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Multiple Normalized Solutions to a Choquard Equation Involving Fractional p-Laplacian in $\mathbb{R}^N$

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Abstract: In this paper, we study the existence of multiple normalized solutions for a Choquard equation involving fractional p-Laplacian in $\mathbb{R}^N$. With the help of variational methods, minimization techniques, and the Lusternik–Schnirelmann category, the existence of multiple normalized solutions is obtained for the above problem.

Keywords: Choquard equation; p-Laplacian operator; normalized solution; Lusternik–Schnirelmann category; variational method

MSC: 35J20; 35J62; 35J92

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1. Introduction

This work concerns the existence of multiple normalized solutions for the following Choquard equation involving fractional p-Laplacian in $\mathbb{R}^N$ of the form:

$$\begin{cases} (-\Delta)_p^s u + B(\varpi x)|u|^{p-2}u = \lambda |u|^{p-2}u + \left[\frac{1}{|x|^{N-\alpha}} * G(u)\right] g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = a^p, \end{cases} \tag{1}$$

where $\varpi > 0$, $a > 0$, $\alpha \in (0, N)$, $\lambda \in \mathbb{R}$ is a Lagrange multiplier, g is a continuous differentiable function with L^p-subcritical growth, and $B : \mathbb{R}^N \to [0, \infty)$ is a continuous function satisfying some appropriate conditions. $\mathfrak{B}_\varepsilon(x)$ is a sphere with the center x and radius ε, and the fractional p-Laplace operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus \mathfrak{B}_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy \ (x \in \mathbb{R}^N).$$

Our research on problem (1) is based on theoretical and practical application research. First, the equations with p-Laplacian occur in fluid dynamics, nonlinear elasticity, glaciology, and so on; please refer to [1,2]. When $p = 2$, problem (1) comes from the research of solitary waves for the following fractional Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + B(x)\psi - G(\psi)\psi(I_\alpha * g(\psi)) \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $0 < s < 1$, $(-\Delta)^s$ is the fractional Laplacian, i denotes the imaginary unit and $\psi(x,t)$ is a complex wave. The standing wave is a solution of the form $\psi(t,x) = e^{-i\lambda t} B(x)$, where $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}$ is a time-independent function that satisfies the following equation:

$$(-\Delta)^s u + B(x)u = \lambda u + \left[\frac{1}{|x|^{N-\alpha}} * G(u)\right] g(u) \quad \text{in } \mathbb{R}^N. \tag{3}$$

Fix λ, the energy functional $\mathcal{E}_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ corresponding to problem (3) is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(|\nabla u|^2 + (B(x)-\lambda)|u|^2\Big)dx - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big]G(u)dx.$$

In recent years, many papers have been published concerning the existence and multiplicity of solutions for this case. For instance, Ambrosio [3] studied the following equation with the potential function of the form:

$$\epsilon^{2s}(-\Delta)^s u + B(x)u = \epsilon^{\mu-N}\Big[\frac{1}{|x|^\mu} * G(u)\Big]g(u), \tag{4}$$

where $0 < s < 1$ and g have subcritical growth. They obtained the concentration and multiplicity of positive solutions to (4) using the Ljusternik–Schnirelmann theory and the penalization method. Moreover, Ambrosio [4] investigated the following fractional Choquard equation involving the fractional p-Laplacian operator:

$$\begin{cases} \epsilon^{sp}(-\Delta)^s_p u + B(x)|u|^{p-2}u = \epsilon^{\mu-N}\Big[\frac{1}{|x|^\mu} * G(u)\Big]g(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N),\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{5}$$

Under the suitable assumptions, they can also obtain the multiplicity and concentration of positive solutions to problem (5). For critical growth cases, please see [5–7]. For the supercritical growth case, Li and Wang [8] obtained the existence of a nontrivial solution to p-Laplacian equations in $\mathbb{R}^N$ using the Moser iteration and perturbation arguments. For more interesting results, see [9–12] and their references.

For another, from a physical point of view, some authors are interested in finding solutions to problem (1) with prescribed mass

$$\int_{\mathbb{R}^N} |u|^p dx = a^p \ \text{ for } a > 0. \tag{6}$$

Under this circumstance, the parameter $\lambda \in \mathbb{R}$ is the Lagrange multiplier, which relies on the solution and is not a priori given. In our study, we intend to establish the existence of multiple weak solutions to problem (1). Here and after, by a solution, we always mean a couple (u, λ), which satisfies problem (1). We refer to this type of solution as a normalized solution since condition (6) imposes a normalization on the L^p-norm of u.

In the local case, i.e., $s = 1$, the fractional Laplace $(-\Delta)^s$ reduces to the local differential operator $-\Delta$. In recent years, for some special forms of problem (1), many authors have obtained the existence, multiplicity, and asymptotic properties of normalized solutions under different conditions by various methods. For example, when $p = 2$, $B(x) = 0$ and $s = 1$, some authors have studied the following nonlinear elliptic problems:

$$\begin{cases} -\Delta u = \lambda u + g(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases} \tag{7}$$

Jeanjean [13] used mountain-pass geometry to study the existence of normalized solutions in purely L^2-supercritical. Cazenave and Lions [14] showed the orbital stability of some standing waves in nonlinear Schrödinger equations when $g(u) = |u|^{p-1}u$ for three cases. For the general case of $g(u)$, with a scaling argument, Shibata [15] studied (7). If $g(u) = \mu|u|^{q-2}u + |u|^{p-2}u$, Soave [16] studied the existence and properties of solutions to the problem with prescribed mass for the L^2-supercritical case with subcritical Sobolev growth. Moreover, they also gave the new criteria for global existence and finite-time blow-up in the associated dispersive equation. For the critical case, Jeanjean and Le [17] considered a class of Sobolev critical Schrödinger equation, and they proved the existence of standing waves that are not ground states while located at a mountain-pass level of the energy functional. Furthermore, when time is finite, these solutions are not stable because of blow-up.

For $s \neq 1$ and $B = 0$, Yu et al. [18] considered the following mass subcritical fractional Schrödinger equations:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ \displaystyle\int_\Omega |u|^2 dx = a^2, \end{cases} \tag{8}$$

where $s \in (0,1)$, $2 < p < 2 + \frac{N}{4}$, $\Omega \subset \mathbb{R}^N$ is an exterior domain with smooth boundary satisfying $\mathbb{R}^N \backslash \Omega$ contained in a small ball. For any $a > 0$, they not only used barycentric functions to show the existence of a positive normalized solution but also used the minimax method and Brouwer degree theory. Moreover, if Ω is the complement of the unit ball in $\mathbb{R}^N$, for any $a > 0$, they established the existence and multiplicity of radial normalized solutions according to genus theory. If we consider the case of whole space, Luo and Zhang [19] studied the fractional nonlinear Schrödinger equations with combined nonlinearities under different assumptions on parameters, and they proved some existence and nonexistence results about the normalized solutions.

Then, for $p = 2$, $B(x) \neq 0$, there is some literature devoted to these problems:

$$\begin{cases} (-\Delta)^s u + B(x)u = \lambda u + g(x,u) & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases} \tag{9}$$

If B and g satisfy some suitable assumptions, Ikoma and Miyamoto [20] proved the L^2-constraint minimization exists and a minimizer to problem (9) does not exist. If $g(x,u) = h(\varepsilon x)f(u)$ in problem (9), Zhang et al. [21] showed the existence of normalized solutions depends on the global maximum points of h when ε is small enough. For $g(x,u) = |u|^{p-2}u$, Peng and Xia [22] used a new min–max argument and splitting lemma for the nonlocal version to overcome the lack of compactness and proved that there exists at least one L^2-normalized solution $(u,\omega) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ of problem (9).

However, for the case $p \neq 2$, as far as we know, the results about the normalized solution of the p-Laplacian equation are relatively few. Wang et al. [23] considered the following p-Laplacian equation:

$$\begin{cases} -\Delta_p u + \lambda|u|^{p-2}u = \mu u + |u|^{s-2}u & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = \rho^2, \end{cases} \tag{10}$$

where $-\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $\mu \in \mathbb{R}$, $s \in (\frac{N+2}{N}p, p^*)$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. They proved that problem (10) has a normalized solution with constrained variational methods. Zhang and Zhang [24] is the first paper to study the following p-Laplacian equation:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}v + \mu|u|^{q-2}u + g(u) & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^p dx = \rho^p, \end{cases} \tag{11}$$

where $N \geq 2$, $\rho > 0$, $1 < p < q \leq \overline{p} := p + \frac{p^2}{N}$, $\mu \in \mathbb{R}$, $g \in C(\mathbb{R},\mathbb{R})$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. They used the Schwarz rearrangement and Ekeland variational principle to prove the existence of positive radial ground states for suitable μ. Recently, Wang and

Sun [25] considered the following p-Laplacian equation with a trapping potential $B(x)$ of the form:

$$\begin{cases} -\Delta_p u + B(x)|u|^{p-2}u = \lambda|u|^{r-2}u + |u|^{q-2}u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^r dx = c, \end{cases} \tag{12}$$

where $1 < p < N$, $\rho > 0$, $r = p$ or 2, $p < q < p^*$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. The trapping potential B is a continuous function with $B \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$, $B(0) = 0$ satisfying

$$(\mathcal{A})0 = \inf_{x\in\mathbb{R}^N} B(x) < \liminf_{|x|\to+\infty} B(x) = B_\infty.$$

When $r = p$, they showed that problem (12) has a ground-state solution with positive energy for c small enough. When $r = 2$, the authors also showed that problem (12) has at least two solutions, both with positive energy, where one is a ground state and the other is a high-energy solution.

Thin and Rădulescu [26] first considered the following fractional p-Laplace problem:

$$\begin{cases} (-\Delta)_p^s u + B(\varpi x)|u|^{p-2}u = \lambda|u|^{p-2}u + g(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = a^p, \end{cases} \tag{13}$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator, $p \geq 2$, $a > 0$, $\varpi > 0$, $\lambda \in \mathbb{R}$ is an uncharted Lagrange multiplier, the potential function B verifies condition (B), and g is a continuous function with L^p-subcritical growth. They proved the existence of multiple normalized solutions using the Lusternik–Schnirelmann category.

Inspired by the above literature, in this paper, we intend to prove the existence of multiple normalized solutions for a Choquard equation involving fractional p-Laplacian and potential functions. As far as we know, there are no results about the existence of multiple normalized solutions to problem (1). In order to give our main results, let us fix some notations and also assume that the nonlinearity g is a continuous differentiable function and satisfies the following growth conditions:

$(\mathfrak{g}_1)$ g is an odd and continuous function, for $q \in (\frac{p(N+\alpha)}{2N}, \frac{p(N+\alpha)+p^2 s}{2N})$, $\alpha_* \in (0, +\infty)$ such that $\lim_{t\to 0} \frac{|g(t)|}{|t|^{q-1}} = \alpha_*$.

$(\mathfrak{g}_2)$ There are two positive constants c_1, $c_2 > 0$ and $\mathfrak{p} \in (\frac{p(N+\alpha)}{2N}, \frac{p(N+\alpha)+p^2 s}{2N})$, $t \in \mathbb{R}$ such that

$$|g(t)| \leq c_1 + c_2|t|^{\mathfrak{p}-1}. \tag{14}$$

$(\mathfrak{g}_3)$ There is $q_1 > \frac{p}{2}$ such that $\left(\frac{g(t)}{t^{q_1-1}}\right)' > 0$ for all $t \in (0, +\infty)$.

$(\mathfrak{g}_4)$ There exists $\theta > 2p$ such that $2g(t)t \geq \theta G(t) > 0$ for all $t > 0$, where $G(t) = \int_0^t g(\tau)d\tau$.

In the present paper, we intend to prove the existence of multiple normalized solutions for problem (1) involving the nonautonomous case, i.e., the case $B \not\equiv 0$, with the Lusternik–Schnirelmann category of the sets M and M_δ given by

$$M = \{x \in \mathbb{R}^N : B(x) = 0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leq \delta\}.$$

Here, we mention that if Y is a closed subset of a topological space X, and the Lusternik–Schnirelmann category $cat_X(Y)$ is the least number of closed and contractible sets in X that cover Y. If $X = Y$, we use $cat(X)$ instead of $cat(Y)$. For more details, see Willem [27].

Now, we are ready to state our main results in this paper.

