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Existence Results Related to a Singular Fractional Double-Phase Problem in the Whole Space

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Abstract: In this paper, we will study a singular problem involving the fractional $(q_1(x,.)$-$q_2(x,.))$-Laplacian operator in the whole space $\mathbb{R}^N, (N \geq 2)$. More precisely, we combine the variational method with monotonicity arguments to prove that the associated functional energy admits a critical point, which is a weak solution for such a problem.

Keywords: fractional Laplacian; singular elliptic problem; variational methods; generalized Sobolev spaces

1. Introduction

In this paper, we study the following singular fractional problem:

$$(Q_\theta) : (-\Delta)^s_{q_1(z,.)}\psi + (-\Delta)^s_{q_2(z,.)}u + |\psi|^{\sigma(z)-2}\psi = b(z)\psi^{-\tau(z)} - \theta g(z,\psi),\, z \in \mathbb{R}^N,$$

where $N \geq 2$, θ is a positive parameter, $0 < s < 1$, δ, σ and g are continuous functions that satisfy some hypotheses described later in Section 3. The operator $(-\Delta)^s_{q_i(x,.)}$ is given by

$$(-\Delta)^s_{q_i(z,.)}\psi(z) = p.v. \int_\Omega \frac{|\psi(z) - \psi(\chi)|^{q_i(z,\chi)-2}(\psi(z) - \psi(\chi))}{|z-\chi|^{N+sq_i(z,\chi)}}\, d\chi, z \in \mathbb{R}^N.$$

$q_i \in C(\mathbb{R} \times \mathbb{R}, (1,\infty))$, where i will denote (throughout this paper) the integers 1 or 2. It is noted that the operator $(-\Delta)^s_{q(z,.)}$ is a generalization of the operator $(-\Delta)^s_{q(z)}$; moreover, regarding the manipulation of this operator in the weak formulation of the proposed problem, it seems that it is more complicated than the fractional $q(z)$-Laplace operator.

Problems involving the non-local fractional operator $(-\Delta)^s_{q(z,.)}$ have received more interest throughout recent years. This is due to their several applications in different fields. To be more precise, this operator appears, for example, in electrorheological fluids (see the paper of Ruzicka [1]); in elastic mechanics (see the work of Zhikov [2]); and in image processing (see the monograph of Chen et al. [3]).

Problems like (Q_θ) have been extensively considered by several authors, about this operator and other particular cases of it, of which we cite as examples the papers by Chammen et al. [4,5], Azroul et al. [6], Bahrouni [7], Bahrouni and Rădulescu [8], Kefi and Saoudi [9], Fan and Zhang [10,11], Ghanmi and Saoudi [12,13], and Rădulescu and Repovš [14]. More precisely, Chammem et al. [4] considered the following singular problem:

$$\begin{cases} (-\Delta)^s_{f_1(z,.)}\psi + (-\Delta)^s_{f_2(z,.)}u + |\psi|^{q(z)-2}\psi = h(z)\psi^{-\tau(z)} + \theta k(z,\psi), \text{in } \Omega, \\ \psi = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\theta \geq 0$; the functions f_i, q are continuous with values in $(1,\infty)$; and the function γ is continuous with values in $(0,1)$. The author used the variational methods and combined them with some monotonicity arguments to prove that the problem (1) admits a nontrivial solution.

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Recently, by combining the variational method with monotonicity arguments, Chammem et al. [4] proved some existing results related to the following problem:

$$\begin{cases} (-\Delta)^s_{f_1(z,.)}\psi + (-\Delta)^s_{f_2(z,.)}\psi + |\psi|^{q(z)-2}\psi = h(z)\psi^{-\tau(z)} + \theta k(z,\psi) \quad \text{in } \Omega, \\ \\ \psi = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $f_i : \overline{\Omega} \times \overline{\Omega} \to (1,\infty), q : \overline{\Omega} \to (1,\infty)), \gamma : \overline{\Omega} \to (0,1)$ are continuous functions and $\theta \geq 0$.

