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Fractional Operators and Fractionally Integrated Random Fields on $\mathbb{Z}^\nu$

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Abstract: We consider fractional integral operators $(I-T)^d, d \in (-1,1)$ acting on functions $g : \mathbb{Z}^\nu \to \mathbb{R}, \nu \geq 1$, where T is the transition operator of a random walk on $\mathbb{Z}^\nu$. We obtain the sufficient and necessary conditions for the existence, invertibility, and square summability of kernels $\tau(\mathbf{s};d), \mathbf{s} \in \mathbb{Z}^\nu$ of $(I-T)^d$. The asymptotic behavior of $\tau(\mathbf{s};d)$ as $|\mathbf{s}| \to \infty$ is identified following the local limit theorem for random walks. A class of fractionally integrated random fields X on $\mathbb{Z}^\nu$ solving the difference equation $(I-T)^d X = \varepsilon$ with white noise on the right-hand side is discussed and their scaling limits. Several examples, including fractional lattice Laplace and heat operators, are studied in detail.

Keywords: fractional differentiation/integration operators; tempered fractional operators; fractional random field; random walk; limit theorems; long-range dependence; negative dependence; conditional autoregression

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1. Introduction

Classical fractional differentiation/integration operators $(I-T)^d$, $d \in (-1,1), d \neq 0$ acting on functions $g : \mathbb{Z} \to \mathbb{R}$, where $(I-T)g(t) = g(t) - g(t-1)$ is a 'discrete derivative' with respect to 'time' $t \in \mathbb{Z}$, are defined through the binomial expansion $(1-z)^d = \sum_{j=0}^{\infty} \psi_j(d) z^j$, $z \in \mathbb{C}$, $|z| < 1$, viz.:

$$(I-T)^d g(t) := \sum_{j=0}^{\infty} \psi_j(d) T^j g(t) = \sum_{j=0}^{\infty} \psi_j(d) g(t-j), \quad t \in \mathbb{Z} \tag{1}$$

with the coefficients

$$\psi_j(d) := \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}, \quad j \in \mathbb{N}. \tag{2}$$

Here, Γ denotes the gamma function $\Gamma(z) := \int_0^\infty t^{z-1} \mathrm{e}^{-t} dt, z > 0$, and $\Gamma(z) := z^{-1}\Gamma(z+1)$, $-1 < z < 0$. Also, see the end of this section for all unexplained notation. The asymptotics

$$\psi_j(d) \sim \Gamma(-d)^{-1} j^{-d-1} \quad (j \to \infty), \quad 0 < |d| < 1 \tag{3}$$

(which follows by application of Stirling's formula to (2)) determines the class of functions g and the summability properties of (1).

Fractional operators in (1) play an important role in the theory of discrete-time stochastic processes—in particular, time series (see, e.g., the monographs [1–5] and the references therein). The autoregressive fractionally integrated moving-average ARFIMA$(0,d,0)$ process $\{X(t); t \in \mathbb{Z}\}$ is defined as a stationary solution of the stochastic difference equation

$$(I-T)^d X(t) = \sum_{j=0}^{\infty} \psi_j(d) X(t-j) = \varepsilon(t), \quad t \in \mathbb{Z} \tag{4}$$

with white noise (a sequence of standardized uncorrelated random variables (r.v.s)) $\{\varepsilon(t); t \in \mathbb{Z}\}$. For $d \in (-1/2, 1/2)$, the solution of (4) is obtained by applying the inverse operator, viz.:

$$X(t) = (I-T)^{-d}\varepsilon(t) = \sum_{j=0}^{\infty} \psi_j(-d)\varepsilon(t-j), \quad t \in \mathbb{Z}. \tag{5}$$

Since (3) implies $\sum_{j=0}^{\infty} \psi_j(d)^2 < \infty$ ($|d| < 1/2$), (5) is a well-defined stationary process with zero mean and finite variance. The ARFIMA$(0,d,0)$ process is the basic parametric model in statistical inference for time series with a long memory property (also referred to as long-range dependence) (see [1–3,5,6] for a discussion of the ARFIMA$(0,d,0)$ and its generalization ARFIMA(p,d,q) models). We note that the ARFIMA$(0,d,0)$ process has an explicit covariance function and the spectral density

$$f(x) = (2\pi)^{-1}|1-\mathrm{e}^{-\mathrm{i}x}|^{-2d}, \quad x \in \Pi := [-\pi, \pi]$$

which explodes or vanishes at the origin $x = 0$ as $(2\pi)^{-1}|x|^{-2d}$, depending on the sign of d.

In this paper, we extend fractional operators in (1) to functions g on a regular ν-dimensional lattice $\mathbb{Z}^\nu, \nu \geq 1$. Whereas generalization of our construction to irregular lattices or more abstract index sets is an interesting and challenging open problem, our choice of $\mathbb{Z}^\nu$ follows the traditional approach in random field theory, which heavily relies on the Fourier transform and spectral representation. We consider a rather general form of the operator T:

$$Tg(\boldsymbol{t}) = \sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} g(\boldsymbol{t}+\boldsymbol{u})p(\boldsymbol{u}) = \mathrm{E}g(S_1+\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

where $\{S_j; j \geq 0\}$ is a random walk on $\mathbb{Z}^\nu$ starting at $S_0 = \mathbf{0}$ with (1-step) probabilities $p = \{p(\boldsymbol{u}) := \mathrm{P}(S_1 = \boldsymbol{u}); \boldsymbol{u} \in \mathbb{Z}^\nu\}$. We assume that $p(\mathbf{0}) < 1$, i.e., the random walk is non-degenerate at $\mathbf{0}$. Clearly, $T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} g(\boldsymbol{t}+\boldsymbol{u})p_j(\boldsymbol{u}) = \mathrm{E}g(S_j+\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$, where $p_j(\boldsymbol{u}) := \mathrm{P}(S_j = \boldsymbol{u}), \boldsymbol{u} \in \mathbb{Z}^\nu$ are the j-step probabilities, $j = 0,1,2,\cdots$ with $p_0(\boldsymbol{u}) = \mathbb{I}(\boldsymbol{u} = \mathbf{0})$. Similarly to (1), we define fractional operators $(I-T)^d, -1 < d < 1, d \neq 0$ acting on $g : \mathbb{Z}^\nu \to \mathbb{R}$ by

$$(I-T)^d g(\boldsymbol{t}) = \sum_{j=0}^{\infty} \psi_j(d) T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} \tau(\boldsymbol{u}; d) g(\boldsymbol{t}+\boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu$$

with coefficients

$$\tau(\boldsymbol{u}; d) := \sum_{j=0}^{\infty} \psi_j(d) p_j(\boldsymbol{u}), \tag{6}$$

expressed through the binomial coefficients $\psi_j(d)$ and random walk probabilities $p_j(\boldsymbol{u})$.

Let us describe the content and results of this paper in more detail. The main result of Section 2 is Theorem 1, which provides the sufficient condition

$$\int_{\Pi^\nu} |1-\widehat{p}(\boldsymbol{x})|^{-2|d|}\mathrm{d}\boldsymbol{x} < \infty \tag{7}$$

for invertibility $(I-T)^d(I-T)^{-d} = I$ and the square summability of fractional coefficients in (6), in terms of the characteristic function $\widehat{p}(\boldsymbol{x}) := \mathrm{E}\exp\{\mathrm{i}\langle \boldsymbol{x}, S_1\rangle\}$ (the Fourier transform) of the random walk. Section 2 also includes a discussion of the asymptotics of (6) as $|\boldsymbol{u}| \to \infty$, which is important in limit theorems and other applications of fractional integrated random fields. Using classical local limit theorems, Propositions 1 and 2 obtain 'isotropic' asymptotics of (6) for a large class of random walk $\{S_j\}$, showing that $\tau(\boldsymbol{u}; d)$ decay as $O(|\boldsymbol{u}|^{-\nu-2d})$; hence, $\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} |\tau(\boldsymbol{u}; -d)| = \infty$ ($d > 0$). The last fact is interpreted as the *long-range dependence* [3,4,7] of the fractionally integrated random field $\{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$, defined as a stationary solution of the difference equation,

$$(I-T)^d X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu \tag{8}$$

with white noise on the r.h.s. and it is studied in Section 3. Corollary 1 obtains conditions for the existence of the stationary solution of (8) given by the inverse operator $X(\boldsymbol{t}) = (I-T)^{-d}\varepsilon(\boldsymbol{t})$, which is detailed in Examples 1 and 2 for fractional Laplacian and fractional heat operators. Sections 2 and 3 also include a discussion of *tempered* fractional operators $(I-rT)^d, r \in (0,1)$ and *tempered* fractional random fields solving the analogous equation $(I-rT)^d X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t})$, which generalize the class of tempered ARFIMA processes [8] and have *short-range dependence* and a summable covariance function.

Section 4 is devoted to the scaling limits of moving-average random fields on $\mathbb{Z}^\nu$ with coefficients satisfying Assumption 1, which includes 'isotropic' fractional coefficients $\tau(\boldsymbol{u};-d)$ as a special case. The scaling limits refer to the integrals $X_\lambda(\phi) = \int_{\mathbb{R}^\nu} X([\boldsymbol{t}])\phi(\boldsymbol{t}/\lambda)\mathrm{d}\boldsymbol{t}$ of random field $\{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ for each $\phi : \mathbb{R}^\nu \to \mathbb{R}$ from a class of (test) functions as scaling parameter $\lambda \to \infty$. The scaling limits are identified in Corollary 3 as self-similar Gaussian random fields with a Hurst parameter $H = (\nu - 4d)/2$. We note that limit theorems for random fields with long-range dependence or negative dependence have been studied in many works. The seminal paper [9] dealt with noncentral limit theorems for Gaussian subordinated fields. Anisotropic scaling limits of linear and subordinated random fields in dimensions $\nu = 2,3$ were discussed in [10–16] and in the references therein, with particular focus on scaling transition arising under critical anisotropy exponents. Whereas most of the abovementioned works considered partial sums on rectangular domains, [17] studied the case of irregular summation regions and 'edge effects' arising under strong negative dependence. Statistical applications for random fields with long-range dependence were discussed in [2,18,19] and other works.

We expect that this study can be extended in several directions, including anisotropic scaling, infinite variance random fields, and fractional operators in $\mathbb{R}^\nu$ (see [20–25] for discussion and the properties of fractional random fields with the continuous argument $\boldsymbol{t} \in \mathbb{R}^\nu$).

Notation. In what follows, C denotes generic positive constants that may be different at different locations. We write $\xrightarrow{\mathrm{d}}$ and $\overset{\mathrm{d}}{=}$ for the weak convergence and equality of probability distributions. Denote by $|\cdot|$ the absolute-value norm on $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ and the Euclidean norm on $\mathbb{R}^\nu$. $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^\nu$. Denote by $\boldsymbol{e}_j$ the vector in $\mathbb{R}^\nu$ with 1 in the jth coordinate and 0's elsewhere. For $p \geq 1$, denote by $L^p(\mathbb{Z}^\nu)$ the space of functions $f : \mathbb{Z}^\nu \to \mathbb{K}$ for which $\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} |f(\boldsymbol{u})|^p < \infty$ and by $L^p(\mathbb{R}^\nu)$ the space of measurable functions $f : \mathbb{R}^\nu \mapsto \mathbb{K}$ for which the p-th power of the absolute value is integrable with respect to the Lebesgue measure $\mathrm{d}\boldsymbol{x}$ on $\mathbb{R}^\nu$: $\|f\|_{L^p(\mathbb{R}^\nu)} := (\int_{\mathbb{R}^\nu} |f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x})^{1/p} < \infty$ with the identification of functions f, g, such that $f = g$ almost everywhere (a.e.). Denote by $L^\infty(\mathbb{R}^\nu)$ the space of measurable functions $f : \mathbb{R}^\nu \to \mathbb{K}$ for which $\|f\|_{L^\infty(\mathbb{R}^\nu)} := \inf\{C \geq 0 : |f| \leq C \text{ a.e.}\} < \infty$, with the identification of functions f, g, such that $f = g$ a.e. Write $\mathbb{I}(A)$ for the indicator function of a set A. Write $\lceil x \rceil$ for the smallest integer greater than or equal to $x \in \mathbb{R}$. $\mathrm{i} := \sqrt{-1} \in \mathbb{C}, \mathbb{Z}^\nu_0 := \mathbb{Z}^\nu \setminus \{\boldsymbol{0}\}$ and $\mathbb{N} := \{0,1,2,\dots\}$.

2. Invertibility and Properties of Fractional Operators

We start with the properties of the binomial coefficients in (2):

$$\begin{aligned} &\psi_j(d) < 0 \quad (j \geq 1), \quad \sum_{j=0}^{\infty} \psi_j(d) = 0 \quad \text{if } 0 < d < 1, \\ &\psi_j(d) > 0 \quad (j \geq 1), \quad \sum_{j=0}^{\infty} \psi_j(d) = \infty \quad \text{if } -1 < d < 0. \end{aligned} \tag{9}$$

The identity $(1-z)^d(1-z)^{-d} = 1$ leads to

$$1 = \sum_{j,k=0}^{\infty} \psi_j(d)\psi_k(-d)z^{j+k} = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \psi_j(d)\psi_{n-j}(-d)$$

and the invertibility relation

$$\sum_{j=0}^{n} \psi_j(d)\psi_{n-j}(-d) = \mathbb{I}(n=0), \quad n \in \mathbb{N}. \tag{10}$$

The following lemma gives some basic properties of the fractional coefficients $\tau(\boldsymbol{u}; d)$ in (6).