Theorem 1. *Let g satisfy conditions $(\mathfrak{g}_1) - (\mathfrak{g}_4)$ and B satisfy condition (B). Then for each $\delta > 0$, there exists $\varpi_0 > 0$ and $B_* > 0$ such that problem (1) admits at least $cat_{M_\delta}(M)$ couples $(u_j, \lambda_j) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $0 < \varpi < \varpi_0$ and $|B|_\infty < B_*$ while $\int_{\mathbb{R}^N} |u_j|^p dx = a^p$, $\lambda_j < 0$ and $\mathcal{J}_\varpi(u_j) < 0$. Additionally, when u_ϖ is one of these solutions, ζ_ϖ is the global maximum of $|u_\varpi|$, satisfies*

$$\lim_{\varpi \to 0} B(\varpi \zeta_\varpi) = 0.$$

Remark 1. *Compared with the previous literature, our paper has the following characteristics:*

(1) *When $p \neq 2$, the operator $-\Delta_p$ is no longer linear, which leads to some quite different properties from the classical Laplacian operator $-\Delta$. For example, for the case $p = 2$, the equation*

$$-\Delta_p u + |u|^{p-2}u = |u|^{q-2}u \ \text{ in } \mathbb{R}^N$$

has a unique positive radially symmetric solution (see [28,29]), but in general cases, we know that this fact holds only for $1 < p < 2$ (see [30,31]), and it is still unknown for $2 < p < N$. Moreover, because of the nonlinear character of $-\Delta_p$, the approach in Moroz and Van Schaftingen [32] becomes not simple for p-Laplacian operator $-\Delta_p$ with $p \neq 2$.

(2) *Unlike Li and Ye [33], we do not consider Hilbert space, and we cannot use some properties. For example, Wang et al. [23] use the workspace $W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ which is a Hilbert space. The workspace is Hilbert space and very important for Wang et al. [23] because they need the direct-sum decomposition.*

(3) *Due to $p \neq 2$, it is difficult to prove that $x \cdot \nabla u \in W^{s,p}(\mathbb{R}^N)$ for the nonlinearity of the operator $(-\Delta)^s_p$ and its nonlocal character. Meanwhile, it is also hard to deal with an integration by parts formula for $(-\Delta)^s_p$. Moreover, we cannot directly adopt these methods in [24–26] due to nonlocal term $[\frac{1}{|x|^{N-\alpha}} * G(u)]g(u)$. Therefore, we need to develop new techniques to overcome this difficulty and the loss of compactness due to the unbounded regions.*

(4) *The nonlinearity g has L^p-subcritical growth, so we need to estimate the mountain-pass level situated in a suitable interval when (PS) condition holds. To use the Ljusternik–Schnirelmann category theory, we establish some lemmas and technical results. Compared to the work by the authors in [26], the difficulties raise the Choquard term. We overcome this using the Hardy–Littlewood–Sobolev inequality and some new technique analysis steps.*

Remark 2. *Our work is independent from [34]. Indeed, Chen and Wang studied problem (1) as $\varpi = 1$ and g has exponential growth in the Trudinger–Moser sense, and they did not study the multiple solutions. In our work, we study the nonautonomous problem and g has subcritical growth. We mainly use the Lusternik–Schnirelmann category to obtain multiple solutions. We also do not use the genus method as in He et al. [35] to obtain multiple solutions.*

Remark 3. *In different research fields, we have different definitions of fractional operators and different applications. Here, we give some examples. In physics, Maheswari and Bakshi [36] mentioned a general time-fractional differential equation defined by $(\partial^\alpha v / \partial t^\alpha) = F[x, v, v_x, v_{xx}, v_{xxx}]$, by using the invariant subspace method. For different equations with different operators of F, they obtain various solutions. In quantum mechanics, by using a method with the parameters of the system and Riemann–Liouville definition of the fractional derivatives, Al-Raeei [37] considered the Schrödinger equation for the electrical screening potential and obtained the amplitude of the wave functions for multiple values of the spatial fractional parameter. In kernel dynamics, Al Baidani et al. [38] considered magneto-acoustic waves in plasma. In a manner of Caputo derivatives, they studied the Caputo–Fabrizio and the Atangana–Baleanu derivatives. Finally, they obtained the solution calculated as a convergent series, and it was demonstrated that the NTDM solutions converge to the exact solutions. There are many applications of fractional order operators, and we will not give any examples here.*

The paper is organized as follows: in Section 2, we consider the autonomous case associated with problem (1). In Section 3, we consider the nonautonomous case and give the corresponding energy functional. Moreover, to obtain the multiplicity consequence, we verify the Palais–Smale condition and establish some tools and lemmas. Finally, in Section 4, we give the proof of Theorem 1.

2. The Autonomous Case

In this section, we consider the autonomous case corresponding to problem (1). First, we list some notations for readers to study.

It is also known that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space equipped with the norm

$$\|u\|^p := \|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p},$$

where

$$[u]^p_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy$$

and

$$\|u\|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^p dx.$$

Then we give a statement of Lions's theorem:

Lemma 1 (Ambrosio [39])**.** *Let $N > sp$ and $r \in [p, p_s^*)$. If (u_n) is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and let*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^r dx = 0$$

for some $R > 0$. Then, $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (p, p_s^*)$.

Lemma 2 (Lieb and Loss [40])**.** *For $r, t > 1$, $0 < \mu < N$ be such that $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{t} = 2$. For $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. We have a constant $C(r, N, \mu, t) > 0$ which does not depend on g and h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dxdy \le C(r, N, \mu, t)\|g\|_{L^r(\mathbb{R}^N)}\|h\|_{L^t(\mathbb{R}^N)}.$$

If $r = t = \frac{2N}{2N-\mu}$, for $G(u) = |u|^q$, we can see

$$\int_{\mathbb{R}^N} [\frac{1}{|x|^\mu} * G(u)]g(u)dx$$

is well defined on $L^t(\mathbb{R}^N)$ for $t = \frac{2N}{2N-\mu}$.

Now, we consider the autonomous case corresponding to problem (1), i.e.,

$$\begin{cases} (-\Delta)^s_p u + \mu|u|^{p-2}u = \lambda|u|^{p-2}u + \left[\frac{1}{|x|^{N-\alpha}} * G(u)\right]g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = a^p \end{cases} \tag{15}$$

where $\mu > 0$, $a > 0$, $\alpha \in (0, N)$, and λ is a Lagrange multiplier, which is an unknown parameter.

We denote the energy functional associated with problem (15) as follows:

$$\mathcal{I}_\mu(u) = \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + \mu\int_{\mathbb{R}^N} |u|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N} \left[\frac{1}{|x|^{N-\alpha}} * G(u)\right]G(u)dx$$

restricted to the sphere $S(a)$ and

$$S(a) = \{u \in W^{s,p}(\mathbb{R}^N) : \|u\|_{L^p(\mathbb{R}^N)} = a\}.$$

Lemma 3. *The energy functional $\mathcal{I}_\mu$ is bounded and coercive on $S(a)$.*

Proof. According to $(\mathfrak{g}_1) - (\mathfrak{g}_3)$, there exist $C_1, C_2 > 0$ such that

$$|G(t)| \le C_1|t|^q + C_2|t|^{\mathfrak{p}}, \ \ \forall t \in \mathbb{R}.$$

Then by the Hardy–Littlewood–Sobolev inequality in Lemma 2, we have

$$\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big] G(u)dx \le C_{N,\alpha}\Big(\int_{\mathbb{R}^N} |G(u)|^{\frac{2N}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}}.$$

Thus, there exists a suitable constant $C > 0$ such that

$$\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big] G(u)dx \le C\Big[\Big(\int_{\mathbb{R}^N} |u|^{\frac{2Nq}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}} + \Big(\int_{\mathbb{R}^N} |u|^{\frac{2N\mathfrak{p}}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}}\Big].$$

Because $C_0^\infty(\mathbb{R}^N)$ is density in $W^{s,p}(\mathbb{R}^N)$ for all $u \in W^{s,p}(\mathbb{R}^N)$, the fractional Gagliardo–Nirenberg inequality (Nguyen and Squassina [41], Lemma 2.1) gives us

$$\|u\|^{\tau}_{L^\tau(\mathbb{R}^N)} \le C\|u\|^{\tau(1-\mathfrak{a})}_{L^p(\mathbb{R}^N)}[u]^{\tau\mathfrak{a}}_{s,p},$$

where $\tau > 0$, $0 \le \mathfrak{a} \le 1$, $C = C(s, N, \tau) \ge 1$ and

$$\frac{1}{\tau} = \mathfrak{a}(\frac{1}{p} - \frac{s}{N}) + \frac{1-\mathfrak{a}}{p}.$$

Then, $\tau = \frac{pN}{N-\mathfrak{a}ps} \in [p, p_s^*]$ and we deduce

$$\|u\|^{\tau\frac{N+\alpha}{N}}_{L^\tau(\mathbb{R}^N)} \le C^{\frac{N+\alpha}{N}}\|u\|^{\tau(1-\mathfrak{a})\frac{N+\alpha}{N}}_{L^p(\mathbb{R}^N)}[u]^{\tau\mathfrak{a}\frac{N+\alpha}{N}}_{s,p}, \tag{16}$$

where $\tau\mathfrak{a}\frac{N+\alpha}{N} = p$, then $\mathfrak{a} = \frac{p}{\tau}\frac{N}{N+\alpha}$ so $\tau = p + \frac{p^2 s}{N+\alpha}$.

We can apply (16) for $\tau = \frac{2Nq}{N+\alpha}$, thus

$$\Big(\int_{\mathbb{R}^N} |u|^{\frac{2Nq}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}} \le C_1[u]^{\frac{2Nq\mathfrak{a}}{N}}_{s,p} \tag{17}$$

on $S(a) = \{u \in W^{s,p}(\mathbb{R}^N) : \|u\|_{L^p(\mathbb{R}^N)} = a\}$, where $C_1 > 0$ is a suitable constant depending on a. Since $q \in \Big(\frac{p(N+\alpha)}{2N}, \frac{p(N+\alpha)+p^2 s}{2N}\Big)$, then $\frac{2Nq\mathfrak{a}}{N} < p$, where $\mathfrak{a} = \frac{N}{s}\Big(\frac{1}{p} - \frac{N+\alpha}{2Nq}\Big)$. Similarly, there exists $C_2 > 0$ such that

$$\Big(\int_{\mathbb{R}^N} |u|^{\frac{2N\mathfrak{p}}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}} \le C_2[u]^{\frac{2N\mathfrak{p}\mathfrak{a}}{N}}_{s,p}. \tag{18}$$

Hence, we have

$$\begin{aligned}
\mathcal{I}_\mu(u) &= \frac{1}{p}\Big([u]_{s,p}^p + \mu\int_{\mathbb{R}^N}|u|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big]G(u)dx \\
&\geq \frac{1}{p}\Big([u]_{s,p}^p + \mu\int_{\mathbb{R}^N}|u|^p dx\Big) - \frac{1}{2}C\Big[\Big(\int_{\mathbb{R}^N}|u|^{\frac{2Nq}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}} + \Big(\int_{\mathbb{R}^N}|u|^{\frac{2N\mathfrak{p}}{N+\alpha}}dx\Big)^{\frac{N+\alpha}{N}}\Big] \\
&\geq \frac{1}{p}\Big([u]_{s,p}^p + \mu\int_{\mathbb{R}^N}|u|^p dx\Big) - \frac{1}{2}CC_1[u]_{s,p}^{\frac{2Nq\mathfrak{a}}{N}} - \frac{1}{2}CC_2[u]_{s,p}^{\frac{2N\mathfrak{p}\mathfrak{a}}{N}} \\
&\geq \frac{1}{p}[u]_{s,p}^p - C_3[u]_{s,p}^{\frac{2Nq\mathfrak{a}}{N}} - C_4[u]_{s,p}^{\frac{2N\mathfrak{p}\mathfrak{a}}{N}}.
\end{aligned} \tag{19}$$

Since $q, \mathfrak{p} \in \Big(\frac{p(N+\alpha)}{2N}, \frac{p(N+\alpha)+p^2 s}{2N}\Big)$, then $0 < \frac{2Nt_1\mathfrak{a}}{N} < p$, for $t_1 \in \{\mathfrak{p}, q\}$. Above all, we prove the coercivity and boundedness of $\mathcal{I}_\mu$ on $S(a)$. $\square$

Thus, we obtain the existence of

$$\mathcal{I}_{\mu,a} = \inf_{u\in S(a)} \mathcal{I}_\mu(u).$$

Then, we show some properties of $\mathcal{I}_\mu$ in relation to the parameter $\mu \geq 0$.