After that, Chammem et al. [15] used diverse versions of the mountain pass theorem to prove several results associated with the following problem:

$$\begin{cases} \zeta_1\psi + \zeta_2\psi = h(z,\psi) + \epsilon k(z,\psi), \text{in } \Omega, \\ \psi = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2}$$

where $\epsilon > 0$ is a positive parameter and the non-local operator ζ_i is defined by

$$\zeta_i\psi(z) = \frac{1}{2}\int_{\mathbb{R}^N} (\psi(z+t) + \psi(z-t) - 2\psi(z))K_i(t)dt, \tag{3}$$

where $K_i : \mathbb{R}^N \setminus \{0\} \longrightarrow (0,\infty)$ is assumed to satisfy some important assumptions. Under supplementary hypotheses on the nonlinearities h and k, the authors proved that problem (2) has a nontrivial solution. Moreover, the multiplicity of solutions is also studied for the problem (2).

There are presently too many papers that have studied problems with regard to the $q(x)$-Laplace operator, of which we cite as examples the papers by Alsaedi et al. [16–18] (several variational methods), Chammem et al. [4,5] (variational method, monotonicity arguments and the method of Nehari), Ghanmi and Saoudi [12,13] (variational method combined with the method of Nehari), Giacomoni and Saoudi [19] (variational and sub-super-solution methods), Kefi and Saoudi [20] (monotonicity arguments method), and Saoudi and Ghanmi [21] (variational techniques). Meanwhile, the problems involving the operator $(-\Delta)^s_{q(z,.)}$ in the whole space $\mathbb{R}^N$ are more complicated than the problems in a bounded domain. In particular, the embedding Λ into $L^{\delta(z)}(\mathbb{R}^N)$ is only continuous for $1 < \tau^- \leq \delta(z) < q_s^*(z)$, which causes the verification of the compactness of the Palai–Smale sequences to become difficult. Consequently, we managed to use the space $\mathbb{R}^N_{loc}$, where Λ, $L^{\delta(z)}(\mathbb{R}^N)$, $q_s^*(z)$, and $\mathbb{R}^N_{loc}$ are defined in Section 2.

Very recently, Ge and Gao [22] proved the existence of a solution for the following $p(.,.)$-Laplacian equation in the whole space:

$$(-\Delta)^s_{p(z,.)}u + |\psi|^{\overline{p}(z)-2}u = \theta\omega_1(z)|\psi|^{q(z)-2}u - \omega_2(z)|\psi|^{r(z)-2}u, \text{ in } \mathbb{R}^N. \tag{4}$$

To find solutions for the problem (4), the authors used compactness results (proved by Ho and Kim (Theorem 3.5 [23])) concerning the compact embedding from the Sobolev space into the space $L^{\delta(z)}_{loc}(\mathbb{R}^N)$.

In this work, we extend the above investigations to a singular problem in the whole space. We note that the singularity and the lack of compactness make the study of the problem (Q_θ) more complicated.

2. Notations and Variational Setting

In this section, we present some necessary properties and important results about Lebesgue and Sobolev spaces with variable exponents. For interested readers, several results and other properties can be found in the papers of Fan and Zhao [24], Harjulehto et al. [25], Mihăilescu and Rădulescu [26], Rădulescu and Repovš [14], and Zhang and Fu [27].

We consider the set $C_+(\mathbb{R}^N)$, constituted by all continuous functions on $\mathbb{R}^N$ with values in $(1,\infty)$. For each $\sigma \in C_+(\mathbb{R}^N)$, the variable exponent Lebesgue space $L^{\sigma(z)}(\mathbb{R}^N)$ is the set of all measurable function ψ, for which $\rho_{\sigma(.)}(\psi)$ is finite, where

$$\rho_{\sigma(.)}(\psi) = \int_{\mathbb{R}^N} |\psi(z)|^{\sigma(z)} dz.$$

The space $L^{\sigma(z)}(\mathbb{R}^N)$ is endowed with with the following norm:

$$|\psi|_{\sigma(z)} = \inf\left\{\mu > 0 : \rho_{\sigma(.)}\left(\frac{\psi(z)}{\mu}\right) \leq 1\right\}.$$

We recall that $L^{\sigma(x)}(\mathbb{R}^N)$ is a Banach space; moreover, if in addition, we have $1 < \sigma^- \leq \sigma^+ < \infty$, then $L^{\sigma(x)}(\mathbb{R}^N)$ becomes reflexive and also separable, where σ^+ and σ^- are given by

$$\sigma^- = \inf_{z\in\mathbb{R}^N} \sigma(z) \text{ and } \sigma^+ = \sup_{z\in\mathbb{R}^N} \sigma(z).$$