Lemma 1. *(i) Let $0 < d < 1$. Then, the series in (6) converges for every $\boldsymbol{u} \in \mathbb{Z}^\nu$ and*

$$\tau(\mathbf{0}; d) > 0, \quad \tau(\boldsymbol{u}; d) \le 0 \ (\boldsymbol{u} \ne \mathbf{0}), \quad \text{and} \quad \sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} \tau(\boldsymbol{u}; d) = 0. \tag{11}$$

(ii) Let $-1 < d < 0$. Then, $0 \le \tau(\boldsymbol{u}; d) \le \infty$ for every $\boldsymbol{u} \in \mathbb{Z}^\nu$ and $\tau(\mathbf{0}; d) \ge 1$ and

$$\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} \tau(\boldsymbol{u}; d) = \infty.$$

Moreover, $\tau(\mathbf{0}; d) < \infty$ implies $\tau(\boldsymbol{u}; d) < \infty$ and

$$-\sum_{\boldsymbol{u}\ne\mathbf{0}} \tau(\boldsymbol{u}; d)\tau(-\boldsymbol{u}; -d) \le \tau(\mathbf{0}; d) < \infty. \tag{12}$$

(iii) Let $0 < d < 1$ and $\tau(\mathbf{0}; -d) < \infty$. Then,

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s}; d)\tau(\boldsymbol{t}-\boldsymbol{s}; -d) = \mathbb{I}(\boldsymbol{t}=\mathbf{0}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu. \tag{13}$$

Proof. (i) From (6) and (9) we obtain

$$\tau(\mathbf{0}; d) = 1 + \sum_{j=1}^{\infty} \psi_j(d)p_j(\mathbf{0}) > 1 + \sum_{j=1}^{\infty} \psi_j(d) = 0$$

since $p_j(\mathbf{0}) = 1 (\forall j \ge 1)$ is not possible. On the other hand, for $\boldsymbol{u} \ne \mathbf{0}$ we have $p_0(\boldsymbol{u}) = 0$ and

$$\tau(\boldsymbol{u}; d) = \sum_{j=1}^{\infty} \psi_j(d)p_j(\boldsymbol{u}) \le 0 \tag{14}$$

in view of (9).
(ii) Since $\psi_j(d)p_j(\boldsymbol{u}) \ge 0$ is obvious from (9), it suffices to show (12), since it implies $\tau(\boldsymbol{u}; d) < \infty$ by (11). We have

$$\begin{aligned}\Sigma_0 := \sum_{\boldsymbol{u}\ne\mathbf{0}} \tau(\boldsymbol{u}; d)(-\tau(-\boldsymbol{u}; -d)) &= \sum_{\boldsymbol{u}\ne\mathbf{0}} \sum_{j,k=1}^{\infty} \psi_j(d)(-\psi_k(-d))p_j(\boldsymbol{u})p_k(-\boldsymbol{u}) \\ &= \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \psi_j(d)(-\psi_{n-j}(-d)) \sum_{\boldsymbol{u}\ne\mathbf{0}} p_j(\boldsymbol{u})p_{n-j}(-\boldsymbol{u})\end{aligned}$$

where exchanging the order of summation is legitimate as all summands are non-negative. Hence, using $\sum_{\boldsymbol{u}\ne\mathbf{0}} p_j(\boldsymbol{u})p_{n-j}(-\boldsymbol{u}) \le p_n(\mathbf{0})$ and (10), we obtain

$$\begin{aligned}\Sigma_0 &\le \sum_{n=2}^{\infty} p_n(\mathbf{0}) \sum_{j=1}^{n-1} \psi_j(d)(-\psi_{n-j}(-d)) = \sum_{n=2}^{\infty} p_n(\mathbf{0})(\psi_n(d) + \psi_n(-d)) \\ &\le \sum_{n=2}^{\infty} p_n(\mathbf{0})\psi_n(d) < \tau(\mathbf{0}; d)\end{aligned}$$

proving part (ii).
(iii) The convergence of the series in (13) and the equality follow as in (12):

$$\begin{aligned}\sum_{\mathbf{s}\in\mathbb{Z}^\nu}\tau(\mathbf{s};d)\tau(\mathbf{t}-\mathbf{s};-d) &= \sum_{j,k=0}^{\infty}\psi_j(d)\psi_k(-d)\sum_{\mathbf{s}\in\mathbb{Z}^\nu}p_j(\mathbf{s})p_k(\mathbf{t}-\mathbf{s})\\ &= \sum_{n=0}^{\infty}p_n(\mathbf{t})\sum_{j=0}^{n}\psi_j(d)\psi_{n-j}(-d)\\ &= p_0(\mathbf{t}) = \mathbb{I}(\mathbf{t}=\mathbf{0}).\end{aligned}$$

Lemma 1 is proved. □

Remark 1. *Let $0<d<1$. Then, the inequalities are strict: $\tau(\mathbf{u};d)<0$ and $\tau(\mathbf{u};-d)>0$, if $p_j(\mathbf{u})>0$ for some j, i.e., $\mathbf{u}$ is accessible from state $\mathbf{0}$. Moreover, if state $\mathbf{0}$ is transient, i.e., the probability of eventual return to $\mathbf{0}$ is strictly less than 1, which is equivalent to $\sum_{j=0}^{\infty}p_j(\mathbf{0})<\infty$, then $\tau(\mathbf{0};-d)<\infty$.*

The main result of this section is Theorem 1, which provides the necessary and sufficient conditions for the square summability of the fractional coefficients in (6), in terms of the characteristic function $\widehat{p}(\mathbf{x})$ (see (7)). Write $\widehat{f}$ for the Fourier transform of a function $f:\mathbb{Z}^\nu\to\mathbb{R}$. For $r\in(0,1), d\in(-1,1)$ introduce the *tempered fractional operators*

$$(I-rT)^d g(\mathbf{t}) = \sum_{j=0}^{\infty} r^j\psi_j(d)T^j g(\mathbf{t}) = \sum_{\mathbf{u}\in\mathbb{Z}^\nu}\tau_r(\mathbf{u};d)g(\mathbf{t}+\mathbf{u}), \quad \mathbf{t}\in\mathbb{Z}^\nu$$

with coefficients

$$\tau_r(\mathbf{u};d) := \sum_{j=0}^{\infty} r^j\psi_j(d)p_j(\mathbf{u}), \tag{15}$$

and the Fourier transform $\widehat{\tau}_r(\mathbf{x};d)=(1-r\widehat{p}(\mathbf{x}))^d$.

Theorem 1. *For $-1<d<1$, the following conditions are equivalent:*

$$\int_{\Pi^\nu}|1-\widehat{p}(\mathbf{x})|^{-2|d|}\mathrm{d}\mathbf{x}<\infty, \tag{16}$$

$$\sum_{\mathbf{u}\in\mathbb{Z}^\nu}\tau(\mathbf{u};-|d|)^2<\infty. \tag{17}$$

Either of these conditions implies

$$\widehat{\tau}(\cdot;-|d|) = (1-\widehat{p}(\cdot))^{-|d|} \quad \text{in } L^2(\Pi^\nu). \tag{18}$$

Moreover, for $0<d<1$, the above conditions (16)–(18) hold with d in place of $-|d|$.

Proof. Let $0<d<1$. Firstly, we consider $\tau(\mathbf{u};d)$ in (6). They satisfy $\sum_{\mathbf{u}\in\mathbb{Z}^\nu}|\tau(\mathbf{u};d)|\le\sum_{j=0}^{\infty}|\psi_j(d)|<\infty$ because of (3) and $\sum_{\mathbf{u}\in\mathbb{Z}^\nu}p_j(\mathbf{u})=1$ with $0\le p_j(\mathbf{u})\le 1$. Then, $\sum_{\mathbf{u}\in\mathbb{Z}^\nu}\tau(\mathbf{u};d)^2<\infty$ is immediate. Moreover, we have the Fourier transform $\widehat{\tau}(\mathbf{x};d)=\sum_{j=0}^{\infty}\psi_j(d)\widehat{p}_j(\mathbf{x})$, where $\widehat{p}_j(\mathbf{x})=\widehat{p}(\mathbf{x})^j$ satisfies $|\widehat{p}(\mathbf{x})|\le 1$. We see that

$$\widehat{\tau}(\mathbf{x};d) = (1-\widehat{p}(\mathbf{x}))^d, \quad \mathbf{x}\in\Pi^\nu, \tag{19}$$

belongs to $L^2(\Pi^\nu)$.

Now let us prove the implication (16) ⇒ (17). We use approximation by the tempered fractional coefficients $\tau_r(\mathbf{u};-d)$ in (15) as $r\nearrow 1$. We ascertain that $\widehat{\tau}_r(\mathbf{x};-d)=(1-r\widehat{p}(\mathbf{x}))^{-d}\to(1-\widehat{p}(\mathbf{x}))^{-d}$ a.e. as $r\nearrow 1$. Next, for $z\in\mathbb{C}$, $|z|\le 1$, $0<r<1$ the inequality $|1-z|\le|1-rz|+|rz-z|\le|1-rz|+1-r$, where $1-r\le 1-|rz|\le|1-rz|$ becomes $|1-z|\le 2|1-rz|$. Using this, we obtain the domination for all $0<r<1$, $\mathbf{x}\in\Pi^\nu$,

$$|\widehat{\tau}_r(\mathbf{x};-d)| \le \frac{1}{|1-r\widehat{p}(\mathbf{x})|^d} \le \frac{2^d}{|1-\widehat{p}(\mathbf{x})|^d}$$

by a function in $L^2(\Pi^\nu)$ according to (16). Hence, by the dominated convergence theorem (DCT), $\widehat{\tau}_r(\cdot;-d) \to (1-\widehat{p}(\cdot))^{-d}$ as $r \nearrow 1$ holds in $L^2(\Pi^\nu)$. As a consequence, $\widehat{\tau}_r(\cdot;-d)$, $0<r<1$ is a Cauchy sequence in $L^2(\Pi^\nu)$. By Parseval's theorem, the inverse Fourier transforms,

$$\tau_r(\boldsymbol{u};-d) = \frac{1}{(2\pi)^\nu}\int_{\Pi^\nu} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{u},\boldsymbol{x}\rangle}\widehat{\tau}_r(\boldsymbol{x};-d)\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{u}\in\mathbb{Z}^\nu,\ 0<r<1,$$

are a Cauchy sequence in $L^2(\mathbb{Z}^\nu)$, and so $\tau_r(\cdot;-d)$ converges in $L^2(\mathbb{Z}^\nu)$ to some $f\in L^2(\mathbb{Z}^\nu)$ as $r\nearrow 1$. This f must be $\tau(\cdot;-d)$ because $\tau_r(\boldsymbol{u};-d)\nearrow\tau(\boldsymbol{u};-d)$ as $r\nearrow 1$ for all $\boldsymbol{u}$. We conclude that $\tau(\cdot;-d)\in L^2(\mathbb{Z}^\nu)$ or (17).

Let us turn to the implication (17) $\Rightarrow$ (16). From (17) and $\tau_r(\boldsymbol{u};-d)\nearrow\tau(\boldsymbol{u};-d)$ for all $\boldsymbol{u}$ it follows that $\tau_r(\cdot;-d)\to\tau(\cdot;-d)$ as $r\nearrow 1$ holds in $L^2(\mathbb{Z}^\nu)$. By Parseval's theorem, $\widehat{\tau}_r(\cdot;-d) = (1-r\widehat{p}(\cdot))^{-d}$, $0<r<1$ is a Cauchy sequence in $L^2(\Pi^\nu)$. It follows that $\lim_{r\nearrow 1}\int_{\Pi^\nu}|\widehat{\tau}_r(\boldsymbol{x};-d)-g(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}=0$ for some $g\in L^2(\Pi^\nu)$. We also have $\lim_{r\nearrow 1}(1-r\widehat{p}(\boldsymbol{x}))^{-d} = (1-\widehat{p}(\boldsymbol{x}))^{-d}$ for each $\boldsymbol{x}\in\Pi^\nu$, such that $\widehat{p}(\boldsymbol{x})\neq 1$. Since $\mathrm{Leb}_\nu(\boldsymbol{x}\in\Pi^\nu : \widehat{p}(\boldsymbol{x})=1)=0$ (see Lemma 2.3.2(a) in [26]) we conclude that $g(\cdot)=(1-\widehat{p}(\cdot))^{-d}$ a.e., proving (16).

The above argument also proves (18). On the one hand, $\widehat{\tau}(\cdot;-d)$ is the limit of $\widehat{\tau}_r(\cdot;-d)$ in $L^2(\Pi^\nu)$ as $r\nearrow 1$ because $\tau_r(\cdot;-d)$ converges in $L^2(\mathbb{Z}^\nu)$ to $\tau(\cdot;-d)$ as $r\nearrow 1$. On the other hand, $\widehat{\tau}_r(\cdot;-d)=(1-r\widehat{p}(\cdot))^{-d}\to(1-\widehat{p}(\cdot))^{-d}$ in $L^2(\Pi^\nu)$ as $r\nearrow 1$. We conclude that $\widehat{\tau}(\cdot;-d)=(1-\widehat{p}(\cdot))$ a.e. Theorem 1 is proved. □

Next, we turn to the asymptotics of the 'fractional coefficients' $\tau(\boldsymbol{u};d)$ in (6). The proof uses the local limit theorem in [26] for random walk probabilities $p_j(\boldsymbol{u})=\mathrm{P}(S_j=\boldsymbol{u})$. Following the latter work, we assume that

$$\mathrm{E}\mathrm{e}^{c|S_1|}<\infty \quad (\exists\, c>0) \quad \text{and } \{S_j\} \text{ is zero mean, aperiodic, irreducible.} \tag{20}$$

For example, if S_1 is symmetric, i.e., $S_1 \overset{\mathrm{d}}{=} -S_1$, and, moreover, has finite support that contains $\mathbf{0}$, $\boldsymbol{e}_i$, $i=1,\dots,\nu$, then the random walk satisfies our assumption (20). The conditions in (20) imply that the random walk has zero mean $\mathrm{E}S_1=\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu}\boldsymbol{u}p(\boldsymbol{u})=\mathbf{0}$ and an invertible covariance matrix

$$\Gamma := \mathrm{E}S_1S_1'.$$

According to the classical (integral) CLT, the normalized sum $S_j/\sqrt{j}$, $j\to\infty$ approaches a Gaussian distribution on $\mathbb{R}^\nu$ with density

$$\phi(\boldsymbol{z}) := \frac{1}{(2\pi)^{\nu/2}\sqrt{\det\Gamma}}\mathrm{e}^{-\langle \boldsymbol{z},\Gamma^{-1}\boldsymbol{z}\rangle/2}, \quad \boldsymbol{z}\in\mathbb{R}^\nu.$$

