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Common Attractors for Generalized F-Iterated Function Systems in G-Metric Spaces

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Abstract: In this paper, we study the generalized F-iterated function system in G-metric space. Several results of common attractors of generalized iterated function systems obtained by using generalized F-Hutchinson operators are also established. We prove that the triplet of F-Hutchinson operators defined for a finite number of general contractive mappings on a complete G-metric space is itself a generalized F-contraction mapping on a space of compact sets. We also present several examples in 2-D and 3-D for our results.

Keywords: common attractor; F-Hutchinson operator; F-iterated function system; G-metric space; common fixed point

MSC: 47H09; 47H10; 54C60; 54H25

Citation: Nazir, T.; Silvestrov, S. Common Attractors for Generalized F-Iterated Function Systems in G-Metric Spaces. *Fractal Fract.* **2024**, *8*, 346. https://doi.org/10.3390/fractalfract8060346

Academic Editor: Alicia Cordero

Received: 7 April 2024
Revised: 4 June 2024
Accepted: 4 June 2024
Published: 10 June 2024

1. Introduction

Fixed point theory has attracted much attention in the past few years with a vast range of applications both within and beyond mathematics [1–4]. Mustafa and Sims [5] generalized metric space by introducing the structure of G-metric space. Several researchers derived some fixed point theorems for maps satisfying a variety of contractive constraints in G-metric space [3,6–14].

In his 1981 seminal work, Hutchinson [15] established mathematical foundations for iterated function systems (IFSs) and showed that the Hutchinson operator defined on $\mathbb{R}^k$ has as its fixed point a bounded and closed subset of $\mathbb{R}^k$ called an attractor of IFS [16,17]. Several researchers have obtained useful results for iterated function systems (see [18,19] and references therein). Nazir, Silvestrov, and Abbas [20] established fractals by employing F-Hutchinson maps in the setup of metric space. Recently, Navascués [21] presented the approximation of fixed points and fractal functions by means of different iterative algorithms. Navascués et al. [22] established some useful results of the collage type for Reich mutual contractions in b-metric and strong b-metric spaces. Thangaraj et al. [23] constructed an iterated function system called Controlled Kannan Iterated Function System based on Kannan contraction maps in a controlled metric space and used it to develop a new kind of invariant set, known as a Controlled Kannan Attractor or Controlled Kannan Fractal. Recently, Nazir and Silvestrov [24] investigated a generalized iterated function system based on pair of self-mappings and obtained the common attractors of these maps in complete dislocated metric spaces, established the well-posedness of the attractor problems of rational contraction maps in the framework of dislocated metric spaces, and obtained the generalized collage theorem in dislocated metric spaces.

In this paper, we consider the triplet of generalized F-contractive operators and define generalized F-Hutchinson operators to obtain the common attractors in complete G-metric spaces. The contractive conditions are different from those in [24], and both dislocated metric spaces and G-metric spaces are independent to each other. We construct some new common attractor point results based on a generalized F-iterated function system in G-metric spaces. We define F-Hutchinson operators with a finite number of general F-contractive operators in the complete G-metric space and show that these operators are themselves general F-contractions. It is worth mentioning that we are obtaining these results without using any type of commuting conditions of selfmaps in non-symmetric G-metric space. At the end, we present several nontrivial examples of common attractors as a result of F-Hutchinson operators.

Mustafa and Sims [5] established the following notion of G-metric.

Definition 1. *Let Z be a non-empty set. A map with three arguments (ternary map) $G : Z \times Z \times Z \to [0, +\infty)$ is called G-metric if*

$G_1:$ $G(\mu,\nu,\omega) = 0$ *if* $\mu = \nu = \omega$,

$G_2:$ $0 < G(\mu,\nu,\nu)$ *for all* $\mu,\nu \in Z$ *with* $\mu \neq \nu$,

$G_3:$ $G(\mu,\mu,\nu) \leq G(\mu,\nu,\omega)$ *for all* $\mu,\nu,\omega \in Z$ *with* $\nu \neq \omega$,

$G_4:$ *G is symmetric mapping in all its variables, meaning that it is invariant under any permutation of its variables, that is,* $G(\sigma(\mu),\sigma(\nu),\sigma(\omega)) = G(\mu,\nu,\omega)$, *for all permutations* σ *of* $\{\mu,\nu,\omega\}$.

$G_5:$ $G(\mu,\nu,\omega) \leq G(\mu,\varkappa,\varkappa) + G(\varkappa,\nu,\omega)$ *for all* $\mu,\nu,\omega,\varkappa \in Z$.

Then, (Z,G) is called G-metric space. Further, (Z,G) is called symmetric G-metric space whenever $G(\mu,\mu,\varkappa) = G(\mu,\varkappa,\varkappa)$ for all $\mu,\varkappa \in Z$, which can be written also as $G(\varkappa,\mu,\varkappa) = G(\mu,\varkappa,\mu)$, using the invariance of G under permutations of variables (axiom G_4).

Example 1 ([5,25,26])**.** *Let (Z,d) be a metric space. Then, $G : Z \times Z \times Z \to [0,+\infty)$, defined by*

$$G(\mu,\nu,\omega) = \max\{d(\mu,\nu), d(\nu,\omega), d(\mu,\omega)\},$$
$$G(\mu,\nu,\omega) = d(\mu,\nu) + d(\nu,\omega) + d(\mu,\omega)$$

for all $\mu,\nu,\omega \in Z$, are G-metrics on Z.

Example 2. *Let (Z,G) be G-metric space and $d_G : Z \times Z \to [0,+\infty)$ defined as*

$$d_G(u,v) = G(u,v,v) + G(v,u,u) \text{ for all } u,v \in Z.$$

Then, (d_G, Z) is a metric space.

Definition 2 ([25])**.** *Let $\{y_n\}$ be a sequence in G-metric space (Z,G). Then,*

(a) *$\{y_n\} \subset Z$ is G-convergent sequence if, for any $\varepsilon > 0$, there is a point $y \in Z$ and a natural number N such that for all $n,m \geq N$, $G(y,y_n,y_m) < \varepsilon$;*

(b) *$\{y_n\} \subset Z$ is G-Cauchy sequence if, for any $\lambda > 0$, there is an $N \in \mathbb{N}$ such that for all $l,n,m \geq N$, $G(y_n,y_m,y_l) < \lambda$;*

(c) *(Z,G) is G-complete when each G-Cauchy sequence in G-metric space is convergent in Z. $\{y_n\}$ converges to $y \in Z$ whenever $G(y_m,y_n,y) \to 0$ as $m,n \to \infty$ and $\{y_n\}$ is Cauchy whenever $G(y_m,y_n,y_l) \to 0$ as $m,n,l \to \infty$.*

Definition 3 ([25])**.** *Let (Z,G) and (Z',G') be two G-metric spaces. Map $h : (Z,G) \to (Z',G')$ is G-continuous at a point $b \in Z$ when for an $\lambda > 0$, there exists $\delta > 0$ such that $u,v \in Z$ and $G(b,u,v) < \delta$ implies $G'(h(b),h(u),h(v)) < \lambda$. Further, h is G-continuous on Z when it is G-continuous on every $b \in Z$.*

Proposition 1 ([25]). *Let (Z,G) be G-metric space. Then,*

(i) *$G(u,v,w)$ is simultaneously continuous map,*

(ii) *$G(w,v,v) \leq 2G(v,w,w)$ for $w,v \in Z$.*

Consider, next, the following subsets of G-metric space (Z,G) (see [27]):

$N(Z) = \{U : U$ is a non empty subset of $Z\}$.

$B(Z) = \{W : U$ is a non empty bounded subset of $Z\}$.

$CL(Z) = \{U : U$ is a non empty closed subset of $Z\}$.

$CB(Z) = \{U : U$ is a non empty closed and bounded subset of $Z\}$.

$\mathcal{C}^G(Z) = \{U : U$ is a non empty compact set in $Z\}$.

Remark 1 ([28]). *In G-metric space (Z,G), let $H_G : CB(Z) \times CB(Z) \times CB(Z) \to [0,+\infty)$ be a mapping defined as*

$$H_G(D,E,F) = \max\{\sup_{u\in D} G(u,E,F), \sup_{v\in E} G(v,F,D), \sup_{w\in F} G(w,D,E)\}$$

for all $E,D,F \in CB(Z)$, where $G(u,E,D) = \inf\{G(u,v,x) : v \in E, x \in D\}$ is called a Hausdorff G-metric on $CB(Z)$.

If (Z,G) is G-complete metric space, then the H_G-complete metric space $(CB(Y),H_G)$ is also complete.

Lemma 1. *In G-metric space (Z,G), for $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{S},\mathcal{U},\mathcal{V} \in \mathcal{C}^G(Z)$, the following are satisfied:*

(i) *If $\mathcal{Q} \subseteq \mathcal{R}$, then $\sup_{k\in\mathcal{P}} G(k,\mathcal{R},\mathcal{R}) \leq \sup_{k\in\mathcal{P}} G(k,\mathcal{Q},\mathcal{Q})$;*

(ii) *$\sup_{x\in\mathcal{P}\cup\mathcal{Q}} G(x,\mathcal{R},\mathcal{U}) = \max\{\sup_{k\in\mathcal{P}} G(k,\mathcal{R},\mathcal{U}), \sup_{\ell\in\mathcal{Q}} G(\ell,\mathcal{R},\mathcal{U})\}$;*

(iii) *$H_G(\mathcal{P}\cup\mathcal{Q},\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V}) \leq \max\{H_G(\mathcal{P},\mathcal{R},\mathcal{U}), H_G(\mathcal{Q},\mathcal{S},\mathcal{V})\}$.*

Proof. (i) Since $\mathcal{Q} \subseteq \mathcal{R}$, for all $r \in \mathcal{P}$,

$$\begin{aligned} G(r,\mathcal{R},\mathcal{R}) &= \inf\{G(r,\mu,\mu) : \mu \in \mathcal{R}\} \\ &\leq \inf\{G(r,\ell,\ell) : \ell \in \mathcal{Q}\} = G(r,\mathcal{Q},\mathcal{Q}), \end{aligned}$$

this implies that

$$\sup_{r\in\mathcal{P}} G(r,\mathcal{R},\mathcal{R}) \leq \sup_{r\in\mathcal{P}} G(r,\mathcal{Q},\mathcal{Q}).$$

(ii) Note that

$$\begin{aligned} \sup_{x\in\mathcal{P}\cup\mathcal{Q}} G(x,\mathcal{R},\mathcal{U}) &= \max\{\sup\{G(x,\mathcal{R},\mathcal{U}) : x \in \mathcal{P}\}, \sup\{G(x,\mathcal{R},\mathcal{U}) : x \in \mathcal{Q}\}\} \\ &= \max\{\sup_{k\in\mathcal{P}} G(k,\mathcal{R},\mathcal{U}), \sup_{\ell\in\mathcal{Q}} G(\ell,\mathcal{R},\mathcal{U})\}. \end{aligned}$$

(iii) Since

$$\begin{aligned} &\sup_{x\in\mathcal{P}\cup\mathcal{Q}} G(x,\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V}) \\ &= \max\{\sup_{k\in\mathcal{P}} G(k,\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V}), \sup_{\ell\in\mathcal{Q}} G(\ell,\mathcal{Q}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V})\} \quad \text{(from (ii))} \end{aligned}$$

$$\leq \max\{\sup_{k\in\mathcal{P}} G(k,\mathcal{R},\mathcal{U}), \sup_{\ell\in\mathcal{Q}} G(\ell,\mathcal{S},\mathcal{V})\} \quad \text{(from (i))}$$
$$\leq \max\left\{\max\{\sup_{k\in\mathcal{P}} G(k,\mathcal{R},\mathcal{U}), \sup_{\mu\in\mathcal{R}} G(\mu,\mathcal{P},\mathcal{U})\}, \max\{\sup_{\ell\in\mathcal{Q}} G(\ell,\mathcal{S},\mathcal{V}), \sup_{\eta\in\mathcal{S}} G(\eta,\mathcal{Q},\mathcal{V})\}\right\}$$
$$\leq \max\{H_G(\mathcal{P},\mathcal{R},\mathcal{U}), H_G(\mathcal{Q},\mathcal{S},\mathcal{V})\}.$$