Also, we recall that if σ' is such that $\frac{1}{\sigma(z)} + \frac{1}{\sigma'(z)} = 1$, then for each $\varphi \in L^{\sigma(z)}(\mathbb{R}^N)$ and each $\psi \in L^{\sigma'(z)}(\mathbb{R}^N)$, we have

$$\left|\int_{\mathbb{R}^N} \varphi\psi dz\right| \leq (\frac{1}{\sigma^-} + \frac{1}{(\sigma')^-})|\varphi|_{\sigma(z)}|\psi|_{\sigma'(z)}.$$

Proposition 1 (see [23,28])**.** *For any $\varphi \in L^{\sigma(x)}(\mathbb{R}^N)$, we have the following:*

(1) $|\varphi|_{\sigma(z)} < 1 \Leftrightarrow \rho_{\sigma(z)}(\varphi) < 1$. *Moreover, the last equivalence holds true if we replace $<$ with $>$ or with $=$.*

(2) *In the case when $|\varphi|_{\sigma(z)} > 1$, we have the following inequality:*

$$|\varphi|^{\sigma^-}_{\sigma(z)} \leq \rho_{\sigma(.)}(\varphi) \leq |\varphi|^{\sigma^+}_{\sigma(z)}.$$

Moreover, in the case when $|\varphi|_{\sigma(z)} < 1$, then we have the following inequality

$$|\varphi|^{\sigma^+}_{\sigma(z)} \leq \rho_{\sigma(.)}(\varphi) \leq |\varphi|^{\sigma^-}_{\sigma(z)}.$$

Proposition 2 (see [23])**.** *Assume that σ and θ are measurable functions such that for all $z \in \mathbb{R}^N$, we have*

$$\theta \in L^\infty(\mathbb{R}^N) \text{ and } 1 \leq \sigma(z)\theta(z) \leq \infty.$$

Then, for any nontrivial function ψ in $L^{\sigma(z)}(\mathbb{R}^N)$, we have

(1) *In the case when $|\psi|_{\theta(z)\sigma(z)} \leq 1$, we have*

$$|\psi|^{\sigma^+}_{\theta(z)\sigma(z)} \leq ||\psi|^{\theta(z)}|_{\sigma(z)} \leq |\psi|^{\sigma^-}_{\theta(z)\sigma(z)}.$$

(2) *In the case when $|\psi|_{\theta(z)\sigma(z)} \geq 1$, we have*

$$|\psi|^{\sigma^-}_{\theta(z)\sigma(z)} \leq ||\psi|^{\theta(z)}|_{\sigma(z)} \leq |\psi|^{\sigma^+}_{\theta(z)\sigma(z)}.$$

Next, for a continuous function σ on $\mathbb{R}^N$ and a continuous symmetric function q on $\mathbb{R}^{2N}$, such that

$$1 < q^- \leq q(z,t) \leq q^+ < \infty, \text{ and } 1 < \sigma^- \leq \sigma(z) \leq \sigma^+ < \infty. \tag{5}$$

We define the space Λ, by

$$\Lambda = \left\{ \psi \in L^{\sigma(z)}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|\psi(z) - \psi(t)|^{q(z,t)}}{|z-t|^{N+sq(z,t)}} dzdt < \infty \right\}.$$

The space Λ is endowed with the following norm:

$$||\psi||_{\Lambda} = |\psi|_{L^{\sigma(z)}(\mathbb{R}^N)} + [\psi]_{s,q(z,t)},$$

where

$$[\psi]_{s,q(z,t)} = \inf \left\{ t > 0 : \int_{\mathbb{R}^{2N}} \frac{|\psi(z) - \psi(t)|^{q(z,t)}}{t^{q(z,t)}|z-t|^{N+sq(z,t)}} dzdt \leq 1 \right\}.$$

We recall that Λ is a Banach space; it is, in addition, separable and reflexive.

