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# Distributed Control for Non-Cooperative Systems Governed by Time-Fractional Hyperbolic Operators

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**Abstract:** This paper studies distributed optimal control for non-cooperative systems involving time-fractional hyperbolic operators. Through the application of the Lax–Milgram theorem, we confirm the existence and uniqueness of weak solutions. Central to our approach is the utilization of the linear quadratic cost functional, which is meticulously crafted to encapsulate the interplay between the system's state and control variables. This functional serves as a pivotal tool in imposing constraints on the dynamic system under consideration, facilitating a nuanced understanding of its controllability. Using the Euler–Lagrange first-order optimality conditions with an adjoint problem defined by means of the right-time fractional derivative in the Caputo sense, we obtain an optimality system for the optimal control. Finally, some examples are analyzed.

**Keywords:** hyperbolic systems; optimal control; fractional integral; Riemann–Liouville derivative; Caputo derivative



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## 1. Introduction

Consider $n \in \mathbb{N}^*$ and $\Psi$ as a limited and open set in $\mathbb{R}^n$ with a boundary limit $\partial\Psi$ of class $C^2$, with $\varkappa \in \Psi$ denoting the space variable and $\varsigma \in (0,T)$ denoting time. For a specific $T \in (0,\infty)$, we consider $\mathfrak{U} = \Psi \times (0,T)$ and $\Upsilon = \partial\Psi \times (0,T)$ and discuss the following $2 \times 2$ non-cooperative systems:

$$\begin{cases} D_\varsigma^\beta \mathfrak{u}_1(\varkappa,\varsigma) - \Delta\mathfrak{u}_1(\varkappa,\varsigma) + \mathfrak{u}_1(\varkappa,\varsigma) - \mathfrak{u}_2(\varkappa,\varsigma) = f_1(\varkappa,\varsigma), & (\varkappa,\varsigma) \in \mathfrak{U}, \\ D_\varsigma^\beta \mathfrak{u}_2(\varkappa,\varsigma) - \Delta\mathfrak{u}_2(\varkappa,\varsigma) + \mathfrak{u}_1(\varkappa,\varsigma) + \mathfrak{u}_2(\varkappa,\varsigma) = f_2(\varkappa,\varsigma), & (\varkappa,\varsigma) \in \mathfrak{U}, \\ \mathfrak{u}_1(\varkappa,\varsigma) = \mathfrak{u}_2(\varkappa,\varsigma) = 0, & (\varkappa,\varsigma) \in \Upsilon, \\ I^{2-\beta}\mathfrak{u}_1(\varkappa,0^+) = \mathfrak{u}_{1,0}(\varkappa), & \varkappa \in \Psi, \\ I^{2-\beta}\mathfrak{u}_2(\varkappa,0^+) = \mathfrak{u}_{2,0}(\varkappa), & \varkappa \in \Psi, \\ \frac{\partial}{\partial\varsigma} I^{2-\beta}\mathfrak{u}_1(\varkappa,0^+) = \mathfrak{u}_{1,1}(\varkappa), & \varkappa \in \Psi, \\ \frac{\partial}{\partial\varsigma} I^{2-\beta}\mathfrak{u}_2(\varkappa,0^+) = \mathfrak{u}_{2,1}(\varkappa), & \varkappa \in \Psi, \end{cases} \tag{1}$$

such that $3/2 < \beta < 2, \mathfrak{u}_{1,0}, \mathfrak{u}_{2,0} \in H^2(\Psi) \cap H_0^1(\Psi), \mathfrak{u}_{1,1}, \mathfrak{u}_{2,1} \in L^2(\Psi), I^{2-\beta}\mathfrak{u}(\varkappa,0^+) = \lim_{\varsigma\to 0} I^{2-\beta}\mathfrak{u}(\varkappa,\varsigma)$ and $\frac{\partial}{\partial\varsigma} I^{2-\beta}\mathfrak{u}(\varkappa,0^+) = \lim_{\varsigma\to 0} \frac{\partial}{\partial\varsigma} I^{2-\beta}\mathfrak{u}(\varkappa,\varsigma)$, where the $\beta$-order integral $I^\beta$ and derivative $D_\varsigma^\beta$ are taken in Riemann–Liouville significance. The external force $f_1, f_2 \in L^2(\mathfrak{U})$.

Fractional partial differential equations (FPDEs) are a generalization of classical partial differential equations (PDEs) in which the derivatives are of fractional order. These equations use fractional derivatives to reflect the non-local behavior of the underlying system. The derivatives in classical PDEs reflect the rate of change of a function in relation to its independent variables. However, fractional derivatives in FPDEs reflect the memory effects, hereditary features, and anomalous diffusion observed in many physical, biological, and engineering systems. The Caputo derivative is the most widely used fractional derivative. Other fractional derivatives, such as the Riemann–Liouville derivative, are used in many settings as well. The fractional derivative used is determined by the problem at hand; for more details, see [1–3].

The non-locality of FPDEs and the absence of analytical solutions in the majority of situations make them difficult to solve. Therefore, numerical methods are typically employed to approximate the solutions. Finite difference, finite element, and spectral methods are commonly used to discretize the equations and transform them into a system of algebraic equations that can be solved numerically [4–8].

FPDEs have applications in a number of disciplines, particularly biological sciences, economics, and technology. They have been used to model diffusion processes, wave propagation in complex media, fractal behavior, and anomalous transport phenomena, among others. Additionally, FPDEs have been applied in image processing, signal processing, and finance to capture long-range dependencies and memory effects; see [9–11].

The intricate domain of optimal control for systems driven by partial differential equations (PDEs) has been extensively examined in prior research [12–15]. These foundational works have provided crucial insights into the fundamental principles governing such systems. Expanding upon this groundwork, subsequent investigations [16,17] have further elucidated the nuanced dynamics of cooperative and non-cooperative systems. By delving deeper into the interplay of various factors and exploring novel methodologies, these studies have enriched our understanding and paved the way for advancements in optimal control theory for PDE-driven systems.

**Definition 1** ([18])**.** *For given numbers $a_{ij}$, the systems*

$$\begin{cases} -\Delta \mathbb{u}_i + \sum_{i=1}^{n} a_{ij} \ \mathbb{u}_j = f_i \quad in\ \Psi, \\ \forall i = 1, 2, 3, \dots, n, \end{cases} \tag{2}$$

*are said to be cooperative if for all $i, j = 1, \dots, n$ we have $a_{ij} > 0$ for $i \neq j$; otherwise, the systems are said to be non-cooperative.*

Optimal control of fractional non-cooperative hyperbolic systems is a crucial problem in fields such as physics, engineering, and economics. It involves finding controls that optimize performance criteria. Pontryagin's maximum principle is a common approach, but analytical solutions can be challenging; thus, numerical methods and optimization algorithms are used to approximate optimal controls. Fractional systems with fractional derivatives present additional challenges. Traditional optimal control techniques can be used. Non-cooperative systems introduce strategic aspects; game theory can be used to model and analyze these interactions. For more details about optimal control of fractional order, see for example [19,20].