Similarly,

$$\sup_{y\in\mathcal{R}\cup\mathcal{S}} G(y,\mathcal{P}\cup\mathcal{Q},\mathcal{U}\cup\mathcal{V}) \leq \max\{H_G(\mathcal{P},\mathcal{R},\mathcal{U}), H_G(\mathcal{Q},\mathcal{S},\mathcal{V})\},$$
$$\sup_{z\in\mathcal{U}\cup\mathcal{V}} G(y,\mathcal{P}\cup\mathcal{Q},\mathcal{R}\cup\mathcal{S}) \leq \max\{H_G(\mathcal{P},\mathcal{R},\mathcal{U}), H_G(\mathcal{Q},\mathcal{S},\mathcal{V})\}.$$

Hence,

$$H_G(\mathcal{P}\cup\mathcal{Q},\mathcal{U}\cup\mathcal{V},\mathcal{R}\cup\mathcal{S}) =$$
$$\max\left\{\sup_{x\in\mathcal{P}\cup\mathcal{Q}} G(x,\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V}), \sup_{y\in\mathcal{R}\cup\mathcal{S}} G(y,\mathcal{P}\cup\mathcal{Q},\mathcal{U}\cup\mathcal{V}), \sup_{z\in\mathcal{U}\cup\mathcal{V}} G(y,\mathcal{P}\cup\mathcal{Q},\mathcal{R}\cup\mathcal{S})\right\}$$
$$\leq \max\{H_G(\mathcal{P},\mathcal{R},\mathcal{U}), H_G(\mathcal{Q},\mathcal{S},\mathcal{V})\}.$$

□

Wardowski [29] defined F-contraction maps for fixed point results as follows. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a continuous map satisfying the following conditions:

(F_1) For $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

(F_2) For $\alpha_k > 0, k = 1,2,3,\ldots, \lim_{k\to\infty} \alpha_k = 0$ and $\lim_{k\to\infty} F(\alpha_k) = -\infty$ are equivalent.

(F_3) There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^\theta F(\alpha) = 0$

We denote a set $\digamma$ as a collection of all F-contractions.

Definition 4. *In G-metric space (Z,G), a self-map $\mathfrak{h} : Z \to Z$ is called an F-contraction on Z if for all $u,v,w \in Z$, there exists $F \in \digamma$ and $\tau > 0$ such that*

$$\tau + F(G(\mathfrak{h}u,\mathfrak{h}v,\mathfrak{h}w)) \leq F(G(u,v,w))$$

whenever $G(\mathfrak{h}u,\mathfrak{h}v,\mathfrak{h}w) > 0$.

We discuss F-iterated function systems in G-metric space. First, we define generalized F-contractive operators as a preliminary result.

Definition 5. *In G-metric space (Z,G), let $\mathfrak{f},\mathfrak{g},\mathfrak{h} : Z \to Z$ be three self-mappings. A triplet $(\mathfrak{f},\mathfrak{g},\mathfrak{h})$ is called a generalized F-contraction mappings if for all $u,v,w \in Z$, there exists $F \in \digamma$ and $\tau > 0$ such that*

$$\tau + F(G(\mathfrak{f}u,\mathfrak{g}v,\mathfrak{h}w)) \leq F(G(u,v,w))$$

whenever $G(\mathfrak{f}u,\mathfrak{g}v,\mathfrak{h}w) > 0$.

Theorem 1. *Consider G-metric space (Z,G) and let $\mathfrak{f},\mathfrak{g},h : Z \to Z$ be continuous maps. If the triplet of mappings $(\mathfrak{f},\mathfrak{g},h)$ is a generalized F-contraction, then*

(i) *the elements in $\mathcal{C}^G(Z)$ are mapped to elements in $\mathcal{C}^G(Z)$ under $\mathfrak{f}$, $\mathfrak{g}$ and h;*

(ii) *if for an arbitrary* $U \in \mathcal{C}^G(Z)$, *the mappings* $\mathfrak{f}, h, \mathfrak{g} : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$ *are defined as*

$$\begin{aligned} \mathfrak{f}(U) &= \{\mathfrak{f}(u) : u \in U\}, \\ \mathfrak{g}(U) &= \{\mathfrak{g}(v) : v \in U\}, \\ h(U) &= \{h(w) : w \in U\}, \end{aligned}$$

then, the triplet $(\mathfrak{f}, \mathfrak{g}, h)$ *is a generalized F-contraction on* $(\mathcal{C}^G(Z), H_G)$.

Proof. (i) Since $\mathfrak{f}$ is a continuous and the image of a compact subset under a continuous mapping, $\mathfrak{f} : Z \to Z$ is compact, then $U \in \mathcal{C}^G(Z)$ gives $\mathfrak{f}(U) \in \mathcal{C}^G(Z)$. Also, $U \in \mathcal{C}^G(Z)$ implies that $\mathfrak{g}(U) \in \mathcal{C}^G(Z)$ and $h(U) \in \mathcal{C}^G(Z)$.
(ii) Let $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$. Since the triplet $(\mathfrak{f}, \mathfrak{g}, h)$ is a generalized F-contraction mappings on Z. Then,

$$G(\mathfrak{f}u, \mathfrak{g}v, hw) < G(u, v, w)$$

for all $u, v, w \in Z$ such that $G(\mathfrak{f}u, \mathfrak{g}v, hw) > 0$. Now,

$$\begin{aligned} G(\mathfrak{f}u, \mathfrak{g}(\mathcal{R}), h(\mathcal{N})) &= \inf\{G(\mathfrak{f}u, \mathfrak{g}v, hw) : v \in \mathcal{R}, w \in \mathcal{N}\} \\ &< \inf\{G(u, v, w) : v \in \mathcal{R}, w \in \mathcal{N}\} \\ &= G(u, \mathcal{R}, \mathcal{N}), \\ G(\mathfrak{g}v, \mathfrak{f}(\mathcal{Q}), h(\mathcal{N})) &= \inf\{G(\mathfrak{g}v, \mathfrak{f}u, htw_1) : u \in \mathcal{Q}, w \in \mathcal{N}\} \\ &< \inf\{G(v, u, w) : u \in \mathcal{Q}, w \in \mathcal{N}\} \\ &= G(v, \mathcal{Q}, \mathcal{N}), \\ G(hw, \mathfrak{f}(\mathcal{Q}), \mathfrak{g}(\mathcal{R})) &= \inf\{G(hw, \mathfrak{f}u, \mathfrak{g}v) : u \in \mathcal{Q}, v \in \mathcal{R}\} \\ &< \inf\{G(w, u, v) : u \in \mathcal{Q}, v \in \mathcal{R}\} \\ &= G(w, \mathcal{Q}, \mathcal{R}), \end{aligned}$$

and hence,

$$\begin{aligned} &H_G(\mathfrak{f}(\mathcal{Q}), \mathfrak{g}(\mathcal{R}), h(\mathcal{N})) \\ &= \max\{\sup_{u \in \mathcal{L}} G(\mathfrak{f}u, \mathfrak{g}(\mathcal{R}), h(\mathcal{N})), \sup_{v \in \mathcal{M}} G(\mathfrak{g}v, \mathfrak{f}(\mathcal{Q}), h(\mathcal{N})), \\ &\sup_{w \in \mathcal{N}} G(hw, \mathfrak{f}(\mathcal{Q}), \mathfrak{g}(\mathcal{R}))\} \\ &< \max\{\sup_{u \in \mathcal{L}} G(u, \mathcal{R}, \mathcal{N}), \sup_{v \in \mathcal{M}} G(v, \mathcal{Q}, \mathcal{N}), \sup_{w \in \mathcal{N}} G(w, \mathcal{Q}, \mathcal{R})\} \\ &= H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}). \end{aligned}$$

By (F_1) of F-contraction,

$$F(H_G(\mathfrak{f}(\mathcal{Q}), \mathfrak{g}(\mathcal{R}), h(\mathcal{N}))) < F(H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})).$$

Consequently, there exists $\tau^* > 0$ such that

$$\tau^* + F(H_G(\mathfrak{f}(\mathcal{Q}), \mathfrak{g}(\mathcal{R}), h(\mathcal{N}))) \leq F(H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})).$$

Thus, the triplet $(\mathfrak{f}, \mathfrak{g}, h)$ is a generalized F-contraction mappings on $(\mathcal{C}^G(Z), H_G)$. □

Proposition 2. *In G-metric space* (Z, G), *suppose the mappings* $\mathfrak{f}_k, \mathfrak{g}_k, h_k : Z \to Z$ *for* $k = 1, \ldots, q$ *are continuous and satisfy*

$$\tau + F(G(\mathfrak{f}_k u, \mathfrak{g}_k v, h_k w)) \leq F(G(u, v, w))$$

for all $u, v, w \in Z$ *such that* $G(\mathfrak{f}_k u, \mathfrak{g}_k v, h_k w) > 0$ *for each* $k \in \{1, \ldots, q\}$*. Then, the mappings* $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$ *defined as*

$$\begin{aligned} \Upsilon(\mathcal{Q}) &= \mathfrak{f}_1(\mathcal{Q}) \cup \ldots \cup \mathfrak{f}_q(\mathcal{Q}), \text{ for each } \mathcal{Q} \in \mathcal{C}^G(Z), \\ \Psi(\mathcal{R}) &= \mathfrak{g}_1(\mathcal{R}) \cup \ldots \cup \mathfrak{g}_q(\mathcal{R}), \text{ for each } \mathcal{R} \in \mathcal{C}^G(Z) \\ \Phi(\mathcal{N}) &= h_1(\mathcal{N}) \cup \ldots \cup h_q(\mathcal{N}), \text{ for each } \mathcal{N} \in \mathcal{C}^G(Z) \end{aligned}$$

also satisfy

$$\tau + H_G(\Upsilon\mathcal{Q}, \Psi\mathcal{R}, \Phi\mathcal{N}) \leq F(H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})) \text{ for all } \mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z),$$

whenever $H_G(\Upsilon\mathcal{Q}, \Psi\mathcal{R}, \Phi\mathcal{N}) > 0$*, that is, the triplet* (Υ, Ψ, Φ) *is also a generalized F-contraction on* $\mathcal{C}^G(Z)$*.*

Proof. We give a proof by induction. If $q = 1$, then, the result is true trivially. For $q = 2$, let $\mathfrak{f}_k, \mathfrak{g}_k, h_k, : Z \to Z$, $k \in \{1, 2\}$ be self-mappings such that $(\mathfrak{f}_1, \mathfrak{g}_1, h_1)$ and $(\mathfrak{f}_2, \mathfrak{g}_2, h_2)$ are triplets of generalized F-contractions. Then, for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$ and from Lemma 1 (iii),

$$\begin{aligned} &\tau + F(H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N}))) \\ &= \tau + F(H_G(\mathfrak{f}_1(\mathcal{Q}) \cup \mathfrak{f}_2(\mathcal{Q}), \mathfrak{g}_1(\mathcal{R}) \cup g_2(\mathcal{R}), h_1(\mathcal{N}) \cup h_2(\mathcal{N}))) \\ &\leq \tau + F(\max\{H_G(\mathfrak{f}_1(\mathcal{Q}), \mathfrak{g}_1(\mathcal{R}), h_1(\mathcal{N})), H_G(\mathfrak{f}_2(\mathcal{Q}), \mathfrak{g}_2(\mathcal{R}), h_2(\mathcal{N}))\}) \\ &\leq F(H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})). \end{aligned}$$

