma(2\delta)}.a.e. \text{for } \vartheta \in (0, \tau_2].$$

From (51), we have

$$\lim_{\tau_2 \to \tau_1} \Delta_{31}$$
$$\leq \lim_{\tau_2 \to \tau_1} ||(\Phi(\tau_2) - \Phi(0))^{2-\gamma} w(\tau_2)$$
$$\times \int_{\tau_1}^{\tau_2} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta||$$
$$= \lim_{\tau_2 \to \tau_1} \Phi(b)^{2-\gamma} \int_{\tau_1}^{\tau_2} |((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1}| \, ||w(\tau_2) K^{\delta,\Phi,w}(\tau_2,\vartheta) f(\vartheta) w(\vartheta)|| \Phi'(\vartheta) d\vartheta$$
$$\leq \frac{M(1 + n_0)\Phi(b)^\delta}{\Gamma(2\delta)} \lim_{\tau_2 \to \tau_1} \int_{\tau_1}^{\tau_2} (\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} \varphi(\vartheta) w(\vartheta) \Phi'(\vartheta) d\vartheta$$
$$= 0.$$

Next, using (51) and the continuity of $\Phi$, it follows that

$$\lim_{\tau_2 \to \tau_1} \Delta_{32}$$

$$\leq \lim_{\tau_2 \to \tau_1} |(\Phi(\tau_2) - \Phi(0))^{2-\gamma} - (\Phi(\tau_1) - \Phi(0))^{2-\gamma}| \times$$

$$\int_0^{\tau_1} |((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1}||w(\tau_2)K^{\delta,\Phi,w}(\tau_2,\vartheta)f(\vartheta)w(\vartheta)|| \Phi'(\vartheta)d\vartheta$$

$$\leq \frac{M(1+n_0)\Phi(b)^\delta}{\Gamma(2\delta)} \lim_{\tau_2 \to \tau_1} |(\Phi(\tau_2) - \Phi(0))^{2-\gamma} - (\Phi(\tau_1) - \Phi(0))^{2-\gamma}| \times$$

$$\int_0^{\tau_1} ((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} \varphi(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta$$

$$= 0.$$

For $\Delta_{33}$, we have

$$\lim_{\tau_2 \to \tau_1} \Delta_{33}$$

$$\leq \lim_{\tau_2 \to \tau_1} (\Phi(\tau_1) - \Phi(0))^{2-\gamma} \int_0^{\tau_1} |((\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1} - (\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1}| \times$$

$$||w(\tau_2)K^{\delta,\Phi,w}(\tau_2,\vartheta)f(\vartheta)w(\vartheta)||\Phi'(\vartheta)d\vartheta$$

$$\leq \frac{M(1+n_0)\Phi(b)^\delta}{\Gamma(2\delta)} \times$$

$$\lim_{\tau_2 \to \tau_1} \int_0^{\tau_1} [((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1}\Phi'(\vartheta) - (\Phi(\tau_2) - \Phi(\vartheta))^{\delta-1}\Phi'(\vartheta)]\varphi(\vartheta)w(\vartheta)d\vartheta$$

$$= 0.$$

Finally, due to (36) and (51), it yields

$$\lim_{\tau_2 \to \tau_1} \Delta_{34}$$

$$\leq \lim_{\tau_2 \to \tau_1} ||(\Phi(\tau_1) - \Phi(0))^{2-\gamma}w(\tau_2) \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\tau_2,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta$$

$$- (\Phi(\tau_1) - \Phi(0))^{2-\gamma}w(\tau_1) \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1}K^{\delta,\Phi,w}(\tau_1,\vartheta)f(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta||$$

$$\leq \Phi(b)^{2-\gamma} \lim_{\tau_2 \to \tau_1} \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} \times$$

$$||w(\tau_2)K^{\delta,\Phi,w}(\tau_2,\vartheta) - w(\tau_1)K^{\delta,\Phi,w}(\tau_1,\vartheta)|| \; ||w(\vartheta)f(\vartheta)||\Phi'(\vartheta)d\vartheta$$