Theorem 3 (see [23]). *Assume that σ is a continuous function on $\mathbb{R}^N$ and q is a continuous function on $\mathbb{R}^{2N}$; satisfying Equation (5) and, in addition, for each $(z,t) \in \mathbb{R}^{2N}$, we have*

$$q(z,z) \leq \sigma(z), \text{ and } sq(z,t) < N.$$

If $\delta \in C_+(\mathbb{R}^N)$ is such that

$$1 < \tau^- \leq \delta(z) < q_s^*(z) := \frac{Nq(z,z)}{N - sq(z,z)}, \quad \forall z \in \mathbb{R}^N,$$

then, we have the following important properties:

(i) We have a compact embedding from Λ into $L_{loc}^{\delta(z)}(\mathbb{R}^N)$.

(ii) If for all $z \in \mathbb{R}^N$, we have

$$q(z,z) \leq \delta(z) \text{ and } \inf_{z \in \mathbb{R}^N} (q_s^*(z) - \delta(z)) > 0,$$

then, we have a continuous embedding from Λ into $L^{\delta(z)}(\mathbb{R}^N)$. So, for each $g \in \Lambda$, we have

$$|g|_{L^{\delta(z)}} \leq C||g||_{\Lambda},$$

for some positive constant C.

Lemma 4 (See [23]). *Let $\psi \in \Lambda$, and put*

$$\chi(u) = \int_{\mathbb{R}^{2N}} \frac{|\psi(z) - \psi(t)|^{q(z,t)}}{|z-t|^{N+sq(z,t)}} dzdt + \int_{\mathbb{R}^N} |\psi(z)|^{p(z)} dz.$$

If $1 \leq ||\psi||_{\Lambda} < \infty$, then

$$||\psi||_{\Lambda}^{q^-} \leq \chi(\psi) \leq ||\psi||_{\Lambda}^{q^+},$$

and if $||\psi||_{\Lambda} \leq 1$ then

$$||\psi||_{\Lambda}^{q^+} \leq \chi(\psi) \leq ||\psi||_{\Lambda}^{q^-}.$$

In the rest of this work, we assume that the functions q_i and σ are continuous with values in $(1, \infty)$, q_i is symmetric and satisfies Equation (5). Also, we assume that the function q, defined by

$$q(z,t) = max\{q_1(z,t), q_2(z,t)\},$$

satisfies (5). We assume further that $b \in L^{\frac{r(z)}{r(z)+\tau(z)-1}}(\mathbb{R}^N) \cap L_{loc}^{r(z)}(\mathbb{R}^N)$ is almost always positive for some function r satisfying for each $(z,t) \in \mathbb{R}^{2N}$:

$$q(z,z) \leq \min(r(z), \sigma(z)) \leq \max(r(z), \sigma(z)) < q_s^*(z), \text{ and } sq(z,t) < N.$$

By a weak solution of problem (Q_θ), we determine a function $\varphi \in \Lambda$ that satisfies for each $\psi \in \Lambda$:

$$\begin{aligned} 0 &= -\theta \int_{\mathbb{R}^N} g(z,\varphi(z))\psi(z)dz \\ &+ \int_{\mathbb{R}^{2N}} \frac{|\varphi(z)-\varphi(t)|^{q_1(z,t)-2}(\varphi(z)-\varphi(t))(\psi(z)-\psi(t))}{|z-t|^{N+sq_1(z,t)}}dzdt \\ &+ \int_{\mathbb{R}^{2N}} \frac{|\varphi(z)-\varphi(t)|^{q_2(z,t)-2}(\varphi(z)-\varphi(t))(\psi(z)-\psi(t))}{|z-t|^{N+sq_2(z,t)}}dzdt \\ &+ \int_{\mathbb{R}^N} |\varphi(z)|^{\sigma(z)-2}\varphi(z)\psi(z)dz - \int_{\mathbb{R}^N} b(z)|\psi|^{-\tau(z)}\psi(z)dz. \end{aligned}$$

Associate to the problem (Q_θ), we define the functional $J_\theta : \Lambda \to \mathbb{R}$, by

$$J_\theta(\psi) = \Phi(\psi) - \int_{\mathbb{R}^N} \frac{b(z)}{1-\tau}|\psi|^{1-\tau(z)}dz - \theta \int_{\mathbb{R}^N} G(z,\psi(z))dz,$$

where $G(z,t) = \int_0^t g(z,s)ds$, and Φ is defined on Λ by

$$\begin{aligned} \Phi(\psi) &= \int_{\mathbb{R}^N} \frac{|\psi(z)-\psi(t)|^{q_1(z,t)}}{q_1(z,t)|z-t|^{N+sq_1(z,t)}}dzdt + \int_{\mathbb{R}^N} \frac{|\psi(z)-\psi(t)|^{q_2(z,t)}}{q_2(z,t)|z-t|^{N+sq_2(z,t)}}dzdt \\ &+ \int_{\mathbb{R}^N} \frac{|\psi(z)|^{\sigma(z)}}{\sigma(z)}dz. \end{aligned}$$

3. Existence Result for Q_θ and Its Proof

In this section, we give and prove the main result of this work. Our proofs are based on the following assumptions.

