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New Multiplicity Results for a Boundary Value Problem Involving a ψ-Caputo Fractional Derivative of a Function with Respect to Another Function

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Abstract: This paper considers a nonlinear impulsive fractional boundary value problem, which involves a ψ-Caputo-type fractional derivative and integral. Combining critical point theory and fractional calculus properties, such as the semigroup laws, and relationships between the fractional integration and differentiation, new multiplicity results of infinitely many solutions are established depending on some simple algebraic conditions. Finally, examples are also presented, which show that Caputo-type fractional models can be more accurate by selecting different kernels for the fractional integral and derivative.

Keywords: critical point theory; differential equation; ψ-Caputo fractional operator; infinitely many solutions

MSC: 35A15; 34B15; 26A33

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1. Introduction

Fractional calculus is an old field in mathematic study fields and an expansion of Newton Leibniz's integral calculus; namely, it is the theory of integrals and derivatives with arbitrary order. This subject dates back to 1695, when mathematician L'Hospital asked Leibniz such a question: what $d^{1/2}/dx^{1/2}$ could mean. After that, fractional calculus was developed only as a pure mathematical idea for a long time. In the most recent decades, it has developed rapidly and shown versatility in different disciplines, such as viscoelasticity [1], neural network [2,3], image processing [4], anomalous diffusion [5,6], etc. Many scholars, like Fourier, Euler, Riemann, Liouville, and Hadamard, among others, made great contributions by proposing new definitions and studying significant properties for this subject. In 1993, Miller et al. established the fractional differential equations theory and introduced several classical fractional operator definitions [7], such as the Hadamard, Caputo, and Riemann–Liouville versions.

Fractional integrals: Define $f(t) : [0, T] \to \mathbb{R}$ as an integrable function, $\gamma > 0$; we have

$$\text{Riemann–Liouville} : {}^{R-L}I^{\gamma}_{0^+} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds, t \in [0, T], \tag{1}$$

$$\text{Hadamard} : {}^{H}I^{\gamma}_{0^+} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \left(\ln \frac{t}{s}\right)^{\gamma-1} \frac{f(s)}{s} ds, t \in [0, T]. \tag{2}$$

Fractional derivatives: Define $f(t):[0,T]\to\mathbb{R}$, $n\in\mathbb{N}$, $n-1\le\gamma<n$; we have

$$\text{Riemann–Liouville}: {}^{R-L}D_{0^+}^{\gamma}f(t)=\left(\frac{d}{dt}\right)^n {}^{R-L}I_{0^+}^{n-\gamma}f(t), \tag{3}$$

$$\text{Caputo}: {}^{C}D_{0^+}^{\gamma}f(t)={}^{R-L}I_{0^+}^{n-\gamma}f^{(n)}(t), \tag{4}$$

$$\text{Hadamard}: {}^{H}D_{0^+}^{\gamma}f(t)=\left(t\frac{d}{dt}\right)^n {}^{H}I_{0^+}^{n-\gamma}f(t). \tag{5}$$

Due to abundant forms of fractional operators [7–9], it is natural for people to put forward new general fractional differentiations and integrations to unify such forms as a single one. For this reason, the general type called thhe ψ-Caputo fractional operator is proposed in some related works [10–12], whose definitions contain a nonsingular kernel depending upon a function, and the classical fractional integrals and derivatives can be acquired by choosing special kernels. This new form can more accurately describe practical problems, for instance, ref. [13] analyzed a population growth model, which showed that different kernels were chosen such that the ψ-Caputo fractional operator could model the process of demographic change more accurately. Hence, this is a very rich topic, and the theory of fractional differential equations with integrals and derivatives depending upon kernels has broad prospects for further study.

Inspired by the groundwork mentioned above, we intend to study a new class of impulsive fractional boundary value problems with a ψ-Caputo-type fractional derivative and integral as follows:

$$\begin{cases} {}^{C}D_{T^-}^{\alpha_i,\psi}({}^{C}D_{0^+}^{\alpha_i,\psi}u_i(t))-\lambda D_{u_i}f(t,u(t))=0, t\in[0,T], t\neq t_j,\\ \Delta({}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i))(t_j)=I_{ij}(u_i(t_j)),\\ u_i(T)=u_i(0)=0, \end{cases} \tag{6}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, $0=t_0<t_1<\ldots<t_{m+1}=T$, $u(t)=(u_1(t),\ldots,u_n(t))$, $0<\alpha_i\le1$. The function $\psi(t)$ is increasing and satisfies with $\psi(t)\in C^1([0,T])$, $\psi'(t)\neq0$ for all $t\in[0,T]$, $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$, with $f(\cdot,u(t))\in C([0,T])$ and $f(t,\cdot,\ldots,\cdot)\in C^1([\mathbb{R}^n])$, $I_{ij}\in C^1([\mathbb{R}])$; $I_{0^+}^{1-\alpha_i,\psi}$ is the ψ-Riemann–Liouville fractional integral with order $1-\alpha_i$; ${}^{C}D_{T^-}^{\alpha_i,\psi}$ and ${}^{C}D_{0^+}^{\alpha_i,\psi}$ are right and left ψ-Caputo fractional derivatives with order α_i, and D_uf is the partial derivative of f with respect to u. We define the notation Δ by

$$\Delta({}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i))(t_j)={}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i)(t_j^+)-{}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i)(t_j^-),$$

where

$${}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i)(t_j^+)=\lim_{t\to t_j^+}{}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i)(t),$$

$${}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i)(t_j^-)=\lim_{t\to t_j^-}{}^{C}D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i)(t).$$

The goal of this work is to deal with a new class of ψ-Caputo-type impulsive fractional boundary value problems. Combining critical point theory and properties of fractional calculus of thee ψ-Caputo fractional integral and derivative, new multiplicity results of infinitely many solutions are established for the problem (6). Recently, some achievements available in the references discussed the existence and multiplicity results for ψ-Caputo-type fractional boundary value problems via fixed point theorems [12,13], while few results were based on variational methods, even though variational methods are effective ways for studying the existence of solutions for fractional differential equations [14–18]. Moreover, some simple algebraic conditions are applied in the paper instead of the conventional asymptotic conditions used in previous articles because most nonlinear functions can not adapted for these asymptotic conditions. It is noted that the ψ-Caputo fractional integral and derivative are able to reduce into some well-known fractional definitions by changing the kernel function $\psi(t)$, such as Hadamard, Riemann–Liouville, and Caputo, etc., which implies that the existence results concentrating on classical fractional operators are generalized.

2. Essential Lemmas and Theorems

Definition 1 ([7,13]). *For any $t \in [a,b]$, $-\infty \leq a < b \leq +\infty$, $\alpha > 0$, $\psi(t)$ is increasing on $[a,b]$ with $\psi(t) \in C^1([a,b])$ and $\psi'(t) \neq 0$; $f(t)$ is integrable on $[a,b]$.*

(i) *Define*

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(x)(\psi(t)-\psi(x))^{\alpha-1} f(x)dx,$$
$$I_{b^-}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(x)(\psi(x)-\psi(t))^{\alpha-1} f(x)dx,$$

where $I_{a^+}^{\alpha,\psi} f(t)$ and $I_{b^-}^{\alpha,\psi} f(t)$, respectively, represent the left and right ψ-Riemann–Liouville (ψ-RL) fractional integrals of a function f with respect to another function ψ.