Our investigation delves into the intricate realm of distributed optimal control concerning time-fractional non-cooperative hyperbolic systems governed by fractional derivatives in the Riemann–Liouville sense. Our contribution lies in the advancement and generalization of prior research elucidated by Lions [13,14] to encompass a broader spectrum of fractional non-cooperative systems. These systems serve as versatile frameworks capable of encapsulating a myriad of phenomena across disciplines spanning physics, chemistry, mathematics, and biology. Initially, we substantiate the existence and uniqueness of the system's state through the lens of classical control theory, establishing a robust theoretical underpinning for subsequent analysis. The crux of our endeavor lies in delineating the adjoint problem, which serves as a pivotal instrument in characterizing fractional optimal control. Through rigorous mathematical treatment, we unveil the requisite conditions for optimality, shedding light on the intricate interplay between system dynamics and control inputs. Our study unveils theoretical insights into the dynamics of fractional non-cooperative systems while providing practical implications for the design and optimization of control strategies across diverse domains. By bridging the theoretical and practical realms, we aim to foster a deeper understanding of the underlying mechanisms governing complex dynamical systems with fractional derivatives, paving the way for enhanced control and manipulation of these systems in real-world applications.

This paper is organized as follows. In Section 2, we recall some definitions and lemmas related to fractional calculus along with some spaces in which our problem is investigated. In Section 3, the weak formulation is represented, and the existence and uniqueness of the solution are proved with the help of the Lax Milgram Theorem. In Section 4, the optimal control problem is formulated with the linear quadratic cost functional and the adjoint problem is used to examine the optimality conditions. In Section 5, we provide some application examples related to our problem. Finally, the conclusion is addressed in Section 6.

## 2. Notations

This section provides a review of some notations in fractional calculus theory, outlining key lemmas for fractional derivatives.

**Definition 2** ([21])**.** *The Riemann–Liouville fractional partial integral operator of order $\beta$ with respect to $\varsigma$ of a function $f(\varkappa,\varsigma)$ is defined by*

$$I^{\beta} f(\varkappa,\varsigma) = \frac{1}{\Gamma(\beta)} \int_0^{\varsigma} \frac{f(\varkappa,s)}{(\varsigma-s)^{1-\beta}} \mathrm{d}s, \tag{3}$$

*where $f(\cdot,\varsigma)$ is an integrable function and $\Gamma(\cdot)$ is the well-known Euler Gamma function.*

For any $n-1<\beta<n, n\in\mathbb{N}$, the Riemann–Liouville and Caputo fractional partial derivative operators are defined as follows:

**Definition 3** ([21])**.** *The left Riemann–Liouville fractional partial derivative of order $\beta$ of a function $f(\varkappa,\varsigma)$ with respect to $\varsigma$ is defined by*

$$D_{\varsigma}^{\beta} f(\varkappa,\varsigma) = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial\varsigma^n} \int_0^{\varsigma} \frac{f(\varkappa,s)}{(\varsigma-s)^{\beta-n+1}} \mathrm{d}s, \tag{4}$$

*where the function $f(\cdot,\varsigma)$ has absolutely continuous derivatives up to order $(n-1)$.*

**Definition 4** ([21])**.** *The right Caputo fractional partial derivative of order $\beta$ with respect to $\varsigma$ of a function $f(\varkappa,\varsigma)$ is defined as*

$$^{C}D_{\varsigma}^{\beta} f(\varkappa,\varsigma) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_{\varsigma}^{T} \frac{1}{(s-\varsigma)^{\beta-n+1}} \frac{\partial^n f(\varkappa,s)}{\partial s^n} \mathrm{d}s, \tag{5}$$

*where the function $f(\cdot,\varsigma)$ has absolutely continuous derivatives up to order $(n-1)$.*

In all of the aforementioned definitions, we presume that integrals exist.

**Lemma 1** ([22]). *Assume $\mathbb{u} \in C^\infty(\overline{\mathfrak{U}})$, $\chi \in C^\infty(\overline{\mathfrak{U}})$, where $\overline{\mathfrak{U}}$ is the closure of $\mathfrak{U}$. Let $A$ be a linear partial differential operator and let $A^*$ be its adjoint. Then, we gain the following equality:*

$$\begin{aligned}\int_{\mathfrak{U}}(D_\varsigma^\beta \mathbb{u}(\varkappa,\varsigma) &+ A\mathbb{u}(\varkappa,\varsigma))\chi(\varkappa,\varsigma)d\varkappa d\varsigma \\ = &\int_\Psi \chi(\varkappa,T)\tfrac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}(\varkappa,T)d\varkappa - \int_\Psi \chi(\varkappa,0)\tfrac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}(\varkappa,0^+)d\varkappa \\ &- \int_\Psi I^{2-\beta}\mathbb{u}(\varkappa,T)\tfrac{\partial\chi}{\partial\varsigma}(\varkappa,T)d\varkappa + \int_\Psi I^{2-\beta}\mathbb{u}(\varkappa,0)\tfrac{\partial\chi}{\partial\varsigma}(\varkappa,0)d\varkappa \\ &+ \int_\Upsilon \mathbb{u}(\varkappa,\varsigma)\tfrac{\partial\chi}{\partial v}(\varkappa,\varsigma)d\varkappa d\varsigma - \int_\Upsilon \tfrac{\partial\mathbb{u}}{\partial v}(\varkappa,\varsigma)\chi(\varkappa,\varsigma)d\sigma d\varsigma \\ &+ \int_{\mathfrak{U}} \mathbb{u}(\varkappa,\varsigma)\left({}^C D_\varsigma^\beta\chi(\varkappa,\varsigma) + A^*\chi(\varkappa,\varsigma)\right)d\varkappa d\varsigma.\end{aligned}$$

*where $\frac{\partial f}{\partial v} = \nabla f.v$ is the normal derivative of $f$.*