Denote

$$\bar{p}_j(\boldsymbol{u}) := \frac{1}{(2\pi j)^{\nu/2}\sqrt{\det\Gamma}}\mathrm{e}^{-\langle \boldsymbol{u},\Gamma^{-1}\boldsymbol{u}\rangle/2j}, \quad \boldsymbol{u}\in\mathbb{R}^\nu.$$

Lemma 2 ([26] Theorem 2.3.11)**.** *Under the conditions of* (20)*, there exists $C>0$, such that*

$$|p_j(\boldsymbol{u})-\bar{p}_j(\boldsymbol{u})| \le C\bar{p}_j(\boldsymbol{u})\Big(\frac{1}{j^{1/2}}+\frac{|\boldsymbol{u}|^3}{j^2}\Big), \quad \forall\, |\boldsymbol{u}|<j^2,\ \boldsymbol{u}\in\mathbb{Z}^\nu. \tag{21}$$

For 'very atypical' values $|S_j|>j$ we use the following bound ([26], Proposition 2.1.2): for any $k\ge 1$ there exists $C>0$, such that

$$\mathrm{P}(|S_j|>z\sqrt{j}) \le Cz^{-k}, \quad \forall\, z>0. \tag{22}$$

Proposition 1. *Let $p=\{p(\boldsymbol{u});\boldsymbol{u}\in\mathbb{Z}^\nu\}$ satisfy* (20)*. The coefficients in* (6) *are well-defined for any $-(1\wedge\frac{\nu}{2})<d<1, d\neq 0$ and satisfy*

$$\tau(\boldsymbol{u};d) = (B_1(d) + o(1))\langle \boldsymbol{u}, \Gamma^{-1}\boldsymbol{u}\rangle^{-(\nu/2)-d}, \quad |\boldsymbol{u}| \to \infty, \tag{23}$$

where

$$B_1(d) := \frac{2^d\Gamma(d+(\nu/2))}{\pi^{\nu/2}\Gamma(-d)\sqrt{\det\Gamma}}.$$

Proof. Let us prove (23). Since Γ is positive-definite, $|\boldsymbol{u}|_\Gamma := \sqrt{\langle \boldsymbol{u}, \Gamma^{-1}\boldsymbol{u}\rangle}$, $\boldsymbol{u} \in \mathbb{R}^\nu$ is a norm. Note that it is equivalent to the Euclidean norm because any two norms are equivalent in finite-dimensional real vector space. In particular, the spectral decomposition $\Gamma^{-1} = U\Lambda U'$—where U is an orthogonal matrix whose columns are the real, orthonormal eigenvectors of Γ^{-1}, U' is the transpose of U, and Λ is a diagonal matrix whose entries are the eigenvalues of Γ^{-1} with $\lambda_{\max}, \lambda_{\min} > 0$ denoting the largest and smallest, respectively—gives $|\boldsymbol{u}|_\Gamma = |\Lambda^{1/2}U'\boldsymbol{u}|^2 \le \lambda_{\max}|\boldsymbol{u}|^2$ and, similarly, $|\boldsymbol{u}|_\Gamma^2 \ge \lambda_{\min}|\boldsymbol{u}|^2$. Using (6) for a large $K > 0$ we decompose $|\boldsymbol{u}|_\Gamma^{\nu+2d}\tau(\boldsymbol{u};d) = \sum_{i=1}^3 J_i(\boldsymbol{u})$, where

$$J_1(\boldsymbol{u}) := |\boldsymbol{u}|_\Gamma^{\nu+2d}\Gamma(-d)^{-1}\sum_{j>|\boldsymbol{u}|_\Gamma^2/K} j^{-d-1}p_j(\boldsymbol{u}),$$
$$J_2(\boldsymbol{u}) := |\boldsymbol{u}|_\Gamma^{\nu+2d}\sum_{j>|\boldsymbol{u}|_\Gamma^2/K}(\psi_j(d) - \Gamma(-d)^{-1}j^{-d-1})p_j(\boldsymbol{u}),$$
$$J_3(\boldsymbol{u}) := |\boldsymbol{u}|_\Gamma^{\nu+2d}\sum_{0\le j\le|\boldsymbol{u}|_\Gamma^2/K}\psi_j(d)p_j(\boldsymbol{u}).$$

It suffices to show that

$$\lim_{K\to\infty}\lim_{|\boldsymbol{u}|\to\infty} J_1(\boldsymbol{u}) = B_1(d), \quad \lim_{K\to\infty}\limsup_{|\boldsymbol{u}|\to\infty} J_i(\boldsymbol{u}) = 0, \quad i = 2,3. \tag{24}$$

To show the first relation in (24), use (21). We have $J_1(\boldsymbol{u}) = J_1'(\boldsymbol{u}) + J_1''(\boldsymbol{u})$, where, for each $K > 0$ fixed, the main term $J_1'(\boldsymbol{u})$ and the remainder term $J_1''(\boldsymbol{u})$ asymptotically behave when $|\boldsymbol{u}| \to \infty$ as

$$\begin{aligned} J_1'(\boldsymbol{u}) &:= |\boldsymbol{u}|_\Gamma^{\nu+2d}\Gamma(-d)^{-1}\sum_{j>|\boldsymbol{u}|_\Gamma^2/K} j^{-d-1}\bar{p}_j(\boldsymbol{u}) \\ &= \frac{|\boldsymbol{u}|_\Gamma^{\nu+2d}}{(2\pi)^{\nu/2}\Gamma(-d)\sqrt{\det\Gamma}}\int_0^\infty \mathbb{I}(|\boldsymbol{u}|_\Gamma^2/K < [y])[y]^{-d-1-(\nu/2)}\mathrm{e}^{-|\boldsymbol{u}|_\Gamma^2/2[y]}\mathrm{d}y \\ &\sim \frac{1}{(2\pi)^{\nu/2}\Gamma(-d)\sqrt{\det\Gamma}}\int_{1/K}^\infty x^{-d-1-(\nu/2)}\mathrm{e}^{-1/2x}\mathrm{d}x \end{aligned}$$

and, for some constants $C, c > 0$,

$$\begin{aligned} |J_1''(\boldsymbol{u})| &\le C|\boldsymbol{u}|_\Gamma^{\nu+2d}K^{3/2}\sum_{j>|\boldsymbol{u}|_\Gamma^2/K} j^{-d-3/2}\bar{p}_j(\boldsymbol{u}) \\ &\le C|\boldsymbol{u}|_\Gamma^{-1}K^{3/2}\int_0^\infty x^{-d-(3/2)-(\nu/2)}\mathrm{e}^{-c/x}\mathrm{d}x = o(1). \end{aligned}$$

Hence, the first relation in (24) follows, using $\int_0^\infty x^{-1-\tau}\mathrm{e}^{-1/x}dx = \Gamma(\tau), \tau > 0$. In view of (3), the same argument also proves the second relation in (24) for $i = 2$.

Consider (24) for $i = 3$. Split $J_3(\boldsymbol{u}) = J_3''(\boldsymbol{u}) + J_3'(\boldsymbol{u})$ into two sums over $j > 0$, where $j^2 \le |\boldsymbol{u}|$ and $j^2 > |\boldsymbol{u}|$, respectively. In the sum $J_3'(\boldsymbol{u})$ we also have $j \le |\boldsymbol{u}|_\Gamma^2/K \le |\boldsymbol{u}|^2$, and Lemma 2 entails the bound

$$p_j(\boldsymbol{u}) \le C\bar{p}_j(\boldsymbol{u})(\frac{|\boldsymbol{u}|^3}{j^2}) \le C|\boldsymbol{u}|^3 j^{-(\nu/2)-2}\mathrm{e}^{-c|\boldsymbol{u}|^2/j}$$

for some constants $C, c > 0$. Hence,

$$|J_3'(\boldsymbol{u})| \le C|\boldsymbol{u}|^{\nu+2d+3} \int_0^{|\boldsymbol{u}|^2} [y]^{-d-3-(\nu/2)} \mathrm{e}^{-c|\boldsymbol{u}|^2/[y]} dy$$
$$\le C|\boldsymbol{u}|^{-1} \int_0^1 x^{-d-3-(\nu/2)} \mathrm{e}^{-c/x} \mathrm{d}x = o(1)$$

since the last integral converges for any d. Finally, by (22), given a large enough $k > 0$, there exists $C > 0$, such that $p_j(\boldsymbol{u}) \le Cj^{k/2}/|\boldsymbol{u}|^k$, which implies $J_3''(\boldsymbol{u}) = o(1)$. This proves (24) and completes the proof of Proposition 1. □

Lemma 2 does not apply to the simple random walk (which is not aperiodic), in which case the local CLT takes a somewhat different form (see [26], Theorem 2.1.3). The application of the latter result and the argument in the proof of Proposition 1 yields the following result:

Proposition 2. *Let $p(\boldsymbol{e}_j) = p(-\boldsymbol{e}_j) = \frac{1}{2\nu}$, $j = 1, \dots, \nu$. The coefficients in (6) are well-defined for any $-(1 \wedge \frac{\nu}{2}) < d < 1, d \neq 0$ and satisfy*

$$\tau(\boldsymbol{u}; d) = (B(d) + o(1))|\boldsymbol{u}|^{-\nu-2d}, \quad |\boldsymbol{u}| \to \infty,$$

where

$$B(d) := \frac{2^d \Gamma(d + (\nu/2))}{\nu^d \Gamma(-d)}.$$

Proposition 1 and Lemma 2 do not apply to random walks with a non-zero mean, as in Example 2 below (fractional heat operator), in which case the fractional coefficients exhibit an anisotropic behavior different from (23). Such behavior is described in the following proposition. We assume that the underlying random walk factorizes into a deterministic drift by 1 in direction $-\boldsymbol{e}_1$ and a random walk on $\mathbb{Z}^{\nu-1}$, as in Lemma 2:

$$p(\boldsymbol{u}) = \begin{cases} 1 - \theta, & \boldsymbol{u} = -\boldsymbol{e}_1, \\ \theta \tilde{q}(\tilde{\boldsymbol{u}}), & \boldsymbol{u} = -\boldsymbol{e}_1 + (0, \tilde{\boldsymbol{u}}), \end{cases} \tag{25}$$

where $\theta \in (0, 1)$ and $\tilde{q}(\tilde{\boldsymbol{u}})$ is a probability distribution concentrated on $\tilde{\boldsymbol{u}} = (u_2, \dots, u_\nu) \in \mathbb{Z}^{\nu-1}$, such that $\tilde{\boldsymbol{u}} \neq \boldsymbol{0}$. Write $\{S_j; j \ge 0\}$ for the random walk starting at $\boldsymbol{0}$ with j-step probabilities $\mathrm{P}(\tilde{S}_j = \tilde{\boldsymbol{u}} | \tilde{S}_0 = \boldsymbol{0}) =: \tilde{q}_j(\tilde{\boldsymbol{u}})$, $j = 0, 1, \dots$, such that $\tilde{q}_1(\tilde{\boldsymbol{u}}) := \tilde{q}(\tilde{\boldsymbol{u}})$, $\tilde{\boldsymbol{u}} \in \mathbb{Z}^{\nu-1}$. In order to apply Lemma 2, we make a similar assumption to (20):

$$\mathrm{E}\mathrm{e}^{c|\tilde{S}_1|} < \infty \quad (\exists\, c > 0) \quad \text{and } \{\tilde{S}_j\} \text{ is zero mean, irreducible} \tag{26}$$

and we denote $\tilde{\Gamma} := \mathrm{E}\tilde{S}_1\tilde{S}_1'$, the respective covariance matrix. Let

$$\rho(\boldsymbol{x}) := \big(x_1^2 + \langle \tilde{\boldsymbol{x}}, \tilde{\Gamma}^{-1}\tilde{\boldsymbol{x}} \rangle^2\big)^{1/2}, \quad \boldsymbol{x} = (x_1, \tilde{\boldsymbol{x}}) \in \mathbb{R}^\nu$$

be a positive function on $\mathbb{R}^\nu$ satisfying the homogeneity property, $\rho(\lambda x_1, \lambda^{1/2}\tilde{\boldsymbol{x}}) = \lambda \rho(\boldsymbol{x})$, $\forall \lambda > 0$. As in Example 2, the fractional coefficients for $p(\boldsymbol{u})$ in (25) we write as

$$\tau(-\boldsymbol{u}; d) = \psi_{u_1}(d) p_{u_1}(-\boldsymbol{u}) \mathbb{I}(u_1 \ge 0), \quad \boldsymbol{u} = (u_1, \tilde{\boldsymbol{u}}) \in \mathbb{Z}^\nu. \tag{27}$$

Proposition 3. *Let (26) hold and $\theta \in (0, 1)$. Then,*

$$\tau(-\boldsymbol{u}; d) = \frac{u_1^{-d-(\nu+1)/2}}{\Gamma(-d)(2\pi\theta)^{(\nu-1)/2}\sqrt{\det \tilde{\Gamma}}} \exp\Big\{ -\frac{\langle \tilde{\boldsymbol{u}}, \tilde{\Gamma}^{-1}\tilde{\boldsymbol{u}} \rangle}{2\theta u_1} \Big\} (1 + o(1)) \tag{28}$$

as $u_1 \to \infty$ and $|\tilde{\boldsymbol{u}}| \to \infty$, $|\tilde{\boldsymbol{u}}| = o(u_1^{2/3})$. We also have

$$\tau(-\boldsymbol{u};d) = \rho(\boldsymbol{u})^{-d-(\nu+1)/2}\Big(L_0\big(\frac{u_1}{\rho(\boldsymbol{u})}\big) + o(1)\Big), \quad |\boldsymbol{u}| \to \infty, \tag{29}$$

where $L_0(z), z \in [-1,1]$ is a continuous function on $[-1,1]$ given by

$$L_0(z) := \frac{z^{-d-(\nu+1)/2}}{\Gamma(-d)(2\pi\theta)^{(\nu-1)/2}\sqrt{\det\tilde{\Gamma}}} \exp\big\{-(1/2\theta)\sqrt{(1/z)^2-1}\big\}$$

for $z \in (0,1]$ and equals 0 for $z \in [-1,0]$.