Moreover, the ψ-RL fractional integrals satisfy the following semigroup properties:

$$I_{a^+}^{\alpha,\psi} I_{a^+}^{\beta,\psi} f(t) = I_{a^+}^{\alpha+\beta,\psi} f(t),\ I_{b^-}^{\alpha,\psi} I_{b^-}^{\beta,\psi} f(t) = I_{b^-}^{\alpha+\beta,\psi} f(t), \forall \alpha, \beta > 0.$$

(ii) *For $0 < \alpha < 1$, define*

$$\begin{aligned} D_{a^+}^{\alpha,\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) I_{a^+}^{1-\alpha,\psi} f(t) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) \int_a^t (\psi(t)-\psi(x))^{-\alpha}\psi'(x) f(x)dx, \\ D_{b^-}^{\alpha,\psi} f(t) &= \left(\frac{-1}{\psi'(t)}\frac{d}{dt}\right) I_{b^-}^{1-\alpha,\psi} f(t) \\ &= \frac{-1}{\Gamma(1-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) \int_t^b (\psi(x)-\psi(t))^{-\alpha}\psi'(x) f(x)dx, \end{aligned}$$

where $D_{a^+}^{\alpha,\psi} f(t)$ and $D_{b^-}^{\alpha,\psi} f(t)$, respectively, represent the left and right ψ-RL fractional derivatives of a function f with respect to another function ψ.

(iii) *For $0 < \alpha < 1$, define*

$$^C D_{a^+}^{\alpha,\psi} f(t) = I_{a^+}^{1-\alpha,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (\psi(t)-\psi(x))^{-\alpha} f'(x)dx,$$
$$^C D_{b^-}^{\alpha,\psi} f(t) = I_{b^-}^{1-\alpha,\psi}\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right) f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (\psi(x)-\psi(t))^{-\alpha} f'(x)dx,$$

where $^C D_{a^+}^{\alpha,\psi} f(t)$ and $^C D_{b^-}^{\alpha,\psi} f(t)$, respectively, represent the left and right ψ-Caputo fractional derivatives of f with respect to another function ψ.

(iv) *For any $\alpha > 0$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$, and $f(t) \in C^n([a,b])$; we have*

$$^C D_{a^+}^{\alpha,\psi} f(t) = D_{a^+}^{\alpha,\psi}\left[f(t) - \Sigma_{k=0}^{n-1}\frac{1}{k!}(\psi(t)-\psi(a))^k \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k f(a)\right], \tag{7}$$
$$^C D_{b^-}^{\alpha,\psi} f(t) = D_{b^-}^{\alpha,\psi}\left[f(t) - \Sigma_{k=0}^{n-1}\frac{(-1)^k}{k!}(\psi(b)-\psi(t))^k \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k f(b)\right]. \tag{8}$$

The ψ-Caputo-type fractional derivative and integral are mainly dealt with in this paper.

Remark 1. *Some classical fractional derivatives are special cases of the ψ-Riemann–Liouville and ψ-Caputo fractional derivatives. For instance, take $\psi(t) = t$ in the ψ-Caputo fractional derivative and ψ-RL fractional derivative: we can obtain the well-known Caputo (4) fractional derivative and RL (3) fractional derivative, respectively. And we can also obtain the Hadamard (5) fractional derivative by choosing $\psi(t) = \ln t$ in the ψ-RL fractional derivative.*

Definition 2. *For any $\frac{1}{2} < \alpha_i \le 1$, $i = 1, 2, \dots, n$, $t \in [0, T]$, we define the fractional derivative space $X_{\alpha_i,\psi}$ by the closure of $C_0^\infty([0,T],\mathbb{R})$ with the weighted norm*

$$\|u_i\|_{\alpha_i,\psi} := \left(\int_0^T \mid u_i(t) \mid^2 dt + \int_0^T \mid {}^C D_{0^+}^{\alpha_i,\psi} u_i(t) \mid^2 dt \right)^{\frac{1}{2}}. \tag{9}$$

Apparently, $X_{\alpha_i,\psi}$ is the space of $u_i(t) \in L^2([0,T])$ with an α_i-order ψ-Caputo fractional derivative ${}^C D_{0^+}^{\alpha_i,\psi} u_i(t) \in L^2([0,T])$ and $u_i(T) = u_i(0) = 0$.

Remark 2. *Based on (7), (8), and $u_i(0) = u_i(T) = 0$, we can obtain*

$${}^C D_{0^+}^{\alpha_i,\psi} u_i(t) = D_{0^+}^{\alpha_i,\psi} u_i(t),\ {}^C D_{T^-}^{\alpha_i,\psi} u_i(t) = D_{T^-}^{\alpha_i,\psi} u_i(t),\ i = 1, 2, \dots, n.$$

Lemma 1 ([19])**.** *For any $i = 1, 2, \dots, n$, $X_{\alpha_i,\psi}$ is a separable and reflexive Banach space.*

Lemma 2. *Let $\frac{1}{2} < \alpha_i \le 1$; for all $u_i(t) \in X_{\alpha_i,\psi}$, we have*

$$\|u_i\|_{L^2} \le \frac{\max_{t\in[0,T]}\{\psi'(t)\}(\psi(T))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \|{}^C D_{0^+}^{\alpha_i,\psi} u_i\|_{L^2}, i = 1, 2, \dots, n, \tag{10}$$

$$\|u_i\|_\infty \le \frac{\max_{t\in[0,T]}\{\psi'(t)\}(\psi(T))^{\alpha_i - \frac{1}{2}}}{\Gamma(\alpha_i)(2(\alpha_i - 1) + 1)^{\frac{1}{2}}} \|{}^C D_{0^+}^{\alpha_i,\psi} u_i\|_{L^2}, i = 1, 2, \dots, n. \tag{11}$$

Proof. From Proposition 2.2 in [19], the conclusions can be easily obtained. □

For the sake of convenience, denote

$$M_i = \frac{(\psi(T))^{\alpha_i} \max_{t\in[0,T]}\{\psi'(t)\}}{\Gamma(\alpha_i + 1)},\ \widehat{M}_i = \frac{(\psi(T))^{\alpha_i - \frac{1}{2}} \max_{t\in[0,T]}\{\psi'(t)\}}{\Gamma(\alpha_i)(2(\alpha_i - 1) + 1)^{\frac{1}{2}}},\ i = 1, 2, \dots, n. \tag{12}$$

We can easily obtain that the norm defined as

$$\|u_i\|_{\alpha_i,\psi} := \left(\int_0^T \mid {}^C D_{0^+}^{\alpha_i,\psi} u_i(t) \mid^2 dt \right)^{\frac{1}{2}},\ \forall\, u_i(t) \in X_{\alpha_i,\psi},\ i = 1, 2, \dots, n, \tag{13}$$

is equivalent to norm (9). The norm (13) comes into effect hereinafter.

Lemma 3 ([19])**.** *Suppose that any sequence $\{u_{k_i}\}$ converges to u_i in $X_{\alpha_i,\psi}$ weakly for $\frac{1}{2} < \alpha_i \le 1$. Then, $u_{k_i} \to u_i$ in $C([0,T])$ as $k \to \infty$, i.e., $\|u_{k_i} - u_i\|_\infty \to 0$ as $k \to \infty$, $i = 1, 2, \dots, n$.*

Now, define $X = \Pi_{i=1}^n X_{\alpha_i,\psi}$ with the weighted norm

$$\|u\|_X = \sum_{i=1}^n \|u_i\|_{\alpha_i,\psi},\ u_i \in X_{\alpha_i,\psi},\ u = (u_1, \dots, u_n) \in X. \tag{14}$$

Evidently, X is a separable and reflexive Banach space.