**Lemma 2** ([22]). *Let $\mathbb{u} \in C^\infty(\overline{\mathfrak{U}})$ and $\chi \in C^\infty(\overline{\mathfrak{U}})$, given $\chi|_\Sigma = 0, \chi(\varkappa,T) = 0, \chi'(\varkappa,T) = 0$, $A$ as a linear partial differential operator, and $A^*$ as its adjoint. Then, we have*

$$\begin{aligned}\int_{\mathfrak{U}}(D_\varsigma^\beta \mathbb{u}(\varkappa,\varsigma) &+ A\mathbb{u}(\varkappa,\varsigma))\chi(\varkappa,\varsigma)d\varkappa d\varsigma \\ = &- \int_\Psi \chi(\varkappa,0)\tfrac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}(\varkappa,0^+)d\varkappa \\ &+ \int_\Psi I^{2-\beta}\mathbb{u}(\varkappa,0)\tfrac{\partial\chi}{\partial\varsigma}(\varkappa,0)d\varkappa + \int_\Upsilon \mathbb{u}(\varkappa,\varsigma)\tfrac{\partial\chi}{\partial v}(\varkappa,\varsigma)d\varkappa d\varsigma \\ &+ \int_{\mathfrak{U}} \mathbb{u}(\varkappa,\varsigma)\left({}^C D_\varsigma^\beta\chi(\varkappa,\varsigma) + A^*\chi(\varkappa,\varsigma)\right)d\varkappa d\varsigma.\end{aligned}$$

We require the following function space in order to fully explore our issue:

$$W_{2,2}^\beta\left(0,T;H_0^1(\Psi),H^{-1}(\Psi)\right) = \left\{\mathbb{u} : \mathbb{u} \in L^2\left(0,T;H_0^1(\Psi)\right), D_\varsigma^\beta\mathbb{u}(\varsigma) \in L^2\left(0,T;H^{-1}(\Psi)\right)\right\}$$

with the embedding given as follows:

$$L^2\left(0,T;H_0^1(\Psi)\right) \hookrightarrow L^2(0,T;L^2(\Psi)) \hookrightarrow L^2\left(0,T;H^{-1}(\Psi)\right) \tag{6}$$

which is continuous and compact, where $H^{-1}(\Psi)$ is the dual of $H_0^1(\Psi)$; see [23,24].

## 3. Existence and Uniqueness of the Solution

Well-posedness problems can be analyzed by transforming them into weak formulations, extending to Sobolev spaces, and interpreting them as abstract variational problems using Lax–Milgram Theorem.

**Definition 5.** *For each $\varsigma \in (0,T), \mathbb{u} = (\mathbb{u}_1,\mathbb{u}_2)$ and $\chi = (\chi_1,\chi_2)$, we define a family of bilinear forms $\mu(\varsigma;\mathbb{u},\chi)$ on $(H_0^1(\Psi))^2$ by*

$$\mu(\varsigma;\mathbb{u},\chi) = (-\Delta\mathbb{u}_1 + \mathbb{u}_1 - \mathbb{u}_2,\chi_1)_{L^2(\Psi)} + (-\Delta\mathbb{u}_2 + \mathbb{u}_2 + \mathbb{u}_1,\chi_2)_{L^2(\Psi)}, \mathbb{u},\chi \in (H_0^1(\Psi))^2,$$

*which can be written as*

$$\mu(\varsigma;\mathbb{u},\chi) = \int_\Psi(\nabla\mathbb{u}_1(\varkappa)\nabla\chi_1(\varkappa) + \nabla\mathbb{u}_2(z)\nabla\chi_2(\varkappa))\mathrm{d}\varkappa + \int_\Psi[\mathbb{u}_1\chi_1 + \mathbb{u}_2\chi_2 - \mathbb{u}_2\chi_1 + \mathbb{u}_1\chi_2]d\varkappa. \tag{7}$$

**Lemma 3.** *The bilinear form $\mu(\varsigma;\mathbb{u},\chi)$ in (7) is bounded, symmetric, and satisfies the coercive condition on $\left(H_0^1(\Psi)\right)^2$, meaning that for $\mathbb{u} = (\mathbb{u}_1,\mathbb{u}_2)$ we have*

$$\mu(\varsigma;\mathbb{u},\mathbb{u}) \geq C \parallel \mathbb{u} \parallel^2_{(H_0^1(\Psi))^2}, C > 0. \tag{8}$$

**Proof.** Replacing $\chi$ with $\mathbb{u}$ $\mu(\varsigma;\mathbb{u},\chi)$ in (7), we obtain

$$\begin{aligned}\mu(\varsigma;\mathfrak{u},\mathfrak{u}) = & \int_\Psi(\nabla\mathfrak{u}_1(\varkappa)\nabla\mathfrak{u}_1(\varkappa)+\nabla\mathfrak{u}_2(\varkappa)\nabla\mathfrak{u}_2(\varkappa)\mathrm{d}\varkappa+\int_\Psi[\mathfrak{u}_1\mathfrak{u}_1+\mathfrak{u}_2\mathfrak{u}_2-\mathfrak{u}_2\mathfrak{u}_1+\mathfrak{u}_1\mathfrak{u}_2]\mathrm{d}\varkappa \\ \geq & K\|\nabla\mathfrak{u}_1(\varkappa)\|^2_{L^2(\Psi)}+K\|\mathfrak{u}_1(z)\|^2_{L^2(\Psi)}+\|\mathfrak{u}_1(\varkappa)\|^2_{L^2(\Psi)} \\ & +K\|\nabla\mathfrak{u}_2(\varkappa)\|^2_{L^2(\Psi)}+K\|\mathfrak{u}_2(\varkappa)\|^2_{L^2(\Psi)}+\|\mathfrak{u}_2(\varkappa)\|^2_{L^2(\Psi)} \\ \geq & C_1\|\mathfrak{u}_1\|^2_{H^1_0(\Psi)}+C_2\|\mathfrak{u}_2\|^2_{H^1_0(\Psi)}\geq C\|\mathfrak{u}\|^2_{(H^1_0(\Psi))^2}, C=\max(C_1,C_2)>0.\end{aligned}$$

□

In addition, we assume that $\forall\mathfrak{u},\chi\in(H^1_0(\Psi))^2$ the function. $\varsigma\to\mu(\varsigma;\mathfrak{u},\chi)$ is continuously differentiable in $[0,T]$, and the bilinear form $\mu(\varsigma;\mathfrak{u},\chi)$ is symmetric,

$$\mu(\varsigma;\mathfrak{u},\chi)=\mu(\varsigma;\chi,\mathfrak{u}) \quad \forall\mathfrak{u},\chi\in\left(H^1_0(\Psi)\right)^2. \tag{9}$$

**Lemma 4.** *If (8) and (9) hold, then the problem in (1) admits a unique solution* $\mathfrak{u}(\varsigma)=(\mathfrak{u}_1(\varsigma),\mathfrak{u}_2(\varsigma))\in(W^\beta_{2,2}(0,T;H^1_0(\Psi),H^{-1}(\Psi))^2$.