$$\leq \Phi(b)^{2-\gamma} \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} \times$$

$$\lim_{\tau_2 \to \tau_1} ||w(\tau_2)K^{\delta,\Phi,w}(\tau_2,\vartheta) - w(\tau_1)K^{\delta,\Phi,w}(\tau_1,\vartheta)||w(\vartheta)\varphi(\vartheta)\Phi'(\vartheta)d\vartheta$$

$$\leq \Phi(b)^{4-2\gamma} \int_0^{\tau_1} ((\Phi(\tau_1) - \Phi(\vartheta))^{\delta-1} \times$$

$$\lim_{\tau_2 \to \tau_1} ||w(\tau_2)K^{\delta,\Phi,w}(\tau_2,\vartheta) - w(\tau_1)K^{\delta,\Phi,w}(\tau_1,\vartheta)||w(\vartheta)\varphi(\vartheta)\Phi'(\vartheta)d\vartheta$$

$$= 0,$$

independently of $v$.

Suppose that $z \in \mathcal{L}_{|\overline{T_i}}; i \in N_1$. Then, there is $v \in \Re(\mho_{n_0}) \subseteq \mho_{n_0}$ with

$$z(\sigma) = \begin{cases} v(\sigma) = g_i(\sigma, v(\sigma_i^-)), \sigma \in T_i \\ z(\sigma_i) = \lim_{\sigma \to \sigma_i^+} z(\sigma). \end{cases}$$

Case 2. Let $\tau_1, \tau_2 \in T_i$ and $\tau_1 < \tau_2$. Since $\mho_{n_0}$, then due to $(H_1)$,

$$\lim_{\tau_2 \to \tau_1} ||z(\tau_2) - z(\tau_1)||$$
$$= \lim_{\tau_2 \to \tau_1} ||g_i(\tau_2, v(\sigma_i^-)) - g_i(\tau_1, v(\sigma_i^-))|| = 0,$$

independently of $v$. If $\tau_1 = \sigma_i$, then

$$\lim_{\tau_2 \to \tau_1} ||z(\tau_2) - z(\tau_1)||$$
$$= \lim_{\tau_2 \to \tau_1} \lim_{\sigma \to \tau_1} ||g_i(\tau_2, v(\sigma_i^-)) - g_i(\sigma, v(\sigma_i^-))|| = 0,$$

independently of $v$.

Case 3. Suppose that $z \in \pounds_{|\overline{\Im_i}}; i \in N_1$. Following the same arguments used in case 1, one can show that $\pounds_{|\overline{\Im_i}}; i \in N_1$ is equicontinuous.

As a result of the above discussion, the proof of the results in this step is complete.

Step 4. Let $\mho = \cap_{n=1}^{\infty} \mho_n$, where $\mho_1 = \Re(\mho_{n_0})$ and $\mho_{n+1} = \Re(\mho_n), n \in N_1$. Then, $(\mho_n)$ is a decreasing sequence of not empty, bounded, convex subsets. In this step, our aim is to show that $\mho$ is not empty or compact. Using the Cantor intersection property, it remains to be shown that

$$\lim_{n \to \infty} \chi_{C_{2-\gamma,\Phi,w}(\Im,E)} \{\mho_n : n \in N_1\} = 0, \tag{54}$$

where $\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}$ is the measure of noncompactness on $C_{2-\gamma,\Phi,w}(\Im, E)$, which is defined in the introduction section.