Lemma 4. *For any $u_i(t), v_i(t) \in X_{\alpha_i,\psi}$, $i = 1, 2, \ldots, n$, we have*

$$\begin{aligned}&\int_0^T {}^C D_{T^-}^{\alpha_i,\psi}({}^C D_{0^+}^{\alpha_i,\psi} u_i(t))\psi'(t) v_i(t)dt \\ &= \int_0^T \psi'(t) {}^C D_{0^+}^{\alpha_i,\psi} u_i(t) {}^C D_{0^+}^{\alpha_i,\psi} v_i(t)dt - \sum_{j=1}^{m} I_{ij}(u_i(t_j))v_i(t_j).\end{aligned} \tag{15}$$

Proof. Drawing upon the definition of the ψ-Caputo fractional derivative in Definition 1, one has

$$\begin{aligned}&\int_0^T \psi'(t) {}^C D_{0^+}^{\alpha_i,\psi} u_i(t) {}^C D_{0^+}^{\alpha_i,\psi} v_i(t)dt \\ =&\frac{1}{\Gamma(1-\alpha_i)} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \int_0^t \psi'(t) {}^C D_{0^+}^{\alpha_i,\psi} u_i(t)(\psi(t)-\psi(x))^{-\alpha_i} v_i'(x)dxdt \\ =&\frac{1}{\Gamma(1-\alpha_i)} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \left[\int_t^T \psi'(x) {}^C D_{0^+}^{\alpha_i,\psi} u_i(x)(\psi(x)-\psi(t))^{-\alpha_i}dx\right] v_i'(t)dt \\ =&\frac{1}{\Gamma(1-\alpha_i)} \sum_{j=0}^{m} \left[\int_t^T \psi'(x) {}^C D_{0^+}^{\alpha_i,\psi} u_i(x)(\psi(x)-\psi(t))^{-\alpha_i}dx\right] v_i(t) \mid_{t=t_j}^{t=t_{j+1}} \\ &- \frac{1}{\Gamma(1-\alpha_i)} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \frac{d}{dt}\left[\int_t^T \psi'(x) {}^C D_{0^+}^{\alpha_i,\psi} u_i(x)(\psi(x)-\psi(t))^{-\alpha_i}dx\right] v_i(t)dt.\end{aligned} \tag{16}$$

Then, taking advantage of the ψ-RL fractional derivative and integral definitions, and based on Remark 2, Equation (16) can be further written as:

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha_i)} \sum_{j=0}^{m} \left[\int_t^T \psi'(x) {}^C D_{0^+}^{\alpha_i,\psi} u_i(x)(\psi(x)-\psi(t))^{-\alpha_i}dx\right] v_i(t) \mid_{t=t_j}^{t=t_{j+1}} \\ &- \frac{1}{\Gamma(1-\alpha_i)} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \frac{d}{dt}\left[\int_t^T \psi'(x) {}^C D_{0^+}^{\alpha_i,\psi} u_i(x)(\psi(x)-\psi(t))^{-\alpha_i}dx\right] v_i(t)dt \\ =& \sum_{j=0}^{m} \left[\frac{1}{\Gamma(1-\alpha_i)} \int_t^T \psi'(x) D_{0^+}^{\alpha_i,\psi} u_i(x)(\psi(x)-\psi(t))^{-\alpha_i}dx\right] v_i(t) \mid_{t=t_j}^{t=t_{j+1}} \\ &+ \int_0^T \left[\frac{-1}{\Gamma(1-\alpha_i)} (\frac{1}{\psi'(t)}\frac{d}{dt}) \int_t^T \psi'(x)(\psi(x)-\psi(t))^{-\alpha_i} D_{0^+}^{\alpha_i,\psi} u_i(x)dx\right] \psi'(t) v_i(t)dt \\ =& \sum_{j=0}^{m} \left[\frac{1}{\Gamma(1-\alpha_i)} \int_t^T \psi'(x)(\psi(x)-\psi(t))^{-\alpha_i} (\frac{1}{\psi'(x)}\frac{d}{dx}) I_{0^+}^{1-\alpha_i,\psi} u_i(x)dx\right] v_i(t) \mid_{t=t_j}^{t=t_{j+1}} \\ &+ \int_0^T D_{T^-}^{\alpha_i,\psi}(D_{0^+}^{\alpha_i,\psi} u_i(t))\psi'(t) v_i(t)dt \\ =& \sum_{j=0}^{m} -{}^C D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi} u_i(t)) v_i(t) \mid_{t=t_j}^{t=t_{j+1}} + \int_0^T D_{T^-}^{\alpha_i,\psi}(D_{0^+}^{\alpha_i,\psi} u_i(t))\psi'(t) v_i(t)dt \\ =& \sum_{j=1}^{m} {}^C D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi} u_i(t_j^+)) v_i(t_j^+) - {}^C D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi} u_i(t_j^-)) v_i(t_j^-) \\ &+ \int_0^T {}^C D_{T^-}^{\alpha_i,\psi}({}^C D_{0^+}^{\alpha_i,\psi} u_i(t))\psi'(t) v_i(t)dt \\ =& \sum_{j=1}^{m} I_{ij}(u_i(t_j)) v_i(t_j) + \int_0^T {}^C D_{T^-}^{\alpha_i,\psi}({}^C D_{0^+}^{\alpha_i,\psi} u_i(t))\psi'(t) v_i(t)dt.\end{aligned} \tag{17}$$

Uniting (16) and (17), we obtain that

$$\int_0^T {}^C D_{T^-}^{\alpha_i,\psi}({}^C D_{0^+}^{\alpha_i,\psi} u_i(t))\psi'(t)v_i(t)dt$$
$$= \int_0^T \psi'(t){}^C D_{0^+}^{\alpha_i,\psi} u_i(t){}^C D_{0^+}^{\alpha_i,\psi} v_i(t)dt - \sum_{j=1}^{m} I_{ij}(u_i(t_j))v_i(t_j),$$

for any $u_i(t), v_i(t) \in X_{\alpha_i,\psi}$, $i = 1, 2, \ldots, n$. □

Lemma 5. *For any $(v_1(t), \ldots, v_n(t)) \in X$, $\psi(t)$ is increasing on $[0, T]$ with $\psi'(t) \neq 0$ and $\psi(t) \in C^1([0, T])$ if the following relationship holds:*

$$\sum_{i=1}^{n}\int_0^T \psi'(t){}^C D_{0^+}^{\alpha_i,\psi} u_i(t){}^C D_{0^+}^{\alpha_i,\psi} v_i(t)dt - \sum_{i=1}^{n}\sum_{j=1}^{m} I_{ij}(u_i(t_j))v_i(t_j)$$
$$= \sum_{i=1}^{n}\lambda\int_0^T D_{u_i} f(t, u(t))\psi'(t)v_i(t)dt, \tag{18}$$

then we say $u(t) = (u_1(t), \ldots, u_n(t)) \in X$ is a weak solution of problem (6).

Proof. Firstly, multiplying both ends of the first equation of (6) with $\psi'(t)v_i(t)$ and integrating both ends from 0 to T simultaneously, then summing from $i = 1$ to $i = n$, we can obtain an equivalent form for problem (6). Combining with (17), we can obtain Equation (18). □

Consider the functional $\phi : X \to \mathbb{R}$ with

$$\phi(u) := \frac{1}{2}\sum_{i=1}^{n}\int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha_i,\psi} u_i(t)|^2 dt - \sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^{u_i(t_j)} I_{ij}(s)ds - \lambda\int_0^T \psi'(t)f(t, u(t))dt. \tag{19}$$

Obviously, $\phi \in C^1(X, \mathbb{R})$ and

$$\phi'(u)(v) = \sum_{i=1}^{n}\int_0^T \psi'(t){}^C D_{0^+}^{\alpha_i,\psi} u_i(t){}^C D_{0^+}^{\alpha_i,\psi} v_i(t)dt - \sum_{j=1}^{m}\sum_{i=1}^{n} I_{ij}(u_i(t_j))v_i(t_j)$$
$$- \lambda\sum_{i=1}^{n}\int_0^T D_{u_i} f(t, u(t))\psi'(t)v_i(t)dt, \forall v(t), u(t) \in X. \tag{20}$$

It is not difficult to observe that the critical points of ϕ are the solutions of problem (6).

Definition 3. *A function*

$$u \in \left\{u(t) = (u_1(t), \ldots, u_n(t) \in AC([0, T], \mathbb{R}^n) : \int_{t_j}^{t_{j+1}} | u_i(t) |^2 + | {}^C D_{0^+}^{\alpha_i,\psi} u_i(t) |^2 dt < +\infty,\right.$$
$$\left. i = 1, 2, \ldots, n, j = 0, 1, \ldots, m\right\}$$

is a classical solution of problem (6) if u satisfies the first equation of (6) a.e. on $[0, T] \setminus \{t_1, \ldots, t_m\}$, the boundary value conditions $u_i(T) = u_i(0) = 0$ hold, and the limits ${}^C D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi} u_i)(t_j^+)$ and ${}^C D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi} u_i)(t_j^-)$ exist and satisfy the impulsive conditions of (6).