Lemma 4. *There is a constant $B_* > 0$ such that $\mathcal{I}_{\mu,a}(u) < 0$ when $0 \leq \mu < B_*$.*

Proof. Fix a function $u_0 \in L^\infty(\mathbb{R}^N) \cap S(a)$, $u_0 > 0$ and let

$$\mathscr{H}(u_0, t)(x) = e^{\frac{Nt}{p}} u_0(e^t x) \text{ for all } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}.$$

Obviously, we have

$$\int_{\mathbb{R}^N} |\mathscr{H}(u_0,t)(x)|^p dx = a^p.$$

For any fixed $t \gg 0$, let

$$L(m) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(mu_0)\Big]G(mu_0)dx \text{ for any } m > 0.$$

By using $(\mathfrak{g}_4)$, for any $m > 0$, we have

$$\begin{aligned}
\frac{dL}{dm} &= \Big(\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(mu_0)\Big]G(mu_0)dx\Big)' \\
&= \frac{2}{m}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(mu_0)\Big]g(mu_0)mu_0 dx \\
&> \frac{\theta}{m}L(m).
\end{aligned}$$

Thus, integrating this on $[1, e^{\frac{Nt}{p}}]$, we have

$$L(e^{\frac{Nt}{p}}) \geq L(1)(e^{\frac{Nt}{p}})^\theta$$

which yields

$$\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(e^{\frac{Nt}{p}}u_0)\Big]G(e^{\frac{Nt}{p}}u_0)dx \geq e^{\frac{Nt\theta}{p}}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_0)\Big]G(u_0)dx. \tag{20}$$

Note that

$$\iint_{\mathbb{R}^{2N}} \frac{|\mathscr{H}(u_0,t)(x) - \mathscr{H}(u_0,t)(y)|^p}{|x-y|^{N+sp}} dxdy = e^{pst} \iint_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^p}{|x-y|^{N+sp}} dxdy$$

and so,

$$\mathcal{I}_\mu(\mathscr{H}(u_0,t)) \le \frac{e^{Nt}}{p}[u_0]^p_{s,p} + \frac{\mu a^p}{p} - \frac{e^{\frac{Nt\theta}{p}}}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^{N-\alpha}} * G(u_0)\right] G(u_0) dx.$$

Since $\theta > 2p$, we have $\frac{Nt\theta}{p} > 2Nt > Nt$, so increasing $|t|$ if necessary, we deduce that

$$\frac{e^{Nt}}{p}[u_0]^p_{s,p} - \frac{e^{\frac{Nt\theta}{p}}}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^{N-\alpha}} * G(u_0)\right] G(u_0) dx = \mathcal{Q}_t < 0.$$

Thus,

$$\mathcal{I}_\mu(\mathscr{H}(u_0,t)) \le \mathcal{Q}_t + \frac{\mu a^p}{p}.$$

Then, take $B_* > 0$ such that

$$\mathcal{Q}_t + \frac{B_* a^p}{p} < 0.$$

Therefore, if $\mu < B_*$, then we derive that for any $\mu \in [0, B_*)$,

$$\mathcal{I}_\mu(\mathscr{H}(u_0,t)) < 0,$$

so $\mathcal{I}_{\mu,a} < 0$. □

Lemma 5. *There exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}} dxdy \ge C \text{ for all } n \ge n_0.$$

Proof. If we assume that there exists a subsequence (u_n), still denoted by itself such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}} dxdy \to 0 \text{ as } n \to +\infty.$$

Then, we have

$$0 > \mathcal{I}_{\mu,a} + o_n(1) = \mathcal{I}_\mu(u_n) \ge -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}} dxdy.$$

This is a contradiction. Therefore, we obtain the proof of Lemma 4. □

Lemma 6. *Let $\mu \in [0, B_*)$ and $0 < a_1 < a_2$. Then, $\frac{a_1^p}{a_2^p} \mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1} < 0$.*

Proof. Since $||u(x)| - |u(y)|| \le |u(x) - u(y)|$ for all $u \in W^{s,p}(\mathbb{R}^N)$, we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{||u(x)| - |u(y)||^p}{|x-y|^{N+sp}} dxdy \le \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy.$$

Therefore, $\mathcal{I}_\mu(u) \ge \mathcal{I}_\mu(|u|)$.

Fixing $\vartheta > 1$ such that $a_2 = \vartheta a_1$ and $(u_n) \subset S(a_1)$ be a nonnegative minimizing sequence with respect to the $\mathcal{I}_{\mu,a_1}$, which exists because $\mathcal{I}_\mu(u) \geq \mathcal{I}_\mu(|u|)$ for all $u \in W^{s,p}(\mathbb{R}^N)$, i.e., when $n \to +\infty$,

$$\mathcal{I}_\mu(u_n) \to \mathcal{I}_{\mu,a_1}.$$

Letting $\mathfrak{u}_n = \vartheta u_n$, then $\mathfrak{u}_n \subset S(a_2)$. From $(\mathfrak{g}_3)$, when $t \geq 1$ and $l > 0$, we obtain

$$G(tl) \geq t^{q_1} G(l).$$

Therefore,

$$\begin{aligned}
\mathcal{I}_{\mu,a_2} \leq \mathcal{I}_\mu(\mathfrak{u}_n) = \mathcal{I}_\mu(\vartheta u_n)
&= \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|\vartheta u_n(x) - \vartheta u_n(y)|^p}{|x-y|^{N+sp}} dxdy + \mu \int_{\mathbb{R}^N} |\vartheta u_n|^p dx \Big) \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\alpha}} * G(\vartheta u_n) \Big] G(\vartheta u_n) dx \\
&= \vartheta^p \mathcal{I}_\mu(u_n) + \frac{1}{2} \vartheta^p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}} dxdy \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(\vartheta u_n(y))G(\vartheta u_n(x))}{|x-y|^{N-\alpha}} dxdy \\
&\leq \vartheta^p \mathcal{I}_\mu(u_n) + \frac{1}{2}(\vartheta^p - \vartheta^{2q_1}) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}} dxdy. \qquad (21)
\end{aligned}$$

When $\vartheta > 1, 2q_1 > p$, we obtain

$$\vartheta^p - \vartheta^{2q_1} < 0.$$

By Lemma 5, fix $n \in \mathbb{N}$ large enough, we have

$$\mathcal{I}_{\mu,a_2} \leq \vartheta^p \mathcal{I}_\mu(u_n) + \frac{1}{2}(\vartheta^p - \vartheta^{2q_1})C.$$

Let $n \to +\infty$, it is easy to obtain

$$\mathcal{I}_{\mu,a_2} \leq \vartheta^p \mathcal{I}_{\mu,a_1} + (\vartheta^p - \vartheta^{2q_1})C < \vartheta^p \mathcal{I}_{\mu,a_1}.$$

Then

$$\frac{a_1^p}{a_2^p} \mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1}.$$

This completes the proof of Lemma 6. □

To overcome the loss of compactness, on $S(a)$, we establish the next theorem that will be used in the autonomous case and the nonautonomous case.

Lemma 7. *We fix $\mu \in [0, B_*)$ and $(u_n) \subset S(a)$ To be a minimizing sequence with respect to $\mathcal{I}_\mu$. Hereafter, either*

(i) (u_n) is a strongly convergent sequence,
or
(ii) for $(y_n) \subset \mathbb{R}^N$ and $|y_n| \to +\infty$, the sequence $\mathfrak{u}_n(x) = u_n(x + y_n)$ is strongly convergent to the function $\mathfrak{u} \in S(a)$ and $\mathcal{I}_\mu(\mathfrak{u}) = \mathcal{I}_{\mu,a}$.

Proof. Let us prove it by contradiction. According to Lemma 3, for some subsequence in $W^{s,p}(\mathbb{R}^N)$ we have $u_n \rightharpoonup u$. If $\|u\|_{L^p(\mathbb{R}^N)} = b \neq a$ and $u \neq 0$ then $b \in (0, a)$ and we use the Brézis-Lieb lemma [27],

$$\|u_n\|^p_{L^p(\mathbb{R}^N)} = \|u_n - u\|^p_{L^p(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)} + o_n(1).$$

Moreover, as the same argument in Chen et al. [42], we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}}dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G((u_n-u)(y))G((u_n-u)(x))}{|x-y|^{N-\alpha}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u(y))G(u(x))}{|x-y|^{N-\alpha}}dxdy + o_n(1).$$

Setting $\mathfrak{u}_n = u_n - u$, $d_n = \|\mathfrak{u}_n\|_{L^p(\mathbb{R}^N)}$ and supposing that $\|\mathfrak{u}_n\|_{L^p(\mathbb{R}^N)} \to d$ for n large enough, we have $a^p = b^p + d^p$ and $d_n \in (0,a)$. Thus,

$$\mathcal{I}_{\mu,a} + o_n(1) = \mathcal{I}_\mu(u_n) = \mathcal{I}_\mu(\mathfrak{u}_n) + \mathcal{I}_\mu(u) + o_n(1) \geq \mathcal{I}_{\mu,d_n} + \mathcal{I}_{\mu,b} + o_n(1)$$

while combining Lemma 6, we deduce

$$\mathcal{I}_{\mu,a} + o_n(1) \geq \frac{d_n^p}{a^p}\mathcal{I}_{\mu,a} + \mathcal{I}_{\mu,b} + o_n(1).$$

Fixing $n \to \infty$, we have

$$\mathcal{I}_{\mu,a} \geq \frac{d^p}{a^p}\mathcal{I}_{\mu,a} + \mathcal{I}_{\mu,b}. \tag{22}$$

Because $b \in (0,a)$, together with (22), Lemma 6, one has

$$\mathcal{I}_{\mu,a} > \frac{d^p}{a^p}\mathcal{I}_{\mu,a} + \frac{b^p}{a^p}\mathcal{I}_{\mu,a} = \mathcal{I}_{\mu,a}$$

which is absurd. Therefore, $\|u\|_{L^p(\mathbb{R}^N)} = a$, that is $u \in S(a)$.

Since $u_n \rightharpoonup u$, $\|u_n\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} = a$ in $L^p(\mathbb{R}^N)$ while $L^p(\mathbb{R}^N)$ is reflexive, so

$$u_n \to u \ \text{ in } \ L^p(\mathbb{R}^N). \tag{23}$$

Then make use of the interpolation theorem in the Lebesgue spaces, $(\mathfrak{g}_1) - (\mathfrak{g}_2)$ leads to

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}}dxdy \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u(y))G(u(x))}{|x-y|^{N-\alpha}}dxdy. \tag{24}$$

From $\mathcal{I}_{\mu,a} = \lim_{n\to+\infty}\mathcal{I}_\mu(u_n)$, we have

$$\mathcal{I}_{\mu,a} \geq \mathcal{I}_\mu(u).$$

Because $u \in S(a)$, it is easy to obtain that $\mathcal{I}_{\mu,a} = \mathcal{I}_\mu(u)$, and

$$\lim_{n\to+\infty}\mathcal{I}_\mu(u_n) = \mathcal{I}_\mu(u) = \mathcal{I}_{\mu,a},$$

then utilize (23) with (24), we have

$$\|u_n\|^p \to \|u\|^p,$$

where $\|\cdot\|$ denotes the usual norm in $W^{s,p}(\mathbb{R}^N)$. Therefore, in $W^{s,p}(\mathbb{R}^N)$, $u_n \to u$.

On the other hand, we suppose that $u_n \rightharpoonup 0$ in $W^{s,p}(\mathbb{R}^N)$. With Lemma 5, we know that there exists $C > 0$ such that for $n \in \mathbb{N}$ large

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}}dxdy \geq C. \tag{25}$$

According to Lemma 1, for $y_n \in \mathbb{R}^N$, we have $R, \beta > 0$ and such that

$$\int_{B_R(y_n)} |u_n|^p dx \geq \beta, \ \ \forall n \in \mathbb{R}^N. \tag{26}$$

Otherwise, by Ambrosio and Isernia [43]'s Lemma 2.1, it is easy to show that for all $t \in (p, p_s^*)$, $u_n \to 0$ in $L^t(\mathbb{R}^N)$. It implies

$$\frac{G(u_n(y))G(u_n(x))}{|x-y|^{N-\alpha}} \to 0$$

which contradicts (25). For $u = 0$, with the inequality (26) and the fractional Sobolev embedding, we understand that (y_n) is unbounded. From this, considering $\tilde{u}_n(x) = u(x + y_n)$, obviously $(\tilde{u}_n) \subset S(a)$, and it is also a minimizing sequence for $\mathcal{I}_{\mu,a}$. Thus,

$$\tilde{u}_n \rightharpoonup \tilde{u} \ \text{ in } \ W^{s,p}(\mathbb{R}^N) \ \text{ and } \ \tilde{u}_n(x) \rightharpoonup \tilde{u}(x) \ \text{ a.e. } \ \text{ in } \ \mathbb{R}^N$$

for $\tilde{u} \in W^{s,p}(\mathbb{R}^N) \backslash \{0\}$. Above all, $\tilde{u}_n \to \tilde{u}$ in $W^{s,p}(\mathbb{R}^N)$. This proves Lemma 7. □

Next, we state the main result of this section.