Assume $n \in \mathbb{N}$ is fixed and $\epsilon > 0$ is arbitrarily small. By Lemma 5 in [54], one can find a sequence $(v_r)_{r \geq 1}$ in $\mho_n$ with

$$\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_n) \leq 2\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(D) + \varepsilon, \tag{55}$$

where $D = \{v_r : r \geq 1\}$. From Step 3, it yields

$$\chi_{PC_{2-\gamma,\Phi,w}(\Im,E)}(D)$$
$$= \max\{\max_{k \in N_0} \max_{\sigma \in \overline{\Im_k}} \chi_E\{v^*(\sigma) : v^* \in D_{|\overline{\Im_k}}, \max_{i \in N_1} \max_{\sigma \in \overline{T_i}} \chi_E\{v^*(\sigma) : v^* \in D_{|\overline{T_i}}\}\},$$

where $\chi_E$ is the measure of non-compactnes in $E$,

$$D_{|\overline{\Im_k}} := \{v^* \in C(\overline{\Im_k}, E) : h^*(\sigma) = (\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma} w(\sigma) v(\sigma), \sigma \in \Im_k,$$
$$v^*(\vartheta_k) = \lim_{\sigma \to \vartheta_k^+} v^*(\sigma), h \in D\},$$

and

$$D_{|\overline{T_i}} := \{v^* \in C(\overline{T_i}, E) : v^*(\sigma) = v(\sigma), \sigma \in T_i, v^*(\sigma_i) = v(\sigma_i^+), v \in D\}.$$

Since, for any $v \in D$, and any $i \in N_1$, $v(\sigma) = g_i(\sigma, v(\sigma_i^-)), \forall \sigma \in \overline{T_i}$, and since $g_i(\sigma, )$ maps bounded sets into relatively compact sets, it follows that $\max_{i \in N_1} \max_{\sigma \in \overline{T_i}} \{v^*(\sigma) : v^* \in D_{|\overline{T_i}}\} = 0$, and hence inequality (55) becomes

$$\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_n)$$
$$\leq 2 \max_{k \in N_0} \max_{\sigma \in \overline{\Im_k}} \chi_E\{(\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma} w(\sigma) v_r : r \geq 1\} + \varepsilon. \tag{56}$$

Let $\sigma \in \Im_0$. Since for any $r \in \mathbb{N}$, $v_r \in \mho_n = \Re(\mho_{n-1})$, there are $u_r \in \mho_{n-1}$ such that

$$v_r(\sigma) = D_{0,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(0))^{\delta-1} K^{\delta,\Phi,w}(\sigma,0))x_0 + (Z_1 *_{\Phi,w} Z_2)(\sigma)$$
$$+ \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta)w(\vartheta)f_r(\vartheta)\Phi'(\vartheta)d\vartheta,$$

where $fr \in L^1((0,b),E)$ with $f_r(\sigma) \in F(\sigma,u_r(\sigma)), a.e.$ In view of $(F_3)$, it holds for a.e.
$\vartheta \in (0,\sigma]$

$$\begin{aligned}
\varrho(\vartheta) = w(\vartheta)\chi\{f_k(\vartheta) : k \geq 1\} &\leq w(\vartheta)\chi_E(\cup_{k\in\mathbb{N}}F(\vartheta,u_r(\vartheta))) \\
&\leq w(\vartheta)(\Phi(\vartheta) - \Phi(0))^{2-\gamma}\varsigma(\vartheta)\chi_E\{u_r(\vartheta) : r \in \mathbb{N}\} \\
&= \varsigma(\vartheta)\chi_E\{(\Phi(\vartheta) - \Phi(0))^{2-\gamma}w(\vartheta)u_r(\vartheta) : r \in \mathbb{N}\} \\
&\leq \varsigma(\vartheta)\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_{n-1}).
\end{aligned} \tag{57}$$

Set

$$L(\sigma)$$
$$= \chi(\{\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta)w(\vartheta)f_r(\vartheta)\Phi'(\vartheta)d\vartheta : r \in \mathbb{N}\}d\vartheta.$$