**Proof.** By implementing Lax–Milgram theorem and the coerciveness criterion, a unique element $\mathfrak{u}$ can be found in $(H^1_0(\Psi))^2$ such that

$$(D^\beta_\varsigma\mathfrak{u},\chi)_{(L^2(\mho))^2}+\mu(\varsigma;\mathfrak{u},\chi)=L(\chi) \quad \forall\chi=(\chi_1,\chi_2)\in(H^1_0(\Psi))^2 \tag{10}$$

where $L(\chi)$ is a continuous linear form on $(H^1_0(\Psi))^2$ and takes the form

$$\begin{aligned}L(\chi)= & \int_\mho(f_1(\varkappa,\varsigma)\chi_1(\varkappa,\varsigma)+f_2(\varkappa,\varsigma)\chi_2(\varkappa,\varsigma))d\varkappa d\varsigma \\ & -\int_\Psi\chi_1(\varkappa,0)\mathfrak{u}_{1,1}d\varkappa+\int_\Psi\mathfrak{u}_{1,0}\frac{\partial\chi_1}{\partial\varsigma}(\varkappa,0)d\varkappa \\ & -\int_\Omega\chi_2(\varkappa,0)\mathfrak{u}_{2,1}d\varkappa+\int_\Omega\mathfrak{u}_{2,0}\frac{\partial\chi_2}{\partial\varsigma}(\varkappa,0)d\varkappa.\end{aligned} \tag{11}$$

Then, Equation (10) is equivalent to the existence of a unique solution $\mathfrak{u}=(\mathfrak{u}_1,\mathfrak{u}_2)\in(H^1_0(\Psi))^2$ for

$$(D^\beta_\varsigma\mathfrak{u}_1-\Delta\mathfrak{u}_1+\mathfrak{u}_1-\mathfrak{u}_2,\chi_1)_{L^2(\mho)}+(D^\beta_\varsigma\mathfrak{u}_2-\Delta\mathfrak{u}_2+\mathfrak{u}_2+\mathfrak{u}_1,\chi_2)_{L^2(\mho)}=L(\chi), \tag{12}$$

which is equivalent to the time-fractional wave equations below.

$$\begin{cases} D^\beta_\varsigma\mathfrak{u}_1-\Delta\mathfrak{u}_1+\mathfrak{u}_1-\mathfrak{u}_2=f_1 \\ D^\beta_\varsigma\mathfrak{u}_2-\Delta\mathfrak{u}_2+\mathfrak{u}_2+\mathfrak{u}_1=f_2\end{cases} \tag{13}$$

Via fractional integration by parts in reverse order, we can determine whether the original formulation can be restored. Multiplying both sides of (13) $\chi=\{\chi_1,\chi_2\}$ and applying the formula for fractional integration by parts, we have

$$\begin{aligned}\int_\mho(D^\beta_\varsigma\mathfrak{u}_1(\varkappa,\varsigma) & -\Delta\mathfrak{u}_1(\varkappa,\varsigma)+\mathfrak{u}_1(\varkappa,\varsigma)-\mathfrak{u}_2(\varkappa,\varsigma))\chi_1(\varkappa,\varsigma)d\varkappa d\varsigma \\ = & -\int_\Psi\chi_1(\varkappa,0)\frac{\partial}{\partial\varsigma}I^{2-\beta}\mathfrak{u}_1(\varkappa,0^+)d\varkappa+\int_\Psi I^{2-\beta}\mathfrak{u}_1(\varkappa,0)\frac{\partial\chi_1}{\partial\varsigma}(\varkappa,0)d\varkappa \\ & +\int_\Upsilon\mathfrak{u}_1(\varkappa,\varsigma)\frac{\partial\chi_1}{\partial v}(\varkappa,\varsigma)d\varkappa d\varsigma+\int_\mho\mathfrak{u}_1(\varkappa,\varsigma)({}^CD^\beta_\varsigma\chi_1(\varkappa,\varsigma)-\Delta\chi_1(\varkappa,\varsigma))d\varkappa d\varsigma \\ & +\int_\mho(\mathfrak{u}_1(\varkappa,\varsigma)-\mathfrak{u}_2(\varkappa,\varsigma))\chi_1(\varkappa,\varsigma)d\varkappa d\varsigma=\int_\mho f_1(\varkappa,\varsigma)\chi_1(\varkappa,\varsigma)d\varkappa d\varsigma\end{aligned} \tag{14}$$

and

$$\begin{aligned}\int_\mho(D^\beta_\varsigma\mathfrak{u}_2(\varkappa,\varsigma) & -\Delta\mathfrak{u}_2(\varkappa,\varsigma)+\mathfrak{u}_2(\varkappa,\varsigma)+\mathfrak{u}_1(\varkappa,\varsigma))\chi_2(\varkappa,\varsigma)d\varkappa d\varsigma \\ = & -\int_\Psi\chi_2(\varkappa,0)\frac{\partial}{\partial\varsigma}I^{2-\beta}\mathfrak{u}_2(\varkappa,0^+)d\varkappa+\int_\Psi I^{2-\beta}\mathfrak{u}_2(\varkappa,0)\frac{\partial\chi_2}{\partial\varsigma}(\varkappa,0)d\varkappa \\ & +\int_\Upsilon\mathfrak{u}_2(\varkappa,\varsigma)\frac{\partial\chi_2}{\partial v}(\varkappa,\varsigma)d\varkappa d\varsigma+\int_\mho\mathfrak{u}_2(\varkappa,\varsigma)({}^CD^\beta_\varsigma\chi_2(\varkappa,\varsigma)-\Delta\chi_2(\varkappa,\varsigma))d\varkappa d\varsigma \\ & +\int_\mho(\mathfrak{u}_2(\varkappa,\varsigma)+\mathfrak{u}_1(\varkappa,\varsigma))\chi_2(\varkappa,\varsigma)d\varkappa d\varsigma=\int_\mho f_2(\varkappa,\varsigma)\chi_2(\varkappa,\varsigma)d\varkappa d\varsigma.\end{aligned} \tag{15}$$

