fractal and fractional

MDPI

Article

The Existence and Ulam Stability Analysis of a Multi-Term Implicit Fractional Differential Equation with Boundary Conditions

Peiguang Wang *, Bing Han and Junyan Bao *

College of Mathematics and Information Science, Hebei University, Baoding 071002, China; hanbing20000119@163.com

* Correspondence: pgwang@hbu.edu.cn (P.W.); jybao@hbu.edu.cn (J.B.)

Abstract: In this paper, we investigate a class of multi-term implicit fractional differential equation with boundary conditions. The application of the Schauder fixed point theorem and the Banach fixed point theorem allows us to establish the criterion for a solution that exists for the given equation, and the solution is unique. Afterwards, we give the criteria of Ulam–Hyers stability and Ulam–Hyers–Rassias stability. Additionally, we present an example to illustrate the practical application and effectiveness of the results.

Keywords: multi-term implicit fractional differential equation; existence; uniqueness; Ulam–Hyers stability; Ulam–Hyers–Rassias stability

MSC: 26A33; 34A09; 34D20

check for updates

Citation: Wang, P.; Han, B.; Bao, J. The Existence and Ulam Stability Analysis of a Multi-Term Implicit Fractional Differential Equation with Boundary Conditions. *Fractal Fract.* **2024**, *8*, 311. https://doi.org/10.3390/fractalfract8060311

Academic Editor: Rodica Luca

Received: 7 May 2024
Revised: 21 May 2024
Accepted: 23 May 2024
Published: 24 May 2024

1. Introduction

In recent years, because of the prevalence of fractional derivatives and integrals in modeling biological systems, such as population dynamics and erythrocyte sedimentation rates, etc. (see [1–3]), the qualitative and stability analyses of fractional differential equations has garnered considerable interest and attention. Regarding the studies of stability problems, based on the Lyapunov method, scholars proposed many different concepts of stability, such as equi-stability, Lipschitz stability, and practical stability, which are documented in the literature [4–6]. However, the difficulty lies in finding and calculating the appropriate Lyapunov functions, which limited the application of this method in a certain sense. Ulam [7] introduced the concept of Ulam stability in 1940. This stability is not only convenient to obtain, but also solves the problem of finding exact solutions of nonlinear differential equations. It ensures the existence of approximate solutions to equations, which is crucial in optimization and numerical analysis. Since then, Hyers [8] refined the Ulam–Hyers stability, and Rassias [9] further developed the Ulam–Hyers–Rassias stability. At present, researchers have made progress in studying the existence and Ulam stability analysis of fractional differential equations (see [10–14]). We note that there are few results on fractional differential equations with multiple terms. For example, Alam et al. [15] conducted a study on the existence and Ulam–Hyers stability of two-term implicit fractional order differential equations as follows:

$$\begin{cases} (K_1{}^cD^{\alpha_1} + K_2{}^RI^{\alpha_2})u(t) = {}^RI^{\alpha_3}\phi_1(t,(K_1{}^cD^{\alpha_1} + K_2{}^RI^{\alpha_2})u(t)) \\ \qquad\qquad + \phi_2(t,(K_1{}^cD^{\alpha_1} + K_2{}^RI^{\alpha_2})u(t)), \\ u(0) = 0,\ {}^cD^1u(0) = 0, \\ \int_0^1 u(s)ds + \int_0^1 {}^cD^1u(s)ds = \int_0^1 \frac{(1-s)^{\alpha_4-1}}{\Gamma(\alpha_4)}u(s)ds, \end{cases}$$

where the functions $\phi_1, \ \phi_2 : J \times \mathbb{R} \to \mathbb{R}$, and the parameters $\alpha_1 \in (2,3), \ \alpha_2, \ \alpha_3, \ \alpha_4 \in (0,1)$, $t \in J = [0,1]$. $K_1, \ K_2$ are nonzero constants. ${}^cD^{\alpha_1}u(t)$ represents the Caputo fractional derivative of the function $u(t)$, and ${}^RI^{\alpha_2}u(t)$ represents the Riemann–Liouville fractional integral of the function $u(t)$.

In 2022, Rahman et al. [16] focused on exploring the existence and Ulam–Hyers–Rassias stability of a class of n-order multi-term fractional differential equations with a delay:

$$\begin{cases} (\sum\limits_{i=1}^{n} \lambda_i {}^cD^{\alpha_i})x(t) = f(t, x(t), x(\sigma t)), \\ x(0) = 0, \ \frac{d^l x(0)}{dx^l} = 0, \ l = 1, 2, \cdots, n-2, \\ x(1) = \sum\limits_{l=1}^{n-2} \delta_l x(\eta_l), \ \delta_l \in \mathbb{R}, \ \eta_l \in (0,1), \ l = 1, 2, \cdots, n-2, \end{cases}$$

where the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the parameters $\alpha_1 \in (n-1, n), \ \alpha_i \in (0,1), \ i = 2, 3, \cdots, n$. $t \in J = [0,1]$. λ_i are positive constants. ${}^cD^{\alpha_i}x(t)$ represents the Caputo fractional derivative of the function $x(t)$.

In this paper, we extend our investigation to address a multi-term implicit fractional differential equation that includes boundary problems:

$$\begin{cases} (\sum\limits_{i=1}^{m} \lambda_i {}^cD^{q_i} + \sum\limits_{j=1}^{m} K_j {}^RI^{q_j})u(s) = f(s, u(s), (\sum\limits_{i=1}^{m} \lambda_i {}^cD^{q_i} + \sum\limits_{j=1}^{m} K_j {}^RI^{q_j}))u(s)), \\ u(0) = 0, \ {}^cD^l u(0) = 0, \ l = 1, 2, \cdots, n-2, \\ u(1) = {}^RI^{\omega}u(\eta), \end{cases} \tag{1}$$

where the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the parameters $q_1 \in (n-1, n), \ q_i \in (0,1), \ i = 2, 3, \cdots, n, \ p_j \in (0,1), \ j = 1, 2, \cdots, n$. $\omega \in (0,1), \ s \in J = [0,1], \ \eta \in [0,1]$. $\lambda_i, \ K_j$ are positive constants. ${}^cD^{q_i}u(s)$ represents the Caputo fractional derivative of the function $u(s)$, and ${}^RI^{p_j}u(s)$ represents the Riemann–Liouville fractional integral of the function $u(s)$.

The purpose of this paper is extend the form of high-order implicit differential equations with integral terms, and obtain results on the existence, uniqueness, and stability of the solutions to such equations. We employ the fixed point theorem to establish the existence results of Equation (1). Additionally, we give the criteria of Ulam–Hyers stability and Ulam–Hyers–Rassias stability for Equation (1). Furthermore, we provide an illustrative example to showcase the practical effectiveness of the obtained results.

2. Existence of Solutions

The following basic definitions, lemmas and theorems are provided first.

Let $C(J, \mathbb{R})$ denote the Banach space of all continuous functions from J to $\mathbb{R}$, where the norm is defined as $||u||_\infty = \sup\limits_{s \in J}\{|u(s)|\}$.