Therefore, from (35), (56) and (57) and the properties of the measure of noncompactness, we obtain

$$\begin{aligned}
&\Phi(\sigma) - \Phi(0))^{2-\gamma}w(\sigma)L(\sigma) \\
&= \Phi(\sigma) - \Phi(0))^{2-\gamma}\chi(\{\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1}||w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta)||\varrho(\vartheta)\Phi'(\vartheta)d\vartheta \\
&\leq \frac{M\Phi(b)^{2-\gamma}}{\Gamma(2\delta)} \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{2\delta-1}\varrho(\vartheta)\Phi'(\vartheta)d\vartheta \\
&\leq \frac{M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)}\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_{n-1}) . \int_0^\sigma \Phi'(\vartheta)\varsigma(\vartheta)w(\vartheta)d\vartheta \\
&= \frac{M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)}||\varsigma||_{L_w^{1,\Phi}(\Im,\mathbb{R}^+)}\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_{n-1}).
\end{aligned} \tag{58}$$

Let $\sigma \in \Im_k, k \in N_1$. As above, for any $r \in \mathbb{N}$,

$$v_r(\sigma) = D_{\vartheta_i,\sigma}^{\mu-\gamma+1,\Phi,w}((\Phi(\sigma) - \Phi(\vartheta_i))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta_i))g_i(\vartheta_i, v_r(\vartheta_i^-))$$
$$+ (Z_{5,i} *_{\Phi,w} Z_{6,i}^{v_r})(\sigma)$$
$$+ \int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta))^{\delta-1} K^{\delta,\Phi,w}(\sigma,\vartheta)f_r(\vartheta)w(\vartheta)\Phi'(\vartheta)d\vartheta,$$

where $fr \in L^1((0,b),E)$ with $f_r(\sigma) \in F(\sigma,u_r(\sigma)), a.e.$ Because both $g_i(\sigma,.)$ and $g_i^*(\sigma,.)$ map bounded sets to relatively compact sets, it yields

$$\begin{aligned}
\chi\{(\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}w(\sigma)v_r : r \geq 1\} \\
= \{(\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}w(\sigma)Y(\sigma) : r \geq 1\},
\end{aligned}$$

where

$$Y(\sigma)$$
$$= \chi(\{\int_0^\sigma (\Phi(\sigma) - \Phi(\vartheta_k))^{\delta-1}w(\sigma)K^{\delta,\Phi,w}(\sigma,\vartheta)w(\vartheta)f_r(\vartheta)\Phi'(\vartheta)d\vartheta : r \in \mathbb{N}\}d\vartheta.$$

As in (58)

$$\chi\{(\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma}w(\sigma)v_r : r \geq 1\}$$
$$\leq \frac{M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)}||\varsigma||_{L^{1,\Phi}_w(\Im,\mathbb{R}^+)}\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_{n-1}). \quad (59)$$

From (56), (58) and (59), one has

$$\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_n)$$
$$\leq \frac{2M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)}||\varsigma||_{L^{1,\Phi}_w(\Im,\mathbb{R}^+)}\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_{n-1}) + \epsilon.$$

Then,

$$\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_n)$$
$$\leq (\frac{2M\Phi(b)^{1-\gamma+2\delta}}{\Gamma(2\delta)}||\varsigma||_{L^{1,\Phi}_w(\Im,\mathbb{R}^+)})^n\chi_{C_{2-\gamma,\Phi,w}(\Im,E)}(\mho_{n_0}) + \epsilon.$$

Step 5. By applying the Cantor intersection property, the set $\mho$ is not empty or compact. Then, the multi-valued function $\Re : \mho \to P_{ck}(\mho)$ satisfies the assumptions in Lemma 8, and hence, the fixed-points set of the function is not empty. Moreover, Using Lemma 9, the set of fixed points of $\Re$ is compact in $C_{2-\gamma,\Phi,w}(\Im, E)$. $\square$