Comparing the sum of (14) and (15) with (11) such that $\chi|_{\partial\Psi} = 0, \chi(T) = 0, \chi'(T) = 0$ in $\Psi$, we obtain

$$\begin{array}{rl} & -\int_\Psi \chi_1(\varkappa,0)\frac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}_1(\varkappa,0^+)d\varkappa + \int_\Psi I^{2-\beta}\mathbb{u}_1(\varkappa,0)\frac{\partial\chi_1}{\partial\varsigma}(\varkappa,0)d\varkappa \\ & + \int_Y \mathbb{u}_1(\varkappa,\varsigma)\frac{\partial\chi_1}{\partial\nu}(\varkappa,\varsigma)d\varkappa d\varsigma - \int_\Psi \chi_2(\varkappa,0)\frac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}_2(\varkappa,0)d\varkappa \\ & + \int_\Psi I^{2-\beta}\mathbb{u}_2(\varkappa,0)\frac{\partial\chi_2}{\partial\varsigma}(\varkappa,0)d\varkappa + \int_Y \mathbb{u}_2(\varkappa,\varsigma)\frac{\partial\chi_2}{\partial\nu}(\varkappa,\varsigma)d\varkappa d\varsigma \\ = & -\int_\Psi \chi_1(\varkappa,0)\mathbb{u}_{1,1}d\varkappa + \int_\Psi \mathbb{u}_{1,0}\frac{\partial\chi_1}{\partial\varsigma}(\varkappa,0)d\varkappa \\ & -\int_\Psi \chi_2(\varkappa,0)\mathbb{u}_{2,1}d\varkappa + \int_\Psi \mathbb{u}_{2,0}\frac{\partial\chi_2}{\partial\varsigma}(\varkappa,0)d\varkappa. \end{array}$$

Then, we can deduce that

$$\begin{cases} \mathbb{u}_1(\varkappa,\varsigma) = \mathbb{u}_2(\varkappa,\varsigma) = 0, & (\varkappa,\varsigma) \in Y, \\ I^{2-\beta}\mathbb{u}_1(\varkappa,0^+) = \mathbb{u}_{1,0}, & \varkappa \in \Psi, \\ \frac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}_1(\varkappa,0^+) = \mathbb{u}_{1,1}, & \varkappa \in \Psi, \\ I^{2-\beta}\mathbb{u}_2(\varkappa,0^+) = \mathbb{u}_{2,0}, & \varkappa \in \Psi, \\ \frac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}_2(\varkappa,0^+) = \mathbb{u}_{2,1}, & \varkappa \in \Psi, \end{cases} \quad (16)$$

This means that for smooth solutions and data, the two formulations (12) and (13) are equivalent (coherence principle). □

## 4. Existence and Uniqueness of an Optimal Control: A First Optimality Condition

In this section, we look at our problem's adjoint state as well as the first-order necessary and sufficient optimality requirements. Let $\mathcal{U} = (L^2(\mathfrak{U}))^2$ be the space of controls. For a control $w \in (L^2(\mathfrak{U}))^2$, the state of the system is represented as $\mathbb{u}(w) \in W^{\beta}_{2,2}(0,T;H^1_0(\Psi),H^{-1}(\Psi))^2$. The observation equation is provided by $x(w) = \mathbb{u}(w)$. Let the set of admissible controls $\mathcal{U}_{ad}$ be a closed convex subset of $\mathcal{U}$. For a given desired state $x_d \in (L^2(\mathfrak{U}))^2$, the control problem is to find the minimization of the following quadratic cost functional:

$$\min J(w) = \frac{1}{2}\| \mathbb{u}(v) - x_d \|^2_{(L^2(\mathfrak{U}))^2} + \frac{\lambda}{2}\| v \|^2_{(L^2(\mathfrak{U}))^2}, \forall v \in \mathcal{U}_{ad}, \lambda \geq 0 \quad (17)$$

subject to

$$\begin{cases} D^\beta_\varsigma \mathbb{u}_1 - \Delta\mathbb{u}_1 + \mathbb{u}_1 - \mathbb{u}_2 = f_1 + w_1, & (\varkappa,\varsigma) \in \mathfrak{U}, \\ D^\beta_\varsigma \mathbb{u}_2 - \Delta\mathbb{u}_2 + \mathbb{u}_1 + \mathbb{u}_2 = f_2 + w_2, & (\varkappa,\varsigma) \in \mathfrak{U}, \\ \mathbb{u}_1 = \mathbb{u}_2 = 0, & (\varkappa,\varsigma) \in Y, \\ I^{2-\beta}\mathbb{u}_1(\varkappa,0^+) = \mathbb{u}_{1,0}(\varkappa), & \varkappa \in \Psi, \\ I^{2-\beta}\mathbb{u}_2(\varkappa,0^+) = \mathbb{u}_{2,0}(\varkappa), & \varkappa \in \Psi, \\ \frac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}_1(\varkappa,0^+) = \mathbb{u}_{1,1}(\varkappa), & \varkappa \in \Psi, \\ \frac{\partial}{\partial\varsigma}I^{2-\beta}\mathbb{u}_2(\varkappa,0^+) = \mathbb{u}_{2,1}(\varkappa), & \varkappa \in \Psi. \end{cases} \quad (18)$$

**Theorem 1.** *Assume that (8) and (9) are satisfied. If the cost functional is provided by (17), then there exists an optimal control $w \in \mathcal{U}_{ad}$, which is characterized by the following equations:*

$$\begin{cases} {}^C D^\beta_\varsigma p_1 - \Delta p_1 + p_1 + p_2 = \mathbb{u}_1 - x_{d1} & in\ \mathfrak{U}, \\ {}^C D^\beta_\varsigma p_2 - \Delta p_2 - p_1 + p_2 = \mathbb{u}_2 - x_{d2} & in\ \mathfrak{U}, \\ p_1 = p_2 = 0 & on\ Y, \\ p_1(\varkappa,T) = p_2(\varkappa,T) = 0 & in\ \Psi, \\ \frac{\partial p_1(\varkappa,T)}{\partial\varsigma} = \frac{\partial p_2(\varkappa,T)}{\partial\varsigma} = 0 & in\ \Psi, \end{cases} \quad (19)$$

*with*

$$\int_0^T \int_\Psi (p(w) + \lambda w)(v - w) d\varkappa d\varsigma \geq 0 \tag{20}$$

*where* $p(w) = \{p_1(w), p_2(w)\} \in (L^2(0,T; (H_0^1(\Psi))^2$ *is the adjoint state.*