Hence, the result is true for $q = 2$. Suppose that for $q = n$, the result holds, that is,

$$\tau + F\left(H_G\left(\bigcup_{l=1}^{n} \mathfrak{f}_l(\mathcal{Q}), \bigcup_{l=1}^{n} \mathfrak{g}_l(\mathcal{Q}), \bigcup_{l=1}^{n} h_l(\mathcal{Q})\right)\right) \leq F(H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})) \quad \text{for all } \mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z),$$

whenever $H_G\left(\bigcup_{l=1}^{n} \mathfrak{f}_l(\mathcal{Q}), \bigcup_{l=1}^{n} \mathfrak{g}_l(\mathcal{Q}), \bigcup_{l=1}^{n} h_l(\mathcal{Q})\right) > 0$. For

$$\Upsilon(\mathcal{Q}) = \bigcup_{l=1}^{n+1} \mathfrak{f}_l(\mathcal{Q}),\ \Psi(\mathcal{Q}) = \bigcup_{l=1}^{n+1} \mathfrak{g}_l(\mathcal{Q}), \Phi(\mathcal{Q}) = \bigcup_{l=1}^{n+1} h_l(\mathcal{Q})$$

for each $\mathcal{Q} \in \mathcal{C}^G(Z)$, and from Lemma 1 (iii), we have

$$\begin{aligned} &\tau + F(H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N}))) \\ &= \tau + F\left(H_G(\bigcup_{l=1}^{n+1} \mathfrak{f}_l(\mathcal{Q}), \bigcup_{l=1}^{n+1} \mathfrak{g}_l(\mathcal{R}), \bigcup_{l=1}^{n+1} h_l(\mathcal{N}))\right) \\ &= \tau + F\left(H_G(\bigcup_{l=1}^{n} \mathfrak{f}_l(\mathcal{Q}) \cup \mathfrak{f}_{n+1}(\mathcal{Q}), \bigcup_{l=1}^{n} \mathfrak{g}_l(\mathcal{R}) \cup g_{n+1}(\mathcal{R}), \bigcup_{l=1}^{n} h_l(\mathcal{N}) \cup h_{n+1}(\mathcal{N}))\right) \\ &\leq \tau + F\left(\max\{H_G(\bigcup_{l=1}^{n} \mathfrak{f}_l(\mathcal{Q}), \bigcup_{l=1}^{n} \mathfrak{g}_l(\mathcal{R}), \bigcup_{l=1}^{n} h_l(\mathcal{N})), H_G(\mathfrak{f}_{n+1}(\mathcal{Q}), \mathfrak{g}_{n+1}(\mathcal{R}), h_{n+1}(\mathcal{N}))\}\right) \\ &\leq F(H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})). \end{aligned}$$

Hence, the result is true for $q = n + 1$. Thus, the triplet (Υ, Ψ, Φ) is also a generalized F-contraction on $\mathcal{C}^G(Z)$. □

Definition 6. *In G-metric space (Z, G), let $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$. The mappings (Υ, Ψ, Φ) are called generalized F-Hutchinson contractive operators if for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$ obeying $H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) > 0$, it holds that*

$$\tau + F(H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N}))) \leq F(M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N})) \tag{1}$$
$$\text{where } M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = \max\{H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}), H_G(\mathcal{Q}, \Upsilon(\mathcal{Q}), \Upsilon(\mathcal{Q})), H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})), H_G(\mathcal{N}, \Phi(\mathcal{N}), \Phi(\mathcal{N}))\}.$$

Definition 7. *In a complete G-metric space (Z, G), let $\mathfrak{f}_k, \mathfrak{g}_k, h_k : Z \to Z$, $k = 1, \ldots, q$ be continuous maps, where each triplet $(\mathfrak{f}_k, \mathfrak{g}_k, h_k)$ for $k = 1, \ldots, q$ is a generalized F-contraction, then $\{Z; (\mathfrak{f}_k, \mathfrak{g}_k, h_k), k = 1, \ldots, q\}$ is called the generalized F-iterated function system.*

Consequently, a generalized F-iterated function system in G-metric space is a finite collection of generalized F-contractions on Z.

Definition 8. *Let (Z, G) be a complete G-metric space and $U \subseteq Z$ a non-empty compact set. Then, U is the common attractor of the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$ if*

(i) $\Upsilon(U) = \Psi(U) = \Phi(U) = U$

(ii) *There exists an open set $V \subseteq Z$ satisfying $U \subseteq V$ and $\lim_{k\to+\infty} \Upsilon^k(\mathcal{Q}) = \lim_{k\to+\infty} \Psi^k(\mathcal{R}) = \lim_{k\to+\infty} \Phi^k(\mathcal{N}) = U$ for any compact sets $\mathcal{Q}, \mathcal{R}, \mathcal{N} \subseteq V$, where the limit is taken relative to the G-Hausdorff metric.*

2. Main Results

Now, we establish the results of common attractors of generalized F-Hutchinson contraction in G-metric spaces.

Theorem 2. *In a complete G-metric space (Z, G), let $\{Z; (\mathfrak{f}_k, \mathfrak{g}_k, h_k), k = 1, \ldots, q\}$ be the generalized F-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$ by*

$$\Upsilon(\mathcal{Q}) = \mathfrak{f}_1(\mathcal{Q}) \cup \ldots \cup \mathfrak{f}_q(\mathcal{Q}),$$
$$\Psi(\mathcal{R}) = \mathfrak{g}_1(\mathcal{R}) \cup \ldots \cup \mathfrak{g}_q(\mathcal{R}),$$
$$\Phi(\mathcal{N}) = h_1(\mathcal{N}) \cup \ldots \cup h_q(\mathcal{N})$$

for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$. If the mappings (Υ, Ψ, Φ) are generalized F-Hutchinson contractive operators, then Υ, Ψ and Φ have a unique common attractor $U^* \in \mathcal{C}^G(Z)$, that is,

$$U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*).$$

Additionally, for any arbitrarily chosen initial set $\mathcal{R}_0 \in \mathcal{C}^G(Z)$, the sequence

$$\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), \ldots\}$$

of compact sets converges to the common attractor U^*.

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $U^* \in \mathcal{C}^G(Z)$ is such that $\Upsilon(U^*) = U^*$. We need to show that $U^* = \Psi(U^*) = \Phi(U^*)$. If not, then as the mappings (Υ, Ψ, Φ) are generalized F-Hutchinson contractive operators, for $H_G(\Upsilon(U^*), \Psi(U^*), \Phi(U^*)) > 0$, by using (G_3), we obtain

$$\begin{aligned} \tau + F(H_G(U^*, \Psi(U^*), \Phi(U^*))) &= \tau + F(H_G(\Upsilon(U^*), \Psi(U^*), \Phi(U^*))) \\ &\leq F(M_{\Upsilon,\Psi,\Phi}(U^*, U^*, U^*))), \end{aligned} \tag{2}$$

where

$$\begin{aligned} M_{\Upsilon,\Psi,\Phi}(U^*,U^*,U^*) &= \max\{H_G(U^*,U^*,U^*), H_G(U^*,\Upsilon(U^*),\Upsilon(U^*)), \\ &\qquad H_G(U^*,\Psi(U^*),\Psi(U^*)), H_G(U^*,\Phi(U^*),\Phi(U^*))\} \\ &= \max\{H_G(U^*,\Psi(U^*),\Psi(U^*)), H_G(U^*,\Phi(U^*),\Phi(U^*))\} \\ &\leq \max\{H_G(U^*,\Psi(U^*),\Phi(U^*)), H_G(U^*,\Phi(U^*),\Psi(U^*))\} \\ &= H_G(U^*,\Psi(U^*),\Phi(U^*)). \end{aligned}$$

From (2), it follows that

$$\tau + F(H_G(U^*,\Psi(U^*),\Phi(U^*))) \leq F(H_G(U^*,\Psi(U^*),\Phi(U^*))),$$

where $\tau > 0$, a contradiction. Thus, $H_G(U^*,\Psi(U^*),\Phi(U^*)) = 0$, and so we obtain $U^* = \Psi(U^*) = \Phi(U^*)$. In an analogous manner, for $U^* = \Phi(U^*)$ or for $U^* = \Psi(U^*)$, we obtain U^* as the common attractor of Υ, Ψ, and Φ.

We proceed by showing that Υ, Ψ, and Φ have a unique common attractor. Let $\mathcal{R}_0 \in \mathcal{C}^G(Z)$ be chosen arbitrary. Define a sequence $\{\mathcal{R}_k\}$ by $\mathcal{R}_{3k+1} = \Upsilon(\mathcal{R}_{3k})$, $\mathcal{R}_{3k+2} = \Psi(\mathcal{R}_{3k+1})$ and $\mathcal{R}_{3k+3} = \Phi(\mathcal{R}_{3k+2})$, $k = 0,1,2,\ldots$. If $\mathcal{R}_k = \mathcal{R}_{k+1}$ for some k, with $k = 3n$, then $U^* = \mathcal{R}_{3k}$ is an attractor of Υ and from the Proof above, U^* is a common attractor for Υ, Ψ and Φ. The same is true for $k = 3n+1$ or $k = 3n+2$. We assume that $\mathcal{R}_k \neq \mathcal{R}_{k+1}$ for all $k \in \mathbb{N}$, then by using (G_3), we have

$$\begin{aligned} \tau + F(H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3})) &= \tau + F(H_G(\Upsilon(\mathcal{R}_{3k}),\Psi(\mathcal{R}_{3k+1}),\Phi(\mathcal{R}_{3k+2}))) \\ &\leq F(M_{\Upsilon,\Psi,\Phi}(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2})), \end{aligned} \tag{3}$$

where

$$\begin{aligned} &M_{\Upsilon,\Psi,\Phi}(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) \\ &= \max\{H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}), H_G(\mathcal{R}_{3k},\Upsilon(\mathcal{R}_{3k}),\Upsilon(\mathcal{R}_{3k})), \\ &\qquad H_G(\mathcal{R}_{3k+1},\Psi(\mathcal{R}_{3k+1}),\Psi(\mathcal{R}_{3k+1})), H_G(\mathcal{R}_{3k+2},\Phi(\mathcal{R}_{3k+2}),\Phi(\mathcal{R}_{3k+2}))\} \\ &= \max\{H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}), H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+1}), \\ &\qquad H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+2}), H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+3})\} \\ &\leq \max\{H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}), H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}), \\ &\qquad H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k}), H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+1})\} \\ &= H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}). \end{aligned}$$

Thus from (3), we have

$$\tau + F(H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3})) \leq F(H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2})).$$

Similarly, one can show that

$$\tau + F(H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+4})) \leq F(H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3}))$$

and

$$\tau + F(H_G(\mathcal{R}_{3k+3},\mathcal{R}_{3k+4},\mathcal{R}_{3k+5})) \leq F(H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+4})).$$

Thus, for all k,

$$\tau + F(H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})) \leq F(H_G(\mathcal{R}_{k},\mathcal{R}_{k+1},\mathcal{R}_{k+2})).$$

Thus,

$$F(H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})) \leq F(H_G(\mathcal{R}_{k},\mathcal{R}_{k+1},\mathcal{R}_{k+2})) - \tau$$

$$\leq F(H_G(\mathcal{R}_{k-1},\mathcal{R}_k,\mathcal{R}_{k+1})) - 2\tau$$
$$\leq \ldots \leq F(H_G(\mathcal{R}_0,\mathcal{R}_1,\mathcal{R}_2)) - k\tau$$

and we obtain that $\lim_{k\to\infty} F(H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})) = -\infty$ which together with (F_2) implies that $\lim_{k\to\infty} H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}) = 0$. Now by (F_3), there exists $h \in (0,1)$ such that

$$\lim_{k\to\infty}[H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})]^h F(H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})) = 0.$$

Thus,

$$\begin{aligned}&[H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})]^h F(H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}))\\ &\qquad - [H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})]^h F(H_G(\mathcal{R}_0,\mathcal{R}_1,\mathcal{R}_2))\\ &\qquad\qquad \leq -k\tau[H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})]^h \leq 0.\end{aligned}$$

On taking limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} k[H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})]^h = 0.$$

As $\lim_{k\to\infty} k^{\frac{1}{h}} H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}) = 0$, there exists $n_1 \in \mathbb{N}$ such that

$$k^{\frac{1}{h}} H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}) \leq 1$$

for all $n \geq n_1$. So we have $H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}) \leq \frac{1}{k^{1/h}}$ for all $k \geq n_1$. Now, for l, m, k, with $l > m > k$,

$$\begin{aligned}H_G(\mathcal{R}_k,\mathcal{R}_m,\mathcal{R}_l) &\leq H_G(\mathcal{R}_k,\mathcal{R}_{k+1},\mathcal{R}_{k+1}) + H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+2})\\ &\quad + \cdots + H_G(\mathcal{R}_{l-1},\mathcal{R}_{l-1},\mathcal{R}_l)\\ &\leq H_G(\mathcal{R}_k,\mathcal{R}_{k+1},\mathcal{R}_{k+2}) + H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3})\\ &\quad + \cdots + H_G(\mathcal{R}_{l-2},\mathcal{R}_{l-1},\mathcal{R}_l) \leq \sum_{i=k}^{\infty} \frac{1}{i^{1/h}}.\end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we obtain $H_G(\mathcal{R}_k,\mathcal{R}_m,\mathcal{R}_l) \to 0$ as $k, m, l \to +\infty$. Thus, $\{\mathcal{R}_k\}$ is G-Cauchy sequence in $\mathcal{C}^G(Z)$. Since $(\mathcal{C}^G(Z), H_G)$ is a complete G-metric space, there is $U^* \in \mathcal{C}^G(Z)$ such that $\lim_{k\to+\infty} \mathcal{R}_k = U^*$, that is, $\lim_{k\to+\infty} H_G(\mathcal{R}_k,\mathcal{R}_k,U^*) = 0$.