Theorem 2. *When g meets the conditions $(\mathfrak{g}_1) - (\mathfrak{g}_3)$, there is $B_* > 0$ such that for $0 \leq \mu < B_*$, problem (15) has a coupled (u, λ) solution and here, u is nonnegative while λ satisfies $\lambda < 0$.*

Proof. First, we prove $\lambda < 0$. With Lemma 3, we have $\mathcal{I}_\mu(u_n) = \mathcal{I}_{\mu,a}$. Then, using Theorem 7, we have $\mathcal{I}_\mu(u) = \mathcal{I}_{\mu,a}$. Thus, for $\lambda_a \in \mathbb{R}$ and by the Lagrange multiplier, we have

$$\mathcal{I}'_\mu(u) = \lambda_a \Psi'_\mu(u) \ \text{ in } \ (W^{s,p}(\mathbb{R}^N))', \tag{27}$$

where $u \in W^{s,p}(\mathbb{R}^N)$ and $\Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$\Psi(u) = \int_{\mathbb{R}^N} |u|^p dx.$$

Therefore, using (27) in $\mathbb{R}^N$, it is obvious to obtain that

$$(-\Delta)_p^s u + \mu |u|^{p-2} u = \lambda_a |u|^{p-2} u + \Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big] g(u).$$

Since $\mathcal{I}_\mu(u) = \mathcal{I}_{\mu,a} < 0$, we obtain $\lambda_a < 0$.

Next, we are going to prove that u is nonnegative. With the definition of the functional $\mathcal{I}_\mu$, i.e., $\mathcal{I}_\mu(u) \geq \mathcal{I}_\mu(|u|)$. Moreover, with $u \in S(a)$, then $|u| \in S(a)$, and

$$\mathcal{I}_{\mu,a} = \mathcal{I}_\mu(u) \geq \mathcal{I}_\mu(|u|) \geq \mathcal{I}_{\mu,a}$$

which implies that $\mathcal{I}_{\mu,a} = \mathcal{I}_\mu(|u|)$; hence, we can replace u by $|u|$. Moreover, we denote u^* by Schwarz's symmetrization of u (Almgren and Lieb [44], Section 9.2) that we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy \geq \iint_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^p}{|x-y|^{N+sp}} dxdy, \ \int_{\mathbb{R}^N} |u|^p dx = \int_{\mathbb{R}^N} |u^*|^p dx$$

and

$$\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big] G(u) dx = \int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big] G(u^*) dx,$$

then $u^* \in S(a)$ with $\mathcal{I}_\mu(u^*) = \mathcal{I}_{\mu,a}$. Therefore, we can replace u by u^*. For some $\alpha \in (0,1)$ by Iannizzotto et al. [45]'s Corollary 5.5, we have that $u \in C^\alpha(\mathbb{R}^N)$. This completes the proof of Theorem 2. □

According to Theorem 2, we deduce the next corollary:

Corollary 1. *Let $a > 0$, $0 \leq \mu_1 < \mu_2 \leq B_*$. Then, $\mathcal{I}_{\mu_1,a} < \mathcal{I}_{\mu_2,a} < 0$.*

Proof. Fix $\mu_{2,a} \in S(a)$ and $\mathcal{I}_{\mu_2}(u_{\mu_2,a}) = \mathcal{I}_{\mu_2,a}$. Afterwards,

$$\mathcal{I}_{\mu_1,a} \leq \mathcal{I}_{\mu_1}(u_{\mu_2,a}) < \mathcal{I}_{\mu_2}(u_{\mu_2,a}) = \mathcal{I}_{\mu_2,a} < 0.$$

This completes the proof of Corollary 1. □

3. The Nonautonomous Case

In this section, we will study the nonautonomous case. The energy function $\mathcal{J}_\varpi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathcal{J}_\varpi(u) = \frac{1}{p}\Big([u]_{s,p}^p + \int_{\mathbb{R}^N} B(\varpi x)|u|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big]G(u)dx,$$

is restricted to the sphere

$$S(a) = \{u \in W^{s,p}(\mathbb{R}^N) : \|u\|_{L^p(\mathbb{R}^N)} = a\}.$$

It and it is easy to prove that $\mathcal{J}_\varpi(u) \in C^1$. Moreover

$$\begin{aligned}\mathcal{J}'_\varpi(u)\varphi = &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\varphi(x)-\varphi(y)}{|x-y|^{N+sp}}dxdy \\ &+ B(\varpi x)\int_{\mathbb{R}^N}|u|^{p-2}u\varphi dx - \iint_{\mathbb{R}^{2N}}\frac{G(u(y))g(u(x))\varphi(x)}{|x-y|^{N-\alpha}}dxdy.\end{aligned}$$

Here, B_* is given in Section 3. We suppose that $\|B\|_{L^\infty(\mathbb{R}^N)} < B_*$.

Then, we give some notations that will be used in the following. Let $\mathcal{J}_0$, $\mathcal{J}_\infty : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ as

$$\mathcal{J}_0(u) = \frac{1}{p}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big]G(u)dx$$

and

$$\mathcal{J}_\infty(u) = \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N} B_\infty|u|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u)\Big]G(u)dx.$$

Additionally, we note $\Phi_{\varpi,a}$, $\Phi_{0,a}$ and $\Phi_{\infty,a}$ by

$$\Phi_{\varpi,a} = \inf_{u\in S(a)}\mathcal{J}_\varpi(u), \ \ \Phi_{0,a} = \inf_{u\in S(a)}\mathcal{J}_0(u), \ \ \Phi_{\infty,a} = \inf_{u\in S(a)}\mathcal{J}_\infty(u).$$

With $0 < B_\infty < +\infty$ and Corollary 1, we have

$$\Phi_{0,a} < \Phi_{\infty,a} < 0. \tag{28}$$

Above all, we can fix $0 < \rho_1 = \frac{1}{2}(\Phi_{\infty,a} - \Phi_{0,a})$.

Then, we give a lemma to show the relationship among $\Phi_{\varpi,a}$, $\Phi_{0,a}$ and $\Phi_{\infty,a}$.

Lemma 8. $\limsup_{\varpi\to 0^+}\Phi_{\varpi,a} \leq \Phi_{0,a}$ *and there is $\varpi_0 > 0$ satisfying $\Phi_{\varpi,a} < \Phi_{\infty,a}$ for all $\varpi \in (0,\varpi_0)$.*

Proof. Note $u_0 \in S(a)$ and $\mathcal{J}_0(u_0) = \Phi_{0,a}$. Then

$$\Phi_{\varpi,a} \leq \mathcal{J}_\varpi(u_0) = \frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^p}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N} B(\varpi x)|u_0|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_0)\Big]G(u_0)dx.$$

Fixing $\varpi \to 0^+$, then

$$\limsup_{\varpi\to 0^+} \Phi_{\varpi,a} \leq \lim_{\varpi\to 0^+} \mathcal{J}_\varpi(u_0) = \mathcal{J}_0(u_0) = \Phi_{0,a}. \tag{29}$$

Combining (28) and (29), when ϖ is small enough, we obtain $\Phi_{\varpi,a} < \Phi_{\infty,a}$. □

Lemma 9. *Let $(u_n) \subset S(a)$, $\varpi \in (0, \varpi_0)$ such that $\mathcal{J}_\varpi(u_n) \to c$ and $c < \Phi_{0,a} + \rho_1 < 0$. For $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$, we have $u \neq 0$.*

Proof. When $u = 0$, it is easy to deduce that

$$\Phi_{0,a} + \rho_1 + o_n(1) > \mathcal{J}_\varpi(u_n) = \mathcal{J}_\infty(u_n) + \frac{1}{p}\int_{\mathbb{R}^N}\Big(B(\varpi x) - B_\infty\Big)|u_n|^p dx.$$

By condition of (B) for $\zeta_1 > 0$, there exists $R > 0$ such that

$$B(x) \geq B_\infty - \zeta_1, \ \ \forall |x| \geq R.$$

Therefore,

$$\Phi_{0,a} + \rho_1 + o_n(1) > \mathcal{J}_\varpi(u_n) \geq \mathcal{J}_\infty(u_n) + \frac{1}{p}\int_{B_{\frac{R}{\varpi}}(0)}\Big(B(\varpi x) - B_\infty\Big)|u_n|^p dx - \frac{\zeta_1}{p}\int_{B^c_{\frac{R}{\varpi}}(0)}|u_n|^p dx.$$

Since (u_n) is bounded in $W^{s,p}(\mathbb{R}^N)$ while for all $\gamma \in [1, p_s^*)$, $u_n \to 0$ in $L^\gamma(B_{\frac{R}{\varpi}}(0))$ for some $K > 0$, we have

$$\Phi_{0,a} + \rho_1 + o_n(1) \geq \mathcal{J}_\infty(u_n) - \zeta_1 K \geq \Phi_{\infty,a} - \zeta_1 K.$$

Because $\zeta_1 > 0$ is arbitrary, we have

$$\Phi_{0,a} + \rho_1 \geq \Phi_{\infty,a}$$

which is a contradiction with the definition of $\rho_1 = \frac{1}{2}(\Phi_{\infty,a} - \Phi_{0,a})$. Therefore, we obtain $u \neq 0$. □

Lemma 10. *Assume that g satisfies condition $(\mathfrak{g}_1) - (\mathfrak{g}_3)$ and (u_n) is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$. Hereafter, there exists $\mathfrak{C}_0 > 0$ such that*

$$\Big|\frac{1}{|x|^\mu} * G(u_n)\Big| \leq \mathfrak{C}_0 \ \ \textit{for all} \ \ n.$$

Proof. According to condition $(\mathfrak{g}_1)$, for all $t \in \mathbb{R}$, we have

$$|G(t)| \leq C(|t|^q + |t|^{\mathfrak{p}}). \tag{30}$$

Thus, with the boundedness of (u_n) and Lemma 2, we have

$$\begin{aligned}\Big|\frac{1}{|x|^{N-\alpha}} * G(u_n)\Big| &\le \Big|\int_{|x-y|\le 1}\frac{G(u_n(y))}{|x-y|^{N-\alpha}}dy\Big| + \Big|\int_{|x-y|\ge 1}\frac{G(u_n(y))}{|x-y|^{N-\alpha}}dy\Big| \\ &\le C\Big(\int_{|x-y|\le 1}\frac{|u_n|^q+|u_n|^{\mathfrak{p}}}{|x-y|^{N-\alpha}}dy + \int_{\mathbb{R}^N}(|u_n|^q+|u_n|^{\mathfrak{p}})dy\Big) \quad (31)\\ &\le C\int_{|x-y|\le 1}\frac{|u_n|^q+|u_n|^{\mathfrak{p}}}{|x-y|^{N-\alpha}}dy + C. \quad (32)\end{aligned}$$

By using the Hölder inequality for $\max\{1,\frac{p}{\mathfrak{p}}\} < t < \frac{p_s^*}{\mathfrak{p}}$, we obtain

$$\begin{aligned}\int_{|x-y|\le 1}\frac{|u_n|^{\mathfrak{p}}}{|x-y|^{N-\alpha}}dy &\le \Big(\int_{|x-y|\le 1}|u_n|^{\mathfrak{p}t}\Big)^{\frac{1}{t}}\Big(\int_{|x-y|\le 1}\frac{1}{|x-y|^{\frac{t(N-\alpha)}{t-1}}}dy\Big)^{\frac{t-1}{t}} \\ &\le C < \Big(\int_0^1 \rho^{N-1-\frac{t(N-\alpha)}{t-1}}d\rho\Big)^{\frac{t-1}{t}} < +\infty, \quad (33)\end{aligned}$$

because of $N-1-\frac{t(N-\alpha)}{t-1} > -1$. Similarly, for $\max\{1,\frac{p}{q}\} < r < \frac{p_s^*}{q}$, we obtain

$$\begin{aligned}\int_{|x-y|\le 1}\frac{|u_n|^{q}}{|x-y|^{N-\alpha}}dy &\le \Big(\int_{|x-y|\le 1}|u_n|^{qr}\Big)^{\frac{1}{r}}\Big(\int_{|x-y|\le 1}\frac{1}{|x-y|^{\frac{r(N-\alpha)}{r-1}}}dy\Big)^{\frac{r-1}{r}} \\ &\le C < \Big(\int_0^1 \rho^{N-1-\frac{r(N-\alpha)}{r-1}}d\rho\Big)^{\frac{r-1}{r}} < +\infty, \quad (34)\end{aligned}$$

due to $N-1-\frac{r(N-\alpha)}{r-1} > -1$. Combining (33) and (34), we prove that

$$\int_{|x-y|\le 1}\frac{|u_n|^q+|u_n|^{\mathfrak{p}}}{|x-y|^{N-\alpha}}dy \le C \ \text{ for all } \ x\in\mathbb{R}^N$$

which in view of (31) yields

$$\Big|\frac{1}{|x|^{\mu}} * G(u_n)\Big| \le \mathfrak{C}_0 \ \text{ for all } \ n.$$