Definition 1 (See [2])**.** *The q order Riemann–Liouville fractional integral of the integrable function $u(s)$ is defined as*

$$ {}^RI^q u(s) = \frac{1}{\Gamma(q)} \int_0^s (s-z)^{q-1} u(z) dz, \ s \in J.$$

Definition 2 (See [2])**.** *The q order Caputo fractional derivative of the differentiable function $u(s)$ is defined as*

$$ {}^cD^q u(s) = \frac{1}{\Gamma(n-q)} \int_0^s (s-z)^{q+n-1} u^{(n)}(z) dz, \ s \in J,$$

where $n = [q]$, i.e., n is the smallest integer not exceeding q.

Lemma 1 (See [2]). *Let $q_1 > q_2 > 0$, and $u(s)$ is a integrable function on $[0,b]$; then, for any $s \in [0,b]$, there is*

$$
\begin{aligned}
&{}^R I_{0^+}^{q_1}({}^R I_{0^+}^{q_2} u(s)) = {}^R I_{0^+}^{q_1+q_2} u(s);\\
&{}^c D_{0^+}^{q_1}({}^c D_{0^+}^{q_2} u(s)) = {}^c D_{0^+}^{q_1+q_2} u(s);\\
&{}^R I_{0^+}^{q_1}({}^c D_{0^+}^{q_2} u(s)) = {}^R I_{0^+}^{q_1-q_2} u(s).
\end{aligned}
$$

Lemma 2 (See [17]). *The solution of fractional differential equation ${}^cD^q u(s) = 0$ $(n-1 < q \le n)$ is given by*

$$u(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1},$$

where $a_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n-1$.

To establish the existence results of Equation (1), we derive an equivalent integral expression of Equation (1).

Theorem 1. *Let $u(s) \in C(J, \mathbb{R})$; the equivalent integral form of Equation (1) is*

$$
\begin{aligned}
u(s) =& \frac{1}{q_1}\Big\{\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-1}\phi(z)dz - \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s (s-z)^{q_1-q_i-1}u(z)dz\\
&- \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s (s-z)^{q_1+p_j-1}u(z)dz + \frac{s^{n-1}}{\Delta}\Big[\frac{1}{\Gamma(q_1)}\int_0^1 (1-z)^{q_1-1}\phi(z)dz\\
&- \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^1 (1-z)^{q_1-q_i-1}u(z)dz - \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1 (1-z)^{q_1+p_j-1}u(z)dz\\
&- \frac{1}{\Gamma(q_1+\omega)}\int_0^\eta (\eta-z)^{\omega+q_1-1}\phi(z)dz\\
&+ \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^\eta (\eta-z)^{\omega+q_1-q_i-1}u(z)dz\\
&+ \sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^\eta (\eta-z)^{\omega+q_1+p_j-1}u(z)dz\Big]\Big\},
\end{aligned}
$$

where $\phi(s) = f(s, u(s), (\sum_{i=1}^{m}\lambda_i {}^c D^{q_i} + \sum_{j=1}^{m} K_j {}^R I^{p_j})u(s))$, and $\Delta = 1 - \frac{\eta^{n+\omega}}{\Gamma(\omega)\omega(\omega+1)\cdots(\omega+n)(n-1)!} \neq 0$.

Proof. After integrating the q_1 order on both sides of Equation (1), according to Lemmas 1 and 2, we obtain

$$
\begin{aligned}
{}^R I^q[{}^c D^q u(s)] =& \frac{1}{\lambda_1}\Big[\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-1}\phi(z)dz - \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s (s-z)^{q_1-q_i-1}u(z)dz\\
&- \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s (s-z)^{q_1+p_j-1}u(z)dz\Big].
\end{aligned}
$$

Then, we use the following relationship that exists between fractional integral and derivative [17]

$$ {}^R I^q[{}^c D^q u(s)] = u(s) + b_0 + b_1 s + b_2 s^2 + \cdots + b_{n-1} s^{n-1}, \tag{2}$$

where $a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1$, and we obtain

$$\begin{aligned} u(s) =& \frac{1}{\lambda_1}\Big[\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-1}\phi(z)dz - \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s (s-z)^{q_1-q_i-1}u(z)dz \\ &- \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s (s-z)^{q_1+p_j-1}u(z)dz\Big] + a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1}s^{n-1}. \end{aligned} \tag{3}$$

Using the boundary condition $u(0) = 0$, we have $a_0 = 0$. Then, by differentiating (3), we obtain

$$\begin{aligned} {}^cD^1u(s) =& \frac{1}{\lambda_1}\Big[\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-2}\phi(z)dz - \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-\alpha_i-1)}\int_0^s (s-z)^{q_1-q_i-2}u(z)dz \\ &- \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j-1)}\int_0^s (s-z)^{q_1+p_j-2}u(z)dz\Big] + a_1 + \cdots + (n-1)a_{n-1}s^{n-2}. \end{aligned} \tag{4}$$

From the boundary condition ${}^cD^1u(0) = 0$, we have $a_1 = 0$. Continuing differentiating (4), we obtain $a_2 = 0$. Repeating the process, we obtain $a_3 = a_4 = \cdots = a_{n-2} = 0$. Then,

$$\begin{aligned} u(s) =& \frac{1}{\lambda_1}\Big[\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-1}\phi(z)dz - \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s (s-z)^{q_1-q_i-1}u(z)dz \\ &- \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s (s-z)^{q_1+p_j-1}u(z)dz\Big] + a_{n-1}s^{n-1}. \end{aligned} \tag{5}$$

By integrating the ω order on both sides of (5), we obtain

$$\begin{aligned} {}^RI^{\omega}u(s) =& \frac{1}{\lambda_1}\Big[\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{\omega+q_1-1}\phi(z)dz \\ &- \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^s (s-z)^{\omega+q_1-q_i-1}u(z)dz \\ &- \sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^s (s-z)^{\omega+q_1+p_j-1}u(z)dz\Big] \\ &+ a_{n-1}\frac{s^{n+\omega}}{\Gamma(\omega)\omega(\omega+1)\cdots(\omega+n)(n-1)!}. \end{aligned} \tag{6}$$

According to the boundary condition $u(1) = {}^RI^{\omega}u(\eta)$, it follows that

$$\begin{aligned} &a_{n-1}\Big(1 - \frac{\eta^{n+\omega}}{\Gamma(\omega)\omega(\omega+1)\cdots(\omega+n)(n-1)!}\Big) \\ =& \frac{1}{q_1}\Big[\frac{1}{\Gamma(q_1)}\int_0^1 (1-z)^{q_1-1}\phi(z)dz - \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^1 (1-z)^{q_1-q_i-1}u(z)dz \\ &- \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1 (1-z)^{q_1+p_j-1}u(z)dz - \frac{1}{\Gamma(q_1+\omega)}\int_0^{\eta} (\eta-z)^{\omega+q_1-1}\phi(z)dz \\ &+ \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^{\eta} (\eta-z)^{\omega+q_1-q_i-1}u(z)dz \\ &+ \sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^{\eta} (\eta-z)^{\omega+q_1+p_j-1}u(z)dz\Big]. \end{aligned}$$