Lemma 6. *The weak solution of problem (6) is also a classical solution of problem (6).*

Proof. If $u(t) = (u_1(t), \ldots, u_n(t)) \in X$ is a classical solution of problem (6), it satisfies the first equation in (6). Owing to the proof of Lemma 5, we can easily see that $u(t)$ is also a weak solution of (6). Conversely, if $u(t) \in X$ is a weak solution of (6), then

$u_i(T) = u_i(0) = 0$ and (18) holds, $i = 1, 2, \ldots, n$. Without losing generality, we choose a test function $v(t) = (v_1(t), \ldots, v_n(t))$ satisfying $v_i(t) \equiv 0$ for $t \in [0, t_j] \bigcup [t_{j+1}, T]$ and $v_i(t) \in C_0^\infty[t_j, t_{j+1}]$, $j = 1, 2, \ldots, m, i = 1, 2, \ldots, n$. Substituting $v(t)$ into (18) and using Lemma 4, we have

$$\int_{t_j}^{t_{j+1}} \psi'(t)^C D_{0^+}^{\alpha_i,\psi} u_i(t)^C D_{0^+}^{\alpha_i,\psi} v_i(t) dt = \lambda \int_{t_j}^{t_{j+1}} D_{u_i} f(t, u(t)) \psi'(t) v_i(t) dt,$$

and

$$\int_{t_j}^{t_{j+1}} {}^C D_{T^-}^{\alpha_i,\psi} ({}^C D_{0^+}^{\alpha_i,\psi} u_i(t)) \psi'(t) v_i(t) dt = \int_{t_j}^{t_{j+1}} \psi'(t)^C D_{0^+}^{\alpha_i,\psi} u_i(t)^C D_{0^+}^{\alpha_i,\psi} v_i(t) dt < \infty,$$

which implies

$$ {}^C D_{T^-}^{\alpha_i,\psi} ({}^C D_{0^+}^{\alpha_i,\psi} u_i(t)) = \lambda D_{u_i} f(t, u(t)), \forall t \in [t_j, t_{j+1}]. \tag{21}$$

Since $u_i \in X_{\alpha_i,\psi} \subset C([0, T])$, one has

$$\int_{t_j}^{t_{j+1}} | u_i(t) |^2 + | {}^C D_{0^+}^{\alpha_i,\psi} u_i(t) |^2 \, dt < +\infty.$$

From Definition 1, we can see

$$\begin{aligned} \psi'(t)^C D_{T^-}^{\alpha_i,\psi} ({}^C D_{0^+}^{\alpha_i,\psi} u_i(t)) =& \psi'(t) D_{T^-}^{\alpha_i,\psi} (D_{0^+}^{\alpha_i,\psi} u_i(t)) \\ =& \psi'(t) D_{T^-}^{\alpha_i,\psi} \left[\frac{1}{\psi'(t)} \frac{d}{dt} I_{0^+}^{1-\alpha_i,\psi} u_i(t) \right] \\ =& \frac{d}{dt} \frac{-1}{\Gamma(1-\alpha_i)} \int_t^T (\psi(x) - \psi(t))^{-\alpha_i} \frac{d}{dx} I_{0^+}^{1-\alpha_i,\psi} u_i(x) dx \\ =& \frac{d}{dt} \left[{}^C D_{T^-}^{\alpha_i,\psi} I_{0^+}^{1-\alpha_i,\psi} u_i(t) \right]. \end{aligned}$$

Since $\psi(t), f(t) \in C^1([0, T])$, from (21), we have ${}^C D_{T^-}^{\alpha_i,\psi} I_{0^+}^{1-\alpha_i,\psi} u_i(t) \in AC([t_j, t_{j+1}])$, namely, the following limits exist

$$\begin{aligned} {}^C D_{T^-}^{\alpha_i,\psi} (I_{0^+}^{1-\alpha_i,\psi} u_i)(t_j^+) &= \lim_{t \to t_j^+} {}^C D_{T^-}^{\alpha_i,\psi} (I_{0^+}^{1-\alpha_i,\psi} u_i)(t), \\ {}^C D_{T^-}^{\alpha_i,\psi} (I_{0^+}^{1-\alpha_i,\psi} u_i)(t_j^-) &= \lim_{t \to t_j^-} {}^C D_{T^-}^{\alpha_i,\psi} (I_{0^+}^{1-\alpha_i,\psi} u_i)(t). \end{aligned}$$

Substituting (21) into (18), one obtains

$$\int_0^T \psi'(t)^C D_{0^+}^{\alpha_i,\psi} u_i(t)^C D_{0^+}^{\alpha_i,\psi} v_i(t) dt - \int_0^T D_{T^-}^{\alpha_i,\psi} (D_{0^+}^{\alpha_i,\psi} u_i(t)) \psi'(t) v_i(t) dt = \sum_{j=1}^m I_{ij}(u_i(t_j)) v_i(t_j), \tag{22}$$

for $i = 1, 2, \ldots, n$. Due to Lemma 4, we know

$$\begin{aligned} & \int_0^T \psi'(t)^C D_{0^+}^{\alpha_i,\psi} u_i(t)^C D_{0^+}^{\alpha_i,\psi} v_i(t) dt \\ =& \sum_{j=1}^m {}^C D_{T^-}^{\alpha_i,\psi} (I_{0^+}^{1-\alpha_i,\psi} u_i(t_j^+)) v_i(t_j^+) - {}^C D_{T^-}^{\alpha_i,\psi} (I_{0^+}^{1-\alpha_i,\psi} u_i(t_j^-)) v_i(t_j^-) \\ & + \int_0^T D_{T^-}^{\alpha_i,\psi} (D_{0^+}^{\alpha_i,\psi} u_i(t)) \psi'(t) v_i(t) dt, \end{aligned}$$

that is,

$$\Delta({}^C D_{T^-}^{\alpha_i,\psi}(I_{0^+}^{1-\alpha_i,\psi}u_i))(t_j) = I_{ij}(u_i(t_j)),$$

for $j = 1, 2, \ldots, m, i = 1, 2, \ldots, n$. Consequently, $u_i(t)$ satisfies the first equation, impulsive conditions, and boundary conditions of problem (6). Thus, $u(t)$ is a classical solution of (6). □

Lemma 7 ([20]). *Let $\widehat{X}$ be any finite dimensional subspace of X. There exists a constant $\eta_0 > 0$ such that*

$$meas\{t \in [0, T] :| u(t) |\geq \eta_0\|u\|\} \geq \eta_0,\ \forall u(t) \in \widehat{X} \setminus \{0\}.$$

Definition 4. *X is a Banach space if functional $\phi \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition; then, for each sequence, $\{u_k\}_{k\in N} \subset X$ such that $\{\phi(u_k)\}$ is bounded and $\lim\limits_{k\to\infty} \phi'(u_k) = 0$ possesses a strongly convergent subsequence in X.*

Theorem 1 ([21]). *Assume X is an infinite dimensional Banach space; $\phi \in C^1(X, \mathbb{R})$ is an even functional and satisfies the Palais–Smale condition and $\phi(0) = 0$. Suppose that*

(i) There exist $r > 0$ and $\theta > 0$, such that $\overline{B}_r \subset \{u \in X \mid \phi(u) \geq 0\}$ and $\phi(u) \geq \theta$ for all $u \in \partial B_r$, where $B_r = \{u \in X \mid \|u\| < r\}$;

(ii) For any finite dimensional subspace $\widehat{X} \subset X$, the set $\widehat{X} \bigcap \{u \in X \mid \phi(u) \geq 0\}$ is a bounded set.

Then, ϕ possesses infinitely many critical points.

3. Multiplicity Results

In this sections, some multiplicity results of infinity many solutions are investigated for a new class of impulsive fractional boundary value problems (6).

Firstly, we introduce some essential assumptions for use in the remaining discussions.