Above all, we end the proof. □

Lemma 11. *Note that $(u_n)\subset S(a)$ is a $(PS)_c$ sequence for $\mathcal{J}_\varpi$ constrained to $S(a)$ satisfying $c < \Phi_{0,a}+\rho_1 < 0$ and $u_n \rightharpoonup u_\varpi$ in $W^{s,p}(\mathbb{R}^N)$, when $n\to\infty$,*

$$\mathcal{J}_\varpi(u_n)\to c \ \text{ and } \ \|\mathcal{J}_\varpi|'_{S(a)}(u_n)\|\to 0.$$

For $\mathrm{u}_n = u_n - u_\varpi \nrightarrow 0$ in $W^{s,p}(\mathbb{R}^N)$, we understand that $\beta^* > 0$ does not depend on $\varpi\in(0,\varpi_0)$ and

$$\liminf_{n\to+\infty}\|u_n-u_\varpi\|^p_{L^p(\mathbb{R}^N)} \ge \beta^*.$$

Proof. Note $\Psi: W^{s,p}(\mathbb{R}^N)\to\mathbb{R}$ is given by

$$\Psi(u) = \frac{1}{p}\int_{\mathbb{R}^N}|u|^p dx, \ \ u\in W^{s,p}(\mathbb{R}^N),$$

we find that $S(a) = \Psi^{-1}(\{a^p/p\})$. Hereafter, according to Willem [27]'s Proposition 5.12, it is easy to obtain a sequence $(\lambda_n)\subset\mathbb{R}$ satisfying

$$\|\mathcal{J}'_\varpi(u_n) - \lambda_n\Psi'(u_n)\|_{(W^{s,p}(\mathbb{R}^N))'} \to 0 \ \text{ as } \ n\to+\infty.$$

Because (u_n) is bounded in $W^{s,p}(\mathbb{R}^N)$ and (λ_n) is also a bounded sequence, so up to a subsequence, when $n \to +\infty$, we can assume that $\lambda_n \to \lambda_\varpi$. Therefore

$$\mathcal{J}'_\varpi(u_n) - \lambda_\varpi \Psi'(u_\varpi) = 0 \ \text{ in } \ (W^{s,p}(\mathbb{R}^N))'. \tag{35}$$

In order to prove (35), the following claims need to be proved.

Claim 1. *For all $\varphi \in W^{s,p}(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u_n(y))g(u_n(x))\varphi(x)}{|x-y|^{N-\alpha}}dxdy \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u_\varpi(y))g(u_\varpi(x))\varphi(x)}{|x-y|^{N-\alpha}}dxdy \tag{36}$$

Since u_n is bounded in $W^{s,p}(\mathbb{R}^N)$, by Lemma 10, there exists $\mathfrak{C}_0 > 0$ such that

$$\Big|\frac{1}{|x|^{N-\alpha}} * G(u_n)\Big| \le \mathfrak{C}_0 \ \text{ for all } \ n.$$

Thus, combining the Vitali's Convergence Theorem and $\varphi \in W^{s,p}(\mathbb{R}^N)$, it is easy to deduce that

$$\Big|\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_n)\Big](g(u_n)-g(u_\varpi))\varphi dx\Big| \le \mathfrak{C}_0\Big|\int_{\mathbb{R}^N}(g(u_n)-g(u_\varpi))\varphi dx\Big| \to 0. \tag{37}$$

By the growth conditions on g and the boundedness of (u_n) in $W^{s,p}(\mathbb{R}^N)$ imply that $G(u_n)$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Observe that $u_n \to u_\varpi$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then, we may assume $G(u_n) \rightharpoonup G(u_\varpi)$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. It is easy to see that $g(u_\varpi)\varphi \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, so $\frac{1}{|x|^{N-\alpha}} * (g(u_n)\varphi) \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Therefore,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * (G(u_n)-G(u_\varpi))\Big]g(u_\varpi)\varphi dx\Big| \\ &= \Big|\int_{\mathbb{R}^N}(G(u_n)-G(u_\varpi))\Big[\frac{1}{|x|^{N-\alpha}} * (g(u_\varpi)\varphi)\Big]dx\Big| \to 0\end{aligned} \tag{38}$$

for any $\varphi \in W^{s,p}(\mathbb{R}^N)$. In view of (37) and (38), we infer

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u_n(y))g(u_n(x))\varphi(x)}{|x-y|^{N-\alpha}}dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u_\varpi(y))g(u_\varpi(x))\varphi(x)}{|x-y|^{N-\alpha}}dxdy\Big| \\ &\le \Big|\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_n(y))\Big]\Big(g(u_n(x)) - g(u_\varpi(x))\Big)\varphi(x)dx\Big| \\ &\quad + \Big|\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * \Big(G(u_n(y)) - G(u_\varpi(y))\Big)\Big]g(u_\varpi(x))\varphi(x)dx\Big| \to 0.\end{aligned}$$

Consequently, Claim 1 is proved.

Claim 2. *We verify that $\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\varphi dx \to \int_{\mathbb{R}^N}|u_\varpi|^{p-2}u_\varpi\varphi dx$ for all $\varphi \in W^{s,p}(\mathbb{R}^N)$.*

Since $W^{s,p}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for all $q \in (p, p_s^*)$, then $\|\varphi\|_{L^q(\mathbb{R}^N)} \le C\|\varphi\| < +\infty$. Then we have $R > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)}|\varphi|^p dx < \varepsilon^p \text{ and } \int_{\mathbb{R}^N\setminus B_R(0)}|\varphi|^q dx < \varepsilon^q. \tag{39}$$

Thus, we obtain

$$\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\varphi dx \to \int_{\mathbb{R}^N}|u_\varpi|^{p-2}u_\varpi\varphi dx \tag{40}$$

as $n \to \infty$.

Claim 3. *We prove that*

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy$$
$$\to \iint_{\mathbb{R}^{2N}} \frac{|u_\varpi(x)-u_\varpi(y)|^{p-2}(u_\varpi(x)-u_\varpi(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy. \quad (41)$$

Using the Hölder inequality, it is easy to obtain

$$\iint_{\mathbb{R}^{2N}} \Big|\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\Big|dxdy$$
$$\leq \Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dxdy\Big)^{(p-1)/p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}dxdy\Big)^{1/p}$$
$$\leq \|u_n\|^{p-1}\|\varphi\| < +\infty. \quad (42)$$

Thus $\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^{2N})$ for all n, and for all $(x,y) \in \mathbb{R}^{2N}$ outside a set with measure zero when we have a constant $K>0$, then

$$\Big|\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\Big| \leq K.$$

For every $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{K}$ such that for all measurable set $E \subset \mathbb{R}^{2N}$, $|E| < \delta$, we have

$$\int_E \Big|\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\Big|dxdy \leq K|E| < \varepsilon.$$

Therefore, $\Big\{\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\Big\}$ is equi-integrable on $\mathbb{R}^{2N}$ and

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}$$
$$\to \frac{|u_\varpi(x)-u_\varpi(y)|^{p-2}(u_\varpi(x)-u_\varpi(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}$$

a.e. on $\mathbb{R}^{2N}$. For $\varphi \in W^{s,p}(\mathbb{R}^N)$, then, there exists $R > 0$ such that

$$\iint_{\mathbb{R}^{2N}\setminus B_R(0)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}dxdy < \varepsilon^p,$$

where $B_R(0)$ is a ball in $\mathbb{R}^{2N}$ with center 0 and radius R. From (42), we know that (u_n) is bounded in $W^{s,p}(\mathbb{R}^N)$, then integrate on $\mathbb{R}^{2N} \setminus B_R(0)$, for a suitable constant $K_* > 0$,

$$\Big|\iint_{\mathbb{R}^{2N}\setminus B_R(0)} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy\Big|$$
$$\leq \Big(\iint_{\mathbb{R}^{2N}\setminus B_R(0)} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dxdy\Big)^{(p-1)/p}$$
$$\times \Big(\iint_{\mathbb{R}^{2N}\setminus B_R(0)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}dxdy\Big)^{1/p} < K_*\varepsilon.$$

Thus, we prove that the Vitali's theorem holds, so (41) holds.

According to Bartsch and Wang ([46], Lemma 2.6),

$$\mathcal{J}'_\varpi(u_n) = \mathcal{J}'_\varpi(u_\varpi) + \mathcal{J}'_\varpi(\mathrm{u}_n) + o_n(1)$$

and

$$\Psi'_\varpi(u_n) = \Psi'_\varpi(u_\varpi) + \Psi'_\varpi(\mathrm{u}_n) + o_n(1).$$

Above all equalities and (35), we deduce that

$$\begin{aligned}\mathcal{J}'_\varpi(u_n) - \lambda_\varpi \Psi'_\varpi(u_n) &= \mathcal{J}'_\varpi(u_\varpi) - \lambda_\varpi \Psi'_\varpi(u_\varpi) + \mathcal{J}'_\varpi(\mathrm{u}_n) - \lambda_\varpi \Psi'_\varpi(\mathrm{u}_n) + o_n(1) \\ &= \mathcal{J}'_\varpi(\mathrm{u}_n) - \lambda_\varpi \Psi'_\varpi(\mathrm{u}_n) + o_n(1),\end{aligned}$$

thus for $n \to +\infty$,

$$\|\mathcal{J}'_\varpi(\mathrm{u}_n) - \lambda_\varpi \Psi'(\mathrm{u}_n)\|_{(W^{s,p}(\mathbb{R}^N))'} \to 0. \tag{43}$$

By $(\mathfrak{g}_3)$ for all $t \geq 0$, we have $q_1 G(t) \leq g(t)t$. Then

$$0 > \Phi_{0,a} + \rho_1 \geq \liminf_{n\to+\infty} \mathcal{J}_\varpi(u_n) = \liminf_{n\to+\infty} \left(\mathcal{J}_\varpi(u_n) - \frac{1}{p}\mathcal{J}'_\varpi(u_n)u_n + \frac{1}{p}\lambda_\varpi a^p \right) \geq \frac{1}{p}\lambda_\varpi a^p$$

and

$$\limsup_{\varpi\to 0} \lambda_\varpi \leq \frac{p(\rho_1 + \Phi_{0,a})}{a^p} < 0.$$

Hence, we have $\lambda_1 < 0$ which does not depend on ϖ such that

$$\lambda_\varpi \leq \lambda_1 < 0, \ \ \forall \varpi \in (0, \varpi_0). \tag{44}$$

According to (43), we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|\mathrm{u}_n(x) - \mathrm{u}_n(y)|^p}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^N} B(\varpi x)|\mathrm{u}_n|^p dx - \lambda_\varpi \int_{\mathbb{R}^N} |\mathrm{u}_n|^p dx \\ &= \iint_{\mathbb{R}^{2N}} \frac{G(\mathrm{u}_n(y))g(\mathrm{u}_n(x))\mathrm{u}_n(x)}{|x-y|^{N-\alpha}} dxdy + o_n(1).\end{aligned} \tag{45}$$

Combining (44) and (45), we deduce

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|\mathrm{u}_n(x) - \mathrm{u}_n(y)|^p}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^N} B(\varpi x)|\mathrm{u}_n|^p dx - \lambda_1 \int_{\mathbb{R}^N} |\mathrm{u}_n|^p dx \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{G(\mathrm{u}_n(y))g(\mathrm{u}_n(x))\mathrm{u}_n(x)}{|x-y|^{N-\alpha}} dxdy + o_n(1).\end{aligned} \tag{46}$$