Proof. Consider the following j-step probabilities of a random walk on $\mathbb{Z}^{\nu-1}$ starting at $\mathbf{0}$: $q_j(\tilde{\boldsymbol{u}}) := p_j(\boldsymbol{u})$, where $\boldsymbol{u} = (-j,\tilde{\boldsymbol{u}})$ for $\tilde{\boldsymbol{u}} \in \mathbb{Z}^{\nu-1}$, $j = 0,1,\cdots$. Let us estimate these by $\bar{q}_j(\tilde{\boldsymbol{u}}) := (2\pi j)^{-(\nu-1)/2}(\det\Gamma)^{-1/2}\exp\{-\langle\tilde{\boldsymbol{u}},\Gamma^{-1}\tilde{\boldsymbol{u}}\rangle/2j\}$, where Γ is the covariance matrix of the 1-step distribution $q_1(\tilde{\boldsymbol{u}})$, $\tilde{\boldsymbol{u}} \in \mathbb{Z}^{\nu-1}$. Note $\Gamma = \theta\tilde{\Gamma}$. By Lemma 2,

$$|q_j(\tilde{\boldsymbol{u}}) - \bar{q}_j(\tilde{\boldsymbol{u}})| \le C\bar{q}_j(\tilde{\boldsymbol{u}})\big(\frac{1}{j^{1/2}} + \frac{|\tilde{\boldsymbol{u}}|^3}{j^2}\big), \quad \forall\, |\tilde{\boldsymbol{u}}| < j^2,\ \tilde{\boldsymbol{u}} \in \mathbb{Z}^{\nu-1}. \tag{30}$$

Relation (28) follows directly from (3), (27), and (30). Relation (29) is written as

$$\rho(\boldsymbol{u})^{d+(\nu+1)/2}\tau(-\boldsymbol{u};d) - L_0\big(\frac{u_1}{\rho(\boldsymbol{u})}\big) \to 0, \quad |\boldsymbol{u}| \to \infty. \tag{31}$$

The asymptotics in (31) is immediate from (28) for $|\boldsymbol{u}|$ tending to ∞ as in (28). The general case of (31) also follows from (28), using the continuity of L_0. For $\nu = 2$, the details can be found in [12] (proof of Proposition 4.1). □

Remark 2. The approximation in (28) compares with the kernel

$$h_{c,-d}(\boldsymbol{t}) = c_1\, t_1^{-d-\frac{1+\nu}{2}} \exp\big\{-ct_1 - \frac{|\tilde{\boldsymbol{t}}|^2}{4t_1}\big\}\mathbb{I}(t_1 > 0), \quad \boldsymbol{t} = (t_1,\tilde{\boldsymbol{t}}) \in \mathbb{R}^\nu \tag{32}$$

of the fractional heat operator $(c + \partial_1 - \widetilde{\Delta})^{-d}, \partial_1 - \widetilde{\Delta} := \partial/\partial t_1 - \sum_{i=2}^{\nu}\partial^2/\partial t_i^2$ for all $c > 0$, $d < 0$, and some $c_1 \in \mathbb{R}$. For $\nu = 2$, Ref. [25] Equation (3.7) has recently derived the analytic form in (32) of the kernel from the absolute square of its Fourier transform:

$$\begin{aligned} |\widehat{h}_{c,-d}(\boldsymbol{z})|^2 &= \Big|\int_{\mathbb{R}^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{z},\boldsymbol{t}\rangle} h_{c,-d}(\boldsymbol{t})\mathrm{d}\boldsymbol{t}\Big|^2 \\ &= c_1^2(4\pi)^{\nu-1}\Gamma(-d)^2\big(z_1^2 + (c+|\tilde{z}|^2)^2\big)^d, \quad \boldsymbol{z} = (z_1,\tilde{\boldsymbol{z}}) \in \mathbb{R}^\nu, \end{aligned} \tag{33}$$

which is the implicit definition of this kernel in [22]. Similarly to derivations in [25], for $\nu \ge 2$, Equations (3.944.5-6) in the table of integrals [27] give

$$\begin{aligned} \widehat{h}_{c,-d}(\boldsymbol{z}) &= c_1 \int_0^\infty \mathrm{e}^{\mathrm{i}z_1t_1 - ct_1} t_1^{d-\frac{1+\nu}{2}}\mathrm{d}t_1 \int_{\mathbb{R}^{\nu-1}} \exp\big\{\mathrm{i}\langle\tilde{\boldsymbol{z}},\tilde{\boldsymbol{t}}\rangle - \frac{|\tilde{\boldsymbol{t}}|^2}{4t_1}\big\}\mathrm{d}\tilde{\boldsymbol{t}} \\ &= c_1(4\pi)^{\frac{\nu-1}{2}} \int_0^\infty \mathrm{e}^{\mathrm{i}z_1t_1 - t_1(c+|\tilde{z}|^2)} t_1^{-d-1}\mathrm{d}t_1 \\ &= c_1(4\pi)^{\frac{\nu-1}{2}}\Gamma(-d)\big(z_1^2 + (c+|\tilde{z}|^2)^2\big)^{\frac{d}{2}} \exp\big\{-\mathrm{i}d\arctan\big(\frac{z_1}{c+|\tilde{z}|^2}\big)\big\}, \end{aligned}$$

yielding (33).

Finally, the tempered fractional coefficients in (15) are summable: $\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu}|\tau_r(\boldsymbol{u};d)| \le \sum_{j=0}^\infty r^j|\psi_j(d)| \le 2(1-r)^{-|d|} < \infty$ for any $d \in (-1,1), r \in (0,1)$ and *any* random walk $\{S_j\}$. Assuming the existence of the exponential moment $\mathrm{E}e^{\kappa|S_1|} < \infty$ for some $\kappa > 0$, (15) decays exponentially,

$$|\tau_r(\boldsymbol{u};d)| \le C e^{-c|\boldsymbol{u}|}, \quad \boldsymbol{u} \in \mathbb{Z}^\nu, \tag{34}$$

for some $C, c > 0$. Indeed, Markov's inequality gives $r^j|\psi_j(d)|p_j(\boldsymbol{u}) \le \mathrm{P}(|S_j| \ge |\boldsymbol{u}|) \le e^{-\kappa|\boldsymbol{u}|}\mathrm{E}e^{\kappa|S_j|} \le e^{-\kappa|\boldsymbol{u}|}(\mathrm{E}e^{\kappa|S_1|})^j \le e^{-(\kappa/2)|\boldsymbol{u}|}$ for any $0 \le j < c|\boldsymbol{u}|$ and large enough $|\boldsymbol{u}|$. Moreover, $\sum_{j \ge c|\boldsymbol{u}|} r^j|\psi_j(d)|p_j(\boldsymbol{u}) \le \sum_{j \ge c|\boldsymbol{u}|} r^j = r^{c|\boldsymbol{u}|}/(1-r)$, proving (34).

3. Fractionally Integrated Random Fields on $\mathbb{Z}^\nu$

Let $\{\varepsilon(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ be a white noise; in other words, a sequence of r.v.s with $\mathrm{E}\varepsilon(\boldsymbol{t}) = 0$, $\mathrm{E}\varepsilon(\boldsymbol{t})\varepsilon(\boldsymbol{s}) = \mathbb{I}(\boldsymbol{t} = \boldsymbol{s}), \boldsymbol{t}, \boldsymbol{s} \in \mathbb{Z}^\nu$. Given a sequence $a \in L^2(\mathbb{Z}^\nu)$ with the above noise we can associate a moving-average random field (RF),

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} a(\boldsymbol{u})\varepsilon(\boldsymbol{t} - \boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu \tag{35}$$

with zero mean and covariance $\mathrm{Cov}(X(\boldsymbol{t}), X(\boldsymbol{s})) = \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} a(\boldsymbol{u})a(\boldsymbol{t} - \boldsymbol{s} + \boldsymbol{u})$, which depends on $\boldsymbol{t} - \boldsymbol{s}$ alone and characterizes the dependence between values of X at distinct points $\boldsymbol{t} \ne \boldsymbol{s}$.

A moving-average RF X in (35) will be said to be

- long-range dependent (LRD) if $\sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} |a(\boldsymbol{u})| = \infty$;
- short-range dependent (SRD) if $\sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} |a(\boldsymbol{u})| < \infty$, $\sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} a(\boldsymbol{u}) \ne 0$;
- negatively dependent (ND) if $\sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} |a(\boldsymbol{u})| < \infty, \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} a(\boldsymbol{u}) = 0$.

The above classification is important in limit theorems and applications of random fields. It is not unanimous; several related but not equivalent classifications of dependence for stochastic processes can be found in [3,4,7,17] and other works.

Many RF models with discrete arguments are defined through linear difference equations involving white noise [28]. In this paper, we deal with fractionally integrated RFs X solving fractional equations on $\mathbb{Z}^\nu$,

$$(I - T)^d X(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} \tau(\boldsymbol{s}; d)X(\boldsymbol{t} + \boldsymbol{s}) = \varepsilon(\boldsymbol{t}), \tag{36}$$

$$(I - rT)^d X(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} \tau_r(\boldsymbol{s}; d)X(\boldsymbol{t} + \boldsymbol{s}) = \varepsilon(\boldsymbol{t}), \quad 0 < r < 1, \qquad \boldsymbol{t} \in \mathbb{Z}^\nu, \tag{37}$$

whose solutions are obtained by inverting these operators (see below).

Definition 1. *Let $d \in (-1, 1)$ and $\tau(\boldsymbol{u}; \pm d)$ in (6) be well-defined. By the stationary solution of equation (36) (respectively, (37)) we mean a stationary RF X, such that for each $\boldsymbol{t} \in \mathbb{Z}^\nu$ the series in (36) converges in mean square and (36) holds (respectively, the series in (37) converges in mean square and (37) holds).*

Corollary 1. *(i) Let $-1 < d < 1$. Then,*

$$X(\boldsymbol{t}) = (I - T)^{-d}\varepsilon(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} \tau(\boldsymbol{u}; -d)\varepsilon(\boldsymbol{t} + \boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu \tag{38}$$

is a stationary solution of equation (36) if condition (16) holds (for $0 < d < 1$, (16) is also necessary for the existence of the above X).
(ii) Let $0 < d < 1$ and (16) hold. Then, X in (38) is LRD. Moreover, it has a non-negative covariance function $\mathrm{Cov}(X(\boldsymbol{0}), X(\boldsymbol{t})) \ge 0$, and $\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} \mathrm{Cov}(X(\boldsymbol{0}), X(\boldsymbol{t})) = \infty$.
(iii) Let $-1 < d < 0$ and (16) hold. Then, X in (38) is ND; moreover, $\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} \mathrm{Cov}(X(\boldsymbol{0}), X(\boldsymbol{t})) = 0$.
(iv) Let $-1 < d < 1, 0 < r < 1$. Then,

$$X(\boldsymbol{t}) = (I - rT)^{-d}\varepsilon(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \mathbb{Z}^\nu} \tau_r(\boldsymbol{u}; -d)\varepsilon(\boldsymbol{t} + \boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu \tag{39}$$

is a stationary solution of equation (37). Moreover, X in (39) is SRD. Furthermore, $\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} |\mathrm{Cov}(X(\boldsymbol{0}), X(\boldsymbol{t}))| < \infty$, $\sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} \mathrm{Cov}(X(\boldsymbol{0}), X(\boldsymbol{t})) = (1 - r)^{-2d} > 0$.

Proof. (i) Let $0 < d < 1$. X in (38) is well-defined if and only if (17) holds, which is, therefore, a necessary condition. Let us show that X in (38) is a stationary solution of (36). We use the spectral representation of white noise,

$$\varepsilon(\boldsymbol{t}) = \int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle} Z(\mathrm{d}\boldsymbol{x}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu, \tag{40}$$

where $Z(\mathrm{d}\boldsymbol{x})$ is a random complex-valued spectral measure on Π^ν with zero mean and variance $\mathrm{E}|Z(\mathrm{d}\boldsymbol{x})|^2 = \mathrm{d}\boldsymbol{x}/(2\pi)^\nu$. Then, $X(\boldsymbol{t})$ is written as

$$X(\boldsymbol{t}) = \int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle} \widehat{\tau}(\boldsymbol{x} \, - d) Z(\mathrm{d}\boldsymbol{x}) = \int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle} \frac{Z(\mathrm{d}\boldsymbol{x})}{(1-\widehat{p}(\boldsymbol{x}))^d} \tag{41}$$

see (18). Then, $(I-T)^d X(\boldsymbol{t}) = \int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle} \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s};d) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle} (1-\widehat{p}(\boldsymbol{x}))^{-d} Z(\mathrm{d}\boldsymbol{x}) = \varepsilon(\boldsymbol{t})$ follows by (19) and absolute summability $\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} |\tau(\boldsymbol{s};d)| < \infty$ (see (11) and (14)).