(H_1) $\lim\limits_{\forall i:|u_i|\to\infty} \frac{f(t,u)}{\sum_{i=1}^n |u_i|^2} = \infty$ uniformly for $t \in [0, T]$, $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$;

(H_2) $0 \leq f(t, u) = o(\sum_{i=1}^n \mid u_i \mid^2)$ as $\sum_{i=1}^n \mid u_i \mid\to 0$ uniformly for $t \in [0, T]$;

(H_3) $I_{ij}(s)$ is odd and satisfies $\int_0^{u_i(t_j)} I_{ij}(s)ds \leq 0$, i = 1, 2, . . . , n, j = 1, 2, . . . , m;

(H_4) There exist constants $a_i, b_i > 0$ and $\tau_i \in [0, 1)$ such that $\mid I_{ij}(s) \mid\leq a_i + b_i \mid s \mid^{\tau_i}$, $\forall s \in \mathbb{R}$, $i = 1, 2, \ldots, n$;

(H_5) For any $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $f(t, u) = \sum_{i=1}^n \frac{\zeta_i}{2} \mid u_i \mid^2 - G(t, u)$ with $G(t, 0) \equiv 0$, and

$$\sum_{i=1}^n (\frac{\zeta_i}{2}) \mid u_i \mid^{\sigma_i}\leq G(t, u) \leq \sum_{i=1}^n \mu_i \mid u_i \mid^{\omega_i},$$

where $\sigma_i \in [0, 2)$, $\omega_i \in [0, 2)$, $\mu_i > 0$, $i = 1, 2, \ldots, n$.

Theorem 2. *Assume $f(t, u) = f(t, -u)$ and (H_1)–(H_4) hold. Then, the problem (6) possesses infinitely many solutions in X.*

Proof. Firstly, we prove that ϕ satisfies the Palais–Smale condition. Suppose sequence $\{\phi(u_k)\}$ is bounded and $\lim\limits_{k\to\infty} \phi'(u_k) = 0$, $u_k(t) = (u_{k,1}(t), \ldots, u_{k,n}(t))$. We claim that $\{u_k\}$ is bounded in X. Indeed, assume $\forall i : \|u_{k,i}\|_{\alpha_i,\psi} \to \infty (k \to \infty)$, $i = 1, 2, \ldots, n$. Based on (H_1), for any constant $\mathcal{K} > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$f(t, u_k(t)) \geq \mathcal{K} \sum_{i=1}^n \|u_{k,i}\|^2_{\alpha_i,\psi},\ \forall\ k > k_0. \tag{23}$$

Then,

$$\int_0^T f(t, u_k(t))\psi'(t)dt \geq (\psi(T) - \psi(0))\mathcal{K} \sum_{i=1}^n \|u_{k,i}\|^2_{\alpha_i,\psi},\ \forall\ k > k_0,\ t \in [0, T]. \tag{24}$$

In view of (H_4) and (12), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^{u_i(t_j)}|I_{ij}(s))|ds &\le \sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^{u_i(t_j)} a_i+b_i\mid s\mid^{\tau_i}ds \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{m}\left[a_i|u_i(t_j)|+\frac{b_i}{\tau_i+1}|u_i(t_j)|^{\tau_i+1}\right] \\ &\le m\sum_{i=1}^{n}\left[a_i\|u_i\|_\infty+\frac{b_i}{\tau_i+1}\|u_i\|_\infty^{\tau_i+1}\right] \\ &\le m\sum_{i=1}^{n}\left[a_i\widehat{M}_i\|u_i\|_{\alpha_i,\psi}+\frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_i\|_{\alpha_i,\psi}^{\tau_i+1}\right].\end{aligned} \tag{25}$$

Combining (19), (24), and (25) yields

$$\begin{aligned}\phi(u_k(t)) \le& \frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2+\sum_{i=1}^{n}m\left[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi}+\frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|_{\alpha_i,\psi}^{\tau_i+1}\right] \\ &-\lambda(\psi(T)-\psi(0))\mathcal{K}\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2,\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\phi(u_k(t))}{\sum_{i=1}^{n}\|u_{k,i}\|_{(\alpha_i,\psi)}^2} \le& \left[\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}-\lambda(\psi(T)-\psi(0))\mathcal{K}\right] \\ &+\frac{\sum_{i=1}^{n}m\left[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi}+\frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|_{\alpha_i,\psi}^{\tau_i+1}\right]}{\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2}.\end{aligned} \tag{26}$$

Since $\tau_i\in[0,1)$, $\|u_{k,i}\|_{(\alpha_i,\psi)}\to\infty$ as $k\to\infty$, $i=1,2,\dots,n$, then

$$\frac{\sum_{i=1}^{n}m\left[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi}+\frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|_{\alpha_i,\psi}^{\tau_i+1}\right]}{\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2}\to 0, k\to\infty. \tag{27}$$

Choose $\mathcal{K}$ large enough such that

$$\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}-\lambda(\psi(T)-\psi(0))\mathcal{K}<-2, \tag{28}$$

based on (26), (27), and (28), we see that $\phi(u_k(t)) < -2\sum_{i=1}^{n}\|u_{k,i}\|_{(\alpha_i,\psi)}^2$, i.e., $\phi(u_k(t))\to -\infty$ as $\|u_{k,i}\|_{(\alpha_i,\psi)}\to\infty$, $i=1,2,\dots,n$, which contradicts that $\{\phi(u_k)\}$ is bounded. Thus, $\{u_k\}$ is bounded in X. Since X is a reflexive and separable Banach space, we can obtain $u_k\rightharpoonup u^0$ in X (up to subsequences); then, $u_k\to u^0$ uniformly in $C([0,T])$ owing to Lemma 3.

Since $\psi(t), f(t)\in C^1([0,T])$, $I_{ij}\in C([\mathbb{R}])$, and $\lim_{k\to\infty}\phi'(u_k)=0$, we can obtain that

$$\begin{cases}(\phi'(u_k)-\phi'(u^0))(u_k-u^0)\le\|\phi'(u_k)\|_X\|u_k-u^0\|_X-\phi'(u^0)(u_k-u^0)\to 0,\ k\to\infty, \\ \sum_{i=1}^{n}\sum_{j=1}^{m}[I_{ij}(u_i^0(t_j))-I_{ij}(u_{k,i}(t_j))][u_{k,i}(t_j)-u_i^0(t_j)]\to 0,\ k\to\infty, \\ \sum_{i=1}^{n}\lambda\int_0^T\psi'(t)(D_{u_i}f(t,u_k(t))-D_{u_i}f(t,u^0(t)))(u_{k,i}(t)-u_i^0(t))dt\to 0,\ k\to\infty.\end{cases} \tag{29}$$

Concerning (20), one has

$$
\begin{aligned}
&(\phi'(u_k)-\phi'(u^0))(u_k-u^0) \qquad (30)\\
=&\sum_{i=1}^{n}\int_0^T \psi'(t)({}^C D_{0^+}^{\alpha_i,\psi}u_{k,i}(t)-{}^C D_{0^+}^{\alpha_i,\psi}u_i^0(t))\,{}^C D_{0^+}^{\alpha_i,\psi}(u_{k,i}(t)-u_i^0(t))dt\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}[u_{k,i}(t_j)-u_i^0(t_j)][I_{ij}(u_i^0(t_j))-I_{ij}(u_{k,i}(t_j))]\\
&-\lambda\sum_{i=1}^{n}\int_0^T \psi'(t)(D_{u_i}f(t,u_k(t))-D_{u_i}f(t,u^0(t)))(u_{k,i}(t)-u_i^0(t))dt.
\end{aligned}
$$

Combining (29) with (30) yields

$$
\sum_{i=1}^{n}\int_0^T \psi'(t)({}^C D_{0^+}^{\alpha_i,\psi}u_{k,i}(t)-{}^C D_{0^+}^{\alpha_i,\psi}u_i^0(t))\,{}^C D_{0^+}^{\alpha_i,\psi}(u_{k,i}(t)-u_i^0(t))dt\to 0,\ k\to\infty,
$$

that is,

$$
\begin{aligned}
&\sum_{i=1}^{n}\int_0^T \psi'(t)({}^C D_{0^+}^{\alpha_i,\psi}u_{k,i}(t)-{}^C D_{0^+}^{\alpha_i,\psi}u_i^0(t))\,{}^C D_{0^+}^{\alpha_i,\psi}(u_{k,i}(t)-u_i^0(t))dt\\
=&\sum_{i=1}^{n}\int_0^T \psi'(t)\left({}^C D_{0^+}^{\alpha_i,\psi}(u_{k,i}(t)-u_i^0(t))\right)^2 dt\\
\geq&\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_{k,i}-u_i^0\|_{\alpha_i,\psi}^2\\
\geq&\min_{t\in[0,T]}\{\psi'(t)\}\frac{1}{n^2}\|u_k-u^0\|_X^2,
\end{aligned}
$$

which means that $\|u_k-u^0\|_X\to 0$, as $k\to\infty$. Thus, the Palais–Smale condition holds.