Let $\Delta = 1 - \frac{\eta^{n+\omega}}{\Gamma(\omega)\omega(\omega+1)\cdots(\omega+n)(n-1)!}$, then we obtain

$$\begin{aligned}
a_{n-1} =&\frac{1}{\lambda_1\Delta}\Big[\frac{1}{\Gamma(q_1)}\int_0^1(1-z)^{q_1-1}\phi(z)dz-\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^1(1-z)^{q_1-q_i-1}u(z)dz\\
&-\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1(1-z)^{q_1+p_j-1}u(z)dz-\frac{1}{\Gamma(q_1+\omega)}\int_0^\eta(\eta-z)^{\omega+q_1-1}\phi(z)dz\\
&+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^\eta(\eta-z)^{\omega+q_1-q_i-1}u(z)dz\\
&+\sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^\eta(\eta-z)^{\omega+q_1+p_j-1}u(z)dz\Big].
\end{aligned} \tag{7}$$

Substituting (7) into (3), there is

$$\begin{aligned}
u(s) =&\frac{1}{\lambda_1}\Big\{\frac{1}{\Gamma(q_1)}\int_0^s(s-z)^{q_1-1}\phi(z)dz-\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s(s-z)^{q_1-q_i-1}u(z)dz\\
&-\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s(s-z)^{q_1+p_j-1}u(z)dz+\frac{s^{n-1}}{\Delta}\Big[\frac{1}{\Gamma(q_1)}\int_0^1(1-z)^{q_1-1}\phi(z)dz\\
&-\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^1(1-z)^{q_1-q_i-1}u(z)dz-\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1(1-z)^{q_1+p_j-1}u(z)dz\\
&-\frac{1}{\Gamma(q_1+\omega)}\int_0^\eta(\eta-z)^{\omega+q_1-1}\phi(z)dz\\
&+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^\eta(\eta-z)^{\omega+q_1-q_i-1}u(z)dz\\
&+\sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^\eta(\eta-z)^{\omega+q_1+p_j-1}u(z)dz\Big]\Big\}.
\end{aligned}$$

In summary, the conclusion is confirmed. □

In order to demonstrate the existence and uniqueness results, it is customary to assume that the following conditions are satisfied.

Hypothesis 1 (H1). *For any $s \in J$, there exist non-negative constants L_1 and L_2, such that*

$$|f(s,u_1,u_2)-f(s,v_1,v_2)| \leq L_1|u_1-u_1|+L_2|v_2-v_2|.$$

Hypothesis 2 (H2). *For any $s \in J$, there exist bounded functions $c_1(s)$, $c_2(s)$ and $c_3(s)$, such that*

$$|f(s,u_1(s),u_2(s))| \leq c_1(s)+c_2(s)|u_1(s)|+c_3(s)|u_2(s)|.$$

Define the operator $\mathcal{F} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as

$$\begin{aligned}
\mathcal{F}x(t) =&\frac{1}{q_1}\Big\{\frac{1}{\Gamma(q_1)}\int_0^s(s-z)^{q_1-1}\phi(z)dz-\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s(s-z)^{q_1-q_i-1}u(z)dz\\
&-\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s(s-z)^{q_1+p_j-1}u(z)dz+\frac{s^{n-1}}{\Delta}\Big[\frac{1}{\Gamma(q_1)}\int_0^1(1-z)^{q_1-1}\phi(z)dz\\
&-\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^1(1-z)^{q_1-q_i-1}u(z)dz-\sum_{j=1}^{n}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1(1-z)^{q_1+p_j-1}u(z)dz\\
&-\frac{1}{\Gamma(q+\omega)}\int_0^\eta(\eta-z)^{\omega+q_1-1}\phi(z)dz
\end{aligned}$$

$$+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^{\eta}(\eta-z)^{\omega+q_1-q_i-1}u(z)dz$$
$$+\sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^{\eta}(\eta-z)^{\omega+q_1+p_j-1}u(z)dz\Big]\Big\},$$

where $\phi(s)=f(s,u(s),(\sum\limits_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum\limits_{j=1}^{m}K_j{}^RI^{p_j})u(s))$.

2.1. Existence Result Using the Shauder Fixed Point Theorem

In this current subsection, we prove the existence result of Equation (1) using the Shauder fixed point theorem.

Theorem 2. *Assume that the conditions (H1) and (H2) are satisfied; then, Equation (1) has at least one solution.*

Proof. We define a subspace $\mathcal{B}=\{u\in C(J,\mathbb{R}):||u||\le d\}$, where

$$d\le\frac{c_1}{\lambda_1|\Delta|}\Big(\frac{1+|\Delta|}{\Gamma(q_1+1)}+\frac{\eta^{\omega+q_1}}{\Gamma(\omega+q_1+1)}\Big)(1-\theta)^{-1}, \tag{8}$$

$$\begin{aligned}\theta=&\frac{1}{\lambda_1|\Delta|}\Big[h\Big(\frac{1+|\Delta|}{\Gamma(q_1+1)}+\frac{\eta^{\omega+q_1}}{\Gamma(\omega+q_1+1)}\Big)+\sum_{i=2}^{m}\lambda_i\Big(\frac{1+|\Delta|}{\Gamma(q_1-q_i+1)}+\frac{\eta^{\omega+q_1-q_i}}{\Gamma(\omega+q_1-q_i)}\Big)\\&+\sum_{j=1}^{m}K_j\Big(\frac{1+|\Delta|}{\Gamma(q_1+p_j+1)}+\frac{\eta^{\omega+q_1+p_j}}{|\Delta|\Gamma(\omega+q_1+p_j)}\Big)\Big].\end{aligned} \tag{9}$$

Step 1. We prove that $\mathcal{F}u\subset\mathcal{B}$. Indeed, for any $u\in\mathcal{B}$, there is

$$\begin{aligned}||\mathcal{F}u||\le&\frac{1}{\lambda_1|\Delta|}\sup_{s\in J}\Big\{\frac{1}{\Gamma(q_1)}\int_0^1(1-z)^{q_1-1}|\phi(z)|dz+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^1(1-z)^{q_1-q_i-1}|u(z)|dz\\&-\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1(1-z)^{q_1+p_j-1}|u(z)|dz+\frac{1}{\Gamma(q_1+\omega)}\int_0^{\eta}(\eta-z)^{\omega+q_1-1}|\phi(z)|dz\\&+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^{\eta}(\eta-z)^{\omega+q_1-q_i-1}|u(z)|dz\\&+\sum_{j=1}^{m}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^{\eta}(\eta-z)^{\omega+q_1+p_j-1}|u(z)|dz\Big\}\\&+\frac{1}{\lambda_1}\sup_{s\in J}\Big\{\frac{1}{\Gamma(q_1)}\int_0^s(s-z)^{q_1-1}|\phi(z)|dz+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s(s-z)^{q_1-q_i-1}|u(z)|dz\\&-\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s(s-z)^{q_1+p_j-1}|u(z)|dz\Big\}.\end{aligned} \tag{10}$$