## 4. Discussion and Conclusions

There are many definitions for the fractional differential operator, and some of them include others. Therefore, it is useful to consider fractional differential equations and fractional differential inclusions that contain a fractional differential operator which includes a large number of other fractional differential operators. Since the $w$-weighted $\Phi$-Hilfer fractional derivative, $D^{\mu,v,\Phi,w}_{0,\sigma}$, interpolates the fractional derivative differential operators that were presented by Riemann–Liouville, Caputo, Hadamard, $\Phi$-Riemann–Liouville, $\Phi$-Caputo, Katugampola, Hilfer–Hadamard, Hilfer, Hilfer–Katugampola and $\Phi$- Hilfer derivatives, it contains a large number of fractional differential operators. In this work, the representation for a mild solution to a semilinear differential inclusion involving the $w$-weighted $\Phi$-Hilfer fractional derivative of order $\mu \in (1,2)$ and of type $v \in (0,1)$ is derived in the presence of non-instantaneous impulses, and then the non-emptiness and compactness of the set of mild solution for the considered problem is proved in infinite dimensional Banach spaces. The nonlinear part of the considered problem is the infinitesimal generator of the strongly continuous cosine family, and the nonlinear part is a multi-valued function. Our results are novel and interesting because no researchers have previously studied such semilinear differential inclusion. Moreover, since the fractional differential operator $D^{\mu,v,\Phi,w}_{0,\sigma}$ interpolates many other known fractional differential operators, our objective problem includes many problems which are considered in many cited papers in the introduction section. In addition, our technique can be used to generalize many cited papers in the introduction to the case when the considered fractional differential operator is replaced by $D^{\mu,v,\Phi,w}_{0,\sigma}$ and the dimension of the space is infinite, and this can be considered as a suggestion for future research work as a result of this paper.

## 5. Example

**Example 1.** *Assume that E is a Hilbert space, D is a non-empty convex compact subset of E, $b = 3$, $\Im = [0,3]$, $m = 1$, $\vartheta_0 = 0$, $\sigma_1 = 1$, $\vartheta_1 = 2$, $\sigma_2 = 3$, $\mu = \frac{4}{3}$, $v = \frac{1}{2}$. Then, $v(2-\mu) = \frac{1}{3}$, $\gamma = \mu + 2v - \mu v = \frac{5}{3}$ and $2 - \gamma = \frac{1}{3}$, $w : \Im \to]0,\infty[$, $\Phi : \Im \to \mathbb{R}$ is a strictly increasing*

*continuously differentiable function with $\Phi'(\vartheta) \neq 0$, and for any $\vartheta \in \Im$, $x_1, x_2$ are two fixed points in E, and $\lambda = \sup\{||x|| : x \in \Omega\}$. The definition of an operator $A : D(A) \subset E \to E$ is given by:*

$$Av = v'', \tag{60}$$

*with*

$$D(A) = \{v \in L^2[0, \pi] : v_{yy} \in L^2[0, 1], v(0) = v(\pi) = 0\}.$$

Note that the representation of the operator $A$ is

$$Ax = \sum_{k=1}^{\infty} -k^2 < x, \ x_k > x_k, x \in D(A),$$

where $x_k(y) = \sqrt{2} \sin ky, k = 1, 2, ...$, is the orthonormal set of eigenfunctions of $A$. In addition, $A$ is the infinitesimal generator of a strongly continuous cosine family $C(\sigma)_{\sigma \in \mathbb{R}}$ which is defined by

$$C(\sigma)(x) = \sum_{k=1}^{\infty} \cos k\sigma < x, \ x_k > x_k, x \in E,$$

and the corresponding sine family $S(\sigma)_{\sigma \in \mathbb{R}}$ on $E$ is given by

$$S(\sigma)(x) = \sum_{k=1}^{\infty} \frac{\sin k\sigma}{k} < x, \ x_k > x_k, x \in E.$$