By means of $f(t,u)=f(t,-u)$ and (H_3), we can see that ϕ is even. Next, we will prove that the geometric structures of Theorem 1 are satisfied with ϕ.

Concerning (H_2), for any $\epsilon>0$, there exists $\delta>0$ such that

$$
f(t,u(t))\leq\epsilon\sum_{i=1}^{n}\mid u_i(t)\mid^2,\ \forall\sum_{i=1}^{n}\mid u_i(t)\mid\leq\delta, t\in[0,T]. \qquad (31)
$$

Choose $r=\frac{\delta}{\max_{1\leq i\leq n}\{\widehat{M}_i\}}$. For any $u=(u_1,u_2,\ldots,u_n)\in\overline{B}_r$, one has $\|u\|_X=\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}\leq r=\frac{\delta}{\max_{1\leq i\leq n}\{\widehat{M}_i\}}$. Then, from (12), it yields that

$$
\frac{\delta}{\max_{1\leq i\leq n}\{\widehat{M}_i\}}\geq\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}\geq\sum_{i=1}^{n}\frac{1}{\widehat{M}_i}\|u_i\|_\infty\geq\frac{1}{\max_{1\leq i\leq n}\{\widehat{M}_i\}}\sum_{i=1}^{n}\|u_i\|_\infty, \qquad (32)
$$

i.e., $\sum_{i=1}^{n}\|u_i\|_\infty\leq\delta$. At this point, by using (19), (H_3), (12), and (31), we obtain

$$
\begin{aligned}
\phi(u(t))\geq&\frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}^2-\lambda\epsilon\sum_{i=1}^{n}\int_0^T\psi'(t)\mid u_i(t)\mid^2 dt \qquad (33)\\
\geq&\frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}^2-\lambda\epsilon(\psi(T)-\psi(0))\sum_{i=1}^{n}\widehat{M}_i^2\|u_i\|_{\alpha_i,\psi}^2\\
\geq&\left[\frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}-\lambda\epsilon(\psi(T)-\psi(0))\max_{1\leq i\leq n}\{\widehat{M}_i^2\}\right]\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}^2,\ \forall u\in\overline{B}_r.
\end{aligned}
$$

Choosing $\epsilon = \frac{\min_{t\in[0,T]}\{\psi'(t)\}}{4\lambda(\psi(T)-\psi(0))\max_{1\le i\le n}\{\widehat{M}_i^2\}}$, from (33), we obtain

$$\phi(u(t)) \geq \frac{1}{4}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}^2 \geq \frac{1}{4}\min_{t\in[0,T]}\{\psi'(t)\}\frac{1}{n^2}(\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi})^2$$
$$= \frac{1}{4n^2}\min_{t\in[0,T]}\{\psi'(t)\}\|u\|_X^2 \geq 0, \forall u \in \overline{B}_r. \quad (34)$$

Hence, $\overline{B}_r \subset \{u \in X \mid \phi(u) \geq 0\}$ and $\phi(u) \geq \frac{r^2}{4n^2}\min_{t\in[0,T]}\{\psi'(t)\}, \forall u \in \partial B_r$.

In what follows, we claim that $\widetilde{X} = \widehat{X}\bigcap\{u \in X \mid \phi(u) \geq 0\}$ is bounded for any finite dimensional space $\widehat{X} \subset X$. Suppose that there exists at least one sequence $\{u_k\} \subset \widetilde{X}$ such that $\|u_k\|_X \to \infty$ as $k \to \infty$. Owing to $\phi(u_k) \geq 0$ and (26), we have

$$0 \leq \frac{\phi(u_k(t))}{\sum_{i=1}^{n}\|u_{k,i}\|_{(\alpha_i,\psi)}^2} \leq \left[\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\} - \lambda(\psi(T)-\psi(0))\mathcal{K}\right]$$
$$+ \frac{\sum_{i=1}^{n} m\left[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi} + \frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|_{\alpha_i,\psi}^{\tau_i+1}\right]}{\sum_{i=1}^{n}\|u_{k,i}\|_{(\alpha_i,\psi)}^2}. \quad (35)$$

Choose $\mathcal{K}$ large enough such that $\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\} - \lambda(\psi(T)-\psi(0))\mathcal{K} < -2$; then, in view of (27), we can see that $0 \leq \frac{\phi(u_k(t))}{\sum_{i=1}^{n}\|u_{k,i}\|_{(\alpha_i,\psi)}^2} < -2$—it is a contradiction. Therefore, $\widetilde{X}$ is bounded. According to Theorem 1, the functional ϕ exists with infinitely many critical points in X, which means that problem (6) exists with infinitely many solutions in X. □

Theorem 3. *Assume that* $(H_3), (H_5)$ *hold and* $G(t,u) = G(t,-u)$*. Then, problem (6) exists with infinitely many solutions with* $\lambda \in [\frac{\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}+1}{\min_{t\in[0,T]}\{\psi'(t)\}\min_{1\le i\le n}\{\frac{\zeta_i}{2}\eta_i^3\}}, \frac{\min_{t\in[0,T]}\{\psi'(t)\}}{2(\psi(T)-\psi(0))\max_{1\le i\le n}\{\frac{\zeta_i}{2}\widehat{M}_i^2\}})$.

Proof. Suppose $\{\phi(u_k)\}$ is bounded and $\lim_{k\to\infty}\phi'(u_k) = 0$, $u_k(t) = (u_{k,1}(t), \ldots, u_{k,n}(t))$. We claim that ϕ satisfies the Palais–Smale condition. Indeed, assume $\forall i : \|u_{k,i}\|_{(\alpha_i,\psi)} \to \infty (k \to \infty)$, $i = 1, 2, \ldots, n$. Combining (19), (H_5), (H_3), and (12), we have

$$\begin{aligned}
\phi(u_k(t)) \geq& \frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2 - \lambda\int_0^T\psi'(t)\left(\sum_{i=1}^{n}\frac{\zeta_i}{2}\mid u_{k,i}(t)\mid^2 - G(t,u_k(t))\right)dt\\
\geq& \frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2 - \lambda(\psi(T)-\psi(0))\sum_{i=1}^{n}\frac{\zeta_i}{2}\|u_{k,i}\|_\infty^2\\
&+ \lambda\int_0^T\psi'(t)G(t,u_k(t))dt\\
\geq& \left[\frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\} - \lambda(\psi(T)-\psi(0))\max_{1\le i\le n}\left\{\frac{\zeta_i}{2}\widehat{M}_i^2\right\}\right]\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2\\
&+ \lambda\int_0^T\psi'(t)G(t,u_k(t))dt. \qquad (36)
\end{aligned}$$

Recall that $\{\phi(u_k)\}$ is bounded and $\lambda < \frac{\min_{t\in[0,T]}\{\psi'(t)\}}{2(\psi(T)-\psi(0))\max_{1\le i\le n}\{\frac{\zeta_i}{2}\widehat{M}_i^2\}}$; a contradiction is generated. Namely, $\{u_k\}$ is bounded in X. The residual proof for the Palais–Smale condition is similar to Theorem 2, so we do not repeat it here.