By condition (H2), we have

$$\begin{aligned}|\phi(s)|=&\Big|f(s,u(s),(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{m}K_j{}^RI^{p_j})u(s))\Big|\\\le&c_1(s)+c_2(s)|u(s)|+c_3(s)\Big|(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{m}K_j{}^RI^{p_j})u(s)\Big|\\\le&c_1(s)+\Big[c_2(s)+c_3(s)\Big(\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(1-q_i)}+\sum_{j=1}^{m}\frac{K_js^{p_j}}{\Gamma(1+p_j)}\Big)\Big]|u(s)|.\end{aligned}$$

Let $h = c_2 + c_3\big(\sum\limits_{i=1}^{m} \frac{\lambda_i}{\Gamma(1-q_i)} + \sum\limits_{j=1}^{m} \frac{K_j}{\Gamma(1+p_j)}\big)$, where $c_2 = \sup\limits_{s\in J} c_2(s),\ c_3 = \sup\limits_{s\in J} c_3(s)$. Then, we obtain

$$|\phi(s)| \le c_1 + h|u(s)|, \tag{11}$$

where $c_1 = \sup\limits_{s\in J} c_1(s)$. Substituting (11) into (10), we have

$$\begin{aligned}
||\mathcal{F}u|| \le & c_1\Big(\frac{1+|\Delta|}{|\Delta|\lambda_1\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{|\Delta|\lambda_1\Gamma(\omega+q_1+1)}\Big) \\
& + \Big[h\Big(\frac{1+|\Delta|}{|\Delta|\lambda_1\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{|\Delta|\lambda_1\Gamma(\omega+q_1+1)}\Big) \\
& + \sum_{i=2}^{m}\frac{\lambda_i}{\lambda_1}\Big(\frac{1+|\Delta|}{|\Delta|\Gamma(q_1-q_i+1)} + \frac{\eta^{\omega+q_1-q_i}}{|\Delta|\Gamma(\omega+q_1-q_i)}\Big) \\
& + \sum_{j=1}^{m}\frac{K_j}{\lambda_1}\Big(\frac{1+|\Delta|}{|\Delta|\Gamma(q_1+p_j+1)} + \frac{\eta^{\omega+q_1+p_j}}{|\Delta|\Gamma(\omega+q_1+p_j)}\Big)\Big]||u(s)|| \\
\le & d.
\end{aligned}$$

Step 2. To establish the continuity of the operator $\mathcal{F}$, we consider a sequence $\{u_n\} \in \mathcal{B}$, when $n \to \infty$, $u_n \to u$. Our aim is to demonstrate that as $n \to \infty$, $\mathcal{F}u_n \to \mathcal{F}u$. Notice that

$$\begin{aligned}
|\mathcal{F}u_n(s) - \mathcal{F}u(s)| \le & \frac{1}{\lambda_1|\Delta|}\sup_{s\in J}\Big\{\frac{1}{\Gamma(q_1)}\int_0^1 (1-z)^{q_1-1}|\phi_n(z)-\phi(z)|dz \\
& + \sum_{i=2}^{m}\frac{\lambda_i}{\lambda_1\Gamma(q_1-q_i)}\int_0^1 (1-z)^{q_1-q_i-1}|u_n(z)-u(z)|dz \\
& + \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^1 (1-z)^{q_1+p_j-1}|u_n(z)-u(z)|dz \\
& + \frac{1}{\Gamma(q_1+\omega)}\int_0^\eta (\eta-z)^{\omega+q_1-1}|\phi_n(z)-\phi(z)|dz \\
& + \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(\omega+q_1-q_i)}\int_0^\eta (\eta-z)^{\omega+q_1-q_i-1}|u_n(z)-u(z)|dz \\
& + \sum_{j=1}^{n}\frac{K_j}{\Gamma(\omega+q_1+p_j)}\int_0^\eta (\eta-z)^{\omega+q_1+p_j-1}|u_n(z)-u(z)|dz\Big\} \\
& + \frac{1}{\lambda_1}\sup_{s\in J}\Big\{\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-1}|\phi_n(z)-\phi(z)|dz \\
& + \sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s (s-z)^{q_1-q_i-1}|u_n(z)-u(z)|dz \\
& + \sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s (s-z)^{q_1+p_j-1}|u_n(z)-u(z)|dz\Big\}.
\end{aligned}$$

By the Lebesgue dominated convergent theorem, when $n \to \infty$, $||\mathcal{F}u_n - \mathcal{F}u|| \to 0$.
Step 3. Now, we prove that $\mathcal{F}$ maps bounded sets into equalcontinuous sets.
Let $s_1 < s_2$; then, we can obtain the following relationship:

$$
\begin{aligned}
|\mathcal{F}u(s_1) - \mathcal{F}u(s_2)| \leq & \frac{|s_2 - s_1|}{\lambda_1 |\Delta|} \sup_{s \in J} \Big\{ \frac{1}{\Gamma(q_1)} \int_0^1 (1-z)^{q_1 - 1} |\phi(z)| dz \\
& + \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(q_1 - q_i)} \int_0^1 (1-z)^{q_1 - q_i - 1} |u(z)| dz \\
& + \sum_{j=1}^{m} \frac{K_j}{\Gamma(q_1 + p_j)} \int_0^1 (1-z)^{q_1 + p_j - 1} |u(z)| dz \\
& + \frac{1}{\Gamma(q_1 + \omega)} \int_0^\eta (\eta - z)^{\omega + q_1 - 1} |\phi(z)| dz \\
& + \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(\omega + q_1 - q_i)} \int_0^\eta (\eta - z)^{\omega + q_1 - q_i - 1} |u(z)| dz \\
& + \sum_{j=1}^{m} \frac{K_j}{\Gamma(\omega + q_1 + p_j)} \int_0^\eta (\eta - z)^{\omega + q_1 + p_j - 1} |u(z)| dz \Big\} \\
& + \frac{1}{\lambda_1} \sup_{s \in J} \Big\{ \frac{1}{\Gamma(q_1)} \int_0^{s_1} \big((s_2 - z)^{q_1 - 1} - (s_1 - z)^{q_1 - 1} \big) |\phi(z)| dz \\
& + \frac{1}{\Gamma(q_1)} \int_{s_1}^{s_2} (s_2 - z)^{q_1 - 1} |\phi(z)| dz \\
& + \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(q_1 - q_i)} \int_0^{s_1} \big((s_2 - z)^{q_1 - q_i - 1} - (s_1 - z)^{q_1 - q_i - 1} \big) |u(z)| dz \\
& + \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(q_1 - q_i)} \int_{s_1}^{s_2} (s_2 - z)^{q_1 - q_i - 1} |u(z)| dz \\
& + \sum_{j=1}^{m} \frac{K_j}{\Gamma(q_1 + p_j)} \int_0^{s_1} \big((s_2 - z)^{q_1 + p_j - 1} - (s_1 - z)^{q_1 + p_j - 1} \big) |u(z)| dz \\
& + \sum_{j=1}^{m} \frac{K_j}{\Gamma(q_1 + p_j)} \int_{s_1}^{s_2} (s_2 - z)^{q_1 + p_j - 1} |u(z)| dz \Big\}.
\end{aligned}
\tag{12}
$$