Next, let $-1 < d < 0$. Then, X in (38) is well-defined and is written as (41), due to $\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} |\tau(\boldsymbol{s};-d)| < \infty$. We need to show that the series in (36) converges in mean square towards $\varepsilon(\boldsymbol{t})$ if and only if (16) or (17) hold. The latter convergence writes as

$$\lim_{M\to\infty} \mathrm{E}|s_M - \varepsilon(\boldsymbol{t})|^2 = 0, \quad \text{where} \quad s_M := \sum_{|\boldsymbol{s}|\le M} \tau(\boldsymbol{s};d) X(\boldsymbol{t}+\boldsymbol{s}).$$

From (41),

$$\begin{aligned}
\mathrm{E}|s_M - \varepsilon(\boldsymbol{t})|^2 &= (2\pi)^{-\nu} \int_{\Pi^\nu} \Big| \sum_{|\boldsymbol{s}|\le M} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle} \tau(\boldsymbol{s};d) - (1-\widehat{p}(\boldsymbol{x}))^d \Big|^2 |1-\widehat{p}(\boldsymbol{x})|^{2|d|} \mathrm{d}\boldsymbol{x} \\
&\le C \int_{\Pi^\nu} \Big| \sum_{|\boldsymbol{s}|\le M} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle} \tau(\boldsymbol{s};d) - (1-\widehat{p}(\boldsymbol{x}))^d \Big|^2 \mathrm{d}\boldsymbol{x} \\
&= C \int_{\Pi^\nu} \Big| \sum_{|\boldsymbol{s}|> M} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle} \tau(\boldsymbol{s};d) \Big|^2 \mathrm{d}\boldsymbol{x} \\
&= C \sum_{|\boldsymbol{s}|> M} \tau(\boldsymbol{s};-|d|)^2 \ \to\ 0 \quad (M\to\infty)
\end{aligned}$$

in view of (17). This proves part (i).
(ii) From (9), (6) we see $\tau(\boldsymbol{s};-d) \ge 0$ are non-negative and $\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s};-d) = \sum_{j=0}^\infty \psi_j(-d) = \infty$. Thus, $\mathrm{Cov}(X(\mathbf{0}), X(\boldsymbol{t})) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s};-d)\tau(\boldsymbol{t}+\boldsymbol{s};-d) \ge 0$ and $\sum_{\boldsymbol{t}\in\mathbb{Z}^\nu} \mathrm{Cov}(X(\mathbf{0}), X(\boldsymbol{t})) = \infty$.
(iii) As in the proof of (i), we obtain $\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} |\tau(\boldsymbol{s};-d)| \le 1 + \sum_{j=1}^\infty \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} |\psi_j(-d)| p_j(\boldsymbol{s}) = 1 + \sum_{j=1}^\infty |\psi_j(-d)| = 2$ (see (9)) and $\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s};-d) = 0$, implying $\sum_{\boldsymbol{t}\in\mathbb{Z}^\nu} \mathrm{Cov}(X(\mathbf{0}), X(\boldsymbol{t})) = \sum_{\boldsymbol{t},\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s};-d)\tau(\boldsymbol{t}+\boldsymbol{s};-d) = 0$.
(iv) Using $\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} |\tau_r(\boldsymbol{u};d)| < \infty$, $\sum_{\boldsymbol{u}\in\mathbb{Z}^\nu} \tau_r(\boldsymbol{u};d) = \sum_{j=0}^\infty r^j \psi_j(d) = (1-r)^d$, the proof is similar to the above. Corollary 1 is proved. □

The ARFIMA(0,d,0) Equation (4) is autoregressive, since the best linear predictor (or conditional expectation in the Gaussian case) of $X(t)$, given the 'past' $X(s), s < t$, is a linear combination $\sum_{j=1}^\infty \psi_j(d) X(t-j)$ of the 'past' observations, due to the fact that $\mathrm{Cov}(X(s), \varepsilon(t)) = 0 \, (s < t)$. For spatial equations, as in (36) or (37), an analogous property given the 'past' $X(\boldsymbol{s}), \boldsymbol{s} \ne \boldsymbol{t}$ does not hold, since $\mathrm{Cov}(X(\boldsymbol{s}), \varepsilon(\boldsymbol{t})) \ne 0 \, (\boldsymbol{s} \ne \boldsymbol{t})$ as a rule. This issue is important in spatial statistics and has been discussed in the literature (see [29,30] and the references therein), distinguishing between 'simultaneous' and 'conditional autoregressive schemes'. A recent work [31] discusses some conditional autoregressive models with LRD property.

Definition 2. *Let X be an RF with $\mathrm{E}X(\boldsymbol{t})^2 < \infty$ for each $\boldsymbol{t} \in \mathbb{Z}^\nu$. We say that X has:*
(i) a simultaneous autoregressive representation with coefficients $b(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{Z}^\nu_0$ if for each $\boldsymbol{t} \in \mathbb{Z}^\nu$

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}_0^\nu} b(\boldsymbol{s})X(\boldsymbol{t}-\boldsymbol{s}) + \xi(\boldsymbol{t}),$$

where the series converges in mean square and the r.v.s $\xi(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$ *satisfy* $\mathrm{Cov}(\xi(\boldsymbol{t}), \xi(\boldsymbol{s})) = 0\,(\forall\, \boldsymbol{s} \neq \boldsymbol{t})$.

(ii) a conditional autoregressive representation with coefficients $c(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{Z}_0^\nu$ *if for each* $\boldsymbol{t} \in \mathbb{Z}^\nu$

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}_0^\nu} c(\boldsymbol{s})X(\boldsymbol{t}-\boldsymbol{s}) + \eta(\boldsymbol{t}), \tag{42}$$

where the series converges in mean square and the r.v.s $\eta(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$ *satisfy* $\mathrm{Cov}(\eta(\boldsymbol{t}), X(\boldsymbol{s})) = 0\,(\forall\, \boldsymbol{s} \neq \boldsymbol{t})$.

Corollary 2. *(i) Let* $d \in (-1,1)$ *and* X *be a fractionally integrated RF in* (38) *and* (16) *holds. Then,* X *has a simultaneous autoregressive representation with coefficients* $b(\boldsymbol{s}) = -\tau(-\boldsymbol{s};d)/\tau(\mathbf{0};d)$, $\boldsymbol{s} \in \mathbb{Z}_0^\nu$ *and* $\xi(\boldsymbol{s}) = \varepsilon(\boldsymbol{s})/\tau(\mathbf{0};d)$, $\boldsymbol{s} \in \mathbb{Z}^\nu$;
(ii) Let $d \in (0,1)$, X *be a fractionally integrated RF in* (38) *and* (16) *holds. Then,* X *has a conditional autoregressive representation with coefficients* $c(\boldsymbol{s}) = -\gamma^*(\boldsymbol{s})/\gamma^*(\mathbf{0}), \boldsymbol{s} \in \mathbb{Z}_0^\nu$ *and* $\eta(\boldsymbol{s}) = \int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle}(1-\widehat{p}(-\boldsymbol{x}))^d Z(\mathrm{d}\boldsymbol{x})/\gamma^*(\mathbf{0})$, *where* $Z(\mathrm{d}\boldsymbol{x})$ *is a complex-valued random measure given in* (40) *with zero mean and variance* $\mathrm{E}|Z(\mathrm{d}\boldsymbol{x})|^2 = \mathrm{d}\boldsymbol{x}/(2\pi)^\nu$ *and*

$$\gamma^*(\boldsymbol{s}) := \frac{1}{(2\pi)^\nu}\int_{\Pi^\nu} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle}|1-\widehat{p}(\boldsymbol{x})|^{2d}\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{s} \in \mathbb{Z}^\nu;$$

(iii) Let $d \in (-1,1), 0 < r < 1$ *and* X *be a (tempered) fractionally integrated RF in* (39)*. Then,* X *has a simultaneous autoregressive representation with* $b(\boldsymbol{s}) = -\tau_r(-\boldsymbol{s};d)/\tau_r(\mathbf{0};d), \xi(\boldsymbol{t}) = \varepsilon(\boldsymbol{t})/\tau_r(\mathbf{0};d)$ *and a conditional autoregressive representation with* $c(\boldsymbol{s}) = -\gamma_r^*(\boldsymbol{s})/\gamma_r^*(\mathbf{0}), \eta(\boldsymbol{t}) = \int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle}(1-r\widehat{p}(-\boldsymbol{x}))^d Z(\mathrm{d}\boldsymbol{x})/\gamma_r^*(\mathbf{0})$, *with the same* $Z(\mathrm{d}\boldsymbol{x})$ *as in part (ii) and*

$$\gamma_r^*(\boldsymbol{s}) := \frac{1}{(2\pi)^\nu}\int_{\Pi^\nu} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{s},\boldsymbol{x}\rangle}|1-r\widehat{p}(\boldsymbol{x})|^{2d}\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{s} \in \mathbb{Z}^\nu.$$

Proof. (i) is obvious from Corollary 1 and (36), $\tau(\mathbf{0};d) \neq 0$.
(ii) By (16), $c(\boldsymbol{s})$ and $\eta(\boldsymbol{t})$ are well-defined, $\eta(\boldsymbol{t}) \in \mathbb{R}$ and $\mathrm{E}\eta(\boldsymbol{t})^2 < \infty$. The orthogonality relation $\mathrm{E}X(\boldsymbol{t})\eta(\boldsymbol{s}) = 0\,(\boldsymbol{t} \neq \boldsymbol{s})$ follows from the spectral representations in (40) and (41):

$$\begin{aligned}\mathrm{E}X(\boldsymbol{t})\eta(\boldsymbol{s}) &= \frac{1}{(2\pi)^\nu\gamma^*(\mathbf{0})}\int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t}-\boldsymbol{s},\boldsymbol{x}\rangle}\frac{\overline{(1-\widehat{p}(-\boldsymbol{x}))^d}}{(1-\widehat{p}(\boldsymbol{x}))^d}\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{(2\pi)^\nu\gamma^*(\mathbf{0})}\int_{\Pi^\nu} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t}-\boldsymbol{s},\boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x} = 0 \quad (\boldsymbol{t} \neq \boldsymbol{s}).\end{aligned}$$

It remains to show (42), including the convergence of the series. In view of the definition of $c(\boldsymbol{s})$, this amounts to showing

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} X(\boldsymbol{t}-\boldsymbol{s})\gamma^*(\boldsymbol{s}) = \gamma^*(\mathbf{0})\eta(\boldsymbol{t})$$

or, in spectral terms, to the convergence of the Fourier series

$$\frac{1}{(1-\widehat{p}(\boldsymbol{x}))^d}\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{x},\boldsymbol{s}\rangle}\gamma^*(\boldsymbol{s}) = (1-\widehat{p}(-\boldsymbol{x}))^d = \frac{|1-\widehat{p}(-\boldsymbol{x})|^{2d}}{(1-\widehat{p}(\boldsymbol{x}))^d} \tag{43}$$

in $L^2(\Pi^\nu)$. Note $\gamma^*(\boldsymbol{s}) = \mathrm{Cov}(X^*(\mathbf{0}), X^*(\boldsymbol{s}))$, where the RF $X^*(\boldsymbol{t}) := (1-T)^d\varepsilon(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$, results from application of the inverse operator. Since X^* has negative dependence (see (41) and the proof of Corollary 1 (iii)) the covariances $\gamma^*(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{Z}^\nu$ are absolutely summable. Therefore, the Fourier series on the l.h.s. of (43) converges uniformly in $\boldsymbol{x} \in \Pi^\nu$ to $|1-\widehat{p}(-\boldsymbol{x})|^{2d}$, proving (43).

(iii) The proof is analogous to (and simpler than) (i)–(ii), using $\sum_{u\in\mathbb{Z}^\nu} |\tau_r(u; d)| < \infty$. □

Example 1. *Fractional Laplacian. The (lattice) Laplace operator on $\mathbb{Z}^\nu$ is defined as*

$$[\Delta]g(\boldsymbol{t}) := \frac{1}{2\nu}\sum_{j=1}^{\nu}(g(\boldsymbol{t}+\boldsymbol{e}_j) + g(\boldsymbol{t}-\boldsymbol{e}_j) - 2g(\boldsymbol{t})), \qquad \boldsymbol{t}\in\mathbb{Z}^\nu$$

so that $[\Delta] = T - I$, where $Tg(\boldsymbol{t}) = \frac{1}{2\nu}\sum_{j=1}^{\nu}(g(\boldsymbol{t}+\boldsymbol{e}_j) + g(\boldsymbol{t}-\boldsymbol{e}_j))$ is the transition operator of the simple random walk $\{S_j; j = 0, 1, \cdots\}$ on $\mathbb{Z}^\nu$ with equal one-step transition probabilities $1/2\nu$ to the nearest-neighbors $\boldsymbol{t} \to \boldsymbol{t} \pm \boldsymbol{e}_j, j = 1, \cdots, \nu$. For $-1 < d < 1$, the fractional Laplace RF can be defined as a stationary solution of the difference equation

$$(-[\Delta])^d X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}), \quad \boldsymbol{t}\in\mathbb{Z}^\nu \tag{44}$$

with weak white noise on the r.h.s., written as a moving-average RF:

$$X(\boldsymbol{t}) = (-[\Delta])^{-d}\varepsilon(\boldsymbol{t}) = \sum_{\boldsymbol{u}\in\mathbb{Z}^\nu}\tau(\boldsymbol{u}; -d)\varepsilon(\boldsymbol{t}+\boldsymbol{u}). \tag{45}$$

We find that $\widehat{p}(\boldsymbol{x}) = (1/\nu)\sum_{j=1}^{\nu}\cos(x_j)$, $\boldsymbol{x} = (x_1, \cdots, x_\nu) \in \Pi^\nu$ and

$$1 - \widehat{p}(\boldsymbol{x}) = \frac{1}{\nu}\sum_{j=1}^{\nu}(1-\cos(x_j)) \ge C|\boldsymbol{x}|^2$$

for some $C > 0$ and $1 - \widehat{p}(\boldsymbol{x}) \sim (1/2\nu)|\boldsymbol{x}|^2$ $(|\boldsymbol{x}| \to 0)$. Hence, condition (16) for (44) translates to

$$\int_{\Pi^\nu}\frac{\mathrm{d}\boldsymbol{x}}{|1-\widehat{p}(\boldsymbol{x})|^{2|d|}} < \infty \quad\Longleftrightarrow\quad |d| < \frac{\nu}{4}.$$

In particular, a stationary solution of Equation (44) on $\nu \ge 4$ exists for all $-1 < d < 1$. Finally, recall that (16) is equivalent to condition (17). We could have verified the latter by using Corollary 2, which gives the asymptotics of coefficients $\tau(\boldsymbol{u}; -d)$ in (45).