**Proof.** If $w \in (L^2(\mathfrak{U}))^2$ is the optimal control, then it is characterized by

$$J'(w)(v - w) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \tag{21}$$

which is equivalent to

$$(\mathbb{u}(w) - x_d, \mathbb{u}(v) - \mathbb{u}(w))_{(L^2(\mathfrak{U}))^2} + (\lambda w, v - w)_{(L^2(\mathfrak{U}))^2} \geq 0. \tag{22}$$

Because $(B^* p, \mathbb{u}) = (p, B\mathbb{u})$, where

$$\begin{aligned} B\mathbb{u} &= B\{\mathbb{u}_1(w), \mathbb{u}_2(w)\} \\ &= \{D_\varsigma^\beta \mathbb{u}_1 - \Delta \mathbb{u}_1 + \mathbb{u}_1 + \mathbb{u}_2, D_\varsigma^\beta \mathbb{u}_2 - \Delta \mathbb{u}_2 - \mathbb{u}_1 + \mathbb{u}_2\} \end{aligned}$$

for $\mathbb{u} \in (W_{2,2}^\beta(0,T; H_0^1(\Psi), H^{-1}(\Psi)))^2$, then

$$\begin{aligned} (p, B\mathbb{u}) &= (p_1, D_\varsigma^\beta \mathbb{u}_1 - \Delta \mathbb{u}_1 + \mathbb{u}_1 - \mathbb{u}_2) + (p_2, D_\varsigma^\beta \mathbb{u}_2 - \Delta \mathbb{u}_2 + \mathbb{u}_2 + \mathbb{u}_1) \\ &= (p_1, D_\varsigma^\beta \mathbb{u}_1 - \Delta \mathbb{u}_1) - (p_1, \mathbb{u}_1) - (p_1, \mathbb{u}_2) + (p_2, D_\varsigma^\beta \mathbb{u}_2 - \Delta \mathbb{u}_2) \\ &\quad -(p_2, \mathbb{u}_1) - (p_2, \mathbb{u}_2) \\ &= ({}^C D_\varsigma^\beta p_1 - \Delta p_1, \mathbb{u}_1) - (p_1, \mathbb{u}_1) - (p_1, \mathbb{u}2) + ({}^C D_\varsigma^\beta p_2 - \Delta p_2, \mathbb{u}_2) \\ &\quad -(p_2, \mathbb{u}_1) - (p_2, \mathbb{u}_2) \\ &= ({}^C D_\varsigma^\beta p_1 - \Delta p_1 + p_1 + p_2, \mathbb{u}_1) + ({}^C D_\varsigma^\beta p_2 - \Delta p_2 + p_2 - p_1, \mathbb{u}_2) \\ &= (B^* p, \mathbb{u}); \end{aligned}$$

hence,

$$\begin{aligned} B^* p(w) &= B^*\{p_1(w), p_2(w)\} \\ &= \{{}^C D_\varsigma^\beta p_1(w) - \Delta p_1(w) + p_1(w) + p_2(w), {}^C D_\varsigma^\beta p_2(w) - \Delta p_2(w) + p_2(w) - p_1(w)\}. \end{aligned}$$

Thus, we can introduce the adjoint systems as follows:

$$\begin{cases} {}^C D_\varsigma^\beta p_1(w) - \Delta p_1(w) + p_1(w) + p_2(w) = \mathbb{u}_1(w) - x_{d1} & \text{in } \mathfrak{U}, \\ {}^C D_\varsigma^\beta p_2(w) - \Delta p_2(w) + p_2(w) - p_1(w) = \mathbb{u}_2(w) - x_{d2} & \text{in } \mathfrak{U}, \end{cases} \tag{23}$$

and by substituting (23) into (22), we obtain

$$\begin{aligned} &({}^C D_\varsigma^\beta p_1(w) - \Delta p_1(w) + p_1(w) + p_2(w), \mathbb{u}_1(v) - \mathbb{u}_1(w))_{L^2(\mathfrak{U})} \\ &+({}^C D_\varsigma^\beta p_2(w) - \Delta p_2(w) + p_2(w) - p_1(w), \mathbb{u}_2(v) - \mathbb{u}_2(w))_{L^2(\mathfrak{U})} \\ &\qquad + (\lambda w, v - w)_{(L^2(\mathfrak{U}))^2} \geq 0. \end{aligned}$$

Now, applying the formula for fractional integration by parts, we obtain

$$
\begin{aligned}
& \int_0^T \int_\Psi ({}^C D_\varsigma^\beta p_1(w) - \Delta p_1(w) + p_1(w) + p_2(w))(\mathbb{u}_1(v) - \mathbb{u}_1(w)) d\varkappa d\varsigma \\
& + \int_0^T \int_\Psi ({}^C D_\varsigma^\beta p_2(w) - \Delta p_2(w) + p_2(w) - p_1(w))(\mathbb{u}_2(v) - \mathbb{u}_2(w)) d\varkappa d\varsigma \\
= & - \int_\Psi \frac{\partial}{\partial \varsigma} I^{2-\beta}(\mathbb{u}_1(v,\varkappa,0) - \mathbb{u}_1(w,\varkappa,0)) \frac{\partial}{\partial \varsigma} p_1(\varkappa,0) d\varkappa + \int_\Psi p_1(\varkappa,0) I^{2-\beta}(\mathbb{u}_1(v,\varkappa,0) - \mathbb{u}_1(w,\varkappa,0)) d\varkappa \\
& - \int_\Psi p_1(\varkappa,T) \frac{\partial}{\partial \varsigma} I^{2-\beta}(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\varsigma)) d\varkappa + \int_\Psi I^{2-\beta}(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\varsigma)) \frac{\partial p_1}{\partial \varsigma}(\varkappa,T) d\varkappa \\
& - \int_0^T \int_{\partial\Psi} p_1(\varkappa,\varsigma) \frac{\partial}{\partial \nu}(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\eta)) d\Upsilon + \int_0^T \int_{\partial\Psi} \frac{\partial}{\partial \nu} p_1(\varkappa,\varsigma)(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\varsigma)) d\Upsilon \\
& - \int_\Psi \frac{\partial}{\partial \varsigma} I^{2-\beta}(\mathbb{u}_2(v,\varkappa,0) - \mathbb{u}_2(w,\varkappa,0)) p_2(\varkappa,0) d\varkappa + \int_\Psi \frac{\partial}{\partial \varsigma} p_2(\varkappa,0) I^{2-\beta}(\mathbb{u}_2(v,\varkappa,0) - \mathbb{u}_2(w,\varkappa,0)) d\varkappa \\
& - \int_\Psi p_2(\varkappa,T) \frac{\partial}{\partial \varsigma} I^{2-\beta}(\mathbb{u}_2(v;\varkappa,T) - \mathbb{u}_2(w;\varkappa,T)) d\varkappa + \int_\Psi I^{2-\beta}(\mathbb{u}_2(v;\varkappa,T) - \mathbb{u}_2(w;\varkappa,T)) \frac{\partial p_2}{\partial \varsigma}(\varkappa,T) d\varkappa \\
& - \int_0^T \int_{\partial\Psi} p_2(\varkappa,\varsigma) \frac{\partial}{\partial \nu}(\mathbb{u}_2(v;\varkappa,\varsigma) - \mathbb{u}_2(w;\varkappa,\varsigma)) d\Upsilon + \int_0^T \int_{\partial\Psi} \frac{\partial}{\partial \nu} p_2(\varkappa,\varsigma)(\mathbb{u}_2(v;\varkappa,\varsigma) - \mathbb{u}_2(w;\varkappa,\varsigma)) d\Upsilon \\
& + \int_0^T \int_\Psi p_1(\varkappa,\varsigma)(D_\varsigma^\beta(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\varsigma)) - \Delta(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\varsigma))) d\varkappa d\varsigma \\
& + \int_0^T \int_\Psi p_2(\varkappa,\varsigma)(D_\varsigma^\beta(\mathbb{u}_2(v;\varkappa,\varsigma) - \mathbb{u}_2(w;\varkappa,\varsigma)) - \Delta(\mathbb{u}_2(v;\varkappa,\varsigma) - \mathbb{u}_2(w;\varkappa,\varsigma))) d\varkappa d\varsigma \\
& + \int_0^T \int_\Psi (p_1(\varkappa,\varsigma) + p_2(\varkappa,\varsigma))(\mathbb{u}_1(v;\varkappa,\varsigma) - \mathbb{u}_1(w;\varkappa,\varsigma)) d\varkappa d\varsigma \\
& + \int_0^T \int_\Psi (p_2(\varkappa,\varsigma) - p_1(\varkappa,\varsigma))(\mathbb{u}_2(v;\varkappa,\varsigma) - \mathbb{u}_2(w;\varkappa,\varsigma)) d\varkappa d\varsigma.
\end{aligned}
$$