(M_1) The function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable such that for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}$, we have

$$g(x,y) \le cf(x)|y|^{\beta(x)-2}y, \quad \text{with}$$

for some non-negative function f in $L^{S(x)}(\mathbb{R}^N)$, where $c > 0$, and the functions S, β are continuous on $\mathbb{R}^N$ and satisfy

$$1 < \beta(x) < q(x,x) < \frac{N}{s} < S(x), \quad \text{and} \quad q(x,x) \le \beta(x)\frac{S(x)}{S(x)-1}.$$

(M_2) For each $t \in \mathbb{R}$, and for each z in some bounded domain Ω, we have $g(z,t) \ge 0$.

Theorem 5. *Assume that Equation (5) holds. If hypotheses (M_1) and (M_2) are satisfied, then, for each $\theta > 0$, problem Q_θ admits a nontrivial weak solution.*

The proof of Theorem 5 is summarized in three lemmas.

Lemma 6. *Under assumption (M_1), J_θ is coercive in Λ.*

Proof. Let $\psi \in F$ such that $\|\psi\| > 1$, then from Lemma 4, we have

$$\begin{aligned}\Phi(\psi) &\geq \frac{1}{q^+}\int_{\mathbb{R}^{2N}} \frac{|\psi(z)-\psi(t)|^{q_1(z,t)}}{|z-t|^{N+sq_1(z,t)}}dzdt \\ &+ \frac{1}{q^+}\int_{\mathbb{R}^{2N}} \frac{|\psi(z)-\psi(t)|^{q_2(z,t)}}{|z-t|^{N+sq_2(z,t)}}dzdt + \frac{1}{p^+}\int_{\mathbb{R}^N}|\psi(z)|^{\sigma(z)}dz \\ &\geq \frac{1}{q^+}\int_{\mathbb{R}^{2N}} \frac{|\psi(z)-\psi(t)|^{q(z,t)}}{|z-t|^{N+sq(z,t)}}dzdt + \frac{1}{p^+}\int_{\mathbb{R}^N}|\psi(z)|^{\sigma(z)}dz \\ &\geq \min\left(\frac{1}{q^+},\frac{1}{p^+}\right)||\psi||_{\Lambda}^{q^-} \end{aligned} \tag{6}$$

Now, the Hölder's inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{b(z)}{1-\tau}\psi^{1-\tau}dz &\leq \frac{1}{1-\tau^+}\int_{\mathbb{R}^N} b(z)\psi^{1-\tau}dz \\ &\leq \frac{1}{1-\tau^+}|b|_{\frac{r(z)}{r(z)+\tau(z)-1}}||\psi|^{1-\tau}|_{\frac{r(z)}{1-\tau}}. \end{aligned} \tag{7}$$

From Proposition 1 and Theorem 3, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{b(z)}{1-\tau}\psi^{1-\tau}dz &\leq \frac{1}{1-\tau^+}|b|_{\frac{r(z)}{r(z)+\tau(z)-1}}\max\left(|\psi|_{r(z)}^{1-\tau^+},|\psi|_{r(z)}^{1-\tau^-}\right) \\ &\leq \frac{c_1}{1-\tau^+}|b|_{\frac{r(z)}{r(z)+\tau(z)-1}}||\psi||_{\Lambda}^{1-\tau^-}, \end{aligned} \tag{8}$$

for some positive constant c_1.