To prove that $\Upsilon(U^*) = U^*$, when assuming the contrary we have

$$\begin{aligned}&\tau + F(H_G(\Upsilon(U^*),\mathcal{R}_{3k+2},\mathcal{R}_{3k+3})) = \tau + F(H_G(\Upsilon(U^*),\Psi(\mathcal{R}_{3k+1}),\Phi(\mathcal{R}_{3k+2})))\\ &\leq F(M_{\Upsilon,\Psi,\Phi}(U^*,\mathcal{R}_{3k+1},\mathcal{R}_{3k+2})),\end{aligned} \tag{4}$$

where

$$\begin{aligned}&M_{\Upsilon,\Psi,\Phi}(U^*,\mathcal{R}_{3k+1},\mathcal{R}_{3k+2})\\ &\quad = \max\{H_G(\Upsilon(U^*),\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}), H_G(U^*,\Upsilon(U^*),\Upsilon(U^*)),\\ &H_G(\mathcal{R}_{3k+1},\Psi(\mathcal{R}_{3k+1}),\Psi(\mathcal{R}_{3k+1})), H_G(\mathcal{R}_{3k+2},\Phi(\mathcal{R}_{3k+2}),\Phi(\mathcal{R}_{3k+2}))\}\\ &\quad \leq \max\{H_G(U^*,\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}), H_G(\Upsilon(U^*),U^*,\mathcal{R}_{3k+1}),\\ &H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+2}), H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+3})\}.\end{aligned}$$

Thus, (4) implies

$$\tau + F(H_G(\Upsilon(U^*),\mathcal{R}_{3k+2},\mathcal{R}_{3k+3}))$$

$$\leq F(\max\{H_G(U^*, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}), H_G(\Upsilon(U^*), U^*, \mathcal{R}_{3k+1}), H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+2}), H_G(\mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}, \mathcal{R}_{3k+3})\})$$

and taking the limit as $k \to +\infty$ yields

$$\begin{aligned} &\tau + F(H_G(\Upsilon(U^*), U^*, U^*)) \\ &\leq F(\max\{H_G(U^*, U^*, U^*), H_G(\Upsilon(U^*), U^*, U^*), H_G(U^*, U^*, U^*), H_G(U^*, U^*, U^*)\}) \\ &= F(H_G(\Upsilon(U^*), U^*, U^*)), \end{aligned}$$

which is a contradiction as $\tau > 0$. Thus, $\Upsilon(U^*) = U^*$. Following the conclusion above, U^* is the common attractor of Υ, Ψ, and Φ.

For uniqueness, we consider V as another common attractor of Υ, Ψ and Φ with $H_G(U^*, V, V) > 0$. Then,

$$\tau + F(H_G(U^*, V, V)) = \tau + F(H_G(\Upsilon(U^*), \Psi(V), \Phi(V))) \leq F(M_{\Upsilon,\Psi,\Phi}(U^*, V, V)) \tag{5}$$

$$\begin{aligned} \text{where } M_{\Upsilon,\Psi,\Phi}(U^*, V, V) &= \max\{H_G(U^*, V, V), H_G(U^*, \Upsilon(U^*), \Upsilon(U^*)), \\ &\qquad H_G(V, \Psi(V), \Psi(V)), H_G(V, \Phi(V), \Phi(V))\} \\ &= \max\{H_G(U^*, V, V), H_G(U^*, U^*, U^*), \\ &\qquad H_G(V, V, V), H_G(V, V, V)\} \\ &= H_G(U^*, V, V). \end{aligned}$$

Thus, (5) implies that $\tau + F(H_G(U^*, V, V)) \leq F(H_G(U^*, V, V))$ from which we conclude that $H_G(U^*, V, V) = 0$, and thus, $U^* = V$. Hence, U^* is a unique common attractor of Υ, Ψ, and Φ. □

Remark 2. *In Theorem 2, take the collection $\mathcal{S}^G(Z)$, of all singleton subsets of Z, then $\mathcal{S}^G(Z) \subseteq \mathcal{C}^G(Z)$. Furthermore, if we take the mappings $(\mathfrak{f}_k, \mathfrak{g}_k, h_k) = (\mathfrak{f}, \mathfrak{g}, h)$ for each k, where $\mathfrak{f} = \mathfrak{f}_1$, $\mathfrak{g} = \mathfrak{g}_1$ and $h = h_1$, then the operators (Υ, Ψ, Φ) become*

$$(\Upsilon(v_1), \Psi(v_2), \Phi(v_3)) = (\mathfrak{f}(v_1), \mathfrak{g}(v_2), h(v_3)).$$

Thus, we obtain the following result on common fixed point.

Corollary 1. *Let $\{Z; (\mathfrak{f}_k, \mathfrak{g}_k, h_k), k = 1, 2, \ldots, q\}$ be a generalized F-iterated function system in a complete G-metric space (Z, G) and define the maps $\mathfrak{f}, \mathfrak{g}, h : Z \to Z$ as in Remark 2. If there exists $\tau > 0$ such that for $v_1, v_2, v_3 \in Z$ having $G(\mathfrak{f}v_1, \mathfrak{g}v_2, hv_3) > 0$, the following holds*

$$\begin{aligned} &\tau + F(G(\mathfrak{f}v_1, \mathfrak{g}v_2, hv_3)) \leq F\big(M_{\mathfrak{f},\mathfrak{g},h}(v_1, v_2, v_3)\big), \\ &\textit{where} \quad M_{\mathfrak{f},\mathfrak{g},h}(v_1, v_2, v_3) = \max\{G(v_1, v_2, v_3), G(v_1, \mathfrak{f}(v_1), \mathfrak{f}(v_1)), \\ &\qquad\qquad G(v_2, \mathfrak{g}(v_2), \mathfrak{g}(v_2)), G(v_3, \mathfrak{h}(v_3), \mathfrak{h}(v_3))\}. \end{aligned}$$

Then, $\mathfrak{f}, \mathfrak{g}$, and h have a unique common fixed point $u \in Z$. Additionally, for an arbitrary element $u_0 \in Z$, the sequence $\{u_0, \mathfrak{f}u_0, \mathfrak{g}\mathfrak{f}u_0, h\mathfrak{g}\mathfrak{f}u_0, \mathfrak{f}h\mathfrak{g}\mathfrak{f}u_0, \cdots\}$ converges to the common fixed point of $\mathfrak{f}, \mathfrak{g}$, and h.

Corollary 2. *In a complete G-metric space (Z, G), let $\{Z; (\mathfrak{f}_k, g_k, h_k), k = 1, \cdots, q\}$ be the generalized F-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$ by*

$$\begin{aligned} \Upsilon(\mathcal{Q}) &= \mathfrak{f}_1(\mathcal{Q}) \cup \cdots \cup \mathfrak{f}_q(\mathcal{Q}), \\ \Psi(\mathcal{R}) &= \mathfrak{g}_1(\mathcal{R}) \cup \cdots \cup \mathfrak{g}_q(\mathcal{R}), \\ \Phi(\mathcal{N}) &= h_1(\mathcal{N}) \cup \cdots \cup h_q(\mathcal{N}) \end{aligned}$$

for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$. If for some $m \in \mathbb{N}$, there exists $\tau > 0$ such that for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$ with $H_G(\Upsilon^m(\mathcal{Q}), \Psi^m(\mathcal{R}), \Phi^m(\mathcal{N})) > 0$ it holds that

$$\begin{aligned} &\tau + F(H_G(\Upsilon^m(\mathcal{Q}), \Psi^m(\mathcal{R}), \Phi^m(\mathcal{N}))) \leq F(M_{\Upsilon^m,\Psi^m,\Phi^m}(\mathcal{Q}, \mathcal{R}, \mathcal{N})), \\ &\text{where } M_{\Upsilon^m,\Psi^m,\Phi^m}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = \max\{H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}), H_G(\mathcal{Q}, \Upsilon^m(\mathcal{Q}), \Upsilon^m(\mathcal{Q})), \\ &\qquad H_G(\mathcal{R}, \Psi^m(\mathcal{R}), \Psi^m(\mathcal{R})), H_G(\mathcal{N}, \Phi^m(\mathcal{N}), \Phi^m(\mathcal{N}))\}. \end{aligned}$$

Then, there exists unique $U^* \in \mathcal{C}^G(Z)$ that satisfies

$$U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*).$$

Additionally, for any arbitrarily chosen initial set $\mathcal{R}_0 \in \mathcal{C}^G(Z)$, the sequence

$$\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), \ldots\}$$

of compact sets converges to the common attractor U^*.

Proof. From Theorem 2, we obtain that there exists unique $U^* \in \mathcal{C}^G(Z)$ that satisfy

$$U^* = \Upsilon^m(U^*) = \Psi^m(U^*) = \Phi^m(U^*).$$

Now, $\Upsilon(U^*) = \Upsilon(\Upsilon^m(U^*)) = \Upsilon^m(\Upsilon(U^*))$, that is, $\Upsilon(U^*)$ is also an attractor of Υ^m. Following the similar steps for those in Proof of Theorem 2, we obtain that $\Upsilon(U^*)$ is also the common attractor of Υ^m, Ψ^m and Φ^m. By the uniqueness of the common attractor, $U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*)$. □

Example 3. *Let $Z = [0,1] \times [0,1]$ and G-metric on Z be defined as*

$$G(u,v,w) = \max\left\{ \left(\sum_{i=1}^{2}(u_i - v_i)^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(v_i - w_i)^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(w_i - u_i)^2\right)^{\frac{1}{2}} \right\}$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$. Define $\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{h}_k : Z \to Z$, $k = 1, 2$ by

$$\mathfrak{f}_1(z_1,z_2) = \left(\frac{z_1+1}{7}, \frac{z_2+3}{9}\right) \text{ for } z_1, z_2 \in [0,1],$$

$$\mathfrak{g}_1(z_1,z_2) = \left(\frac{2z_1+3}{20}, \frac{3(z_2+1)}{11}\right) \text{ for } z_1, z_2 \in [0,1],$$

$$\mathfrak{h}_1(z_1,z_2) = \left(\frac{3z_1+2}{15}, \frac{4z_2+3}{12}\right) \text{ for } z_1, z_2 \in [0,1],$$

$$\mathfrak{f}_2(z_1,z_2) = \left(\frac{2z_1+5}{12}, \frac{z_2+4}{8}\right) \text{ for } z_1, z_2 \in [0,1],$$

$$\mathfrak{g}_2(z_1,z_2) = \left(\frac{2(z_1+1)}{6}, \frac{2z_2+4}{9}\right) \text{ for } z_1, z_2 \in [0,1],$$

$$\mathfrak{h}_2(z_1,z_2) = \left(\frac{5z_1+2}{9}, \frac{3z_2+4}{10}\right) \text{ for } z_1, z_2 \in [0,1].$$

The maps $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1$, and $\mathfrak{h}_2$ are continuous and non commutative.