According to condition (H2), there is

$$
\begin{aligned}
|\mathcal{F}u(s_1) - \mathcal{F}u(s_2)| \leq & \frac{|s_2 - s_1|}{\lambda_1 |\Delta|} \Big\{ (c_1 + hd) \Big(\frac{1}{\Gamma(q_1)} + \frac{\eta^{\omega + q_1}}{\Gamma(\omega + q_1 + 1)} \Big) \\
& + \frac{d}{\lambda_1} \Big[\sum_{i=2}^{m} \lambda_i \Big(\frac{1}{\Gamma(q_1 - q_i + 1)} + \frac{\eta^{\omega + q_1 - q_i}}{\Gamma(\omega + q_1 - q_i + 1)} \Big) \\
& + \sum_{j=1}^{m} K_j \Big(\frac{1}{|\Delta| \Gamma(q_1 + p_j + 1)} + \frac{\eta^{\omega + q_1 + p_j}}{\Gamma(\omega + q_1 + p_j + 1)} \Big) \Big] \Big\} \\
& + \frac{1}{\lambda_1} \Big[\frac{(c_1 + hd)(s_2 - s_1)^{q_1}}{\Gamma(q_1)} \\
& + d \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(q_1 - q_i)} \Big(\frac{s_2^{q_1 - q_i} - (s_2 - s_1)^{q_1 - q_i}}{q_1 - q_i} - \frac{s_1^{q_1 - q_i} - (s_1 - s_2)^{q_1 - q_i}}{q_1 - q_i} \Big) \\
& + d \sum_{j=1}^{m} \frac{K_j}{\Gamma(q_1 + p_j)} \Big(\frac{s_2^{q_1 + p_j} - (s_2 - s_1)^{q_1 + p_j}}{q_1 + p_j} - \frac{s_1^{q_1 + p_j} - (s_1 - s_2)^{q_1 + p_j}}{q_1 + p_j} \Big) \Big],
\end{aligned}
$$

when $s_1 \to s_2$, we obtain $|\mathcal{F}u(s_1) - \mathcal{F}u(s_2)| \to 0$.

In conclusion, by applying the Arzelà–Ascoli theorem, we can deduce that the operator $\mathcal{F}$ is completely continuous.

Step 4. Lastly, we demonstrate that the set

$$\mathcal{E} = \{u \in C(J, \mathbb{R}) : u = \lambda^* \mathcal{F} u,\ \lambda^* \in (0,1)\}$$

is bounded.

Suppose $u \in \mathcal{E}$; then, we obtain

$$\begin{aligned} ||u|| = ||\lambda^* \mathcal{F} u|| \leq ||\mathcal{F} u|| \leq & \frac{1}{\lambda_1 |\Delta|} \Big\{ c_1 \Big(\frac{1 + |\Delta|}{\Gamma(q_1 + 1)} + \frac{\eta^{\omega + q_1}}{\Gamma(\omega + q_1 + 1)} \Big) \\ & + \Big[h \Big(\frac{1 + |\Delta|}{\Gamma(q_1 + 1)} + \frac{\eta^{\omega + q_1}}{\Gamma(\omega + q_1 + 1)} \Big) \\ & + \sum_{i=2}^{m} \lambda_i \Big(\frac{1 + |\Delta|}{\Gamma(q_1 - q_i + 1)} + \frac{\eta^{\omega + q_1 - q_i}}{\Gamma(\omega + q_1 - q_i)} \Big) \\ & + \sum_{j=1}^{m} K_j \Big(\frac{1 + |\Delta|}{\Gamma(q_1 + p_j + 1)} + \frac{\eta^{\omega + q_1 + p_j}}{\Gamma(\omega + q_1 + p_j)} \Big) \Big] \Big\} ||u|| \\ \leq & d. \end{aligned}$$

By the Shauder fixed point theorem, Equation (1) has at least one solution. □

2.2. Existence Result Using the Banach Fixed Point Theorem

Subsequently, we employ the Banach fixed point theorem to establish the existence and uniqueness result of Equation (1).

Theorem 3. *Assume that the condition (H1) is satisfied, and the inequality*

$$\begin{aligned} & \Big[\Big(\frac{1 + |\Delta|}{|\Delta| \lambda_1 \Gamma(q_1 + 1)} + \frac{\eta^{\omega + q_1}}{|\Delta| \lambda_1 \Gamma(\omega + q_1 + 1)} \Big) L_3 \\ & \quad + \sum_{i=2}^{m} \frac{\lambda_i}{\lambda_1} \Big(\frac{1 + |\Delta|}{|\Delta| \Gamma(q_1 - q_i + 1)} + \frac{\eta^{\omega + q_1 - q_i}}{|\Delta| \Gamma(\omega + q_1 - q_i)} \Big) \\ & \quad + \sum_{j=1}^{m} \frac{K_j}{\lambda_1} \Big(\frac{1 + |\Delta|}{|\Delta| \Gamma(q_1 + p_j + 1)} + \frac{\eta^{\omega + q_1 + p_j}}{|\Delta| \Gamma(\omega + q_1 + p_j)} \Big) \Big] \\ & < 1 \end{aligned} \tag{13}$$

holds, then Equation (1) has a unique solution.

Proof. For any $u_1,\ u_2 \in C(J, \mathbb{R})$, there is

$$\begin{aligned} ||\mathcal{F} u_1 - \mathcal{F} u_2|| \leq & \frac{1}{\lambda_1 |\Delta|} \sup_{s \in J} \Big\{ \frac{1}{\Gamma(q_1)} \int_0^1 (1 - z)^{q_1 - 1} |\phi_1(z) - \phi_2(z)| dz \\ & + \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(q_1 - q_i)} \int_0^1 (1 - z)^{q_1 - q_i - 1} |u_1(z) - u_2(z)| dz \\ & + \sum_{j=1}^{m} \frac{K_j}{\Gamma(q_1 + p_j)} \int_0^1 (1 - z)^{q_1 + p_j - 1} |u_1(z) - u_2(z)| dz \\ & + \frac{1}{\Gamma(q_1 + \omega)} \int_0^\eta (\eta - z)^{\omega + q_1 - 1} |\phi_1(z) - \phi_2(z)| dz \\ & + \sum_{i=2}^{m} \frac{\lambda_i}{\Gamma(\omega + q_1 - q_i)} \int_0^\eta (\eta - z)^{\omega + q_1 - q_i - 1} |u_1(z) - u_2(z)| dz \\ & + \sum_{j=1}^{m} \frac{K_j}{\Gamma(\omega + q_1 + p_j)} \int_0^\eta (\eta - z)^{\omega + q_1 + p_j - 1} |u_1(z) - u_2(z)| dz \Big\} \end{aligned} \tag{14}$$

$$+\frac{1}{\lambda_1}\sup_{s\in J}\Big\{\frac{1}{\Gamma(q_1)}\int_0^s (s-z)^{q_1-1}|\phi_1(z)-\phi_2(z)|dz$$
$$+\sum_{i=2}^{m}\frac{\lambda_i}{\Gamma(q_1-q_i)}\int_0^s (s-z)^{q_1-q_i-1}|u_1(z)-u_2(z)|dz$$
$$+\sum_{j=1}^{m}\frac{K_j}{\Gamma(q_1+p_j)}\int_0^s (s-z)^{q_1+p_j-1}|u_1(z)-u_2(z)|dz\Big\}.$$