Example 2. *Fractional heat operator. For a parameter $0 < \theta < 1$, we can extend the definition of the (lattice) heat operator on $\mathbb{Z}^\nu$ from $\nu = 2$ in [12] to $\nu \ge 2$ as follows:*

$$\begin{aligned}\Delta_{1,2}g(\boldsymbol{t}) &:= (1-\theta)(g(\boldsymbol{t}) - g(\boldsymbol{t}-\boldsymbol{e}_1)) \\ &\quad - \frac{\theta}{2(\nu-1)}\sum_{j=2}^{\nu}(g(\boldsymbol{t}-\boldsymbol{e}_1+\boldsymbol{e}_j) + g(\boldsymbol{t}-\boldsymbol{e}_1-\boldsymbol{e}_j) - 2g(\boldsymbol{t})).\end{aligned}$$

Thus, $\Delta_{1,2} = I - T$ corresponds to the random walk on $\mathbb{Z}^\nu$ with 1-step distribution $p(-\boldsymbol{e}_1) = 1-\theta$, $p(-\boldsymbol{e}_1 \pm \boldsymbol{e}_j) = \frac{\theta}{2(\nu-1)}$, $j = 2, \cdots, \nu$. We find that

$$|1-\widehat{p}(\boldsymbol{x})|^2 = \big(\cos(x_1) - 1 + \frac{\theta}{\nu-1}\sum_{j=2}^{\nu}(1-\cos(x_j))\big)^2 + \sin^2(x_1), \quad \boldsymbol{x} = (x_1, \cdots, x_\nu) \in \Pi^\nu.$$

By the Taylor expansion,

$$|1-\widehat{p}(\boldsymbol{x})|^2 \sim \big(\frac{\theta}{2(\nu-1)}\big)^2|\tilde{\boldsymbol{x}}|^4 + x_1^2, \quad \boldsymbol{x}\to\boldsymbol{0}, \quad \tilde{\boldsymbol{x}} := (0, x_2, \cdots, x_\nu).$$

We also find that outside the origin $|1-\widehat{p}(\boldsymbol{x})|^2 \ge C$ for some $C > 0$ since $0 < \theta < 1$. Therefore,

$$\int_{\Pi^\nu}\frac{\mathrm{d}\boldsymbol{x}}{|1-\widehat{p}(\boldsymbol{x})|^{2|d|}} \le C\int_0^1\int_0^1\frac{y^{\nu-2}\mathrm{d}x\mathrm{d}y}{(x^2+y^4)^{|d|}} < \infty \quad \text{if } |d| < \frac{\nu+1}{4}$$

and $\int_{\Pi^\nu} |1 - \widehat{p}(\boldsymbol{x})|^{-2|d|} \mathrm{d}\boldsymbol{x} = \infty$ *if* $|d| \geq \frac{\nu+1}{4}$*. The above result agrees with [12] for* $\nu = 2$, $0 < d < \frac{3}{4}$ *and extends it to the arbitrary* $\nu \geq 2$, $-1 < d < 1$.

Example 3. *Fractionally integrated time series models (case* $\nu = 1$*). As noted above, the ARFIMA*$(0, d, 0)$ *process is a particular case of* (38) *corresponding to the backward shift* $Tg(t) := g(t-1)$ *or the deterministic random walk* $t \to t-1$*. Another fractionally integrated time series model is given in Example 1 and corresponds to the symmetric nearest-neighbor random walk on* $\mathbb{Z}$ *with probabilities 1/2. It is of interest to compare these two processes and their properties. Let* $T_1 g(t) := g(t-1), T_2 g(t) := (1/2)(g(t+1) + g(t-1)), t \in \mathbb{Z}$ *be the corresponding operators,*

$$X_1(t) := (I - T_1)^{-d_1} \varepsilon(t) = \sum_{u=0}^{\infty} \psi_u(-d_1) \varepsilon(t-u),$$
$$X_2(t) := (I - T_2)^{-d_2} \varepsilon(t) = \sum_{u \in \mathbb{Z}} \tau(u; -d_2) \varepsilon(t+u), \quad t \in \mathbb{Z}.$$

For $|d_1| < 1/2$ *and* $|d_2| < 1/4$*, processes* X_1 *and* X_2 *are well-defined; moreover, they are stationary solutions of the respective equations* $(I - T_1)^{d_1} X(t) = \varepsilon(t)$ *and* $(I - T_2)^{d_2} X(t) = \varepsilon(t)$*. The spectral densities of* X_1 *and* X_2 *are given by*

$$f_1(x) = \frac{1}{2\pi |1 - \mathrm{e}^{-\mathrm{i}x}|^{2d_1}} = \frac{1}{2\pi \cdot 2^{d_1} |1 - \cos(x)|^{d_1}},$$
$$f_2(x) = \frac{1}{2\pi |1 - (1/2)(\mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{\mathrm{i}x})|^{2d_2}} = \frac{1}{2\pi |1 - \cos(x)|^{2d_2}}$$

We see that when $d_1 = 2d_2$ *the processes* X_1 *and* X_2 *have the same 2nd order properties up to a multiplicative constant, so that in the Gaussian case* X_2 *is a noncausal representation of the ARFIMA*$(0, 2d_2, 0)$.

4. Scaling Limits

As explained in the Introduction, the isotropic scaling limits refer to the limit distribution of the integrals

$$X_\lambda(\phi) := \int_{\mathbb{R}^\nu} X([\boldsymbol{t}]) \phi(\boldsymbol{t}/\lambda) \mathrm{d}\boldsymbol{t}, \quad \text{as } \lambda \to \infty, \tag{46}$$

where $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ is a given stationary random field (RF) for each $\phi : \mathbb{R}^\nu \to \mathbb{R}$ from a class of (test) functions Φ. We choose the latter class to be

$$\Phi := L^1(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu).$$

In the following, X is a linear or moving-average RF on $\mathbb{Z}^\nu$:

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} a(\boldsymbol{t} - \boldsymbol{s}) \varepsilon(\boldsymbol{s}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu, \tag{47}$$

where $\{\varepsilon(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ are independent identically distributed (i.i.d.) r.v.s, with $\mathrm{E}\varepsilon(\boldsymbol{t}) = 0$, $\mathrm{E}\varepsilon(\boldsymbol{t})^2 = 1$, and $a \in L^2(\mathbb{Z}^\nu)$ being deterministic coefficients. Obviously, stationary solution (38) of Equation (36) satisfying Corollary 1 is a particular case of linear RF with $a(\boldsymbol{t}) = \tau(-\boldsymbol{t}; -d)$. Our limits results assume an 'isotropic' behavior of $a(\boldsymbol{t})$ as $|\boldsymbol{t}| \to \infty$, detailed as follows. Let $C(\mathbb{S}_{\nu-1})$ denote the class of all continuous functions on $\mathbb{S}_{\nu-1} = \{\boldsymbol{t} \in \mathbb{R}^\nu : |\boldsymbol{t}| = 1\}$.

Assumption 1. *Let* $\{a(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^\nu\}$ *be a sequence of real numbers satisfying the following properties:*
(i) Let $0 < d < \nu/4$*. Then,*

$$a(\boldsymbol{t}) = \frac{1}{|\boldsymbol{t}|^{\nu - 2d}} \Big(\ell\Big(\frac{\boldsymbol{t}}{|\boldsymbol{t}|}\Big) + o(1) \Big), \quad |\boldsymbol{t}| \to \infty, \tag{48}$$

where $\ell(\cdot) \in C(\mathbb{S}_{\nu-1})$ is not identically zero.
(ii) Let $-\nu/4 < d < 0$. Then, $a(\mathbf{t})$ satisfies (48) with the same $\ell(\mathbf{t})$ and, moreover, $\sum_{\mathbf{t}\in\mathbb{Z}^\nu} a(\mathbf{t}) = 0$.
(iii) Let $d = 0$. Then, $\sum_{\mathbf{t}\in\mathbb{Z}^\nu} |a(\mathbf{t})| < \infty$ and $\sum_{\mathbf{t}\in\mathbb{Z}^\nu} a(\mathbf{t}) \neq 0$.

The class of RFs in (47) with coefficients satisfying Assumption 1 is related but not limited to the fractionally integrated RFs in (36) and (37). Note that the parameter d is no longer restricted to being in $(-1, 1)$. By easy observation, Assumption 1 implies the LRD, ND, and SRD properties of Section 3 in the respective cases $d > 0, d < 0$, and $d = 0$. Following the terminology in time series [3], the parameter d in (48) may be called the *memory parameter* of the linear RF X in (47), except that for $\nu = 1$ the memory parameter is usually defined as $2d \in (-1/2, 1/2)$.

In particular, the covariance function $r(\mathbf{t}) := \mathrm{Cov}(X(\mathbf{0}), X(\mathbf{t}))$ of the linear RF X in (47) is written as

$$r(\mathbf{t}) = \sum_{\mathbf{u}\in\mathbb{Z}^\nu} a(\mathbf{u})a(\mathbf{t}+\mathbf{u}), \quad \mathbf{t} \in \mathbb{Z}^\nu$$

or the lattice convolution of $a(\mathbf{t})$ with itself. We will use the notation $[a_1 \star a_2]$ for the lattice convolution and $(a_1 \star a_2)$ for continuous convolution, viz.:

$$[a_1 \star a_2](\mathbf{t}) := \sum_{\mathbf{u}\in\mathbb{Z}^\nu} a_1(\mathbf{u})a_2(\mathbf{t}+\mathbf{u}), \quad \mathbf{t} \in \mathbb{Z}^\nu,$$

$$(a_1 \star a_2)(\mathbf{t}) := \int_{\mathbb{R}^\nu} a_1(\mathbf{u})a_2(\mathbf{t}+\mathbf{u})\mathrm{d}\mathbf{u}, \quad \mathbf{t} \in \mathbb{R}^\nu$$

which is well-defined for any $a_i \in L^2(\mathbb{Z}^\nu), i = 1, 2$ (respectively, for any $a_i \in L^2(\mathbb{R}^\nu)$, $i = 1, 2$).

Proposition 4. *Let $a_i \in L^2(\mathbb{Z}^\nu)$ satisfy Assumption 1 with $0 < d < \nu/4$ and some $\ell_i \in C(\mathbb{S}_{\nu-1}), i = 1, 2$. Then,*

$$[a_1 \star a_2](\mathbf{t}) = |\mathbf{t}|^{4d-\nu}\big(L_{12}\big(\frac{\mathbf{t}}{|\mathbf{t}|}\big) + o(1)\big), \quad |\mathbf{t}| \to \infty, \tag{49}$$

where the (angular) function $L_{12}(\cdot) \in C(\mathbb{S}_{\nu-1})$ is given by

$$L_{12}(\mathbf{t}) := \int_{\mathbb{R}^\nu} \frac{\ell_1(\mathbf{s}/|\mathbf{s}|)\ell_2((\mathbf{t}-\mathbf{s})/|\mathbf{t}-\mathbf{s}|)}{|\mathbf{s}|^{\nu-2d}|\mathbf{t}-\mathbf{s}|^{\nu-2d}}\mathrm{d}\mathbf{s}, \quad \mathbf{t} \in \mathbb{S}_{\nu-1}.$$

Proof. The existence and continuity of L_{12} follow from the finiteness of the integrals $\int_{|\mathbf{s}|<1} |\mathbf{s}|^{2d-\nu}\mathrm{d}\mathbf{s} < \infty$, $\int_{|\mathbf{s}|>1} |\mathbf{s}|^{2(2d-\nu)}\mathrm{d}\mathbf{s} < \infty$. For (49), it suffices to show that

$$|\mathbf{t}|^{\nu-4d}[a_1 \star a_2](\mathbf{t}) - L_{12}(\mathbf{t}/|\mathbf{t}|) \to 0, \quad |\mathbf{t}| \to \infty. \tag{50}$$

Let $|\mathbf{t}|_+ := |\mathbf{t}| \vee 1$ and $a_i^0(\mathbf{t}) := |\mathbf{t}|_+^{2d-\nu}\ell_i(\mathbf{t}/|\mathbf{t}|_+), a_i^1(\mathbf{t}) := a_i(\mathbf{t}) - a_i^0(\mathbf{t}) = o(|\mathbf{t}|^{2d-\nu})$, $i = 1, 2$ (see (48)). Then, $[a_1 \star a_2](\mathbf{t}) = \sum_{i,j=0}^1 [a_1^i \star a_2^j](\mathbf{t})$. Clearly, (50) follows from

$$|\mathbf{t}|^{\nu-4d}[a_1^0 \star a_2^0](\mathbf{t}) - L_{12}(\mathbf{t}/|\mathbf{t}|) \to 0, \quad |\mathbf{t}| \to \infty \tag{51}$$

and

$$[a_1^i \star a_2^j](\mathbf{t}) = o(|\mathbf{t}|^{4d-\nu}), \quad |\mathbf{t}| \to \infty, \quad (i,j) \neq (0,0),\ i,j = 0,1. \tag{52}$$

To prove (51), rewrite $[a_1^0 \star a_2^0](\mathbf{t}) = \int_{\mathbb{R}^\nu} a_1^0([\mathbf{u}])a_2^0(\mathbf{t}+[\mathbf{u}])\mathrm{d}\mathbf{u}$ as an integral and change the variable $\mathbf{u} \to |\mathbf{t}|\mathbf{u}$ in it. This leads to $|\mathbf{t}|^{\nu-4d}[a_1^0 \star a_2^0](\mathbf{t}) = \widetilde{L}_{\mathbf{t}}(\mathbf{t}/|\mathbf{t}|)$, where

$$\widetilde{L}_{\mathbf{t}}(\mathbf{z}) := \int_{\mathbb{R}^\nu} a_{1,\mathbf{t}}(\tilde{\mathbf{u}})a_{2,\mathbf{t}}(\mathbf{z}+\tilde{\mathbf{u}})\mathrm{d}\mathbf{u}, \quad \mathbf{z} \in \mathbb{S}_{\nu-1}, \tag{53}$$

where
$$a_{i,\boldsymbol{t}}(\tilde{\boldsymbol{u}}) := \frac{1}{(|\boldsymbol{t}|^{-1} \vee |\tilde{\boldsymbol{u}}|)^{\nu-2d}} \ell_i\Big(\frac{\tilde{\boldsymbol{u}}}{|\boldsymbol{t}|^{-1} \vee |\tilde{\boldsymbol{u}}|}\Big), \quad \tilde{\boldsymbol{u}} := \frac{[|\boldsymbol{t}|\boldsymbol{u}]}{|\boldsymbol{t}|}.$$