Using (18), we obtain

$$
\begin{cases}
p_1 = p_2 = 0 & \text{on } \Upsilon, \\
p_1(\varkappa,T) = p_2(\varkappa,T) = 0 & \text{in } \Psi, \\
\frac{\partial p_1(\varkappa,T)}{\partial \varsigma} = \frac{\partial p_2(\varkappa,T)}{\partial \varsigma} = 0 & \text{in } \Psi
\end{cases}
$$

and

$$
\begin{aligned}
& \int_0^T \int_\Psi ({}^C D_\varsigma^\beta p_1(w) - \Delta p_1(w) + p_1(w) + p_2(w))(\mathbb{u}_1(v) - \mathbb{u}_1(w)) d\varkappa d\varsigma \\
& + \int_0^T \int_\Psi ({}^C D_\varsigma^\beta p_2(w) - \Delta p_2(w) + p_2(w) - p_1(w))(\mathbb{u}_2(v) - \mathbb{u}_2(w)) d\varkappa d\varsigma \\
= & \int_0^T \int_\Psi p_1(w)(v_1 - w_1) d\varkappa + \int_0^T \int_\Psi p_2(w)(v_2 - w_2) d\varkappa d\varsigma;
\end{aligned}
$$

hence, (22) is equivalent to

$$
\int_0^T \int_\Psi p_1(w)(v_1 - w_1) d\varkappa d\varsigma + \int_0^T \int_\Psi p_2(w)(v_2 - w_2) d\varkappa d\varsigma + (\lambda w, v - w) \geq 0, \tag{24}
$$

which is reduced to

$$
\int_0^T \int_\Psi (p(w) + \lambda w)(v - w) d\varkappa d\varsigma \geq 0. \tag{25}
$$

Thus, the proof is complete. □

**Remark 1.** *We can generalize our results to $n \times n$ non-cooperative fractional hyperbolic systems as follows:*

$$
\min J(w) = \frac{1}{2} \sum_{i=1}^{n} \| \mathbb{u}_i(v) - x_{di} \|^2_{L^2(\mathfrak{U})} + \frac{\lambda}{2} \| v \|^2_{(L^2(\mathfrak{U}))^n} \forall v \in \mathcal{U}_{ad}, \tag{26}
$$

*subject to*

$$
\begin{cases}
D_\varsigma^\beta \mathbb{u}_i(\varkappa,\varsigma) - \Delta \mathbb{u}_i(\varkappa,\varsigma) + \sum_{j=1}^{k} a_{ij} \mathbb{u}_j = f_i + w_i, & (\varkappa,\varsigma) \in \mathfrak{U}, \\
\mathbb{u}_i(\varkappa,\varsigma) = 0, & (\varkappa,\varsigma) \in \Upsilon, \\
I^{2-\beta} \mathbb{u}_i(\varkappa,0^+) = \mathbb{u}_{i,0}(\varkappa), & \varkappa \in \Psi, \\
\frac{\partial}{\partial \varsigma} I^{2-\beta} \mathbb{u}_i(\varkappa,0^+) = \mathbb{u}_{i,1}(\varkappa), & \varkappa \in \Psi,
\end{cases}
$$

*where*

$$
a_{ij} = \begin{cases} -1, & i < j \\ 1, & i \geq j. \end{cases}
$$

*The optimal control is characterized by the following equations:*

$$\begin{cases} {}^{C}D_{\varsigma}^{\beta} p_i - \Delta p_i + \sum_{j=1}^{n} a_{ji} p_i = \mathfrak{u}_i - x_{di} & \text{in } \mathfrak{U}, \\ p_i = 0 & \text{on } \Upsilon, \\ p_i(\varkappa, T) = 0 & \text{in } \Psi, \\ \frac{\partial p_i(\varkappa,T)}{\partial \varsigma} = 0 & \text{in } \Psi, \end{cases} \tag{27}$$

*with*

$$\int_0^T \int_{\Psi} (p(w) + \lambda w)(v - w) d\varkappa d\varsigma \geq 0. \tag{28}$$

## 5. Applications

In this section, we provide specific application examples to demonstrate the effectiveness of our results and justify the real contribution of these results.