By condition (H1), it follows that

$$|\phi_1(z)-\phi_2(z)| \le \Big[L_1 + L_2\Big(\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(1-q_i)} + \sum_{j=1}^{m}\frac{K_j}{\Gamma(1+p_j)}\Big)\Big]|u_1(z)-u_2(z)|. \tag{15}$$

Let $L_3 = L_1 + L_2\big(\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(1-q_i)} + \sum_{j=1}^{m}\frac{K_j}{\Gamma(1+p_j)}\big)$. Substituting (15) into (14), we obtain

$$\begin{aligned}||\mathcal{F}u_1-\mathcal{F}u_2|| \le &\frac{1}{\lambda_1|\Delta|}\Big[\Big(\frac{1+|\Delta|}{\Gamma(q_1+1)}+\frac{\eta^{\omega+q_1}}{\Gamma(\omega+q_1+1)}\Big)L_3\\ &+\sum_{i=2}^{m}\lambda_i\Big(\frac{1+|\Delta|}{\Gamma(q_1-q_i+1)}+\frac{\eta^{\omega+q_1-q_i}}{\Gamma(\omega+q_1-q_i)}\Big)\\ &+\sum_{j=1}^{m}K_j\Big(\frac{1+|\Delta|}{\Gamma(q_1+p_j+1)}+\frac{\eta^{\omega+q_1+p_j}}{\Gamma(\omega+q_1+p_j)}\Big)\Big]||u_1-u_2||.\end{aligned} \tag{16}$$

Consequently, by utilizing Inequality (13), we can conclude that the operator $\mathcal{F}$ is contractive. As a result, we can assert that there exists a unique solution for Equation (1) based on the Banach fixed point theorem. □

3. Ulam Stability

In this section, we give the criterion of Ulam stability for Equation (1). To begin, we provide the definitions of Ulam–Hyers stability and Ulam–Hyers–Rassias stability for Equation (1).

Definition 3. *Equation (1) has Ulam–Hyers stability if, given a unique solution $u(s) \in C(J,\mathbb{R})$, there exists a positive real number $n_f > 0$, such that, for any $\epsilon > 0$ and $v(s) \in C(J,\mathbb{R})$ satisfying the inequality*

$$\Big|(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{m}K_j{}^RI^{p_j})v(s) - f(s,v(s),(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{m}K_j{}^RI^{p_j})v(s))\Big| \le \epsilon, \tag{17}$$

there is

$$|v(s)-u(s)| \le n_f\epsilon.$$

Definition 4. *Equation (1) is Ulam–Hyers–Rassias stable with respect to $\xi(s) \in C(J,\mathbb{R})$ if, given a unique solution $u(s) \in C(J,\mathbb{R})$, there exists a positive real number $n_f > 0$, such that for any $\epsilon > 0$ and $v(s) \in C(J,\mathbb{R})$ satisfying the inequality*

$$\Big|(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{m}K_j{}^RI^{p_j})v(s) - f(s,v(s),(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{m}K_j{}^RI^{p_j})v(s))\Big| \le \epsilon\xi(s), \tag{18}$$

there is

$$|v(s)-u(s)| \le n_f\epsilon\xi(s).$$

Remark 1. *A function $v(s)$ is a solution of the inequality (17), if and only if there exists $\sigma(s) \in C(J,\mathbb{R})$ that satisfies the following conditions:*

(S_1) $|\sigma(s)| \leq \epsilon$,
(S_2) $(\sum\limits_{i=1}^{m} \lambda_i{}^c D^{q_i} + \sum\limits_{j=1}^{m} K_j{}^R I^{p_j})v(s) = f(s, v(s), (\sum\limits_{i=1}^{m} \lambda_i{}^c D^{q_i} + \sum\limits_{j=1}^{m} K_j{}^R I^{p_j})v(s)) + \sigma(s)$.

Remark 2. *A function $v(s)$ is a solution of the inequality (18), if and only if there exists $\sigma(s) \in C(J, \mathbb{R})$ that satisfies the following conditions:*
(S_3) $|\sigma(s)| \leq \epsilon\xi(s)$,
(S_4) $(\sum\limits_{i=1}^{m} \lambda_i{}^c D^{q_i} + \sum\limits_{j=1}^{m} K_j{}^R I^{p_j})v(s) = f(s, v(s), (\sum\limits_{i=1}^{m} \lambda_i{}^c D^{q_i} + \sum\limits_{j=1}^{m} K_j{}^R I^{p_j})v(s)) + \sigma(s)$.

Theorem 4. *Assume that condition (H1) is satisfied; then, Equation (1) has Ulam–Hyers stability.*

Proof. Given that $v(s)$ is a solution of the inequality (17), and $u(s)$ is the unique solution of Equation (1), then $v(s)$ satisfies the following equation:

$$\begin{cases} (\sum\limits_{i=1}^{m} \lambda_i{}^c D^{q_i} + \sum\limits_{j=1}^{m} K_j{}^R I^{p_j})v(s) = f(s, v(s), (\sum\limits_{i=1}^{m} \lambda_i{}^c D^{q_i} + \sum\limits_{j=1}^{m} K_j{}^R I^{p_j})v(s)) + \sigma(s), \\ v(0) = 0,\ {}^c D^l v(0) = 0,\ l = 1, 2, \cdots, n-2, \\ v(1) = {}^R I^{\omega} v(\eta). \end{cases} \tag{19}$$

There is

$$\begin{aligned} v(s) =& \mathcal{F}v(s) + \frac{s^{n-1}}{\lambda_1|\Delta|}\Big[\frac{1}{\Gamma(q_1)}\int_0^1 (1-z)^{q_1-1}\sigma(z)dz \\ &- \frac{1}{\Gamma(q_1+\omega)}\int_0^{\eta} (\eta-z)^{\omega+q_1-1}\sigma(z)dz\Big] + \frac{1}{\lambda_1\Gamma(q_1)}\int_0^{s} (s-z)^{q_1-1}\sigma(z)dz. \end{aligned}$$

By Remark 1, it follows that

$$|v(s) - \mathcal{F}v(s)| \leq \frac{\epsilon}{\lambda_1}\Big[\frac{1}{|\Delta|}(\frac{1}{\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{\Gamma(q_1+\omega+1)}) + \frac{1}{\Gamma(q_1+1)}\Big].$$

Denoting

$$\delta = \frac{1}{\lambda_1}\Big[\frac{1}{|\Delta|}\Big(\frac{1}{\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{\Gamma(q_1+\omega+1)}\Big) + \frac{1}{\Gamma(q_1+1)}\Big].$$