Choose $\widetilde{r} \in (0, \frac{1}{\max_{1\leq i\leq n}\{\widehat{M}_i\}})$. For any $u = (u_1, u_2, \ldots, u_n) \in \overline{B}_{\widetilde{r}}$, we have $\|u\|_X = \sum_{i=1}^n \|u_i\|_{\alpha_i,\psi} \leq \widetilde{r} < \frac{1}{\max_{1\leq i\leq n}\{\widehat{M}_i\}}$. A similar analysis with (32) results in $\sum_{i=1}^n \|u_i\|_\infty < 1$. From (36) and (H_5), we obtain

$$\begin{aligned}
\phi(u(t)) \geq& \frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^n \|u_i\|^2_{\alpha_i,\psi} - \lambda\int_0^T \psi'(t)\Big(\sum_{i=1}^n \frac{\zeta_i}{2} \mid u_i(t)\mid^2 - \frac{\zeta_i}{2}\mid u_i(t)\mid^{\sigma_i}\Big)dt \\
\geq& \frac{1}{2}\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^n \|u_i\|^2_{\alpha_i,\psi} - \lambda\int_0^T \psi'(t)\Big(\sum_{i=1}^n \frac{\zeta_i}{2} \mid u_i(t)\mid^2 - \frac{\zeta_i}{2}\mid u_i(t)\mid^{2}\Big)dt \\
\geq& \frac{1}{2n^2}\min_{t\in[0,T]}\{\psi'(t)\}\Big(\sum_{i=1}^n \|u_i\|_{\alpha_i,\psi}\Big)^2 \\
=& \frac{1}{2n^2}\min_{t\in[0,T]}\{\psi'(t)\}\|u\|_X^2 \geq 0, \ \forall\, u \in \overline{B}_{\widetilde{r}}.
\end{aligned}$$

Apparently, $\overline{B}_{\widetilde{r}} \subset \{u \in X \mid \phi(u) \geq 0\}$ and $\phi(u) \geq \frac{\widetilde{r}^2}{2n^2}\min_{t\in[0,T]}\{\psi'(t)\}, \forall u \in \partial B_{\widetilde{r}}$.

We claim that $\widetilde{X}' = \widehat{X}' \bigcap \{u \in X \mid \phi(u) \geq 0\}$ is bounded for any finite dimensional space $\widehat{X}' \subset X$. Suppose that there exists at least one sequence $\{u_k\} \subset \widetilde{X}'$ such that $\|u_k\|_X \to \infty$ as $k \to \infty$. Due to (H_5), we obtain

$$\begin{aligned}
\int_0^T \psi'(t)f(t,u(t))dt =& \int_0^T \psi'(t)\Big(\sum_{i=1}^n \frac{\zeta_i}{2}\mid u_i\mid^2 - G(t,u(t))\Big)dt \\
\geq& \int_0^T \psi'(t)\Big(\sum_{i=1}^n \frac{\zeta_i}{2}\mid u_i\mid^2 - \sum_{i=1}^n \mu_i \mid u_i\mid^{\omega_i}\Big)dt. \qquad (37)
\end{aligned}$$

Hence, consider (19), (25), and (37) and Lemma 7 yields

$$\begin{aligned}
0 \leq \phi(u_k(t)) \leq& \frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^n \|u_{k,i}\|^2_{\alpha_i,\psi} + \sum_{i=1}^n m\Big[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi} + \frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|^{\tau_i+1}_{\alpha_i,\psi}\Big] \\
&- \lambda\int_0^T \psi'(t)\Big(\sum_{i=1}^n \frac{\zeta_i}{2}\mid u_{k,i}\mid^2 - \sum_{i=1}^n \mu_i\mid u_{k,i}\mid^{\omega_i}\Big)dt \\
\leq& \frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^n \|u_{k,i}\|^2_{\alpha_i,\psi} + \sum_{i=1}^n m\Big[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi} + \frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|^{\tau_i+1}_{\alpha_i,\psi}\Big] \\
&+ (\psi(T)-\psi(0))\lambda\sum_{i=1}^n \mu_i\widehat{M}_i^{\omega_i}\|u_{k,i}\|^{\omega_i}_{\alpha_i,\psi} - \lambda\sum_{i=1}^n \frac{\zeta_i}{2}\int_{\Omega_{u_{k,i}}}\psi'(t)\eta_i^2\|u_{k,i}\|^2_{\alpha_i,\psi} \\
\leq& \frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^n \|u_{k,i}\|^2_{\alpha_i,\psi} + \sum_{i=1}^n m\Big[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi} + \frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|^{\tau_i+1}_{\alpha_i,\psi}\Big] \\
&+ (\psi(T)-\psi(0))\lambda\sum_{i=1}^n \mu_i\widehat{M}_i^{\omega_i}\|u_{k,i}\|^{\omega_i}_{\alpha_i,\psi} - \lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=1}^n \frac{\zeta_i}{2}\eta_i^3\|u_{k,i}\|^2_{\alpha_i,\psi}, \qquad (38)
\end{aligned}$$

where $\Omega_{u_{k,i}} = \{t \in [0,T] :\mid u_{k,i}(t)\mid \geq \eta_i\|u_{k,i}\|_{\alpha_i,\psi}\}$ with meas$\{\Omega_{u_{k,i}}\} \geq \eta_i$, which shows that

$$\begin{aligned}
0 \leq \frac{\phi(u_k(t))}{\sum_{i=1}^n \|u_i\|^2_{\alpha_i,\psi}} \leq& \frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\} + \frac{\sum_{i=1}^n m\Big[a_i\widehat{M}_i\|u_{k,i}\|_{\alpha_i,\psi} + \frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|^{\tau_i+1}_{\alpha_i,\psi}\Big]}{\sum_{i=1}^n \|u_i\|^2_{\alpha_i,\psi}} \\
&+ \frac{(\psi(T)-\psi(0))\lambda\sum_{i=1}^n \mu_i\widehat{M}_i^{\omega_i}\|u_{k,i}\|^{\omega_i}_{\alpha_i,\psi}}{\sum_{i=1}^n \|u_{k,i}\|^2_{\alpha_i,\psi}} - \lambda\min_{t\in[0,T]}\{\psi'(t)\}\min_{1\leq i\leq n}\{\frac{\zeta_i}{2}\eta_i^3\}.
\end{aligned}$$

Since $\tau_i \in [0,1), \omega_i \in [0,2), i = 1,2,\ldots,n$ and $\|u_k\|_X \to \infty$ as $k \to \infty$, then

$$\frac{\sum_{i=1}^{n} m\left[a_i \widehat{M}_i \|u_{k,i}\|_{\alpha_i,\psi} + \frac{b_i}{\tau_i+1}\widehat{M}_i^{\tau_i+1}\|u_{k,i}\|_{\alpha_i,\psi}^{\tau_i+1}\right]}{\sum_{i=1}^{n}\|u_i\|_{\alpha_i,\psi}^2} \to 0,\ k \to \infty, \tag{39}$$

$$\frac{(\psi(T)-\psi(0))\lambda\sum_{i=1}^{n}\mu_i\widehat{M}_i^{\omega_i}\|u_{k,i}\|_{\alpha_i,\psi}^{\omega_i}}{\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2} \to 0,\ k \to \infty. \tag{40}$$

Recall that $\lambda \geq \frac{\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\}+1}{\min_{t\in[0,T]}\{\psi'(t)\}\min_{1\leq i\leq n}\{\frac{\zeta_i}{2}\eta_i^3\}}$; then,

$$\frac{1}{2}\max_{t\in[0,T]}\{\psi'(t)\} - \lambda\min_{t\in[0,T]}\{\psi'(t)\}\min_{1\leq i\leq n}\{\frac{\zeta_i}{2}\eta_i^3\} \leq -1. \tag{41}$$