From Lemma 2, for suitable constants $\mathcal{C} > 0$ and $D_3 > 0$, we have

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{G(\mathrm{u}_n(y))g(\mathrm{u}_n(x))\mathrm{u}_n(x)}{|x-y|^{N-\alpha}} dxdy &\leq \mathcal{C}\|G(\mathrm{u}_n)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|g(\mathrm{u}_n(x))\mathrm{u}_n(x)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &\leq \frac{2\mathcal{C}}{\theta}\|g(\mathrm{u}_n(x))\mathrm{u}_n(x)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &\leq D_3(\|\mathrm{u}_n(x)\|^{2\mathrm{p}}_{L^{\frac{2N\mathrm{p}}{N+\alpha}}(\mathbb{R}^N)} + \|\mathrm{u}_n\|^{2q}_{L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)})\end{aligned}$$

Above all, we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|\mathfrak{u}_n(x)-\mathfrak{u}_n(y)|^p}{|x-y|^{N+sp}}dxdy + C_0\int_{\mathbb{R}^N}|\mathfrak{u}_n|^p dx\\ &\le \iint_{\mathbb{R}^{2N}} \frac{G(\mathfrak{u}_n(y))g(\mathfrak{u}_n(x))\mathfrak{u}_n(x)}{|x-y|^{N-\alpha}}dxdy + o_n(1)\\ &\le D_3(\|\mathfrak{u}_n(x)\|^{2\mathfrak{p}}_{L^{\frac{2N\mathfrak{p}}{N+\alpha}}(\mathbb{R}^N)} + \|\mathfrak{u}_n\|^{2q}_{L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)}) + o_n(1),\end{aligned}$$

where C_0 is a constant and does not depend on $\varpi \in (0,\varpi_0)$. With the Sobolev embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{\mathfrak{p}}(\mathbb{R}^N)$,

$$\begin{aligned}\|\mathfrak{u}_n\|^p_{W^{s,p}(\mathbb{R}^N)} &\le D_3(\|\mathfrak{u}_n(x)\|^{2\mathfrak{p}}_{L^{\frac{2N\mathfrak{p}}{N+\alpha}}(\mathbb{R}^N)} + \|\mathfrak{u}_n\|^{2q}_{L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)}) + o_n(1)\\ &\le D_4(\|\mathfrak{u}_n\|^{2\mathfrak{p}}_{W^{s,p}(\mathbb{R}^N)} + \|\mathfrak{u}_n\|^{2q}_{W^{s,p}(\mathbb{R}^N)}) + o_n(1),\end{aligned} \tag{47}$$

where D_3 and D_4 are two constants that do not depend on $\varpi \in (0,\varpi_0)$. For $\mathfrak{u}_n \nrightarrow 0$ in $W^{s,p}(\mathbb{R}^N)$, for a subsequence of $(\mathfrak{u}_n)$, we assume that $\liminf_{n\to+\infty}\|\mathfrak{u}_n\|_{W^{s,p}(\mathbb{R}^N)} > 0$. Since $2\mathfrak{p} > p, 2q > p$, by (47), there is a suitable constant $D_5 > 0$ such that

$$\liminf_{n\to+\infty}\|\mathfrak{u}_n\|_{W^{s,p}(\mathbb{R}^N)} \ge D_5 > 0. \tag{48}$$

Combining (47) and (48), we have

$$\liminf_{n\to+\infty}\|\mathfrak{u}_n\|^{2\mathfrak{p}}_{L^{\frac{2N\mathfrak{p}}{N+\alpha}}(\mathbb{R}^N)} \ge D_6, \tag{49}$$

or

$$\liminf_{n\to+\infty}\|\mathfrak{u}_n\|^{2q}_{L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)} \ge D_6, \tag{50}$$

where D_6 is a constant independent of $\varpi \in (0,\varpi_0)$. Indeed, if

$$\liminf_{n\to+\infty}\|\mathfrak{u}_n\|^{2\mathfrak{p}}_{L^{\frac{2N\mathfrak{p}}{N+\alpha}}(\mathbb{R}^N)} = 0 \tag{51}$$

and

$$\liminf_{n\to+\infty}\|\mathfrak{u}_n\|^{2q}_{L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)} = 0, \tag{52}$$

then $\liminf_{n\to+\infty}\|\mathfrak{u}_n\|^p_{W^{s,p}(\mathbb{R}^N)} = 0$ via (47). This is a contradiction. Please note that

$$\mathfrak{p}, q \in (\frac{p(N+\alpha)}{2N}, \frac{p(N+\alpha)+p^2 s}{2N}),$$

then $\frac{2N\mathfrak{p}}{N+\alpha}, \frac{2Nq}{N+\alpha} \in (p, p_s^*)$. We denote $\mathfrak{p}_1 = \frac{2N\mathfrak{p}}{N+\alpha}, \mathfrak{q}_1 = \frac{2Nq}{N+\alpha} \in (p, p_s^*)$. Applying the fractional Gagliardo–Nirenberg inequality, we obtain

$$\|\mathfrak{u}_n\|^{\mathfrak{t}}_{L^{\mathfrak{t}}(\mathbb{R}^N)} \le C_{s,N,\mathfrak{t}}\|\mathfrak{u}_n\|^{\mathfrak{t}(1-\mathfrak{a})}_{L^p(\mathbb{R}^N)}[\mathfrak{u}_n]^{\mathfrak{t}\mathfrak{a}}_{s,p},$$

for all $\mathfrak{t} \in \{\mathfrak{p}_1, \mathfrak{q}_1\}$, then for all $n \in \mathbb{N}$, we have

$$\liminf_{n\to+\infty}\|\mathfrak{u}_n\|^{\mathfrak{t}}_{L^{\mathfrak{t}}(\mathbb{R}^N)} \le C_{s,N,\mathfrak{t}}(\liminf_{n\to+\infty}\|\mathfrak{u}_n\|_{L^p(\mathbb{R}^N)})^{\mathfrak{t}(1-\mathfrak{a})}G^{\mathfrak{t}\mathfrak{a}}, \tag{53}$$

where G is a positive constant independent of $\varpi \in (0, \varpi_0)$ and $[\mathrm{u}_n]_{s,p} \leq G$. With (49), (50) and (53), we understand that there exists $\beta^* > 0$ independent of $\varpi \in (0, \varpi_0)$ such that

$$\liminf_{n \to +\infty} \|u_n - u_\varpi\|^p_{L^p(\mathbb{R}^N)} \geq \beta^*.$$

This completes the proof of Lemma 11. □

From here, we let ρ satisfy $0 < \rho < \min\{\frac{1}{p}, \frac{\beta^*}{a^p}\}(\Phi_{\infty,a} - \Phi_{0,a}) \leq \rho_1$.

Lemma 12. *Let $\varpi \in (0, \varpi_0)$, $\mathcal{J}_\varpi$ satisfy the $(PS)_c$ condition restricted on $S(a)$ with $c < \Phi_{0,a} + \rho$.*

Proof. Note that $(u_n) \subset S(a)$ is a $(PS)_c$ sequence for $\mathcal{J}_\varpi$ constrained to $S(a)$ and $u_n \rightharpoonup u_\varpi$ in $W^{s,p}(\mathbb{R}^N)$ while $c < \Phi_{0,a} + \rho < 0$. Note that $\Psi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$. As

$$\Psi(u) = \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx, \;\; u \in W^{s,p}(\mathbb{R}^N),$$

we find $S(a) = \Psi^{-1}(\{a^p/p\})$. According to Willem [27]'s Proposition 5.12 as $n \to +\infty$, for a sequence $(\lambda_n) \subset \mathbb{R}$, we have

$$\|\mathcal{J}'_\varpi(u_n) - \lambda_n \Psi'(u_n)\|_{(W^{s,p}(\mathbb{R}^N))'} \to 0.$$

According to Lemma 11, for $\mathrm{u}_n = u_n - u_\varpi \nrightarrow 0$ in $W^{s,p}(\mathbb{R}^N)$, and there exists $\beta^* > 0$, which does not depend on $\varpi \in (0, \varpi_0)$, such that

$$\liminf_{n \to +\infty} \|u_n - u_\varpi\|^p_{L^p(\mathbb{R}^N)} = \liminf_{n \to +\infty} \|\mathrm{u}_n\|^p_{L^p(\mathbb{R}^N)} \geq \beta^*,$$

where ϖ_0 is given in Lemma 8.

Setting $d_n = \|\mathrm{u}_n\|_{L^p(\mathbb{R}^N)}$, while assuming that $\|\mathrm{u}_n\|_{L^p(\mathbb{R}^N)} \to d > 0$ and $\|u_\varpi\|_{L^p(\mathbb{R}^N)} = b$, we have $a^p = b^p + d^p$. According to Lemma 9, for large enough n we have $b > 0$ and $\mathcal{J}_\varpi(u_n) \geq \Phi_{\infty,d_n} + o_n(1)$, so we must obtain $d_n \in (0, a)$. Hence,

$$c + o_n(1) = \mathcal{J}_\varpi(u_n) = \mathcal{J}_\varpi(\mathrm{u}_n) + \mathcal{J}_\varpi(u_\varpi) + o_n(1) \geq \Phi_{\infty,d_n} + \Phi_{0,b} + o_n(1)$$

as well as Lemma 6,

$$\rho + \Phi_{0,a} \geq \frac{d_n^p}{a^p}\Phi_{\infty,a} + \frac{b^p}{a^p}\Phi_{0,a}.$$

Fixing $n \to \infty$, we deduce

$$\rho \geq \frac{d^p}{a^p}(\Phi_{\infty,a} - \Phi_{0,a}) \geq \frac{\beta^*}{a^p}(\Phi_{\infty,a} - \Phi_{0,a}) \tag{54}$$

which is absurd when $\rho < \frac{\beta^*}{a^p}(\Phi_{\infty,a} - \Phi_{0,a})$. Thus, $\mathrm{u}_n \to 0$ in $W^{s,p}(\mathbb{R}^N)$, and $\mathrm{u}_n \to u_\varpi$ in $W^{s,p}(\mathbb{R}^N)$. Thus, $\|u_\varpi\|_{L^p(\mathbb{R}^N)} = a$ and when λ_ϖ is the limit of some subsequence of (λ_n),

$$(-\Delta)^s_p u_\varpi + B(\varpi x)|u_\varpi|^{p-2}u_\varpi = \lambda_\varpi |u_\varpi|^{p-2}u_\varpi + \left[\frac{1}{|x|^{N-\alpha}} * G(u_\varpi)\right]g(u_\varpi) \;\; \text{in } \mathbb{R}^N.$$

This completes the proof of Lemma 12. □

4. Multiplicity Result of Problem (1)

Let $\delta > 0$ and $w > 0$ be the solution of the following problem

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2}u + \left[\frac{1}{|x|^{N-\alpha}} * G(u)\right]g(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = a^p, \end{cases} \tag{55}$$

with $\mathcal{J}_0(w) = \Phi_{0,a}$. Let $\eta : [0, +\infty) \to [0, 1]$ be a smooth non-increasing cut-off function satisfying $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$ as well as $\eta(t) = 0$ if $t \geq \delta$. For any $y \in M$, we set the function $\Psi_{\varpi,y} : \mathbb{R}^N \to \mathbb{R}$

$$\Psi_{\varpi,y}(x) := \eta(|\varpi x - y|) w\left(\frac{\varpi x - y}{\varpi}\right),$$

$$\tilde{\Psi}_{\varpi,y}(x) = a \frac{\Psi_{\varpi,y}(x)}{\|\Psi_{\varpi,y}\|_{L^p(\mathbb{R}^N)}}$$

while denoting by $\Xi_\varpi : M \to S(a)$ the function

$$\Xi_\varpi(y) = \tilde{\Psi}_{\varpi,y}.$$

Therefore, for every $y \in M$, $\Xi_\varpi(y)$ has compact support.