Suppose $F : \Im \times E \to P_{ck}(E)$ is a multivalued function given by:

$$F(\sigma, u) = \begin{cases} \xi(\Phi(\sigma) - \Phi(0))^{\frac{1}{3}}||u|| \ \Omega, \ \sigma \in \Im_0 = [0, 1], \\ \xi \ \Omega, \sigma \in T_1 = (1, \ 2], \\ \xi(\Phi(\sigma) - \Phi(2))^{\frac{1}{3}}||u|| \ \Omega, \ \sigma \in \Im_1 = (2, 3], \end{cases} \tag{61}$$

where $\xi > 0$. Clearly, for any $u \in E$, $F(., u)$ is measurable and, for any $y \in F(\sigma, u)$,

$$||y|| = \begin{cases} \lambda\xi(\Phi(\sigma) - \Phi(0))^{\frac{1}{3}}||u|| \ , \ \sigma \in \Im_0, \\ \xi \ \lambda, \sigma \in T_1, \\ \lambda\xi(\Phi(\sigma) - \Phi(2))^{\frac{1}{3}}||u||, \ \sigma \in \Im_1, \end{cases}$$

$$\leq \begin{cases} \lambda\xi(\Phi(\sigma) - \Phi(0))^{\frac{1}{3}}(1 + ||u||), \ \sigma \in \Im_0, \\ \xi \ \lambda, \sigma \in T_1, \\ \lambda\xi(\Phi(\sigma) - \Phi(2))^{\frac{1}{3}}(1 + ||u||), \ \sigma \in \Im_1, \end{cases}$$

$$\leq \begin{cases} \lambda\xi(\Phi(b) - \Phi(0))^{\frac{1}{3}}(1 + ||u||), \ \sigma \in \Im_0, \\ \xi \ \lambda, \sigma \in T_1, \\ \lambda\xi(\Phi(b) - \Phi(0))^{\frac{1}{3}}(1 + ||u||), \ \sigma \in \Im_1. \end{cases}$$

So, assumption $(F_2)$ is verified with

$$\varphi(\sigma) = \lambda\xi(\Phi(b) - \Phi(0))^{\frac{1}{3}}, \forall \sigma \in \cup_{k \in \{0,1\}} \Im_k. \tag{62}$$

Moreover, let $u_1, u_2 \in E$ and if $y_1 \in F(\sigma, u_1)$, then

$$y_1 = \begin{cases} \xi(\Phi(\sigma) - \Phi(0))^{\frac{1}{3}}||u_1|| \ e_1, \ \sigma \in [0, 1], \\ \xi \ \kappa_1, \sigma \in (1, \ 2], \\ \xi(\Phi(\sigma) - \Phi(2))^{\frac{1}{3}}||u|| \ \varrho_1, \ \sigma \in (2, 3], \end{cases}$$

where $e_1, \kappa_1, \varrho_1$ are elements in $\Omega$. Set

$$
y_2 = \begin{cases} \xi(\Phi(\sigma) - \Phi(0))^{\frac{1}{3}} w(\sigma) ||u_2|| \, e_1, \ \sigma \in [0, 1], \\ \xi \, \kappa_1, \sigma \in (1, \, 2], \\ \xi(\Phi(\sigma) - \Phi(2))^{\frac{1}{3}} w(\sigma) ||u|| \, \varrho_1. \ \sigma \in (2, 3]. \end{cases}
$$

Due to the definition of $F$, it yields $y_2 \in F(\sigma, u_2)$ and

$$
||y_2 - y_1|| = \begin{cases} \lambda\xi(\Phi(\sigma) - \Phi(0))^{\frac{1}{3}} w(\sigma) ||u_2 - u_1|| \, , \ \sigma \in [0, 1], \\ 0, \sigma \in (1, \, 2], \\ \lambda\xi(\Phi(\sigma) - \Phi(2))^{\frac{1}{3}} w(\sigma) ||u_2 - u_1||, \ \sigma \in (2, 3]. \end{cases}
$$

Therefore,

$$
h(F(\sigma, u_2), F(\sigma, u_1) \le \lambda\xi(\Phi(\sigma) - \Phi(\vartheta_k))^{\frac{1}{3}} ||u_2 - u_1||, \forall \sigma \in \Im_k, k = 0, 1. \tag{63}
$$