Combining (39)–(41), we obtain

$$0 \leq \frac{\phi(u_k(t))}{\sum_{i=1}^{n}\|u_{k,i}\|_{\alpha_i,\psi}^2} \leq -1,$$

as $k \to \infty$; a contradiction is generated here. Namely, $\widetilde{X}'$ is bounded. We can obtain that the functional ϕ exists infinitely many critical points from Theorem 1, which shows that problem (6) possesses infinitely many solutions in X. $\square$

4. Examples

Example 1. *Consider the system*

$$\begin{cases} {}^{C}D_{1^-}^{0.6,t}({}^{C}D_{0^+}^{0.6,t}u_1(t)) = \lambda D_{u_1}f(t,u_1(t),u_2(t)), t\in[0,1], t\neq t_1, \\ {}^{C}D_{1^-}^{0.8,t}({}^{C}D_{0^+}^{0.8,t}u_2(t)) = \lambda D_{u_2}f(t,u_1(t),u_2(t)), t\in[0,1], t\neq t_1, \\ \Delta({}^{C}D_{1^-}^{0.6,t}(I_{0^+}^{0.4,t}u_1))(t_1) = I_{11}(u_1(t_1)), \\ \Delta({}^{C}D_{1^-}^{0.8,t}(I_{0^+}^{0.2,t}u_2))(t_1) = I_{21}(u_2(t_1)), \\ u_1(0) = u_1(1) = 0, u_2(0) = u_2(1) = 0, \end{cases} \tag{42}$$

where $\alpha_1 = 0.6, \alpha_2 = 0.8$, $\psi(t) = t$, $t \in [0,1]$. From Definition 1, it is easy to observe that system (42) is equivalent to the classical Caputo fractional differential equation. Define

$$f(t,u_1,u_2) = (1+t)\begin{cases} (u_1^2+u_2^2)^2,\ u_1^2+u_2^2 \leq 1, \\ 2(u_1^2+u_2^2)^2 - (u_1^2+u_2^2)^{\frac{1}{2}},\ u_1^2+u_2^2 > 1, \end{cases}$$

and

$$I_{11}(u_1(t_1)) = -\frac{1}{2}u_1^{\frac{1}{5}},\ I_{21}(u_2(t_1)) = -\frac{1}{3}u_2^{\frac{1}{5}}.$$

Obviously, I_{11} and I_{21} are continuous odd functions and satisfy (H_3) and (H_4), $f(t,u_1,u_2)$ is continuous with respect to t and continuously differentiable with respect to u_1, u_2, and satisfies (H_1) and (H_2). From Theorem 2, we can say that system (42) exists with infinitely many solutions.

Example 2. *Consider the system*

$$\begin{cases} {}^{C}D_{1^-}^{0.6,e^{\frac{1}{10}t}}({}^{C}D_{0^+}^{0.6,e^{\frac{1}{10}t}}u_1(t)) = \lambda D_{u_1}f(t,u_1(t),u_2(t),u_3(t)), t\in[0,1], t\neq t_1, \\ {}^{C}D_{1^-}^{0.75,e^{\frac{1}{10}t}}({}^{C}D_{0^+}^{0.75,e^{\frac{1}{10}t}}u_2(t)) = \lambda D_{u_2}f(t,u_1(t),u_2(t),u_3(t)), t\in[0,1], t\neq t_1, \\ {}^{C}D_{1^-}^{0.8,e^{\frac{1}{10}t}}({}^{C}D_{0^+}^{0.8,e^{\frac{1}{10}t}}u_3(t)) = \lambda D_{u_3}f(t,u_1(t),u_2(t),u_3(t)), t\in[0,1], t\neq t_1, \\ \Delta({}^{C}D_{1^-}^{0.6,e^{\frac{1}{10}t}}(I_{0^+}^{0.4,e^{\frac{1}{10}t}}u_1))(t_1) = I_{11}(u_1(t_1)), \\ \Delta({}^{C}D_{1^-}^{0.75,e^{\frac{1}{10}t}}(I_{0^+}^{0.25,e^{\frac{1}{10}t}}u_2))(t_1) = I_{21}(u_2(t_1)), \\ \Delta({}^{C}D_{1^-}^{0.8,e^{\frac{1}{10}t}}(I_{0^+}^{0.2,e^{\frac{1}{10}t}}u_3))(t_1) = I_{31}(u_3(t_1)), \\ u_1(0)=u_1(1)=u_2(0)=u_2(1)=u_3(0)=u_3(1)=0, \end{cases} \tag{43}$$

where $\alpha_1 = 0.6, \alpha_2 = 0.75, \alpha_3 = 0.8$, $\psi(t) = e^{\frac{1}{10}t}$, $t \in [0,1]$. *Define* $f(t,u_1,u_2,u_3) = \sum_{i=1}^{3} 2|u_i|^2 - \sum_{i=1}^{3} 3|u_i|^{\frac{2}{3}}$, *where* $\zeta_1 = \zeta_2 = \zeta_3 = 4$, $G(t,u_1,u_2,u_3) = \sum_{i=1}^{3} 3|u_i|^{\frac{2}{3}}$. *Obviously, f is continuous with respect to t and continuously differentiable with respect to* u_1,u_2,u_3, $G(t,-u_1,-u_2,-u_3) = G(t,u_1,u_2,u_3)$. *Choosing* $\sigma_i = \omega_i = \frac{2}{3}$, $\mu_i = 4$, $i = 1,2,3$, *then* $\sum_{i=1}^{3}(\frac{\zeta_i}{2}) \mid u_i \mid^{\sigma_i} \leq G(t,u_1,u_2,u_3) \leq \sum_{i=1}^{3} \mu_i \mid u_i \mid^{\omega_i}$. *Define* $I_{11}(u_1(t_1)) = -\frac{1}{2}u_1^{\frac{1}{5}}$, $I_{21}(u_2(t_1)) = -\frac{1}{3}u_2^{\frac{1}{5}}$, $I_{31}(u_3(t_1)) = -\frac{1}{4}u_3^{\frac{1}{5}}$. *Then,* I_{11}, I_{21}, *and* I_{31} *are continuous odd functions and satisfy* (H_3).

By direct calculations, $\max_{t\in[0,1]} \psi'(t) = \frac{1}{10}e^{\frac{1}{10}}$, $\min_{t\in[0,1]} \psi'(t) = \frac{1}{10}$, *and*

$$\widehat{M}_1 = \frac{\frac{1}{10}e^{\frac{1}{10}}(e^{\frac{1}{10}})^{\frac{1}{10}}}{\Gamma(0.6)(\frac{1}{5})^{\frac{1}{2}}} \approx 0.6762,\ \widehat{M}_2 = \frac{\frac{1}{10}e^{\frac{1}{10}}(e^{\frac{1}{10}})^{\frac{1}{4}}}{\Gamma(0.75)(\frac{1}{2})^{\frac{1}{2}}} \approx 0.1308,\ \widehat{M}_3 = \frac{\frac{1}{10}e^{\frac{1}{10}}(e^{\frac{1}{10}})^{\frac{3}{10}}}{\Gamma(0.8)(\frac{3}{5})^{\frac{1}{2}}} \approx 0.1263.$$

Take $\eta_1 = \eta_2 = \eta_3 = 10$, *then*

$$\lambda \in \left[\frac{\frac{1}{20}e^{\frac{1}{10}}+1}{200}, \frac{\frac{1}{10}}{2(e^{\frac{1}{10}}-1)2\widehat{M}_1^2}\right) \approx [1\times10^{-2}, 5.2\times10^{-1}).$$

Consequently, from Theorem 3, we can see that system (43) exists with infinitely many solutions with $\lambda \in [1\times10^{-2}, 5.2\times10^{-1})$.

5. Conclusions

This article dealt with a nonlinear impulsive boundary value problem involving the more general ψ-Caputo-type fractional derivative and integral. Drawing upon some simple and easily verifiable algebraic conditions, we established relationships between a ψ-Caputo-type fractional boundary value problem and critical point theory, and obtained new multiplicity results of infinitely many solutions for problem (6). This work completed the extension of some existing results in terms of the equation form and assumption conditions.

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