Lemma 13. *The functional* Ξ_ϖ *meets*

$$\lim_{\varpi \to 0^+} \mathcal{J}_\varpi(\Xi_\varpi(y)) = \Phi_{0,a} \quad \textit{uniformly in} \ \ y \in M.$$

Proof. We assume by contradiction that there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varpi_n \to 0$ such that

$$|\mathcal{J}_{\varpi_n}(\Xi_{\varpi_n}(y_n)) - \Phi_{0,a}| \geq \delta_0, \quad \forall n \in \mathbb{N}. \tag{56}$$

Let $z = \dfrac{\varpi_n x - y_n}{\varpi_n}$ and $\bar{z} = \dfrac{\varpi_n y - y_n}{\varpi_n}$, we can obtain

$$\begin{aligned}\mathcal{J}_{\varpi_n}(\Xi_{\varpi_n}(y_n)) = &\frac{1}{p}\left([\eta(|\varpi_n z|)w(z)]^p_{s,p} + \int_{\mathbb{R}^N} B(\varpi_n z + y_n)(\eta(|\varpi_n z|)w(z))^p dz\right) \\ &- \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\eta(|\varpi_n \bar{z}|)w(\bar{z}))G(\eta(|\varpi_n z|)w(z))}{|z - \bar{z}|^{N-\alpha}} dz d\bar{z}.\end{aligned}$$

By the Lebesgue-dominated convergence theorem as in Molica Bisci et al. [47]'s Lemma 17 and Palatucci and Pisante [48]'s Lemma 5,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\Psi_{\varpi_n, y_n}|^p dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\eta(|\varpi_n z|)w(z)|^p dx = \int_{\mathbb{R}^N} |w|^p dx = a^p,$$

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\Xi_{\varpi_n}(y_n)(x) - \Xi_{\varpi_n}(y_n)(y)|^p}{|x - y|^{N+sp}} dx dy = [w]^p_{s,p}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\eta(|\varpi_n \bar{z}|)w(\bar{z}))G(\eta(|\varpi_n z|)w(z))}{|z - \bar{z}|^{N-\alpha}} dz d\bar{z} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(w(\bar{z}))G(w(z))}{|z - \bar{z}|^{N-\alpha}} dz d\bar{z}.$$

From Ambrosio and Isernia [43]'s Lemmas 2.2 and 2.5, we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} B(\varpi_n x)|\Xi_{\varpi_n}(y_n)|^p dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{a^p}{\|\Psi_{\varpi_n,y}\|^p_{L^p(\mathbb{R}^N)}} B(\varpi_n z + y_n)|\eta(|\varpi_n z|)w(z)|^p dz = 0.$$

As a consequence, this yields

$$\lim_{n\to\infty} \mathcal{J}_{\varpi_n}(\Xi_{\varpi_n}(y_n)) = \mathcal{J}_0(w) = \Phi_{0,a}$$

which contradicts (56). □

Now, we fix $R = R(\delta) > 0$ and choose $\delta > 0$ such that $M_\delta \subset B_R(0)$. Then note $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by letting $\Upsilon(x) = x$ for $|x| \leq R$ as well as $\Upsilon(x) = \frac{Rx}{|x|}$ for $|x| \geq R$. Hereafter, let $\beta_\varpi : S(a) \to \mathbb{R}^N$ given by

$$\beta_\varpi(u) = \frac{\int_{\mathbb{R}^N} \Upsilon(\varpi x)|u|^p dx}{a^p}.$$

Lemma 14 (Ambrosio [4])**.** *The function β_ϖ meets*

$$\lim_{\varpi \to 0} \beta_\varpi(\Xi_\varpi(y)) = y \text{ uniformly in } y \in M.$$

Proof. With Ambrosio and Isernia [43]'s Lemma 4.18, assuming that there exists $\delta_0 > 0$, $(y_n) \subset M$ and $\varpi_n \to 0$ such that

$$|\beta_{\varpi_n}(\Xi_{\varpi_n}(y_n)) - y_n| \geq \delta_0. \tag{57}$$

By using the definitions of $\Xi_{\varpi_n}(y_n), \beta_{\varpi_n}$ and η, we deduce that

$$\beta_{\varpi_n}(\Xi_{\varpi_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varpi_n z + y_n) - y_n]|\eta(|\varpi_n z|)w(z)|^p dz}{a^p}.$$

With the Dominated Convergence Theorem and $(y_n) \subset M \subset B_R(0)$,

$$|\beta_{\varpi_n}(\Xi_{\varpi_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (57). □

Arguing as in the proof of Ambrosio [3], we have the next lemma.

Lemma 15. *Fix $\varpi_n \to 0$ and $(u_n) \subset S(a)$ be such that $\mathcal{J}_{\varpi_n}(u_n) \to \Phi_{0,a}$. Then, there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $\mathrm{u}_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{s,p}(\mathbb{R}^N)$. Additionally, up to a subsequence, $y_n = \varpi_n \tilde{y}_n \to y \in M$.*

Proof. Since $\langle \mathcal{J}'_{\varpi_n}(u_n), u_n \rangle = 0$, $\mathcal{J}_{\varpi_n}(u_n) \to \Phi_{0,a}$ and we can argue that (u_n) is bounded. Then, for two constants $R > 0$, $\beta > 0$, and a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ we have

$$\liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} |u_n|^p dx \geq \beta > 0.$$

We surmise that the assumption is invalid. Hence, for all $R > 0$,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(0)} |u_n|^p dx = 0.$$

Since (u_n) is bounded in $W^{s,p}(\mathbb{R}^N)$, according to Lemma 1, for any $q \in (p, p_s^*)$, $u_n \to 0$ in $L^q(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} \left[\frac{1}{|x|^{N-\alpha}} * G(u_n)\right] G(u_n) dx \to 0$. Therefore, according to $\lim_{n \to +\infty} \mathcal{J}_{\varpi_n}(u_n) = \Phi_{0,a} < 0$, we have the contradiction with $\lim_{n \to +\infty} \mathcal{J}_{\varpi_n}(u_n) \geq 0$.

Then, we set $\mathrm{u}_n(x) = u_n(x + \hat{y}_n)$. There exists $u \in W^{s,p}(\mathbb{R}^N) \backslash \{0\}$ and for a subsequence, $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$. Because

$$\Phi_{0,a} \leq \mathcal{J}_0(u_n) = \mathcal{J}_0(\mathrm{u}_n) \leq \mathcal{J}_{\varpi_n}(u_n) \text{ and } (u_n) \subset S(a),$$

we have $\mathcal{J}_0(\mathrm{u}_n) \to \Phi_{0,a}$. Using Theorem 7, it is easy to deduce that $u_n \to u$ in $W^{s,p}(\mathbb{R}^N)$ and $(u_n) \subset S(a)$. Then, we show that (y_n) is bounded.

Using rebuttals of evidence, we may hypothesize that there exists a subsequence of (y_n) such that $|y_n| \to \infty$ when $n \to \infty$, and we have

$$\begin{aligned}\Phi_{0,a} &= \lim_{n\to+\infty} \mathcal{J}_{\varpi_n}(u_n)\\ &= \liminf_{n\to+\infty} \Big(\frac{1}{p}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N} B(\varpi_n x + y_n)|u_n|^p dx\Big)\\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_n)\Big]G(u_n)dx\Big),\end{aligned}$$

i.e.,

$$\Phi_{0,a} \geq \frac{1}{p}\Big([u_n]^p_{s,p} + \int_{\mathbb{R}^N} B_\infty |u_n|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_n)\Big]G(u_n)dx \geq \Phi_{\infty,a}$$

which gives an absurd with (28). Therefore, we understand that (y_n) is bounded, so in $\mathbb{R}^N$, we suppose that $y_n \to y$.

Above all, it is easy to derive that

$$\Phi_{0,a} \geq \frac{1}{p}\Big([u_n]^p_{s,p} + \int_{\mathbb{R}^N} B_\infty |u_n|^p dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\alpha}} * G(u_n)\Big]G(u_n)dx \geq \Phi_{B(y),a}.$$

If $y \notin M$, with Corollary 1 and $B(y) > 0$, we have $\Phi_{B(y),a} > \Phi_{0,a}$, which is a contradiction, then $B(y) = 0$, and $y \in M$. □

We set $h(\varpi)$ as a positive function satisfying $h(\varpi) \to 0$ as $\varpi \to 0$. We define

$$\tilde{S}(a) = \{u \in S(a) : \mathcal{J}_\varpi(u) \leq \Phi_{0,a} + h(\varpi)\}. \tag{58}$$

For any $y \in M$, according to Lemma 13, when $\varpi \to 0$, $h(\varpi) = |\mathcal{J}_\varpi(\Xi_\varpi(y)) - \Phi_{0,a}| \to 0$. Thus, $\Phi_\varpi(y) \in \tilde{S}(a)$ for any $\varpi > 0$.

Inspired by Alves and Figueiredo [49] and Alves and Thin [50]'s Lemma 4.5, then we give the next Lemma.

Lemma 16. *We set $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leq \delta\}$, so it holds that*

$$\lim_{\varpi\to 0} \sup_{u\in\tilde{S}(a)} \inf_{z\in M_\delta} |\beta_\varpi(u) - z| = 0.$$

Proof. We fix $\varpi_n \to 0$ as $n \to \infty$. Then, there exists $u_n \in \tilde{S}(a)$ such that

$$\inf_{z\in M_\delta} |\beta_\varpi(u) - z| = \sup_{u\in\tilde{S}(a)} \inf_{z\in M_\delta} |\beta_\varpi(u) - z| + o_n(1).$$

Hence, there exists $(\tilde{y}_n) \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varpi_n}(u_n) - \tilde{y}_n| = 0.$$

Since $u_n \in \tilde{S}(a)$,

$$\Phi_{0,a} \leq \mathcal{J}_0(u_n) \leq \mathcal{J}_{\varpi_n}(u_n) \leq \Phi_{0,a} + h(\varpi_n),$$

i.e.,

$$u_n \in S(a) \ \text{ and } \ \mathcal{J}_{\varpi_n}(u_n) \to \Phi_{0,a}.$$

Using Lemma 15, for large enough n, we obtain that there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $y_n = \varpi_n \tilde{y}_n \in M_\delta$. By setting $\mathrm{u}_n = u_n(x + \tilde{y}_n)$ and using a change in variable, we deduce

$$\beta_{\varpi_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varpi_n z + y_n) - y_n] |\mathrm{u}_n|^p dz}{a^p}$$

and

$$\beta_{\varpi_n}(u_n) - y_n = \frac{\int_{\mathbb{R}^N} [\Upsilon(\varpi_n z + y_n) - y_n] |\mathrm{u}_n|^p dz}{a^p} \to 0 \text{ when } n \to \infty.$$

This completes the proof of Lemma 16. □

Proof of Theorem 1. To prove Theorem 1, we use two steps.

Step 1. We first verify the existence of multiple normalized solutions to problem (1).

For $\varpi \in (0, \varpi_0)$: According to Lemma 13, Lemma 14 and Lemma 16 and the arguments in Cingolani and Lazzo [51], we can understand that $\beta_\varpi \circ \Xi_\varpi$ is a homotopic to the inclusion map id: $M \to M_\delta$. Combining Ambrosio [39]'s Lemma 6.3.21, it is clear we have

$$cat(\tilde{S}(a)) \geq cat_{M_\delta}(M). \tag{59}$$

Furthermore , let us choose a function $h(\varpi) > 0$ such that $h(\varpi) \to 0$ as $\varpi \to 0$ as well as $\Phi_{0,a} + h(\varpi)$ is not a critical level for $\mathcal{J}_\varpi$. According to arguments as in Lemma 4, on $S(a)$, we deduce that $\mathcal{J}_\varpi$ is bounded. For small enough $\varpi > 0$, we deduce from Lemma 12 that $\mathcal{J}_\varpi$ satisfies the Palais–Smale condition in $S(a)$. With the Lusternik–Schnirelman category theorem for critical points in Ghoussoub [52] and Wang et al. [23], it is easy to obtain that $\mathcal{J}_\varpi$ admits at least $cat_{M_\delta}(M)$ critical points on $S(a)$.

Step 2. We study the behavior of maximum points of $|u_\varpi|$.

For h, given in (58), we fix u_ϖ as a solution of problem (1) with $\mathcal{J}_\varpi(u_\varpi) \leq \Phi_{0,a} + h(\varpi)$. Using the proof of Lemma 15, for each $\varpi_n \to 0$, there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $y_n = \varpi_n \tilde{y}_n \to y_0 \in M$ and $\mathrm{u}_n(x) = u_{\varpi_n}(x + \tilde{y}_n)$ has a convergent subsequence in $W^{s,p}(\mathbb{R}^N)$ and $\mathrm{u} \neq 0$. Then, u_n is a solution of

$$(-\Delta)_p^s \mathrm{u}_n + B(\varpi_n x + y_n)|\mathrm{u}_n|^{p-2}\mathrm{u}_n = \lambda_n |\mathrm{u}_n|^{p-2}\mathrm{u}_n + \frac{1}{|x|^{N-\alpha}} * g(\mathrm{u}_n) \text{ in } \mathbb{R}^N$$

and

$$\limsup_{\varpi \to 0} \lambda_\varpi \leq \frac{p(\rho_1 + \Phi_{0,a})}{a^p} < 0.$$

Since $\mathrm{u}_n \to \mathrm{u}$ in $W^{s,p}(\mathbb{R}^N)$, then, applying the same arguments found in Alves and Figueiredo [49]'s Lemma 4.5, we obtain

$$\lim_{|x| \to \infty} \mathrm{u}_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Thus, there are $R_1 > 0$, $n_0 \in \mathbb{N}$ and $\tau > 0$ such that

$$|\mathrm{u}_n(x)| \leq \frac{1}{2}\left(\frac{\tau}{2|B_{R_1}(0)|}\right)^{\frac{1}{p}} \text{ for } |x| \geq R_1 \text{ and } n \geq n_0.$$

We know that $\|u_n\|_{L^\infty(\mathbb{R}^N)} \nrightarrow 0$, because $\|u_n\|_{L^2} = a$, which contradicts $u_n \to 0$ in $W^{s,p}(\mathbb{R}^N)$.