Now, we show that for $F \in \digamma$ and $\tau > 0$, the mappings $\mathfrak{f}_k, \mathfrak{g}_k, h_k : Z \to Z, k = 1,2$ satisfy

$$\begin{aligned} &\tau + F(G(\mathfrak{f}_k(u), \mathfrak{g}_k(v), \mathfrak{h}_k(w))) \leq F\big(m_{\mathfrak{f}_k,\mathfrak{g}_k,\mathfrak{h}_k}(u,v,w)\big) \\ &\text{where } m_{\mathfrak{f}_k,\mathfrak{g}_k,\mathfrak{h}_k}(u,v,w) = \max\{G(u,v,w), G(u, \mathfrak{f}_k(u), \mathfrak{f}_k(u)), \\ &\qquad G(v, \mathfrak{g}_k(v), \mathfrak{g}_1(v)), G(w, \mathfrak{h}_k(w), \mathfrak{h}_k(w))\}. \end{aligned}$$

for all $u, v, w \in Z$ *obeying* $G(\mathfrak{f}_k u, \mathfrak{g}_k v, h_k w) > 0$ *for each* $k \in \{1,2\}$. *As*

$$G(\mathfrak{f}_1(u), \mathfrak{g}_1(v), \mathfrak{h}_1(w))$$
$$= \max\left\{ \sum_{i=1}^{2} (\mathfrak{f}_1 u_i - \mathfrak{g}_1 v_i)^2, \left(\sum_{i=1}^{2} (\mathfrak{g}_1 v_i - \mathfrak{h}_1 w_i)^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2} (\mathfrak{h}_1 w_i - \mathfrak{f}_1 u_i)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \max\left\{ \left((\mathfrak{f}_1 u_1 - \mathfrak{g}_1 v_1)^2 + (\mathfrak{f}_1 u_2 - \mathfrak{g}_1 v_2)^2 \right)^{\frac{1}{2}}, \left((\mathfrak{g}_1 v_1 - \mathfrak{h}_1 w_1)^2 + (\mathfrak{g}_1 v_2 - \mathfrak{h}_1 w_2)^2 \right)^{\frac{1}{2}}, \left((\mathfrak{h}_1 w_1 - \mathfrak{f}_1 u_1)^2 + (\mathfrak{h}_1 w_2 - \mathfrak{f}_1 u_2)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \max\left\{ \left(\left(\frac{u_1+1}{7} - \frac{2v_1+3}{20} \right)^2 + \left(\frac{u_2+3}{9} - \frac{3(v_2+1)}{11} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{2v_1+3}{20} - \frac{3w_1+2}{15} \right)^2 + \left(\frac{3(v_2+1)}{11} - \frac{4w_2+3}{12} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{3w_1+2}{15} - \frac{u_1+1}{7} \right)^2 + \left(\frac{4w_2+3}{12} - \frac{u_2+3}{9} \right)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \max\left\{ \left(\left(\frac{(20u_1+20)-(14v_1+21)}{140} \right)^2 + \left(\frac{(11u_2+33)-(27v_2+27)}{99} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{(6v_1+9)-(12w_1+8)}{60} \right)^2 + \left(\frac{(36v_2+36)-(44w_2+33)}{132} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{(21w_1+14)-(15u_1+15)}{105} \right)^2 + \left(\frac{(12w_2+9)-(4u_2+12)}{36} \right)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \max\left\{ \left(\left(\frac{20u_1-14v_1-1}{140} \right)^2 + \left(\frac{27v_1-11u_2-6}{99} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{12w_1-6v_1-1}{60} \right)^2 + \left(\frac{44w_2-36v_2-3}{132} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{21w_1-15u_1-1}{105} \right)^2 + \left(\frac{12w_2-4u_2-3}{36} \right)^2 \right)^{\frac{1}{2}} \right\};$$

$$G(u,v,w) = \max\left\{ \left(\sum_{i=1}^{2} (u_i - v_i)^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2} (v_i - w_i)^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2} (w_i - u_i)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \max\left\{ \left((u_1 - v_1)^2 + (u_2 - v_2)^2 \right)^{\frac{1}{2}}, \left((v_1 - w_1)^2 + (v_2 - w_2)^2 \right)^{\frac{1}{2}}, \left((w_1 - u_1)^2 + (w_2 - u_2)^2 \right)^{\frac{1}{2}} \right\};$$

$$G(u, \mathfrak{f}_1(u), \mathfrak{f}_1(u))$$
$$= \max\left\{ \left(\sum_{i=1}^{2} (u_i - \mathfrak{f}_1(u_i))^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2} (\mathfrak{f}_1(u_i) - \mathfrak{f}_1(u_i))^2 \right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2} (\mathfrak{f}_1(u_i) - u_i)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \max\left\{ \left((u_1 - \mathfrak{f}_1(u_1))^2 + (u_2 - \mathfrak{f}_1(u_2))^2 \right)^{\frac{1}{2}}, \left((\mathfrak{f}_1(u_1) - \mathfrak{f}_1(u_1))^2 + (\mathfrak{f}_1(u_2) - \mathfrak{f}_1(u_2))^2 \right)^{\frac{1}{2}}, \right.$$

$$\left((u_1-\mathfrak{f}_1(u_1))^2+(u_2-\mathfrak{f}_1(u_2))^2\right)^{\frac{1}{2}}\Bigg\}$$
$$=\left((u_1-\mathfrak{f}_1(u_1))^2+(u_2-\mathfrak{f}_1(u_2))^2\right)^{\frac{1}{2}}$$
$$=\left(\left(u_1-\frac{u_1+1}{7}\right)^2+\left(u_2-\frac{u_2+3}{9}\right)^2\right)^{\frac{1}{2}}=\left(\left(\frac{6u_1-1}{7}\right)^2+\left(\frac{8u_2-3}{9}\right)^2\right)^{\frac{1}{2}};$$

$$G(v,\mathfrak{g}_1(v),\mathfrak{g}_1(v))$$
$$=\max\left\{\left(\sum_{i=1}^{2}(v_i-\mathfrak{g}_1(v_i))^2\right)^{\frac{1}{2}},\left(\sum_{i=1}^{2}(\mathfrak{g}_1(v_i)-\mathfrak{g}_1(v_i))^2\right)^{\frac{1}{2}},\left(\sum_{i=1}^{2}(\mathfrak{g}_1(v_i)-v_i)^2\right)^{\frac{1}{2}}\right\}$$
$$=\max\Bigg\{\left((v_1-\mathfrak{g}_1(v_1))^2+(v_2-\mathfrak{g}_1(v_2))^2\right)^{\frac{1}{2}},\left((\mathfrak{g}_1(v_1)-\mathfrak{g}_1(v_1))^2+(\mathfrak{g}_1(v_2)-\mathfrak{g}_1(v_2))^2\right)^{\frac{1}{2}},$$
$$\left((v_1-\mathfrak{g}_1(v_1))^2+(v_2-\mathfrak{g}_1(v_2))^2\right)\Bigg\}^{\frac{1}{2}}$$
$$=\left((v_1-\mathfrak{g}_1(v_1))^2+(v_2-\mathfrak{g}_1(v_2))^2\right)^{\frac{1}{2}}$$
$$=\left(\left(v_1-\frac{2v_1+3}{20}\right)^2+\left(v_2-\frac{3(v_2+1)}{11}\right)^2\right)^{\frac{1}{2}}=\left(\left(\frac{18v_1-3}{20}\right)^2+\left(\frac{8v_2-3}{11}\right)^2\right)^{\frac{1}{2}};$$

$$G(w,\mathfrak{h}_1(w),\mathfrak{h}_1(w))$$
$$=\max\left\{\left(\sum_{i=1}^{2}(w_i-\mathfrak{h}_1(w_i))^2\right)^{\frac{1}{2}},\left(\sum_{i=1}^{2}(\mathfrak{h}_1(w_i)-\mathfrak{h}_1(w_i))^2\right)^{\frac{1}{2}},\left(\sum_{i=1}^{2}(\mathfrak{h}_1(w_i)-w_i)^2\right)^{\frac{1}{2}}\right\}$$
$$=\max\Bigg\{\left((w_1-\mathfrak{h}_1(w_1))^2+(w_2-\mathfrak{h}_1(w_2))^2\right)^{\frac{1}{2}},$$
$$\left((\mathfrak{h}_1(w_1)-\mathfrak{h}_1(w_1))^2+(\mathfrak{h}_1(w_2)-\mathfrak{h}_1(w_2))^2\right)^{\frac{1}{2}},$$
$$\left((w_1-\mathfrak{h}_1(w_1))^2+(w_2-\mathfrak{h}_1(w_2))^2\right)^{\frac{1}{2}}\Bigg\}$$
$$=\left((w_1-\mathfrak{h}_1(w_1))^2+(w_2-\mathfrak{h}_1(w_2))^2\right)^{\frac{1}{2}}$$
$$=\left(\left(w_1-\frac{3w_1+2}{15}\right)^2+\left(v_2-\frac{4w_2+3}{12}\right)^2\right)^{\frac{1}{2}}=\left(\left(\frac{12w_1-2}{15}\right)^2+\left(\frac{8w_2-3}{12}\right)^2\right)^{\frac{1}{2}}.$$

Now, by taking $F(\lambda)=\ln(\lambda)$ *for* $\lambda>0$, $\tau=\ln(\frac{20}{19})$, *and for* $u,v,w\in Z$ *having* $G(\mathfrak{f}_1(u),\mathfrak{g}_1(v),\mathfrak{h}_1(w))>0$, *we have*

$$G(\mathfrak{f}_1(u),\mathfrak{g}_1(v),\mathfrak{h}_1(w))$$
$$=\max\Bigg\{\left(\left(\frac{20u_1-14v_1-1}{140}\right)^2+\left(\frac{27v_1-11u_2-6}{99}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{12w_1-6v_1-1}{60}\right)^2+\left(\frac{44w_2-36v_2-3}{132}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{21w_1-15u_1-1}{105}\right)^2+\left(\frac{12w_2-4u_2-3}{36}\right)^2\right)^{\frac{1}{2}}\Bigg\}$$

$$\leq \frac{19}{20}\max\Bigg\{\Big((u_1-v_1)^2+(u_2-v_2)^2\Big)^{\frac{1}{2}},\Big((v_1-w_1)^2+(v_2-w_2)^2\Big)^{\frac{1}{2}},$$
$$\Big((w_1-u_1)^2+(w_2-u_2)^2\Big)^{\frac{1}{2}},\left(\left(\frac{6u_1-1}{7}\right)^2+\left(\frac{8u_2-3}{9}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{18v_1-3}{20}\right)^2+\left(\frac{8v_2-3}{11}\right)^2\right)^{\frac{1}{2}},\left(\left(\frac{12w_1-2}{15}\right)^2+\left(\frac{8w_2-3}{12}\right)^2\right)^{\frac{1}{2}}\Bigg\}$$
$$=\frac{19}{20}\max\Bigg\{\max\Big\{\Big((u_1-v_1)^2+(u_2-v_2)^2\Big)^{\frac{1}{2}},\Big((v_1-w_1)^2+(v_2-w_2)^2\Big)^{\frac{1}{2}},$$
$$\Big((w_1-u_1)^2+(w_2-u_2)^2\Big)^{\frac{1}{2}}\Big\},\left(\left(\frac{6u_1-1}{7}\right)^2+\left(\frac{8u_2-3}{9}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{18v_1-3}{20}\right)^2+\left(\frac{8v_2-3}{11}\right)^2\right)^{\frac{1}{2}},\left(\left(\frac{12w_1-2}{15}\right)^2+\left(\frac{8w_2-3}{12}\right)^2\right)^{\frac{1}{2}}\Bigg\}$$
$$=e^{-\tau}\max\{G(u,v,w),G(u,\mathfrak{f}_1(u),\mathfrak{f}_1(u)),G(v,\mathfrak{g}_1(v),\mathfrak{g}_1(v)),G(w,\mathfrak{h}_1(w),\mathfrak{h}_1(w))\}.$$