Then, it follows that

$$|v(s) - \mathcal{F}v(s)| \leq \epsilon\delta.$$

Thus, we can deduce that

$$\begin{aligned} |v(s) - u(s)| \leq& |v(s) - \mathcal{F}v(s)| + |\mathcal{F}v(s) - u(s)| \\ \leq& \epsilon\delta + \frac{|v(s)-u(s)|}{\lambda_1|\Delta|}\Big[\Big(\frac{1+|\Delta|}{\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{\Gamma(\omega+q_1+1)}\Big)L_3 \\ &+ \sum_{i=2}^{m} \lambda_i\Big(\frac{1+|\Delta|}{\Gamma(q_1-q_i+1)} + \frac{\eta^{\omega+q_1-q_i}}{\Gamma(\omega+q_1-q_i)}\Big) \\ &+ \sum_{j=1}^{m} K_j\Big(\frac{1+|\Delta|}{\Gamma(q_1+p_j+1)} + \frac{\eta^{\omega+q_1+p_j}}{\Gamma(\omega+q_1+p_j)}\Big)\Big]. \end{aligned}$$

Let

$$M = \Big[\Big(\frac{1+|\Delta|}{\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{\Gamma(\omega+q_1+1)}\Big)L_3 + \sum_{i=2}^{m}\lambda_i\Big(\frac{1+|\Delta|}{\Gamma(q_1-q_i+1)} + \frac{\eta^{\omega+q_1-q_i}}{\Gamma(\omega+q_1-q_i)}\Big) + \sum_{j=1}^{m}K_j\Big(\frac{1+|\Delta|}{\Gamma(q_1+p_j+1)} + \frac{\eta^{\omega+q_1+p_j}}{\Gamma(\omega+q_1+p_j)}\Big)\Big].$$

Then, we obtain

$$|v(s)-u(s)| \le \epsilon\delta + M|v(s)-u(s)|.$$

Thus, we have

$$|v(s)-u(s)| \le \frac{\epsilon\delta}{1-M}.$$

Based on Definition 3, we can conclude that Equation (1) has Ulam–Hyers stability. □

To obtain the Ulam–Hyers–Rassias stability of Equation (1), assume that the following condition holds:

Hypothesis 3 (H3). *There exist constants M_ξ, $N_\xi > 0$ and a nondecreasing function $\xi(s) \in C(J,\mathbb{R})$, such that*

$$^{R}I^{q_1}\xi(s) \le M_\xi\xi(s),\ ^{R}I^{\omega+q_1}\xi(s) \le N_\xi\xi(s),\ s \in J.$$

Theorem 5. *Assume that the conditions (H1) and (H3) are satisfied, then Equation (1) has Ulam–Hyers–Rassias stability.*

Proof. Given that $v(s)$ is a solution of the inequality (18), and $u(s)$ is the unique solution of Equation (1), then $v(s)$ satisfies the following equation:

$$\begin{cases} (\sum_{i=1}^{m}\lambda_i{}^cD^{q_i} + \sum_{j=1}^{m}K_j{}^RI^{p_j})v(s) = f(s,v(s),(\sum_{i=1}^{m}\lambda_i{}^cD^{q_i} + \sum_{j=1}^{m}K_j{}^RI^{p_j})v(s)) + \sigma(s), \\ v(0)=0,\ {}^cD^lv(0)=0,\ l=1,2,\cdots,n-2, \\ v(1) = {}^RI^{\omega}v(\eta). \end{cases} \quad (20)$$

There is

$$v(s) = \mathcal{F}v(s) + \frac{s^{n-1}}{\lambda_1|\Delta|}\Big[\frac{1}{\Gamma(q_1)}\int_0^1(1-z)^{q_1-1}\xi(z)dz - \frac{1}{\Gamma(q_1+\omega)}\int_0^{\eta}(\eta-z)^{\omega+q_1-1}\xi(z)dz\Big] + \frac{1}{\lambda_1\Gamma(q_1)}\int_0^s(s-z)^{q_1-1}\xi(z)dz.$$

By Remark 2 and (H3), it follows that

$$|v(s)-\mathcal{F}v(s)| \le \Big(\frac{M_\xi+N_\xi}{\lambda_1|\Delta|} + \frac{M_\xi}{\lambda_1}\Big)\epsilon\xi(s) = \epsilon\delta_\xi\xi(s).$$

Denoting

$$\delta_\xi = \frac{M_\xi+N_\xi}{\lambda_1|\Delta|} + \frac{M_\xi}{\lambda_1}.$$

Then, it follows that

$$|v(s)-\mathcal{F}v(s)| \le \epsilon\delta_\xi\xi(s).$$

Thus, we can deduce that

$$\begin{aligned}|v(s)-u(s)| \leq &|v(s)-\mathcal{F}v(s)|+|\mathcal{F}v(s)-u(s)|\\ \leq &\epsilon\delta_{\xi}\xi(s)+\frac{|v(s)-u(s)|}{\lambda_1|\Delta|}\Big[\Big(\frac{1+|\Delta|}{\Gamma(q_1+1)}+\frac{\eta^{\omega+q_1}}{\Gamma(\omega+q_1+1)}\Big)L_3\\ &+\sum_{i=2}^{m}\lambda_i\Big(\frac{1+|\Delta|}{\Gamma(q_1-q_i+1)}+\frac{\eta^{\omega+q_1-q_i}}{\Gamma(\omega+q_1-q_i)}\Big)\\ &+\sum_{j=1}^{m}K_j\Big(\frac{1+|\Delta|}{\Gamma(q_1+p_j+1)}+\frac{\eta^{\omega+q_1+p_j}}{\Gamma(\omega+q_1+p_j)}\Big)\Big].\end{aligned}$$

Then, we obtain

$$|v(s)-u(s)| \leq \epsilon\delta_{\xi}\xi(s)+M|v(s)-u(s)|.$$

Hence, we have

$$|v(s)-u(s)| \leq \frac{\epsilon\delta_{\xi}\xi(s)}{1-M}.$$

Based on Definition 4, we can conclude that Equation (1) has Ulam–Hyers–Rassias stability. □