Next, from hypothesis (M_1), Proposition 2, and the Hölder inequality, one has

$$\begin{aligned}\int_{\mathbb{R}^N} G(z,\psi(z))dz &\leq c\int_{\mathbb{R}^N} f(z)|\psi(z)|^{\delta(z)}dz \\ &\leq c|f|_{\epsilon(z)}||\psi|^{\delta(z)}|_{\epsilon'(z)} \\ &\leq c|f|_{\epsilon(z)}\max\left(|\psi|^{\delta^+}_{\epsilon'(z)\delta(z)},|\psi|^{\delta^-}_{\epsilon'(z)\delta(z)}\right). \end{aligned} \tag{9}$$

On the other hand, using hypothesis (M_1), we obtain

$$q_s^*(z)-\delta(z)\epsilon'(z) = \frac{N\epsilon(z)(q(z,z)-\delta(z))+q(z,z)(\epsilon(z)\delta(z)s-N)}{(\epsilon(z)-1)(N-sq(z,z))} > 0,$$

and

$$\delta(z)\epsilon'(z) = \delta(z)\frac{\epsilon(z)}{\epsilon(z)-1} \geq q(z,z).$$

So, Theorem 3 implies the existence of $c_2 > 0$, for which

$$\int_{\mathbb{R}^N} G(z,\psi(z))dz \leq c_2|f|_{\epsilon(z)}||\psi||_{\Lambda}^{\delta^+}. \tag{10}$$

Therefore, by combining Equations (6) and (8) with Equation (10), we obtain

$$J_\theta(\psi) \geq \min\left(\frac{1}{q^+},\frac{1}{p^+}\right)||\psi||_{\Lambda}^{q^-} - \frac{c_1}{1-\tau^+}|b|_{\frac{r(z)}{r(z)+\tau(z)-1}}||\psi||_{\Lambda}^{1-\tau^-} - c_2\theta|f|_{\epsilon(z)}||\psi||_{\Lambda}^{\delta^+}.$$

Since $1-\tau^- < \delta^+ < q^-$, then, we conclude that

$$\lim_{||\psi||\to\infty} J_\theta(\psi) = \infty.$$

That is, J_θ is coercive. $\square$

Lemma 7. *Assume that hypothesis (M_2) holds. Then, we have a non-negative nontrivial function $\phi \in \Lambda$ such that for a small enough t, $J_\theta(t\phi) < 0$.*

Proof. Let ϕ be a non-negative function in $C_0^\infty(\mathbb{R}^N)$, such that $\phi \leq 1$ in $\mathbb{R}^N$. Let $t \in (0,1)$; so, by Lemma 4, we obtain

$$\Phi(t\phi) \quad \leq \quad t^{\min(q_1^-, q_2^-, p^-)} \left(\frac{1}{q_1^-}[\phi]_{s,q_1}^{q_1^-} + \frac{1}{q_2^-}[\phi]_{s,q_2}^{q_2^-} + \frac{1}{p^-}\int_\Omega |\phi|^{\sigma(z)} dz \right).$$

Therefore, it follows that

$$\begin{aligned} J_\theta(t\phi) \quad &\leq \quad t^{\min(q_1^-, q_2^-, p^-)} \left(\frac{1}{q_1^-}[\phi]_{s,q_1}^{q_1^-} + \frac{1}{q_2^-}[\phi]_{s,q_2}^{q_2^-} + \frac{1}{p^-}\int_\Omega |\phi|^{\sigma(z)} dz \right) \\ &\quad -t^{1-\tau^-} \int_\Omega \frac{b(z)}{1-\tau} \phi^{1-\tau} dz. \end{aligned} \tag{11}$$

We point out that

$$\frac{1}{q_1^-}[\phi]_{s,q_1}^{q_1^-} + \frac{1}{q_2^-}[\phi]_{s,q_2}^{q_2^-} + \frac{1}{p^-}\int_\Omega |\phi|^{\sigma(z)} dz > 0.$$