Then, with (28), when n is large enough, we pick $R_1 > \tau_0$ such that

$$0 < \frac{\tau}{2} \leq \int_{B_{R_1}(0)} |u_n|^p dx \leq |B_{R_1}(0)| \cdot \|u_n\|^p_{L^\infty(\mathbb{R}^N)} \tag{60}$$

when we choose $\delta = \left(\frac{\tau}{2|B_{R_1}(0)|}\right)^{\frac{1}{p}}$. We have that there exists $\delta > 0$ such that $\|u_n\|_{L^\infty} \geq \delta$. In the following, let us consider $\zeta_n \in \mathbb{R}^N$ such that $|u_n(\zeta_n)| = \|u_n\|_{L^\infty(\mathbb{R}^N)}$ for all $n \in \mathbb{N}$. Then, $\zeta_n = z_n + \widetilde{y}_n$ and

$$\lim_{n\to+\infty} B(\varpi_n\zeta_n) = \lim_{n\to+\infty} B(\varpi_n z_n + \varpi_n\widetilde{y}_n) = B(y) = 0.$$

□

5. Conclusions and Future Studies

In our study above, by using variational methods, minimization techniques, and the Lusternik–Schnirelmann category, we obtain the existence of multiple normalized solutions. Moreover, under the autonomous case and nonautonomous case, we prove Theorem 1. In future studies, we will change the growth of g to exponential growth. In this process, forced proof will be affected, and we need to explore new methods to solve this problem. At the same time, we will try our best to explore the practical application of the research problem.

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References

1. Díaz, J.I. *Nonlinear Partial Differential Equations and Free Boundaries*; Elliptic Equations. Pitman Res. Notes Math. Ser. 106; Pitman: Boston, MA, USA, 1985; Volume 1.
2. Mastorakis, N.; Fathabadi, H. On the solution of p-Laplacian for non-Newtonian fluid flow. *WSEAS Trans. Math.* **2009**, *8*, 238–245.
3. Ambrosio, V. Multiplicity and concentration results for a fractional Choquard equation via penalization method. *Potential Anal.* **2019**, *50*, 55–82. [CrossRef]
4. Ambrosio, V. On the multiplicity and concentration of positive solutions for a p-fractional Choquard equation in $\mathbb{R}^N$. *Comput. Math. Appl.* **2019**, *78*, 2593–2617. [CrossRef]
5. Degiovanni, M.; Lancelotti, S. Linking solutions for p-Laplace equations with nonlinearity at critical growth. *J. Funct. Anal.* **2009**, *256*, 3643–3659. [CrossRef]
6. Guedda, M.; Veron, L. Quasilinear elliptic equations involving critical Sobolev exponents. *Nonlinear Anal.* **1989**, *13*, 879–902. [CrossRef]
7. Silva, K.; Macedo, A. Local minimizers over the Nehari manifold for a class of concave-convex problems with sign changing nonlinearity. *J. Differ. Equ.* **2018**, *265*, 1894–1921. [CrossRef]
8. Li, G.; Wang, C. The existence of a nontrivial solution to p-Laplacian equations in $\mathbb{R}^N$ with supercritical growth. *Math. Meth. Appl. Sci.* **2013**, *36*, 69–79. [CrossRef]
9. Alves, C.O.; Miyagaki, O.H. Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method. *Calc. Var. Partial Differ. Equ.* **2016**, *55*, 47. [CrossRef]
10. Correia, J.; Oliveira, C. Existence of a positive solution for a class of fractional elliptic problems in exterior domains involving critical growth. *J. Math. Anal. Appl.* **2022**, *506*, 125543. [CrossRef]
11. Pucci, P.; Xiang, M.; Zhang, B. Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. *Calc. Var. Partial Differ. Equ.* **2015**, *54*, 2785–2806. [CrossRef]
12. Zhang, W.; Wen, Y.; Zhao, P. Multiplicity and concentration of solutions for fractional Kirchhoff-Choquard equation with critical growth. *J. Math. Phys.* **2022**, *63*, 081507. [CrossRef]
13. Jeanjean, L. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.* **1997**, *28*, 1633–1659. [CrossRef]

14. Cazenave, T.; Lions, P.L. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Commun. Math. Phys.* **1982**, *85*, 549–561. [CrossRef]
15. Shibata, M. Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term. *Manuscr. Math.* **2014**, *143*, 221–237. [CrossRef]
16. Soave, N. Normalized ground states for the NLS equation with combined nonlinearities. *J. Differ. Equ.* **2020**, *269*, 6941–6987. [CrossRef]
17. Jeanjean, L.; Le, T.T. Multiple normalized solutions for a Sobolev critical Schrödinger equation. *Math. Ann.* **2022**, *384*, 101–134. [CrossRef]
18. Yu, S.; Tang, C.; Zhang, Z. Normalized solutions of mass subcritical fractional Schrödinger equations in Exterior domains. *J. Geom. Anal.* **2023**, *33*, 162. [CrossRef]
19. Luo, H.; Zhang, Z. Normalized solutions to the fractional Schrödinger equations with combined nonlinearities. *Calc. Var. Partial Differ. Equ.* **2020**, *59*, 143. [CrossRef]
20. Ikoma, N.; Miyamoto, Y. Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities. *Calc. Var. Partial Differ. Equ.* **2020**, *59*, 48. [CrossRef]
21. Zhang, X.; Squassina, M.; Zhang, J. Multiplicity of normalized solutions for the fractional Schrödinger equation with potentials. *Mathematics* **2024**, *12*, 772. [CrossRef]
22. Peng, S.; Xia, A. Normalized solutions of suppercritical nonlinear fractional Schrödinger equation with potential. *Commun. Pure Appl. Math.* **2021**, *20*, 3707–3728.
23. Wang, W.; Li, Q.; Zhou, J.; Li, Y. Normalized solutions for p-Laplacian equations with a L^2-supercritical growth. *Ann. Funct. Anal.* **2021**, *12*, 9. [CrossRef]
24. Zhang, Z.; Zhang, Z. Normalized solutions to p-Laplacian equations with combined nonlinearities. *Nonlinearity* **2022**, *35*, 5621–5663. [CrossRef]
25. Wang, C.; Sun, J. Normalized solutions for the p-Laplacian equation with a trapping potential. *Adv. Nonlinear Anal.* **2023**, *12*, 20220291. [CrossRef]
26. Thin, N.V.; Rădulescu, V.D. Multiple normalized solutions for fractional elliptic problems. *Forum Math.* **2024**. [CrossRef]
27. Willem, M. *Minimax Theorems*; Birkhäuser: Basel, Switzerland, 1996.
28. Kwong, M.K. Uniqueness of positive solutions of $\Delta u - u + |v|^p = 0$ in $\mathbb{R}^N$. *Arch. Ration. Mech. Anal.* **1989**, *105*, 243–266. [CrossRef]
29. Li, Y.; Ni, W.M. Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$. *Commun. Partial Differ. Equ.* **1993**, *18*, 1043–1054. [CrossRef]
30. Damascelli, L.; Pacella, F.; Ramaswamy, M. Symmetry of ground states of p-laplace equations via the moving plane method. *Arch. Ration. Mech. Anal.* **1999**, *148*, 291–308. [CrossRef]
31. Serrin, J.; Tang, M.X. Uniqueness of ground states for quasilinear elliptic equations. *Indiana Univ. Math. J.* **2000**, *49*, 897–923. [CrossRef]
32. Moroz, V.; Schaftingen, J.V. A guide to the Choquard equation. *J. Fixed Point Theory Appl.* **2017**, *19*, 773–813. [CrossRef]
33. Li, G.; Ye, H. The existence of positive solutions with prescribed L^2-norm for nonlinear Choquard equations. *J. Math. Phys.* **2014**, *55*, 121501. [CrossRef]
34. Chen, W.; Wang, Z. Normalized solutions for a fractional N/s-Laplacian Choquard equation with exponential critical nonlinearities. *arXiv* **2023**, arXiv:2310.05477.
35. Feng, Z.; He, X.; Meng, Y. Normalized solution of fractional Choquard equation with critical nonlinearity. *Differ. Integral Equ.* **2023**, *36* , 593–620. [CrossRef]
36. Maheswari, C.U.; Bakshi, S.K. Invariant subspace method for time-fractional nonlinear evolution equations of the third order. *Pramana J. Phys.* **2022**, *96*, 173. [CrossRef]
37. Al-Raeei, M. Applying fractional quantum mechanics to systems with electrical screening effects. *Chaos Solitons Fractals* **2021**, *150* 111209. [CrossRef]
38. AlBaidani, M.M.; Ganie, A.H.; Khan, A. The dynamics of fractional KdV type equations occurring in magneto-acoustic waves through non-singular kernel derivatives. *AIP Adv.* **2023**, *13*, 115215. [CrossRef]
39. Ambrosio, V. Nonlinear fractional Schrödinger equations in $\mathbb{R}^N$. In *Frontiers in Elliptic and Parabolic Problems*; Birkhäuser/Springer: Cham, Switzerland, 2021; xvii+662p.
40. Lieb, E.; Loss, M. *Analysis, Graduate Studies in Mathematics*; American Mathematical Society: Providence, RI, USA, 1997; Volume 14, xviii+278p.
41. Nguyen, H.M.; Squassina, M. Fractional Caffarelli-Kohn-Nirenberg inequalities. *J. Funct. Anal.* **2018**, *274*, 2661–2672. [CrossRef]
42. Chen, S.; Li, Y.; Yang, Z. Multiplicity and concentration of nontrivial nonnegative solutions for a fractional Choquard equation with critical exponent. *Rev. Real Acad. Cienc. Exactas FÏsicasy Nat. Ser. A Mat.* **2020**, *114*, 33. [CrossRef]
43. Ambrosio, V.; Isernia, T. Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional p-Laplacian. *Discrete Contin. Dyn. Syst.* **2018**, *38*, 5835–5881. [CrossRef]
44. Almgren, F.J.; Lieb, E.H. Symmetric decreasing rearrangement is sometimes continuous. *J. Am. Math. Soc.* **1989**, *2*, 683–773. [CrossRef]

45. Iannizzotto, A.; Mosconi, S.; Squassina, M. Global Hölder regularity for the fractional p-Laplacian. *Rev. Mat. Iberoam.* **2016**, *32*, 1353–1392. [CrossRef]
46. Bartsch, T.; Wang, Z. Multiple positive solutions for a nonlinear Schrödinger equation. *Z. Angew. Math. Phys.* **2000**, *51*, 366–384.
47. Bisci, G.M.; Thin, N.; Vilasi, L. On a class of nonlocal Schrödinger equations with exponential growth. *Adv. Differ. Equ.* **2022**, *27*, 571–610.
48. Palatucci, G.; Pisante, A. Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces. *Calc. Var. Partial Differ. Equ.* **2014**, *50*, 799–829. [CrossRef]
49. Alves, C.O.; Figueiredo, G.M. Existence and multiplicity of positive solutions to a p-Laplacian equation in $\mathbb{R}^N$. *Differ. Integral Equ.* **2006**, *19*, 143–162. [CrossRef]
50. Alves, C.O.; Thin, N.V. On existence of multiple normalized solution to a class of elliptic problems in whole $\mathbb{R}^N$ via Lusternik-Schnirelmann category. *SIAM J. Math. Anal.* **2023**, *55*, 1264–1283.
51. Cingolani, S.; Lazzo, M. Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations. *Topol. Methods Nonlinear Anal.* **1997**, *10*, 1–13. [CrossRef]
52. Ghoussoub, N. *Duality and Pertubation Methods in Critical Point Theory*; Cambridge University Press: Cambridge, UK, 1993.