Example 1. *Let us consider a four-term implicit fractional order differential equation with boundary conditions:*

$$\begin{cases} {}^cD^{1.9}u(s)+\frac{1}{10}{}^cD^{0.8}u(s)+{}^RI^{0.3}u(s)+{}^RI^{0.2}u(s)=\\ \frac{sinu(s)}{100}+\frac{\left|{}^cD^{1.9}u(s)+\frac{1}{10}{}^cD^{0.8}u(s)+{}^RI^{0.3}u(s)+{}^RI^{0.2}u(s)\right|}{50\left(1+\left|{}^cD^{1.9}u(s)+\frac{1}{10}{}^cD^{0.8}u(s)+{}^RI^{0.3}u(s)+{}^RI^{0.2}u(s)\right|\right)},\\ u(0)=0,\\ u(1)={}^RI^{0.1}u(0.5),\end{cases} \tag{21}$$

where $q_1=1.9$, $q_2=0.8$, $p_1=0.3$, $p_2=0.2$, $\omega=0.1$, $\eta=0.5$, $\lambda_1=1$, $\lambda_2=\frac{1}{10}$, $K_1=K_2=1$, *and*

$$\begin{aligned}f\Big(s,u(s),(\sum_{i=1}^{2}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{2}K_j{}^RI^{p_j})u(s)\Big)=&\frac{sinu(s)}{100}\\ &+\frac{\left|{}^cD^{1.9}u(s)+\frac{1}{10}{}^cD^{0.8}u(s)+{}^RI^{0.3}u(s)+{}^RI^{0.2}u(s)\right|}{50\left(1+\left|{}^cD^{1.9}u(s)+\frac{1}{10}{}^cD^{0.8}u(s)+{}^RI^{0.3}u(s)+{}^RI^{0.2}u(s)\right|\right)}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\Big|f(s,u_1(s),(\sum_{i=1}^{2}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{2}K_j{}^RI^{p_j})x_1(s))-f(s,u_2(s),(\sum_{i=1}^{2}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{2}K_j{}^RI^{p_j})u_2(s))\Big|\\ \leq&\frac{|sinu_1(s)-sinu_2(s)|}{100}+\frac{\left|{}^cD^{1.9}(u_1(s)-u_2(s))+\frac{1}{10}{}^cD^{0.8}(u_1(s)-u_2(s))\right|}{50}\\ &+\frac{|{}^RI^{0.3}(u_1(s)-u_2(s))+{}^RI^{0.2}(u_1(s)-u_2(s))|}{50}\\ \leq&\frac{|u_1(s)-u_2(s)|}{100}+\frac{\left|(\sum_{i=1}^{2}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{2}K_j{}^RI^{p_j})u_1(s)-(\sum_{i=1}^{2}\lambda_i{}^cD^{q_i}+\sum_{j=1}^{2}K_j{}^RI^{p_j})u_2(s)\right|}{50}.\end{aligned}$$

Denoting $L_1 = \frac{1}{100}$, $L_2 = \frac{1}{50}$, *and* $\Delta = 1 - \frac{\eta^{n+\omega}}{\Gamma(\omega)\omega(\omega+1)\cdots(\omega+n)(n-1)!} = 0.8939$. *When* $u_1,\ u_2 > 0$, *it follows that*

$$\Big[\Big(\frac{1+|\Delta|}{|\Delta|\lambda_1\Gamma(q_1+1)} + \frac{\eta^{\omega+q_1}}{|\Delta|\lambda_1\Gamma(\omega+q_1+1)}\Big)L_3 + \sum_{i=2}^{m}\frac{\lambda_i}{\lambda_1}\Big(\frac{1+|\Delta|}{|\Delta|\Gamma(q_1-q_i+1)} + \frac{\eta^{\omega+q_1-q_i}}{|\Delta|\Gamma(\omega+q_1-q_i)}\Big) + \sum_{j=1}^{m}\frac{K_j}{\lambda_1}\Big(\frac{1+|\Delta|}{|\Delta|\Gamma(q_1+p_j+1)} + \frac{\eta^{\omega+q_1+p_j}}{|\Delta|\Gamma(\omega+q_1+p_j)}\Big)\Big] = 0.5383 < 1. \tag{22}$$

Consequently, based on Theorem 3, we can conclude that Equation (21) has a unique solution. Moreover, the conditions of Theorem 4 are also satisfied. As a result, we can assert that Equation (21) has Ulam–Hyers stability.

4. Conclusions

In this paper, we focused on investigating the existence results and Ulam stability of multi-term implicit fractional differential equations with boundary conditions. We established an equivalent integral expression of Equation (1), and proved the existence and uniqueness results via the Schauder fixed point theorem and the Banach fixed point theorem. Additionally, we provided the criteria for the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of Equation (1). Finally, we presented an example to demonstrate the effectiveness and validity of the obtained results. Similarly, we can consider the existence results and stability of the solutions to multi-term implicit differential equations with Riemann–Liouville, Hilfer, and other fractional-order derivatives.

Author Contributions: Conceptualization, P.W., B.H. and J.B.; writing, B.H.; writing—review and editing, P.W. and J.B.; Supervision, P.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (No. 12171135).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
2. Lakshmikantham, V.; Vatsala, A.S. Basic theory of fractional differential equations. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 2677–2682. [CrossRef]
3. Diethelm, K.; Ford, N.J. Analysis of fractional differential equations. *J. Math. Anal. Appl.* **2002**, *265*, 229–248. [CrossRef]
4. Agarwal, R.; O'Regan, D.; Hristova, S. Stability of Caputo fractional differential equations by Lyapunov functions. *Appl. Math.* **2015**, *60*, 653–676. [CrossRef]
5. Hristova, S.; Tersian, S.; Terzieva, R. Lipschitz Stability in Time for Riemann–Liouville Fractional Differential Equations. *Fractal Fract.* **2021**, *5*, 37. [CrossRef]
6. Agarwal, R.; Almeida, R.; Hristova, S.; O'Regan, D. Caputo fractional differential equation with state dependent delay and practical stability. *Dyn. Syst. Appl.* **2019**, *28*, 715–742.
7. Ulam, S.M. *Problems in Modern Mathematics;* Wiley: New York, NY, USA, 1940.
8. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef]
9. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
10. Benchohra, M.; Bouriah, S.; Nieto, J.J. Existence and Ulam stability for nonlinear implicit differential equations with Riemann-Liouville fractional derivative. *Demonstr. Math.* **2019**, *52*, 437–450. [CrossRef]
11. Chen, C.; Li, M. Existence and Ulam type stability for impulsive fractional differential systems with pure delay. *Fractal Fract.* **2022**, *6*, 742. [CrossRef]

12. Subramanian, M.; Aljoudi, S. Existence and Ulam–Hyers Stability Analysis for Coupled Differential Equations of Fractional-Order with Nonlocal Generalized Conditions via Generalized Liouville–Caputo Derivative. *Fractal Fract.* **2022**, *6*, 629. [CrossRef]
13. Alam, M.; Khan, A.; Asif, M. Analysis of implicit system of fractional order via generalized boundary conditions. *Math. Methods Appl. Sci.* **2023**, *46*, 10554–10571. [CrossRef]
14. Ali, G.; Shah, K.; Rahman, G.U. Existence of solution to a class of fractional delay differential equation under multi-points boundary conditions. *Arab. J. Basic Appl. Sci.* **2020**, *27*, 471–479. [CrossRef]
15. Alam, M.; Zada, A.; Abdeljawad, T. Stability analysis of an implicit fractional integro-differential equation via integral boundary conditions. *Alex. Eng. J.* **2024**, *87*, 501–514. [CrossRef]
16. Rahman, G.; Agarwal, R.P.; Ahmad, D. Existence and stability analysis of nth order multi term fractional delay differential equation. *Chaos Solitons Fractals* **2022**, *155*, 111709. [CrossRef]
17. Derbazi, C.; Baitiche, Z.; Abdo, M.S. Extremal solutions of generalized Caputo-type fractional-order boundary value problems using monotone iterative method. *Fractal Fract.* **2022**, *6*, 146. [CrossRef]