Indeed, if

$$\frac{1}{q_1^-}[\phi]_{s,q_1}^{q_1^-} + \frac{1}{q_2^-}[\phi]_{s,q_2}^{q_2^-} + \frac{1}{p^-}\int_{\mathbb{R}^N} |\phi|^{\sigma(z)} dz = 0,$$

then, we obtain that $[\phi]_{s,q_1} = 0$, and $\int_{\mathbb{R}^N} |\phi|^{\sigma(z)} dz = 0$, which implies that $||\phi|| = 0$, that is, $\phi = 0$ in $\mathbb{R}^N$, and we obtain a contradiction.
Since $\min(q_1^-, q_2^-, p^-) > 1 - \tau^-$, then, from (11), we can see that t satisfies

$$0 < t < \min\left(1, \left(\frac{\frac{1}{1-\tau^-}\int_\Omega b(z)\phi^{1-\tau} dz}{\frac{1}{q_1^-}[\phi]_{s,q_1}^{q_1^-} + \frac{1}{q_2^-}[\phi]_{s,q_2}^{q_2^-} + \frac{1}{p^-}\int_\Omega |\phi|^{\sigma(z)} dz} \right)^{\frac{1}{\min(q_1^-, q_2^-, p^-)}} \right);$$

then, we conclude that $J_\theta(t\phi) < 0$. $\square$

Next, we set

$$L_\theta^+ = \inf_{\psi \in \Lambda} J_\theta(\psi). \tag{12}$$

Lemma 8. *Under assertions (M_1) and (M_2), there exists $\psi_* \in \Lambda$ such that*

$$J_\theta(\psi_*) = L_\theta^+ < 0.$$

Proof. Let $\{\psi_n\}$ be a sequence that satisfies

$$J_\theta(\psi_n) \to L_\theta^+, \quad \text{as} \quad n \to \infty.$$

From the coercivity of J_θ, we can deduce the boundedness of $\{\psi_n\}$ in Λ. Indeed, if this is not true, without loss of generality, we can assume that $||\psi_n||_\Lambda \to \infty$. Therefore, the coercivity of J_θ implies that

$$J_\theta(\psi_n) \to \infty, \text{ as, } n \to \infty,$$

which is a contradiction. Hence, $\{\psi_n\}$ is bounded in a reflexive space F. So, by Theorem 3, there exists a subsequence still denoted by $\{\psi_n\}$ and there exists $\psi_* \in F$ such that, as n tends to infinity, we have

$$\begin{cases} \psi_n \rightharpoonup \psi_* \ \text{ weakly } \ in \ \ F \\ \psi_n \to \psi_* \ \text{ strongly } \ in \ \ L^{\alpha(z)}_{loc}(\mathbb{R}^N), \ \ 1 \leq \alpha(z) < q^*_s(z) \\ \psi_n \to \psi_* \ \text{ a.e } \ \text{ in } \ \ \mathbb{R}^N. \end{cases}$$

We begin by proving that

$$\lim_{n\to+\infty} \int_\Omega b(z)|\psi_n|^{1-\tau} dz = \int_\Omega b(z)|\psi_*|^{1-\tau} dz. \tag{13}$$

By combining the boundedness of the sequence $\{\psi_n\}$ is Λ with Theorem 3, we deduce that $\{\psi_n\}$ is also bounded in $L^{r(z)}_{loc}(\mathbb{R}^N)$. By this fact and using Vitali's Theorem, it suffices to ensure the absolute continuity of the following set:

$$\left\{ \int_{\mathbb{R}^N} b(z)|\psi_n|^{1-\tau} dz, n \in \mathbf{N} \right\}.$$

Let $\eta > 0$. Since $\int_{\mathbb{R}^N} |b(z)|^{\frac{r(z)}{r(z)+\tau(z)-1}} dz$ is absolutely continuous; then, from Proposition 2, we obtain that for any $|\Omega'| < \eta$, there exist $\xi, \mu > 0$, such that

$$|b|^{\mu}_{\frac{r(z)}{r(z)+\tau(z)-1}} \leq \int_{\Omega'} |b(z)|^{\frac{r(z)}{r(z)+\tau(z)-1}} dz \leq \eta^{\mu}.$$

Consequently, from Equation (8), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} b(z)|\psi_n|^{1-\tau(z)} dz &\leq |b|_{\frac{r(z)}{r(z)+\tau(z)-1}} |\psi_n|^{\chi}_{r(z)} \\ &\leq \eta|\psi_n|^{\chi}_{r(z)}, \end{aligned}$$

where $\chi = 1 - \tau^+$ if $|\psi_n|_{r(z)} < 1$ and $1 - \tau^-$ if $|\psi_n|_{r(z)} > 1$.

Since $r(z) < q^*_s(z)$, $|\psi_n|_{r(z)}$ is bounded, and then Equation (13) holds true.