Relation (51) follows once we prove the uniform convergence $\sup_{\boldsymbol{z}\in\mathbb{S}_{\nu-1}} |\widetilde{L}_{\boldsymbol{t}}(\boldsymbol{z}) - L_{12}(\boldsymbol{z})| \to 0$ $(|\boldsymbol{t}| \to \infty)$. Since $\mathbb{S}_{\nu-1}$ is a compact set and L_{12} is continuous, the last relation is implied by the sequential convergence

$$|\widetilde{L}_{\boldsymbol{t}}(\boldsymbol{z}_{\boldsymbol{t}}) - L_{12}(\boldsymbol{z})| \to 0 \quad (|\boldsymbol{t}| \to \infty) \tag{54}$$

for any $\boldsymbol{z} \in \mathbb{S}_{\nu-1}$ and any $\{\boldsymbol{z}_{\boldsymbol{t}}\}$ convergent to $\boldsymbol{z}$: $|\boldsymbol{z}_{\boldsymbol{t}} - \boldsymbol{z}| \to 0$ $(|\boldsymbol{t}| \to \infty)$. The proof of (54) uses the bound
$$|a_{i,\boldsymbol{t}}(\tilde{\boldsymbol{u}})| \le C|\boldsymbol{u}|^{2d-\nu}, \quad \boldsymbol{u} \in \mathbb{R}^\nu, \ i = 1,2, \tag{55}$$
which follows from the boundedness of ℓ_i and $|\boldsymbol{u}| \le |\tilde{\boldsymbol{u}}| + |\boldsymbol{u} - \tilde{\boldsymbol{u}}|$ with $|\boldsymbol{u} - \tilde{\boldsymbol{u}}| \le \nu^{1/2}/|\boldsymbol{t}|$; hence, $|\boldsymbol{u}| \le \nu^{1/2}(|\tilde{\boldsymbol{u}}| + |\boldsymbol{t}|^{-1}) \le 2\nu^{1/2}(|\tilde{\boldsymbol{u}}| \vee |\boldsymbol{t}|^{-1})$. Note $a_{1,\boldsymbol{t}}(\tilde{\boldsymbol{u}})a_{2,\boldsymbol{t}}(\boldsymbol{z}+\tilde{\boldsymbol{u}}) \to a_1^0(\boldsymbol{u})a_2^0(\boldsymbol{z}+\boldsymbol{u})$ $(|\boldsymbol{t}| \to \infty)$ for any $\boldsymbol{u} \ne \boldsymbol{0}, \boldsymbol{z}$ and $|a_{1,\boldsymbol{t}}(\tilde{\boldsymbol{u}})a_{2,\boldsymbol{t}}(\boldsymbol{z}+\tilde{\boldsymbol{u}})| \le C|\boldsymbol{u}|^{2d-\nu}|\boldsymbol{z}+\boldsymbol{u}|^{2d-\nu}$ according to (55). Since $h(\boldsymbol{u}) := C|\boldsymbol{u}|^{2d-\nu}|\boldsymbol{z}+\boldsymbol{u}|^{2d-\nu}$ does not depend on $\boldsymbol{t}$ and $\int_{\mathbb{R}^\nu} h(\boldsymbol{u})\mathrm{d}\boldsymbol{u} < \infty$, Pratt's lemma [32] applies to the integral in (53), resulting in (54) and (51). The proof of (52) is similar and simpler and is omitted. □

The question about the asymptotics of the variance of (46) arises, assuming the power-law asymptotics of the covariance admitting power-law behavior at large lags, which is tackled in the following proposition:

Proposition 5. *(i) For any $\beta > 0, \phi_i \in \Phi, i = 1,2$ as $\lambda \to \infty$*

$$\int_{\mathbb{R}^{2\nu}} |\phi_1(\boldsymbol{t}_1/\lambda)\phi_2(\boldsymbol{t}_2/\lambda)|(1 \wedge |\boldsymbol{t}_1 - \boldsymbol{t}_2|^{-\beta})\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2 = \begin{cases} O(\lambda^\nu), & \beta > \nu, \\ O(\lambda^{2\nu-\beta}), & \beta < \nu, \\ O(\lambda^\nu \log \lambda), & \beta = \nu. \end{cases} \tag{56}$$

(ii) Let $r(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{Z}^\nu$ satisfy

$$r(\boldsymbol{t}) = |\boldsymbol{t}|^{4d-\nu}\big(L\big(\frac{\boldsymbol{t}}{|\boldsymbol{t}|}\big) + o(1)\big), \quad |\boldsymbol{t}| \to \infty, \tag{57}$$

where $0 < d < \nu/4$ and $L \in C(\mathbb{S}_{\nu-1})$. Then, for any $\phi_i \in \Phi, i = 1,2$

$$\lim_{\lambda\to\infty} \lambda^{-\nu-4d} \int_{\mathbb{R}^{2\nu}} \phi_1(\boldsymbol{t}_1/\lambda)\phi_2(\boldsymbol{t}_2/\lambda)r([\boldsymbol{t}_1] - [\boldsymbol{t}_2])\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2 = c(\phi_1,\phi_2), \tag{58}$$

where
$$c(\phi_1,\phi_2) := \int_{\mathbb{R}^{2\nu}} \phi_1(\boldsymbol{t}_1)\phi_2(\boldsymbol{t}_2)L\big(\frac{\boldsymbol{t}_1 - \boldsymbol{t}_2}{|\boldsymbol{t}_1 - \boldsymbol{t}_2|}\big)\frac{\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2}{|\boldsymbol{t}_1 - \boldsymbol{t}_2|^{\nu-4d}}. \tag{59}$$

(iii) Let $r \in L^1(\mathbb{Z}^\nu)$. Then, for any $\phi_i \in \Phi, i = 1,2$,

$$\lim_{\lambda\to\infty} \lambda^{-\nu} \int_{\mathbb{R}^{2\nu}} \phi_1(\boldsymbol{t}_1/\lambda)\phi_2(\boldsymbol{t}_2/\lambda)r([\boldsymbol{t}_1] - [\boldsymbol{t}_2])\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2 = \int_{\mathbb{R}^\nu} \phi_1(\boldsymbol{t})\phi_2(\boldsymbol{t})\mathrm{d}\boldsymbol{t} \times \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} r(\boldsymbol{s}). \tag{60}$$

Proof. (i) Write $I_{\lambda,\beta}$ for the l.h.s. of (56). First, let $\beta > \nu$. Then, $I_{\lambda,\beta} \le C\int_{\mathbb{R}^\nu} |\phi_1(\boldsymbol{t}_1/\lambda)|\mathrm{d}\boldsymbol{t}_1 \times \int_{\mathbb{R}^\nu} 1 \wedge |\boldsymbol{t}_2 - \boldsymbol{t}_1|^{-\beta}\mathrm{d}\boldsymbol{t}_2 \le C\int_{\mathbb{R}^\nu} |\phi_1(\boldsymbol{t}_1/\lambda)|\mathrm{d}\boldsymbol{t}_1 = C\lambda^\nu \int_{\mathbb{R}^\nu} |\phi_1(\boldsymbol{t})|\mathrm{d}\boldsymbol{t} = O(\lambda^\nu)$ as $\int_{\mathbb{R}^\nu} 1 \wedge |\boldsymbol{t}|^{-\beta}\mathrm{d}\boldsymbol{t} < \infty$. Next, let $\beta < \nu$; then, $I_{\lambda,\beta} \le \lambda^{2\nu-\beta}J_\beta$, where $J_\beta := \int_{\mathbb{R}^{2\nu}} |\phi_1(\boldsymbol{t}_1)\phi_2(\boldsymbol{t}_2)||\boldsymbol{t}_1 - \boldsymbol{t}_2|^{-\beta}\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2 < \infty$ is followed by $J_\beta \le C\int_{\mathbb{R}^\nu} |\phi_1(\boldsymbol{t}_1)|\mathrm{d}\boldsymbol{t}_1 \int_{|\boldsymbol{t}_2-\boldsymbol{t}_1|\le 1} |\boldsymbol{t}_2 - \boldsymbol{t}_1|^{-\beta}\mathrm{d}\boldsymbol{t}_2 + \int_{\mathbb{R}^{2\nu}} |\phi_1(\boldsymbol{t}_1) \times \phi_2(\boldsymbol{t}_2)|\mathrm{d}\boldsymbol{t}\mathrm{d}\boldsymbol{t}_2 < \infty$. Finally, for $\beta = \nu$ we have $I_{\lambda,\nu} = \lambda^\nu J_{\lambda,\nu}$, where $J_{\lambda,\nu} := \int_{\mathbb{R}^{2\nu}} |\phi_1(\boldsymbol{t}_1) \times \phi_2(\boldsymbol{t}_2)|(\lambda^{-1} \vee |\boldsymbol{t}_1 - \boldsymbol{t}_2|)^{-\nu}\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2 = O(\log \lambda)$ follows similarly.

(ii) The convergence of the integral in (59) follows from that of J_β in part (i), with $\beta = \nu - 4d$. Let $c_\lambda(\phi_1, \phi_2)$ denote the integral on the l.h.s. of (58). By a change of variables,

$$\frac{c_\lambda(\phi_1, \phi_2)}{\lambda^{\nu+4d}} = \int_{\mathbb{R}^{2\nu}} \frac{\phi(\boldsymbol{t}_1)\phi(\boldsymbol{t}_2)}{|\boldsymbol{t}_1 - \boldsymbol{t}_2|^{\nu-4d}} \tilde{L}_\lambda(\boldsymbol{t}_1, \boldsymbol{t}_2)\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2,$$

where $\tilde{L}_\lambda(\boldsymbol{t}_1, \boldsymbol{t}_2) \to L((\boldsymbol{t}_1 - \boldsymbol{t}_2)/|\boldsymbol{t}_1 - \boldsymbol{t}_2|)\ (\lambda \to \infty)$ for any $\boldsymbol{t}_1 \neq \boldsymbol{t}_2$. Using Pratt's lemma [32], it suffices to prove (58) for $L \equiv 1$. In the latter case, and with $\tilde{\boldsymbol{t}}_i := [\lambda \boldsymbol{t}_i]/\lambda, i = 1, 2$, we see that $|\tilde{L}_\lambda(\boldsymbol{t}_1, \boldsymbol{t}_2)| \le C\big(|\boldsymbol{t}_1 - \boldsymbol{t}_2|/(|\tilde{\boldsymbol{t}}_1 - \tilde{\boldsymbol{t}}_2| \vee (1/\lambda))\big)^{\nu-4d} \le C$ as in the proof of Proposition 4. Thus, (58) follows from the DCT.
(iii) Let $c_\lambda(\phi_1, \phi_2)$ be the same as in the proof of (ii). For a large $K > 0$, write $c_\lambda(\phi_1, \phi_2) = \sum_{i=1}^3 c_{i,\lambda}$, where $c_{3,\lambda} := \int_{|\boldsymbol{t}_1 - \boldsymbol{t}_2| \le K} \phi_1(\boldsymbol{t}_1/\lambda)(\phi_2(\boldsymbol{t}_2/\lambda) - \phi_2(\boldsymbol{t}_1/\lambda))r([\boldsymbol{t}_1] - [\boldsymbol{t}_2])\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2$, and $c_{2,\lambda} := \int_{|\boldsymbol{t}_1 - \boldsymbol{t}_2| \le K} \phi_1(\boldsymbol{t}_1/\lambda)\phi_2(\boldsymbol{t}_1/\lambda)r([\boldsymbol{t}_1] - [\boldsymbol{t}_2])\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2$, and $c_{1,\lambda} := \int_{|\boldsymbol{t}_1 - \boldsymbol{t}_2| > K} \phi_1(\boldsymbol{t}_1/\lambda) \times \phi_2(\boldsymbol{t}_2/\lambda)r([\boldsymbol{t}_1] - [\boldsymbol{t}_2])\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2$. Here, $\lambda^{-\nu}|c_{1,K}| \le C\lambda^{-\nu}\int_{\mathbb{R}^\nu} |\phi_1(\boldsymbol{t}/\lambda)|\mathrm{d}\boldsymbol{t} \times \sum_{|\boldsymbol{s}|>K} |r(\boldsymbol{s})| \le C\sum_{|\boldsymbol{s}|>K} |r(\boldsymbol{s})|$ can be made arbitrarily small uniformly in $\lambda \ge 1$ by choosing K large enough. Next,

$$\lambda^{-\nu}|c_{3,\lambda}| \le C\int_{\mathbb{R}^\nu} |\phi_1(\boldsymbol{t})|\mathrm{d}\boldsymbol{t} \int_{|\boldsymbol{s}| \le K} \big|\phi_2\big(\boldsymbol{t} + \frac{\boldsymbol{s}}{\lambda}\big) - \phi_2(\boldsymbol{t})\big|\mathrm{d}\boldsymbol{s}.$$

By the boundedness of ϕ_2, we see that the integral $\int_{|\boldsymbol{s}| \le K} |\phi_2(\boldsymbol{t} + \frac{\boldsymbol{s}}{\lambda}) - \phi_2(\boldsymbol{t})|\mathrm{d}\boldsymbol{s} \to 0\ (\lambda \to \infty)$ a.e. in $\mathbb{R}^\nu$, and is bounded in $\boldsymbol{t} \in \mathbb{R}^\nu$. Then, since $\phi_1 \in L^1(\mathbb{R}^\nu)$ we conclude $\lim_{\lambda\to\infty} \lambda^{-\nu}|c_{3,\lambda}| = 0$ by the DCT. Finally, $\lambda^{-\nu}c_{2,\lambda} = \int_{\mathbb{R}^\nu} \phi_1(\boldsymbol{t})\phi_2(\boldsymbol{t})\mathrm{d}\boldsymbol{t} \int_{|\boldsymbol{s}+[\lambda\boldsymbol{t}]-\lambda\boldsymbol{t}| \le K} r(-[\boldsymbol{s}])\mathrm{d}\boldsymbol{s}$, and we can replace the last integral by the r.h.s. of (60) uniformly in λ provided K is large enough. □

Proposition 5 does not apply to ND covariances satisfying (57) with negative $d < 0$. This case is more delicate, since it requires additional regularity conditions of the test functions and the occurrence of 'edge effects'. A detailed analysis of this issue in dimension $\nu = 2$ and for indicator (test) functions of rectangles in $\mathbb{R}^2_+$ can be found in [16]. Below, we present a result in this direction and sufficient conditions on $d, \phi_i, i = 1, 2$ when the limits take a similar form to (58). We introduce a subclass of test functions:

$$\Phi_- := \Big\{\phi \in \Phi : \int_{\mathbb{R}^\nu} \big(\int_{\mathbb{R}^\nu} |\phi(\boldsymbol{t} + \boldsymbol{s}) - \phi(\boldsymbol{s})|^2\mathrm{d}\boldsymbol{s}\big)^{1/2}|\boldsymbol{t}|^{2d-\nu}\mathrm{d}\boldsymbol{t} < \infty\Big\}. \tag{61}$$