Again for $u, v, w \in Z$, we have

$$G(\mathfrak{f}_2(u),\mathfrak{g}_2(v),\mathfrak{h}_2(w))$$
$$=\max\left\{\sum_{i=1}^{2}(\mathfrak{f}_2u_i-\mathfrak{g}_2v_i)^2,\left(\sum_{i=1}^{2}(\mathfrak{g}_2v_i-\mathfrak{h}_2w_i)^2\right)^{\frac{1}{2}},\left(\sum_{i=1}^{2}(\mathfrak{h}_2w_i-\mathfrak{f}_2u_i)^2\right)^{\frac{1}{2}}\right\}$$
$$=\max\Big\{\Big((\mathfrak{f}_2u_1-\mathfrak{g}_2v_1)^2+(\mathfrak{f}_2u_2-\mathfrak{g}_2v_2)^2\Big)^{\frac{1}{2}},\Big((\mathfrak{g}_2v_1-\mathfrak{h}_2w_1)^2+(\mathfrak{g}_2v_2-\mathfrak{h}_2w_2)^2\Big)^{\frac{1}{2}},$$
$$\Big((\mathfrak{h}_2w_1-\mathfrak{f}_2u_1)^2+(\mathfrak{h}_2w_2-\mathfrak{f}_2u_2)^2\Big)^{\frac{1}{2}}\Big\}$$
$$=\max\Bigg\{\left(\left(\frac{2u_1+5}{12}-\frac{2(v_1+1)}{6}\right)^2+\left(\frac{u_2+4}{8}-\frac{2v_2+4}{9}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{2(v_1+1)}{6}-\frac{5w_1+2}{9}\right)^2+\left(\frac{2v_2+4}{9}-\frac{3w_2+4}{10}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{5w_1+2}{9}-\frac{2u_1+5}{12}\right)^2+\left(\frac{3w_2+4}{10}-\frac{u_2+4}{8}\right)^2\right)^{\frac{1}{2}}\Bigg\}$$
$$=\max\Bigg\{\left(\left(\frac{(2u_1+5)-(4v_1+4)}{12}\right)^2+\left(\frac{(9u_2+36)-(16v_2+32)}{72}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{(6v_1+3)-(10w_1+4)}{18}\right)^2+\left(\frac{(20v_2+40)-(27w_2+36)}{90}\right)^2\right)^{\frac{1}{2}},$$
$$\left(\left(\frac{(20w_1+8)-(6u_1+15)}{36}\right)^2+\left(\frac{(12w_2+16)-(5u_2+20)}{40}\right)^2\right)^{\frac{1}{2}}\Bigg\}$$
$$=\max\Bigg\{\left(\left(\frac{4v_1-2u_1-1}{12}\right)^2+\left(\frac{16v_2-9u_2-4}{72}\right)^2\right)^{\frac{1}{2}},\left(\left(\frac{10w_1-6v_1+1}{18}\right)^2\right.$$

$$+\left(\frac{27w_2-20v_2-4}{90}\right)^2\Bigg)^{\frac{1}{2}}, \left(\left(\frac{20w_1-6u_1-7}{36}\right)^2+\left(\frac{12w_2-5u_2-4}{40}\right)^2\right)^{\frac{1}{2}}\Bigg\};$$

$$\begin{aligned}
G(u,v,w) &= \max\left\{\left(\sum_{i=1}^{2}(u_i-v_i)^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(v_i-w_i)^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(w_i-u_i)^2\right)^{\frac{1}{2}}\right\} \\
&= \max\Big\{\left((u_1-v_1)^2+(u_2-v_2)^2\right)^{\frac{1}{2}}, \left((v_1-w_1)^2+(v_2-w_2)^2\right)^{\frac{1}{2}}, \\
&\quad \left((w_1-u_1)^2+(w_2-u_2)^2\right)^{\frac{1}{2}}\Big\};
\end{aligned}$$

$$\begin{aligned}
&G(u,\mathfrak{f}_2(u),\mathfrak{f}_2(u)) \\
&= \max\left\{\left(\sum_{i=1}^{2}(u_i-\mathfrak{f}_2(u_i))^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(\mathfrak{f}_2(u_i)-\mathfrak{f}_2(u_i))^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(\mathfrak{f}_2(u_i)-u_i)^2\right)^{\frac{1}{2}}\right\} \\
&= \max\Big\{\left((u_1-\mathfrak{f}_2(u_1))^2+(u_2-\mathfrak{f}_2(u_2))^2\right)^{\frac{1}{2}}, \left((\mathfrak{f}_2(u_1)-\mathfrak{f}_2(u_1))^2+(\mathfrak{f}_2(u_2)-\mathfrak{f}_2(u_2))^2\right)^{\frac{1}{2}}, \\
&\qquad \left((u_1-\mathfrak{f}_2(u_1))^2+(u_2-\mathfrak{f}_2(u_2))^2\right)\Big\}^{\frac{1}{2}} \\
&= \left((u_1-\mathfrak{f}_2(u_1))^2+(u_2-\mathfrak{f}_2(u_2))^2\right)^{\frac{1}{2}} = \left(\left(u_1-\frac{2u_1+5}{12}\right)^2+\left(u_2-\frac{u_2+4}{8}\right)^2\right)^{\frac{1}{2}} \\
&= \left(\left(\frac{10u_1-5}{12}\right)^2+\left(\frac{7u_2-4}{8}\right)^2\right)^{\frac{1}{2}};
\end{aligned}$$

$$\begin{aligned}
&G(v,\mathfrak{g}_2(v),\mathfrak{g}_2(v)) \\
&= \max\left\{\left(\sum_{i=1}^{2}(v_i-\mathfrak{g}_2(v_i))^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(\mathfrak{g}_2(v_i)-\mathfrak{g}_2(v_i))^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(\mathfrak{g}_2(v_i)-v_i)^2\right)^{\frac{1}{2}}\right\} \\
&= \max\Big\{\left((v_1-\mathfrak{g}_2(v_1))^2+(v_2-\mathfrak{g}_2(v_2))^2\right)^{\frac{1}{2}}, \Big((\mathfrak{g}_2(v_1)-\mathfrak{g}_2(v_1))^2 \\
&\qquad +(\mathfrak{g}_2(v_2)-\mathfrak{g}_2(v_2))^2\Big)^{\frac{1}{2}}, \left((v_1-\mathfrak{g}_2(v_1))^2+(v_2-\mathfrak{g}_2(v_2))^2\right)^{\frac{1}{2}} \\
&= \left((v_1-\mathfrak{g}_2(v_1))^2+(v_2-\mathfrak{g}_2(v_2))^2\right)^{\frac{1}{2}} \\
&= \left(\left(v_1-\frac{2(v_1+1)}{6}\right)^2+\left(v_2-\frac{2v_2+4}{9}\right)^2\right)^{\frac{1}{2}} = \left(\left(\frac{4v_1-2}{6}\right)^2+\left(\frac{7v_2-4}{9}\right)^2\right)^{\frac{1}{2}};
\end{aligned}$$

$$\begin{aligned}
&G(w,\mathfrak{h}_2(w),\mathfrak{h}_2(w)) \\
&= \max\left\{\left(\sum_{i=1}^{2}(w_i-\mathfrak{h}_2(w_i))^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(\mathfrak{h}_2(w_i)-\mathfrak{h}_2(w_i))^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{2}(\mathfrak{h}_2(w_i)-w_i)^2\right)^{\frac{1}{2}}\right\} \\
&= \max\Big\{\left((w_1-\mathfrak{h}_2(w_1))^2+(w_2-\mathfrak{h}_2(w_2))^2\right)^{\frac{1}{2}}, \Big((\mathfrak{h}_2(w_1)-\mathfrak{h}_2(w_1))^2 \\
&\qquad +(\mathfrak{h}_2(w_2)-\mathfrak{h}_2(w_2))^2\Big)^{\frac{1}{2}}, \left((w_1-\mathfrak{h}_2(w_1))^2+(w_2-\mathfrak{h}_2(w_2))^2\right)^{\frac{1}{2}}\Big\}
\end{aligned}$$

$$= \left((w_1 - \mathfrak{h}_2(w_1))^2 + (w_2 - \mathfrak{h}_2(w_2))^2 \right)^{\frac{1}{2}}$$
$$= \left(\left(w_1 - \frac{5w_1 + 2}{9} \right)^2 + \left(v_2 - \frac{3z_2 + 4}{10} \right)^2 \right)^{\frac{1}{2}} = \left(\left(\frac{4w_1 - 2}{9} \right)^2 + \left(\frac{7w_2 - 4}{10} \right)^2 \right)^{\frac{1}{2}}.$$

Thus, by taking $F(\lambda) = \ln(\lambda)$ for $\lambda > 0$, $\tau = \ln(\frac{20}{19})$, and for $u, v, w \in Z$ having $G(\mathfrak{f}_2(u), \mathfrak{g}_2(v), \mathfrak{h}_2(w)) > 0$, we have

$$G(\mathfrak{f}_2(u), \mathfrak{g}_2(v), \mathfrak{h}_2(w)) = \max\left\{ \left(\left(\frac{4v_1 - 2u_1 - 1}{12} \right)^2 + \left(\frac{16v_2 - 9u_2 - 4}{72} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{10w_1 - 6v_1 + 1}{18} \right)^2 + \left(\frac{27w_2 - 20v_2 - 4}{90} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{20w_1 - 6u_1 - 7}{36} \right)^2 + \left(\frac{12w_2 - 5u_2 - 4}{40} \right)^2 \right) \right\}^{\frac{1}{2}}$$
$$\leq \frac{19}{20} \max\left\{ \left((u_1 - v_1)^2 + (u_2 - v_2)^2 \right)^{\frac{1}{2}}, \left((v_1 - w_1)^2 + (v_2 - w_2)^2 \right)^{\frac{1}{2}}, \left((w_1 - u_1)^2 + (w_2 - u_2)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{10u_1 - 5}{12} \right)^2 + \left(\frac{7u_2 - 4}{8} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{4v_1 - 2}{6} \right)^2 + \left(\frac{7v_2 - 4}{9} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{4w_1 - 2}{9} \right)^2 + \left(\frac{7w_2 - 4}{10} \right)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \frac{19}{20} \max\left\{ \max\left\{ \left((u_1 - v_1)^2 + (u_2 - v_2)^2 \right)^{\frac{1}{2}}, \left((v_1 - w_1)^2 + (v_2 - w_2)^2 \right)^{\frac{1}{2}}, \left((w_1 - u_1)^2 + (w_2 - u_2)^2 \right)^{\frac{1}{2}} \right\}, \left(\left(\frac{10u_1 - 5}{12} \right)^2 + \left(\frac{7u_2 - 4}{8} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{4v_1 - 2}{6} \right)^2 + \left(\frac{7v_2 - 4}{9} \right)^2 \right)^{\frac{1}{2}}, \left(\left(\frac{4w_1 - 2}{9} \right)^2 + \left(\frac{7w_2 - 4}{10} \right)^2 \right)^{\frac{1}{2}} \right\}$$
$$= e^{-\tau} \max\{G(u, v, w), G(u, \mathfrak{f}_1(u), \mathfrak{f}_1(u)), G(v, \mathfrak{g}_1(v), \mathfrak{g}_1(v)), G(w, \mathfrak{h}_1(w), \mathfrak{h}_1(w))\}.$$