Next, we shall prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} G(z, \psi_n(z)) dz = \int_{\mathbb{R}^N} G(z, \psi_*(z)) dz. \tag{14}$$

Let $\varepsilon > 0$; then, from condition (M_1), we deduce the existence of $c_\varepsilon > 0$ for which we have

$$|G(z, \psi_n(z))| \leq \frac{c_\varepsilon}{\tau^-} |f(z)||\psi_n|^{\delta(z)}.$$

By combining the facts that $\psi_n \rightharpoonup \psi_*$ in Λ and $1 \leq \epsilon'(z)\delta(z) < q^*_s$, with the compact embedding results, we deduce the existence of a sub-sequence (denoted also by $\{\psi_n\}$) that converges strongly to ψ_* in $L^{\epsilon'(z)\delta(z)}_{loc}(\mathbb{R}^N)$. Therefore, there exists $h \in L^{\delta(z)\epsilon'(z)}_{loc}(\mathbb{R}^N)$ such that, $|\psi_n(z)| \leq h(z)$. This implies that

$$|G(z, \psi_n(z))| \leq \frac{c_\varepsilon}{\tau^-} |f(z)||h(z)|^{\delta(z)}.$$

So, we obtain

$$\int_{\mathbb{R}^N} |G(z, \psi_n(z))| dz \leq \frac{c_\varepsilon}{\delta^-} |f|_{\epsilon(z)} \max\left(|h|^{\delta^+}_{\epsilon'(z)\delta(z)}, |h|^{\delta^-}_{\epsilon'(z)\delta(z)} \right).$$

Hence, Equation (14) is a consequence from the combination of Proposition 2 with the Lebesgue-dominated convergence theorem.

Finally, since $\psi_n \to \psi_*$ a.e in $\mathbb{R}^N$, and by Fatou's lemma, we obtain

$$\Phi(\psi_*) \leq \lim_{n\to\infty} \inf \Phi(\psi_n). \tag{15}$$

So, by combining Equations (13) and (14) with Equation (15), we deduce that J_θ is lower semi-continuous. Therefore, using (12), we obtain

$$L_\theta^+ \leq J_\theta(\psi_*) \leq \lim_{n\to\infty} \inf J_\theta(\psi_n) = L_\theta^+.$$

which ends the proof of Lemma 8, □

Proof of Theorem 5. We begin the proof by remarking that Lemma 8 implies the existence of a global minimizer ψ_* for J_θ. Therefore, for each $x > 0$ and each $\psi \in \Lambda$, we obtain

$$0 \leq J_\theta(\psi_* + x\psi) - J_\theta(\psi_*).$$

By dividing the last inequality by $x > 0$ and by letting x tend to zero, we obtain

$$\begin{aligned} - & \theta \int_{\mathbb{R}^N} g(z, \psi_*(z))\psi(z)dz \\ + & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi_*(z) - \psi_*(t)|^{q_1(z,t)-2}(\psi_*(z) - \psi_*(t))(\psi(z) - \psi(t))}{|z-t|^{N+sq_1(z,)}} dzdt \\ + & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi_*(z) - \psi_*(t)|^{q_2(z,t)-2}(\psi_*(z) - \psi_*(t))(\psi(z) - \psi(t))}{|z-t|^{N+sq_2(z,)}} dzdt \\ + & \int_{\mathbb{R}^N} |\psi_*(z)|^{\sigma(z)-2}\psi_*(z)\psi(z)dz - \int_{\mathbb{R}^N} \frac{b(z)}{\psi_*^{\tau(z)}(z)\psi(z)} \geq 0. \end{aligned}$$

The fact that ψ is arbitrary in Λ, implies that the last inequality holds if we replace ψ by $-\psi$. This means that the last inequality becomes an equality, and consequently, ψ_* is a weak solution for problem (Q_θ). Finally, since $J_\theta(\psi_*) < 0$, then ψ_* is nontrivial. □

4. Conclusions

In this paper, we have investigated the existence of solutions. More precisely, we have studied the energy functional in generalized Sobolev spaces with variable exponents and fractional order and proved that this functional has a global minimizer, which is a weak solution to the studied problem. Our main tools are based essentially on the combination of the variational method with some monotony arguments. This is a very interesting study that we will develop and extend to a double-phase problem in the future, with a singularity of the Hardy type and with logarithmic nonlinearity.

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