Proposition 6. *Let $a \in L^2(\mathbb{Z}^\nu)$ satisfy Assumption 1 with $-\nu/4 < d < 0$. Then, for any $\phi_i \in \Phi_-, i = 1, 2$ we have*

$$\lim_{\lambda\to\infty} \lambda^{-\nu-4d} \int_{\mathbb{R}^{2\nu}} \phi_1(\boldsymbol{t}_1/\lambda)\phi_2(\boldsymbol{t}_2/\lambda)[a \star a]([\boldsymbol{t}_1] - [\boldsymbol{t}_2])\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2 = c_-(\phi_1, \phi_2), \tag{62}$$

where

$$c_-(\phi_1, \phi_2) := \int_{\mathbb{R}^\nu} \prod_{i=1}^2 \big(\int_{\mathbb{R}^\nu} (\phi_i(\boldsymbol{t} + \boldsymbol{s}) - \phi_i(\boldsymbol{s}))|\boldsymbol{t}|^{2d-\nu}\ell\big(\frac{\boldsymbol{t}}{|\boldsymbol{t}|}\big)\mathrm{d}\boldsymbol{t}\big)\mathrm{d}\boldsymbol{s}. \tag{63}$$

Proof. The convergence of the integral on the r.h.s. of (63) follows from (61) and the Minkowski integral inequality: $\{\int_{\mathbb{R}^\nu}(\int_{\mathbb{R}^\nu} |\phi(\boldsymbol{t} + \boldsymbol{s}) - \phi(\boldsymbol{s})||\boldsymbol{t}|^{2d-\nu}\mathrm{d}\boldsymbol{t})^2\mathrm{d}\boldsymbol{s}\}^{1/2} \le \int_{\mathbb{R}^\nu} \|\phi(\boldsymbol{t} + \cdot) - \phi(\cdot)\|_{L^2(\mathbb{R}^\nu)} \times |\boldsymbol{t}|^{2d-\nu}\mathrm{d}\boldsymbol{t}$.

The proof of the convergence in (62) resembles that of (58). Write $c_\lambda(\phi_1, \phi_2)$ for the integral on the l.h.s. of (62). Using $\sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} a(\boldsymbol{s}) = 0$ we rewrite $\int_{\mathbb{R}^\nu} \phi_i(\boldsymbol{t}_i/\lambda)a([\boldsymbol{t}_i] - [\boldsymbol{s}])\mathrm{d}\boldsymbol{t}_i = \int_{\mathbb{R}^\nu}(\phi_i((\boldsymbol{t}_i + \boldsymbol{s})/\lambda) - \phi_i(\boldsymbol{s}/\lambda))a([\boldsymbol{t}_i + \boldsymbol{s}] - [\boldsymbol{s}])\mathrm{d}\boldsymbol{t}_i, i = 1, 2, \boldsymbol{s} \in \mathbb{R}^\nu$, and

$$\frac{c_\lambda(\phi_1, \phi_2)}{\lambda^{\nu+4d}} = \int_{\mathbb{R}^\nu} \mathrm{d}\boldsymbol{s} \prod_{i=1}^2 \int_{\mathbb{R}^\nu} (\phi_i(\boldsymbol{t}_i + \boldsymbol{s}) - \phi_i(\boldsymbol{s}))\lambda^{\nu-2d}a([\lambda(\boldsymbol{t}_i + \boldsymbol{s})] - [\lambda\boldsymbol{s}])\mathrm{d}\boldsymbol{t}_i,$$

where the inner integrals tend to those on the r.h.s. of (63) at each $\boldsymbol{s}$, such that $\int_{\mathbb{R}^\nu} |\phi_i(\boldsymbol{t}+\boldsymbol{s}) - \phi_i(\boldsymbol{s})||\boldsymbol{t}|^{2d-\nu}\mathrm{d}\boldsymbol{t} < \infty$, $i = 1, 2$. The remaining details are similar to (58) and are omitted. □

Remark 3. *The restriction $d > -\nu/4$ in Proposition 6 is not necessary for (63). Indeed, if $\phi \in \Phi$ satisfies the uniform Lipschitz condition $|\phi(\boldsymbol{t}) - \phi(\boldsymbol{s})| < C(|\boldsymbol{t}| < 1, \boldsymbol{s} \in \mathbb{R}^\nu)$ then the integral in (61) converges for $0 > d > -\nu/2$, implying $\phi \in \Phi_-$. On the other hand, for the indicator functions $\phi(\boldsymbol{t}) = \mathbb{I}(\boldsymbol{t} \in A)$ of a bounded Borel set $A \subset \mathbb{R}^\nu$ with a 'regular' boundary, we typically have $\|\phi(\boldsymbol{t}+\cdot) - \phi(\cdot)\|_{L^2(\mathbb{R}^\nu)} = O(|\boldsymbol{t}|^{1/2})$ leading to $d > -\nu/4$.*

Relation (48) entails the existence of the scaling limit

$$\lim_{\lambda\to\infty} \lambda^{\nu-2d} a([\lambda \boldsymbol{t}]) = a_\infty(\boldsymbol{t}) := |\boldsymbol{t}|^{2d-\nu}\ell\Big(\frac{\boldsymbol{t}}{|\boldsymbol{t}|}\Big), \qquad \lambda \to \infty, \quad \forall\, \boldsymbol{t} \in \mathbb{R}^\nu \setminus \{\boldsymbol{0}\}, \tag{64}$$

which is a continuous homogeneous function on $\mathbb{R}^\nu$: for any $\lambda > 0$ we have

$$a_\infty(\lambda \boldsymbol{t}) = \lambda^{2d-\nu} a_\infty(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^\nu \setminus \{\boldsymbol{0}\}. \tag{65}$$

With the limit function in (64) we associate a Gaussian RF:

$$W_d(\phi) := \begin{cases} \int_{\mathbb{R}^\nu} (a_\infty \star \phi)(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u}), & 0 < d < \nu/4, \phi \in \Phi \\ \int_{\mathbb{R}^\nu} (a_\infty \star \phi)_{\mathrm{reg}}(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u}), & -\nu/4 < d < 0, \phi \in \Phi_-, \\ \int_{\mathbb{R}^\nu} \phi(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u}), & d = 0, \phi \in \Phi, \end{cases} \tag{66}$$

where $W(\mathrm{d}\boldsymbol{u})$ is a real-valued Gaussian white noise (also called the real-valued Gaussian random measure) with zero mean and where variance $\mathrm{d}\boldsymbol{u}$, $(a_\infty \star \phi)(\boldsymbol{u}) = \int_{\mathbb{R}^\nu} a_\infty(\boldsymbol{t})\phi(\boldsymbol{t} + \boldsymbol{u})\mathrm{d}\boldsymbol{t}$ is the usual and

$$(a_\infty \star \phi)_{\mathrm{reg}}(\boldsymbol{u}) := \int_{\mathbb{R}^\nu} a_\infty(\boldsymbol{t})(\phi(\boldsymbol{t} + \boldsymbol{u}) - \phi(\boldsymbol{u}))\mathrm{d}\boldsymbol{t}, \quad \boldsymbol{u} \in \mathbb{R}^\nu$$

the 'regularized' convolution. For the indicator test function $\phi(\boldsymbol{t}) = \mathbb{I}(\boldsymbol{t} \in B)$ of a Borel set $B \subset \mathbb{R}^\nu$ (belonging to Φ_-) we see that the latter convolution equals

$$(a_\infty \star \phi)_{\mathrm{reg}}(\boldsymbol{u}) = \begin{cases} \int_B a_\infty(\boldsymbol{t} - \boldsymbol{u})\mathrm{d}\boldsymbol{t}, & \boldsymbol{u} \notin B, \\ -\int_{\mathbb{R}^\nu \setminus B} a_\infty(\boldsymbol{t} - \boldsymbol{u})\mathrm{d}\boldsymbol{t}, & \boldsymbol{u} \in B. \end{cases}$$

This paper uses the elementary properties of the white noise integrals in (66) only. Namely, $\int_{\mathbb{R}^\nu} \phi(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u})$ is well-defined for each $\phi \in L^2(\mathbb{R}^\nu)$ and has a Gaussian distribution with zero mean and variance $\|\phi\|^2_{L^2(\mathbb{R}^\nu)}$ (see, e.g., [5,7]), implying that $\int_{\mathbb{R}^\nu} \phi(\boldsymbol{u}/\lambda) W(\mathrm{d}\boldsymbol{u}) \overset{\mathrm{d}}{=} \lambda^{\nu/2} \int_{\mathbb{R}^\nu} \phi(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u})$ for each $\lambda > 0$. The interested reader is referred to [24] on white noise calculus on the Schwartz space and to [33] for fractional calculus with respect to fractional Brownian motion. The existence of stochastic integrals in (66) follows from Propositions 5 and 6. Particularly, the variances $\mathrm{E}W_d^2(\phi) = c(\phi, \phi)$ $(0 < d < \nu/4)$ and $\mathrm{E}W_d^2(\phi) = c_-(\phi, \phi)$ $(-\nu/4 < d < 0)$ agree with (59) and (63).

Let $\mathcal{S}(\mathbb{R}^\nu)$ be the Schwartz space of all infinitely differentiable rapidly decreasing functions $\phi : \mathbb{R}^\nu \to \mathbb{R}$, i.e., for each $p \in \mathbb{N}$ and each multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}^\nu$,

$$\sup_{\boldsymbol{x}\in\mathbb{R}^\nu} (1 + |\boldsymbol{x}|)^p |\partial^{\boldsymbol{\alpha}} \phi(\boldsymbol{x})| < \infty,$$

where $\partial^{\boldsymbol{\alpha}}\phi(\boldsymbol{x}) := \partial^{\sum_{i=1}^\nu \alpha_i}\phi(\boldsymbol{x}) / \prod_{i=1}^\nu \partial x_i$ (see, e.g., [34] (Section 7) for the properties of $\mathcal{S}(\mathbb{R}^\nu)$ and the dual space $\mathcal{S}'(\mathbb{R}^\nu)$ of tempered Schwartz distributions). Following [35], we say that a generalized RF $Y = \{Y(\phi); \phi \in \mathcal{S}(\mathbb{R}^\nu)\}$ is *stationary* if $Y(\phi) \overset{\mathrm{d}}{=} Y(\phi(\cdot + \boldsymbol{a}))$ $(\forall \phi \in \mathcal{S}(\mathbb{R}^\nu), \boldsymbol{a} \in \mathbb{R}^\nu)$ and *H-self-similar* $(H \in \mathbb{R})$ if $Y(\phi) \overset{\mathrm{d}}{=} \lambda^{H-\nu} Y(\phi(\cdot/\lambda))$ $(\forall \phi \in \mathcal{S}(\mathbb{R}^\nu), \lambda > 0)$. As noted in Remark 3, $\mathcal{S}(\mathbb{R}^\nu) \subset \Phi_- \subset \Phi$; hence, (66) is well-defined for any $\phi \in \mathcal{S}(\mathbb{R}^\nu)$ and represents stationary generalized RFs on $\mathcal{S}(\mathbb{R}^\nu)$. By the scaling property in (65) and a change of variables, we see that $W_d(\phi) \stackrel{d}{=} \lambda^{H(d)-\nu} W_d(\phi(\cdot/\lambda))$ $(\forall \phi \in \mathcal{S}(\mathbb{R}^\nu))$; hence, RF W_d in (66) is $H(d)$-self-similar, with

$$H(d) := (\nu - 4d)/2 \in (0, \nu), \quad -\nu/4 < d < \nu/4.$$

The RF in (66) appear as scaling limits in the following corollary:

Corollary 3. *Let X be a linear RF satisfying Assumption 1 and $X_\lambda(\phi)$ be defined in (46). Then,*

$$\lambda^{-(\nu+4d)/2} X_\lambda(\phi) \xrightarrow{\mathrm{d}} \begin{cases} W_d(\phi), & 0 < d < \nu/4, \ \forall \phi \in \Phi, \\ W_d(\phi), & -\nu/4 < d < 0, \ \forall \phi \in \Phi_-, \\ \sigma W_0(\phi), & d = 0, \ \forall \phi \in \Phi, \end{cases}$$

where $\sigma^2 := \big(\sum_{t \in \mathbb{Z}^\nu} a(t)\big)^2$.

Proof. Since (46) writes as a linear form $X_\lambda(\phi) = \sum_{u \in \mathbb{Z}^\nu} \varepsilon(u) \int_{\mathbb{R}^\nu} \phi(t/\lambda) a([t] - u) \mathrm{d}t$ in i.i.d. r.v.s, we can use the Lindeberg-type condition (see also [3] (Corollary 4.3.1)). Accordingly, it suffices to show that

$$\sup_{u \in \mathbb{Z}^\nu} \Big| \int_{\mathbb{R}^\nu} \phi(t/\lambda) a([t] - u) \mathrm{d}t \Big| = o\Big(\sqrt{\mathrm{Var}(X_\lambda(\phi))}\Big), \quad \lambda \to \infty \tag{67}$$

holds in each case, $d > 0, d < 0, d = 0$, of the corollary. The behavior of the last variance is detailed in Propositions 5 and 6, and it grows to infinity in each case of d. On the other hand, the l.h.s. of (67) does not exceed $\|\phi\|_{L^\infty(\mathbb{R}^\nu)} \|a\|_{L^1(\mathbb{Z}^\nu)}$, which is bounded in cases $d < 0$ and $d = 0$. Finally, in case $d > 0$ we see that the l.h.s. of (67) does not exceed $\|\phi(\cdot/\lambda)\|_{L^2(\mathbb{R}^\nu)} \|a\|_{L^2(\mathbb{Z}^\nu)} = O(\lambda^{\nu/2})$ and (67) holds, since $d > 0$. □

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