Thus, for all $u, v, w \in Z$ satisfying $G(\mathfrak{f}_k(u), \mathfrak{g}_k(v), \mathfrak{h}_k(w)) > 0$ for $k = 1, 2$, we have

$$G(\mathfrak{f}_k(u), \mathfrak{g}_k(v), \mathfrak{h}_k(w)) \leq e^{-\tau} m_{\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{h}_k}(u, v, w)$$
$$\text{where } m_{\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{h}_k}(u, v, w) = \max\{G(u, v, w), G(u, \mathfrak{f}_k(u), \mathfrak{f}_k(u)), G(v, \mathfrak{g}_k(v), \mathfrak{g}_1(v)), G(w, \mathfrak{h}_k(w), \mathfrak{h}_k(w))\}.$$

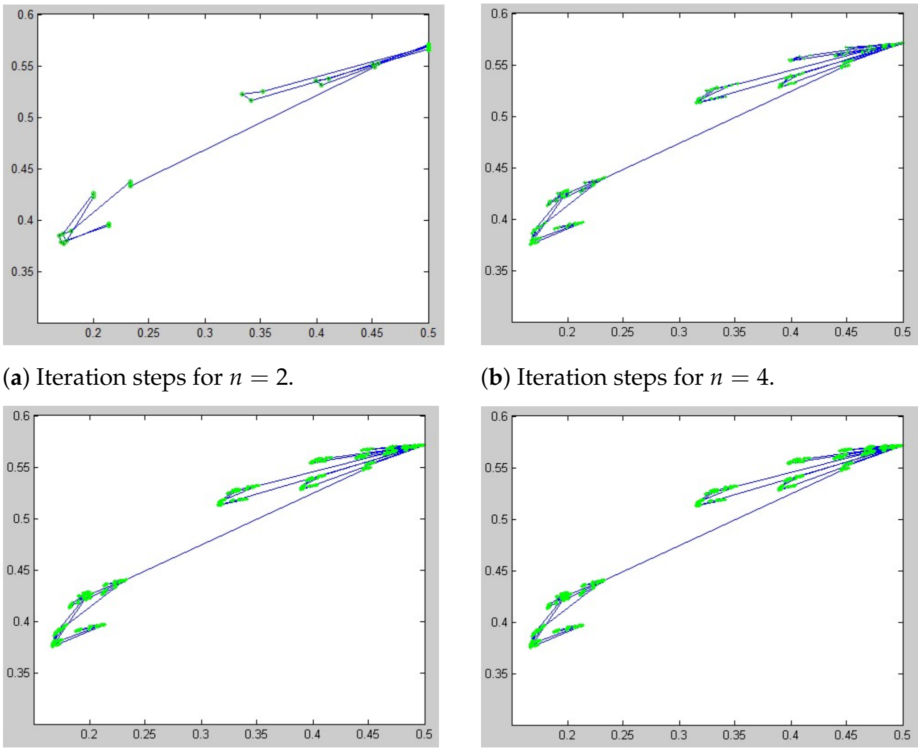
That is, $\{Z; (\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{h}_k), k = 1, 2\}$ is the generalized F-iterated function system. Now, we define the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Z) \to \mathcal{C}^G(Z)$ for all $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$ by

$$\Upsilon(\mathcal{Q}) = \mathfrak{f}_1(\mathcal{Q}) \cup \mathfrak{f}_2(\mathcal{Q}), \quad \Psi(\mathcal{R}) = \mathfrak{g}_1(\mathcal{R}) \cup \mathfrak{g}_2(\mathcal{R}), \quad \Phi(\mathcal{N}) = \mathfrak{h}_1(\mathcal{N}) \cup \mathfrak{h}_2(\mathcal{N})$$

By Proposition 2, for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Z)$ satisfying $H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) > 0$, the condition

$$H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) \leq e^{-\tau} M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N})$$
$$\text{where } M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = \max\{H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}), H_G(\mathcal{Q}, \Upsilon(\mathcal{Q}), \Upsilon(\mathcal{Q})), H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})), H_G(\mathcal{N}, \Phi(\mathcal{N}), \Phi(\mathcal{N}))\}.$$

holds. Thus, all the conditions of Theorem 2 are satisfied, and moreover, for any initial set $\mathcal{R}_0 \in \mathcal{C}^G(Y)$, the sequence $\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), \cdots\}$ of compact sets is convergent and has a limit, the common attractor of Υ, Ψ, and Φ. Figure 1 shows the convergence process of sequence steps at $n = 2, 4, 6$, and 8 in (a), (b), (c), and (d), respectively. The green points in the figures show the data points of convergence steps and the blue lines show the movements of data points in the different places.



(**a**) Iteration steps for $n = 2$. (**b**) Iteration steps for $n = 4$.

(**c**) Iteration steps for $n = 6$. (**d**) Iteration steps for $n = 8$.

Figure 1. Iteration steps of the convergence to the common attractor of Υ, Ψ, and Φ.

Example 4. *Let $\mathbf{Z} = [0,1] \times [0,1]$ and G-metric on Z be defined as*

$$G(u, v, w) = \max\left\{\sum_{i=1}^{2}|u_i - v_i|, \sum_{i=1}^{2}|v_i - w_i|, \sum_{i=1}^{2}|w_i - u_i|\right\}$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbf{Z}$. Define $\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{h}_k : \mathbf{Z} \to \mathbf{Z}, k = 1, 2$ by

$$\begin{aligned}
f_1(z_1, z_2) &= \left(\frac{z_1}{2} + \frac{1}{5}, \frac{z_2}{3} + \frac{1}{3}\right), \quad f_2(z_1, z_2) = \left(\frac{z_1}{4} + \frac{3}{10}, \frac{2z_2}{5} + \frac{3}{10}\right), \\
g_1(z_1, z_2) &= \left(\frac{1}{10}(2.5z_1 - z_2 + 3.5), \frac{1}{10}(2.5z_1 + z_2 + 3.5)\right), \\
g_2(z_1, z_2) &= \left(\frac{1}{10}(2.5z_1 + 3z_2 + 1.5), \frac{1}{10}(-2.5z_1 + 3z_2 + 4.5)\right), \\
h_1(z_1, z_2) &= \left(\frac{3z_1 + 2}{8}, \frac{3z_2 + 2}{7}\right), \quad h_2(z_1, z_2) = \left(\frac{z_1 + 2}{6}, \frac{2z_2 + 3}{8}\right).
\end{aligned}$$

The maps $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1, \mathfrak{h}_2$ are continuous and non-commutative. With $F(\lambda) = \ln(\lambda) + \lambda$ for some $\lambda > 0$ and $\tau > 0$, for $v_1, v_2, v_3 \in \mathbf{Z}$ obeying $G(\mathfrak{f}_k v_1, \mathfrak{g}_k v_2, h_k v_3) > 0$, for $k = 1, 2$,

$$G(\mathfrak{f}_k v_1, \mathfrak{g}_k v_2, h_k v_3) e^{G(\mathfrak{f}_k v_1, \mathfrak{g}_k v_2, h_k v_3) - m_{\mathfrak{f}_k, \mathfrak{g}_k, h_k}(v_1, v_2, v_3)} \le e^{-\tau} m_{\mathfrak{f}_k, \mathfrak{g}_k, h_k}(v_1, v_2, v_3),$$
$$\text{where} \quad m_{\mathfrak{f}_k, \mathfrak{g}_k, h_k}(v_1, v_2, v_3) = \max\{G(v_1, v_2, v_3), G(v_1, \mathfrak{f}_k(v_1), \mathfrak{f}_k(v_1)),$$

$$G(v_2, \mathfrak{g}_k(v_2), \mathfrak{g}_k(v_2)), G(v_3, \mathfrak{h}_k(v_3), \mathfrak{h}_k(v_3))\}$$

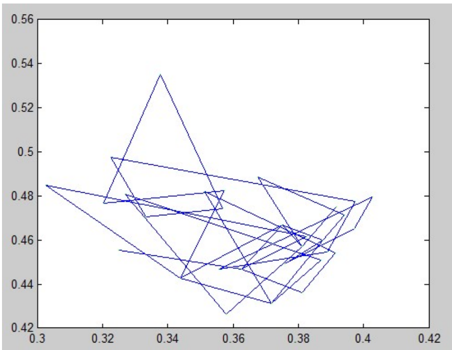
Now, from the generalized F-iterated function system $\{\mathbf{Z}; (\mathfrak{f}_1, \mathfrak{f}_2, g_1, g_2, h_1, h_2)\}$, *we define the mappings* $\Upsilon, \Psi, \Phi : \mathcal{C}^G(\mathbf{Z}) \to \mathcal{C}^G(\mathbf{Z})$ *for* $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(\mathbf{Z})$ *by*

$$\Upsilon(\mathcal{Q}) = \mathfrak{f}_1(\mathcal{Q}) \cup \mathfrak{f}_2(\mathcal{Q}), \quad \Psi(\mathcal{R}) = g_1(\mathcal{R}) \cup g_2(\mathcal{R}), \quad \Phi(\mathcal{N}) = h_1(\mathcal{N}) \cup h_2(\mathcal{N}).$$

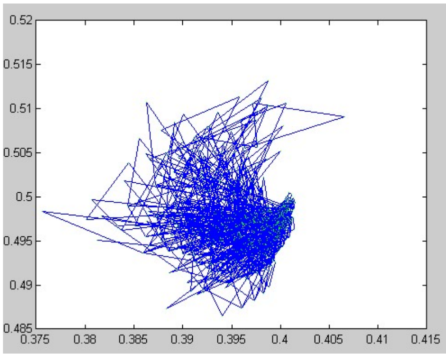
Then, for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(\mathbf{Z})$ *having* $H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) > 0$,

$$H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N}))e^{H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) - M_{\Upsilon,\Psi,\Phi}(\mathcal{Q},\mathcal{R},\mathcal{N})} \leq e^{-\tau} M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N})$$
$$\text{where } \tau > 0 \text{ and } M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = \max\{H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}), H_G(\mathcal{Q}, \Upsilon(\mathcal{Q}), \Upsilon(\mathcal{Q})), H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})), H_G(\mathcal{N}, \Phi(\mathcal{N}), \Phi(\mathcal{N}))\}.$$

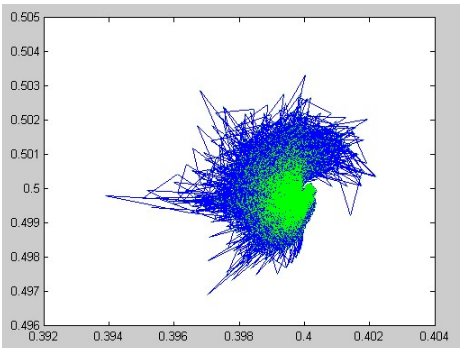
holds. Thus, all the conditions of Theorem 2 are satisfied, and moreover, for any initial set $\mathcal{R}_0 \in \mathcal{C}^G(\mathbf{Z})$, *the sequence* $\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), \cdots\}$ *of compact sets is convergent and has a limit, the common attractor of* Υ, Ψ, *and* Φ. *Figure 2 shows the convergence process of sequence steps at* $n = 2, 4, 6$, *and* 8 *in (a), (b), (c), and (d), respectively. The green points in the figures show the data points of convergence steps and the blue lines show the movements of data points in the different places.*



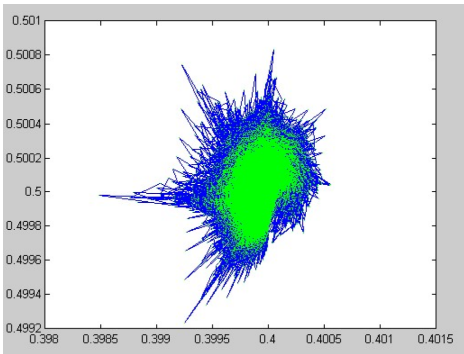
(**a**) Iteration steps for $n = 2$.



(**b**) Iteration steps for $n = 4$.