It follows from (63) that $F(\sigma, .)$ is upper semicontinuous for every $\sigma \in \Im_k, k = 0, 1$, and for almost $\sigma \in \Im_k, k = 0, 1$,

$$
\chi_E(F(\sigma, u) : u \in D))
$$
$$
\le (\Phi(\sigma) - \Phi(\vartheta_k))^{2-\gamma} \varsigma(\sigma) \chi_E \{ (u(\sigma) : z \in D \}.
$$

Then, $(F_3)$ holds with

$$
\zeta(\sigma) = \lambda\xi, \text{ for almost } \sigma \in \Im_k, k = 0, 1. \tag{64}
$$

$\zeta(\sigma) = \lambda\xi, \forall \sigma \in \Im.$

Next, let $g_1 : [1, 2] \times E \to E$ , $g_1^* : [1, 2] \times E \to E$ as follows:

$$
g_1(\sigma, x) = \xi Y(x), g_1^*(\sigma, x) = \xi Y^*(x), \tag{65}
$$

where $Y, Y^* : E \to \Omega$ are the projection operator on $\Omega$. Notice that $||g_1(\sigma, x)|| \le \lambda\xi ||x||$ and $||g_1(\sigma, x)|| \le \xi ||Y^*|| \, ||x||, \forall (\sigma, x) \in [1, 2] \times E$. So, conditions $(H_1)$ and $(H_1)$ are satisfied, where $h_1 = \lambda\xi$ and $h_1^* = \xi ||Y^*||$.

As a result of Theorem (1), we have the following problem:

$$
\begin{cases} D_{\vartheta_i, \sigma}^{\frac{4}{3}, \frac{1}{2}, \Phi, w} x(\sigma) \in Ax(\sigma) + F(\sigma, x(\sigma)), \ a.e., \ \sigma \in \Im_1, i \in \{0, 1\}, \\ x(\sigma_i^+) = g_i(\sigma_i, x(\sigma_i^-)), i = 1, \\ x(\sigma) = g_i(\sigma, x(\sigma_i^-)), \sigma \in (\sigma_i, \vartheta_i], \sigma \in T_1, \\ \lim_{\sigma \to 0^+} w(\sigma) I_{0,\sigma}^{2-\gamma, \Phi, w} x(\sigma) = x_0, \ \lim_{\sigma \to 0^+} \frac{1}{\Phi'(\sigma)} \frac{d}{d\sigma} (w(\sigma) I_{0,\sigma}^{2-\gamma, \Phi, w} x(\sigma)) = x_1, \\ \lim_{\sigma \to \vartheta_i^-} (w(\sigma) I_{0,\sigma}^{2-\gamma} x(\sigma)) = g_i(\vartheta_i, x(\vartheta_i^-)), i = 1 \\ \lim_{\sigma \to \vartheta_i^-} \frac{1}{\Phi'(\sigma)} \frac{d}{d\sigma} (w(\sigma) I_{0,\sigma}^{2-\gamma} x(\sigma)) = g_i^*(\vartheta_i, x(\vartheta_i^-)), i = 1 \end{cases}
$$

which has a mild solution where $A, F$ , $g$ and $g^*$ are as in (60), (61) and (65) given that

$$
\frac{3hM\Phi(3)}{\Gamma(\gamma - 2\delta + 1)\Gamma(2\delta)} + \frac{M\Phi(3)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varphi||_{L_w^{1,\Phi}(\Im, \mathbb{R}^+)} + h < 1. \tag{66}
$$

and

$$
\frac{2M\Phi(3)^{1-\gamma+2\delta}}{\Gamma(2\delta)} ||\varsigma||_{L_w^{1,\Phi}(\Im, \mathbb{R}^+)} < 1, \tag{67}
$$

where $M = \sup_{\sigma \in \Im} ||C(\sigma)||$, $\delta = \frac{\mu}{2} = \frac{2}{3}$, $\varphi$ and $\varsigma$ are as in (62) and (64). By choosing sufficiently small $\Phi, \lambda, \rho$ and $\xi$, inequalities (66) and (67) are satisfied.

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