**Example 1.** *In the case with no constraints on the control, i.e., $\mathcal{U}_{ad} = \mathcal{U}$, (20) reduces to*

$$P + \lambda w = 0,$$

*and we may put $w = -\frac{1}{\lambda} p$ from Equations (18) and (19). Then, the optimal control is provided by the following system of fractional partial differential equations:*

$$\begin{cases} D_{\varsigma}^{\beta} \mathfrak{u}_1 - \Delta \mathfrak{u}_1 + \mathfrak{u}_1 - \mathfrak{u}_2 = f_1 - \frac{1}{\lambda} p_1, & (\varkappa, \varsigma) \in \mathfrak{U}, \\ D_{\varsigma}^{\beta} \mathfrak{u}_2 - \Delta \mathfrak{u}_2 + \mathfrak{u}_1 + \mathfrak{u}_2 = f_2 - \frac{1}{\lambda} p_2, & (\varkappa, \varsigma) \in \mathfrak{U}, \\ \mathfrak{u}_1 = \mathfrak{u}_2 = 0, & (\varkappa, \varsigma) \in \Upsilon, \\ I^{2-\beta} \mathfrak{u}_1(\varkappa, 0^+) = \mathfrak{u}_{1,0}(\varkappa), & \varkappa \in \Psi, \\ I^{2-\beta} \mathfrak{u}_2(\varkappa, 0^+) = \mathfrak{u}_{2,0}(\varkappa), & \varkappa \in \Psi, \\ \frac{\partial}{\partial \varsigma} I^{2-\beta} \mathfrak{u}_1(\varkappa, 0^+) = \mathfrak{u}_{1,1}(\varkappa), & \varkappa \in \Psi, \\ \frac{\partial}{\partial \varsigma} I^{2-\beta} \mathfrak{u}_2(\varkappa, 0^+) = \mathfrak{u}_{2,1}(\varkappa), & \varkappa \in \Psi, \\ {}^{C}D_{\varsigma}^{\beta} p_1 - \Delta p_1 + p_1 + p_2 = \mathfrak{u}_1 - x_{d1} & \text{in } \mathfrak{U}, \\ {}^{C}D_{\varsigma}^{\beta} p_2 - \Delta p_2 - p_1 + p_2 = \mathfrak{u}_2 - x_{d2} & \text{in } \mathfrak{U}, \\ p_1 = p_2 = 0 & \text{on } \Upsilon, \\ p_1(\varkappa, T) = p_2(\varkappa, T) = 0 & \text{in } \Psi, \\ \frac{\partial p_1(\varkappa,T)}{\partial \varsigma} = \frac{\partial p_2(\varkappa,T)}{\partial \varsigma} = 0 & \text{in } \Psi, \end{cases}$$

*a system that provides a rigorous framework for determining the optimal control strategy when there are no constraints on the control. By reducing the problem to a set of equations involving the Lagrange multiplier $\lambda$, we arrive at a solution that balances the system dynamics with the given objective function and boundary conditions. This solution, characterized by the equations governing the controls $u_1$ and $u_2$, as well as the associated state variables $p_1$ and $p_2$, offers insights into the optimal allocation of resources over time. Further analysis and numerical techniques can make it possible to explore the implications of this optimal control strategy in various real-world scenarios, contributing to the advancement of control theory and its applications.*

**Example 2.** *If we take*

$$\mathcal{U}_{ad} = \left\{ v \middle| v \in L^2(\mathfrak{U}), v \geq 0 \text{ almost everywhere in } \mathfrak{U} \right\},$$

*Then (20) is equivalent to*

$$\begin{cases} w \geq 0, & \textit{almost everywhere in } \mathfrak{U}, \\ p(w) + \lambda w \geq 0, & \textit{almost every where in } \mathfrak{U}, \\ w(p(w) + \lambda w) = 0, & \textit{almostevery where in } \mathfrak{U}, \end{cases} \tag{29}$$

*a formulation highlighting the interplay between the control input $w$ and the state variable $p$. This formulation demonstrates that the optimal control strategy must not only satisfy the system dynamics but also adhere to the specified non-negativity constraint. By addressing such constraints in the optimization process, this example contributes to a more comprehensive understanding of optimal control problems in practical applications where physical or operational constraints need to be considered. Further analysis and numerical techniques can help to elucidate the implications of these constraints on the optimal control strategy, facilitating the design of effective and feasible control solutions.*

**Example 3.** *Considering the case where $\mathcal{U}_{ad} = \{v|\xi_0(\varkappa,\varsigma) \leq v(\varkappa,\varsigma) \leq \xi_1(\varkappa,\varsigma)$ almost everywhere in $\mathfrak{U}, \xi_0, \xi_1$ given functions in $L^\infty((\mathfrak{U})\}$, (20) is equivalent to the local condition*

$$(p(\varkappa,\varsigma;w) + \lambda w(\varkappa,\varsigma))(\xi - w(\varkappa,\varsigma)) \geq 0, \forall \xi \in [\xi_0(\varkappa,\varsigma)), \xi_1(\varkappa,\varsigma))], \tag{30}$$

*which simplifies to the following conditions:*

$$\begin{cases} p(\varkappa,\varsigma;w) + \lambda w(\varkappa,\varsigma) > 0, & w(\varkappa,\varsigma) = \xi_0(\varkappa,\varsigma), \\ p(\varkappa,\varsigma;w) + \lambda w(\varkappa,\varsigma) < 0, & w(\varkappa,\varsigma) = \xi_1(\varkappa,\varsigma), \\ p(\varkappa,\varsigma;w) + \lambda w(\varkappa,\varsigma) = 0, & w(\varkappa,\varsigma) = -\frac{1}{\lambda} p(\varkappa,\varsigma;w). \end{cases} \tag{31}$$

*this example contributes to a deeper understanding of optimal control problems in scenarios where control inputs are subject to specific bounds. Such insights are crucial for designing control strategies that optimize performance while adhering to practical constraints, thereby enhancing the applicability and effectiveness of optimal control techniques in real-world systems. Further analysis and computational techniques can facilitate the application of these findings to diverse control problems, fostering advancements in control theory and practice.*

## 6. Conclusions

In the current article, the problem of distributed optimal control for non-cooperative systems involving time-fractional hyperbolic operators has been investigated. Using the Lax–Milgram theorem, the solution of the considered system has been shown to exist and to be unique. The optimal conditions are established for these systems using Euler–Lagrange equations with the assistance of the adjoint problem and the quadratic cost functional. Moreover, the extension of classical control theory forms the foundation upon which we build our analysis of fractional non-cooperative systems. As discussed in [13,14], classical results provide invaluable insights into system dynamics and control strategies. However, the fractional nature of our systems necessitates adaptations and extensions of these classical tools to guarantee further applications. Finally, the results obtained from our fractional problems tend to be the same as in the classical case when the fractional order $\beta$ approaches 2.

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