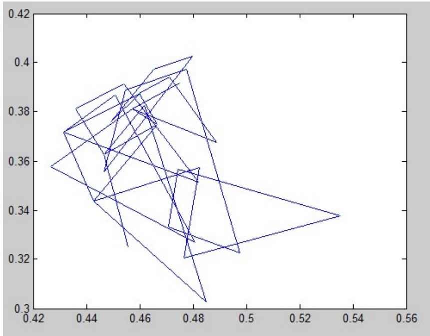
(**c**) Iteration steps for $n = 6$.



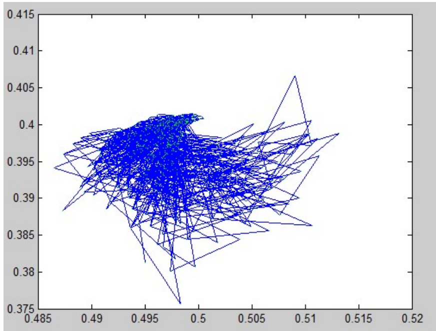
(**d**) Iteration steps for $n = 8$.

Figure 2. Iteration steps of the convergence to the common attractor of Υ, Ψ, and Φ.

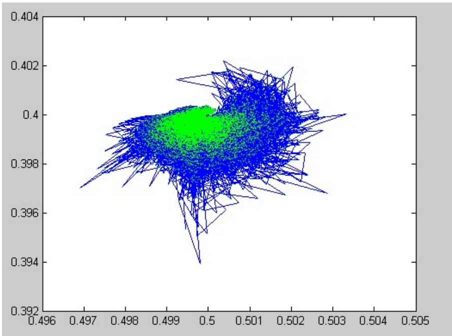
If we are interchanging the order of variables in maps, then we obtain a new form of common attractor of Υ, Ψ, *and* Φ, *see for example in Figure 3. The green points in the figures show the data points of convergence steps and the blue lines show the movements of data points in the different places.*



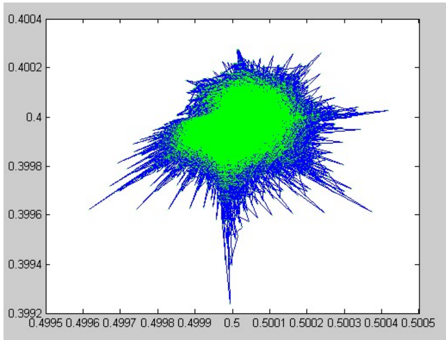
(**a**) Iteration steps for $n = 2$.



(**b**) Iteration steps for $n = 4$.



(**c**) Iteration steps for $n = 6$.



(**d**) Iteration steps for $n = 8$.

Figure 3. Iteration steps of the convergence to the common attractor of Υ, Ψ, and Φ.

Example 5. *Let $\mathbf{Z}_3 = [0,1]^3$ and the G-metric on $\mathbf{Z}_3$ is defined as*

$$G(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \max\{\sum_{i=1}^{3}|x_i - x_{i+1}|, \sum_{i=1}^{3}|y_i - y_{i+1}|, \sum_{i=1}^{3}|z_i - z_{i+1}|\} \text{ for } \mathbf{u}_i = (x_i, y_i, z_i) \in \mathbf{Z}_3$$

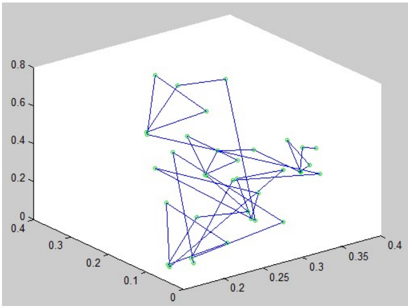
for $i \in \{1,2,3\}$, where $\mathbf{u}_4 = \mathbf{u}_1$. Define $\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{h}_k : \mathbf{Z}_3 \to \mathbf{Z}_3$, $k = 1,2$ by

$$\begin{aligned}
f_1(u_1,u_2,u_3) &= (0.8u_1 + 0.1,\ 0.8u_2 + 0.02,\ 0.8u_3 + 0.04) \text{ for } u_1,u_2,u_3 \in [0,1], \\
f_2(u_1,u_2,u_3) &= (0.5u_1 + 0.2,\ 0.3u_2 + 0.3,\ 0.5u_3 + 0.4) \text{ for } u_1,u_2,u_3 \in [0,1], \\
g_1(u_1,u_2,u_3) &= (0.35u_1 - 0.35u_2 + 0.26, 0.35u_1 + 0.35u_2 + 0.07, 0.35u_2 + 0.35u_3 + 0.76), \\
g_2(u_1,u_2,u_3) &= (0.5u_1 + 0.1,\ 0.4u_2 + 0.03,\ 0.5u_3 + 0.06) \text{ for } u_1,u_2,u_3 \in [0,1], \\
h_1(u_1,u_2,u_3) &= (0.3u_1 + 0.2,\ 0.4u_2 + 0.1,\ 0.2u_3 + 0.4) \text{ for } u_1,u_2,u_3 \in [0,1], \\
h_2(u_1,u_2,u_3) &= (0.1u_1 + 0.3,\ 0.2u_2 + 0.02,\ 0.3u_3 + 0.4) \text{ for } u_1,u_2,u_3 \in [0,1].
\end{aligned}$$

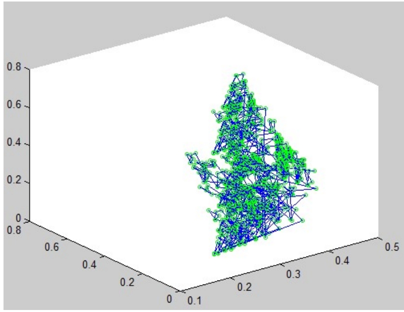
The maps $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{h}_1, \mathfrak{h}_2$ are continuous and non-commutative, and $\{\mathbf{Z}_3; (\mathfrak{f}_1, \mathfrak{f}_2, g_1, g_2, h_1, h_2)\}$ is a generalized F-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(\mathbf{Z}_3) \to \mathcal{C}^G(\mathbf{Z}_3)$ by $\Upsilon(\mathcal{Q}) = \mathfrak{f}_1(\mathcal{Q}) \cup \mathfrak{f}_2(\mathcal{Q})$, $\Psi(\mathcal{R}) = g_1(\mathcal{R}) \cup g_2(\mathcal{R})$, $\Phi(\mathcal{N}) = h_1(\mathcal{N}) \cup h_2(\mathcal{N})$ for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(\mathbf{Z}_3)$. Then for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(\mathbf{Z}_3)$ having $H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) > 0$, the condition

$$\begin{aligned}
H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N}))e^{H_G(\Upsilon(\mathcal{Q}),\Psi(\mathcal{R}),\Phi(\mathcal{N})) - M_{\Upsilon,\Psi,\Phi}(\mathcal{Q},\mathcal{R},\mathcal{N})} &\leq e^{-\tau} M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) \\
\text{where} \quad M_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = \max\{H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}), H_G(\mathcal{Q}, \Upsilon(\mathcal{Q}), \Upsilon(\mathcal{Q})),& \\
H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})), H_G(\mathcal{N}, \Phi(\mathcal{N}), \Phi(\mathcal{N}))\}.&
\end{aligned}$$

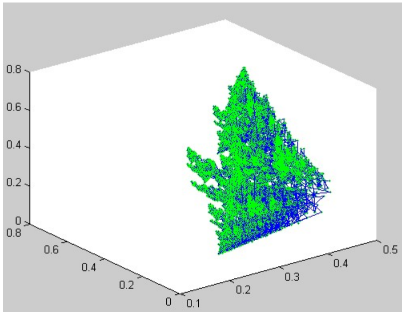
holds. Thus, all the conditions of Theorem 2 are satisfied, and moreover, for any initial set $\mathcal{R}_0 \in \mathcal{C}^G(\mathbf{Z}_3)$, the sequence $\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), \cdots\}$ of compact sets is convergent and has a limit, the common attractor of Υ, Ψ, and Φ (see Figure 4). The Figure 4 shows the convergence process of sequence steps at $n = 2, 4, 6$, and 8 in (a), (b), (c), and (d), respectively. The green points in the figures show the data points of convergence steps and the blue lines show the movements of data points in the different places.



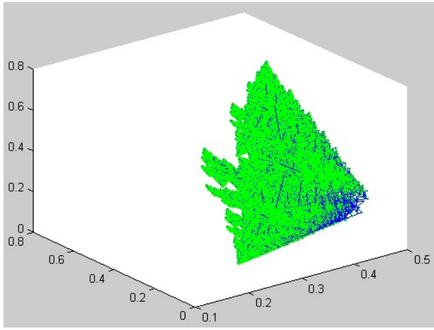
(**a**) Iteration steps for $n = 2$.



(**b**) Iteration steps for $n = 4$.



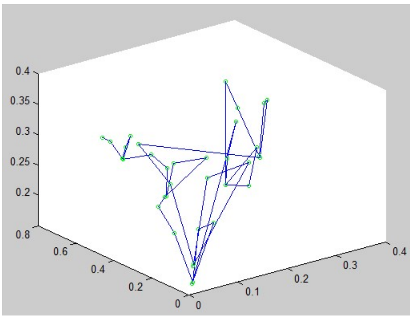
(**c**) Iteration steps for $n = 6$.



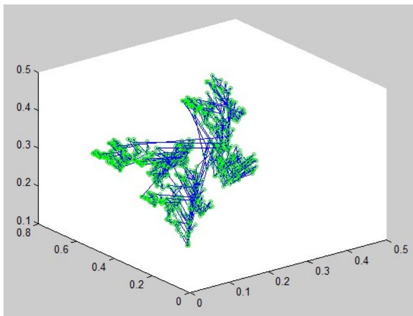
(**d**) Iteration steps for $n = 8$.

Figure 4. Iteration steps to the convergence of the common attractor of Υ, Ψ, and Φ.

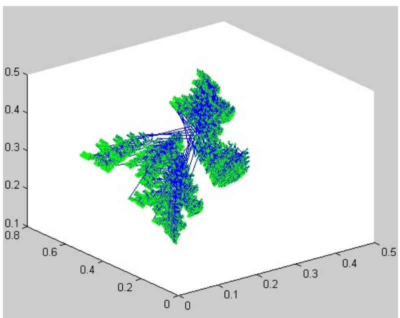
Interchanging the order of variables in maps yields a new form of common attractor of Υ, Ψ, *and* Φ *(see Figure 5). The green points in the figures show the data points of convergence steps and the blue lines show the movements of data points in the different places.*



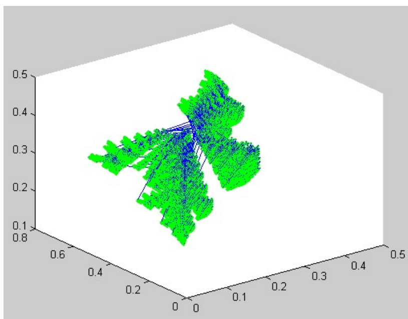
(**a**) Iteration steps for $n = 2$.



(**b**) Iteration steps for $n = 4$.



(**c**) Iteration steps for $n = 6$.



(**d**) Iteration steps for $n = 8$.

Figure 5. Iteration steps to the convergence of the common attractor of Υ, Ψ, and Φ.

3. Conclusions

In this paper, we investigated a method of a generalized F-iterated function system for common attractors based on a finite family of generalized F-contractions in G-metric spaces. We obtained the fractals as a common attractor of the generalized F-iterated function system. We showed that the triplet of F-Hutchinson operators defined by the finite number of general F-contractions on a complete G-metric space is itself a generalized F-contraction mapping. We also presented several examples in 2-D and 3-D applying our results. While the figures in the examples are for the illustration of the main results of the paper, rather than the investigation of numerical aspects of convergence of iterations or its dependence on the iterated maps, they hint that the further numerical analysis of the convergence of iterations to attractors would be an interesting direction of investigation for the generalised iterated function systems and maps considered in this paper.

Author Contributions: Conceptualization, methodology, T.N. and S.S.; writing—original draft preparation, T.N.; writing—review and editing, T.N. and S.S.; supervision, S.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors are grateful to the reviewers for their useful remarks and comments that helped to improve the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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