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Stability Analysis of a Fractional-Order Time-Delayed Solow Growth Model with Environmental Pollution

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Abstract: Economic growth is resulting in serious environmental problems. Effectively establishing an economic growth model that considers environmental pollution is an important topic. To analyze the interplay between economic growth and environmental pollution, a fractional-order time-delayed economic growth model with environmental purification is proposed in this paper. The established model considers not only the environment and economic production but also the labor force population and total factor productivity. Furthermore, the asymptotic stability conditions and parameter stability interval are provided. Finally, in numerical experiments, the correctness of the theory is verified.

Keywords: economic growth model; environmental purification; fractional order; time-delayed

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1. Introduction

Fractional calculus theory analysis and applications have become topics of great interest in current scientific research. The wider application background of fractional calculus has attracted the attention of many scholars from various fields, resulting in abundant research results [1–4]. In the field of finance, because of the “memory effect” of fractional calculus, fractional order equations can describe the long-term logarithmic prices of some financial assets well [5]. Compared with integer calculus, the main advantage of fractional calculus is its memory, and it has been proven to be a very suitable tool for describing the memory and genetic characteristics of various materials and processes. Financial and economic variables have a longer memory, and therefore, it is more appropriate to use fractional differential equation models to portray the dynamic behavior of economic systems, such as, exchange rates, gross domestic product (GDP), interest rates, production, and stock market prices, which are changing in terms of the financial and economic system. This provides a scientific approach to predict economic growth.

The study of the complex dynamics of economic systems has become a prominent issue in economics and macroeconomics in recent years. Several nonlinear continuous models have been proposed to explicate the core features of economic data based on the dynamic behavior of the system [6–15]. The results of investigating the dynamics of an economic system with chaotic behavior and a suboptimal control proposal to suppress the chaotic behavior are presented [8]. A fractional order economic quantity model with time-varying holding cost is discussed in detail with the help of numerical computations [9]. Based on the definition of Atangana–Baleanu–Caputo fractional derivative, the integer-order financial chaotic system with nonconstant demand elasticity is extended to a fractional-order system, and its nonlinear dynamic properties are analyzed [10]. The chaotic complexity of a financial mathematical model in terms of a new generalized Caputo fractional derivative is analyzed [11]. The conditions for the structural stability of a fractional order IS-LM-AS dynamic model with adaptive expectations are given [12]. A dynamic fractional-order discrete gray model for forecasting China’s total renewable energy capacity is proposed [13].

A fault-tolerant prescribed performance control approach for fractional-order economic and supply chain systems is presented [14].

Predicting economic growth is an important subject in economics, and accurate prediction can facilitate integrated economic planning and the development of rational economic policies to promote healthy economic growth [16–18]. In previous studies, scholars mainly focused on the chaotic motion of the financial system, and domestic scholars have achieved certain research results on economic growth by introducing delays. The Solow growth model [19] provides a theoretical foundation for breakthroughs in economic growth and a research framework that can be applied in subsequent work. And the Solow growth model demonstrates how saving rates, capital stock, the labor force growth rate, technological progress, and capital depreciation influence a country's total output. The economy tends towards a stable state and emphasizes that technological progress is the ultimate driving force for long-term growth based on Solow's theory. Because most economic processes are not only influenced by current states but also greatly rely on past relevant factors and indicators, mathematical models with time delay are more suitable for describing economic phenomena. Recently, the global attractivity of the quasi-periodicity of a new class of delayed classical growth models are proved [20]. The fractional order models serve to forecast the economic growth of Group of Twenty countries [21]. Based on the Solow model, a fractional-order time-delayed economic growth model is established to effectively capture memory characteristics in the economic growth path and explore the underlying growth factors [22]. Many results have been achieved using the Solow economic model [22–25], and it still has significant theoretical and practical value, particularly when considering fractional calculus theory, which is expected to yield meaningful results.

The economic system is an organic whole composed of interconnected and interactive economic elements. When addressing the issues related to the economic system, it is necessary to consider not only its economic benefits but also the impact of such benefits on the ecological environment [26]. The environment system has a certain self-regulation and self-recovering capacity. However, excessive pollution beyond its self-regulating capacity can cause irreversible damage. Therefore, the stability of the economic system still depends on the capacity of the environment system, external material exchange, and energy flow [27,28]. Furthermore, environmental pollution will inevitably have an impact on the economy, such as water pollution affecting crops [29,30]. Serious economic losses caused by nuclear pollution in Fukushima, Japan have affected economic development [31,32]. To better characterize the dynamic laws of the operation of economic and environmental systems, economic and environment systems are integrated to form an environmental economic growth system. Economic growth has always been an issue of great interest in macroeconomic research. However, with rapid economic growth, environmental pollution has posed a serious threat to human social development. Using the interplay between economic growth and environmental quality to analyze that negative impact has a certain time lag; that is, it is not instantaneous [26]. Economic growth, environmental pollution, and studies on the interactive influence between economic growth and environmental pollution are considered in Wuhan [33]. The importance and significance of the fractional order derivatives in the nonlinear environmental and economic model are provided [34].

Therefore, in this paper, environmental factors are incorporated into the classic Solow economic growth model, with the consideration that the main indicators in the economic environment system have the characteristic of "memory", and a novel fractional-order time-delayed economic growth model with environmental purification is established. According to the Solow model, a fractional time-delayed economic model with environmental purification is provided to characterize the relationship between economic growth and environmental factors. Based on the strict assumption that technological progress is completely exogenous, the traditional Solow economic growth model is inconsistent with practical experience. Furthermore, environmental pollution will inevitably have an impact on the economy. Accordingly, with the consideration of environmental purification, a fractional economic growth model related to both capital and population is established in which technological progress is endogenous. Furthermore, a detailed stability analysis of the proposed model based on environmental purification is performed. The asymptotic stability condition of the equilibrium point is obtained and a stable parameter interval is provided. The influence of parameter variations on the stability of the established model is investigated.

The paper is structured as follows. In Section 2, some preliminaries and model descriptions are presented. The asymptotic stability conditions and parameter stability interval of the fractional-order time-delayed growth model with environmental pollution are provided in Section 3. Finally, a numerical simulation and discussion are given in Sections 4 and 5.

2. Model Description

2.1. Definitions and Lemmas

The definition of the Caputo derivative has more advantages in actual applications, because its initial value has a measurable physical meaning [1–3]. So the definition of the Caputo derivative is adopted in this paper. In addition, according to its good explanation of the memory characteristics of economic variables [35], it can well capture economic processes.

Definition 1 ([2])**.** *The Caputo fractional derivative is defined as*

$$D_\iota^q f(\iota) = \frac{1}{\Gamma(n-q)} \int_0^\iota \frac{f^{(n)}(\tau)}{(\iota-\tau)^{q-n+1}} \mathrm{d}\tau,$$

where $n-1 < q < n, n \in Z^+, \Gamma(n-q) = \int_0^\infty x^{n-q-1} e^{-x} dx$ *is the Gamma function [36].*

Definition 2 ([2,37])**.** *The Laplace transform of the Caputo derivative is*

$$\{D_\iota^q f(\iota); \varsigma\} = s^q F(\varsigma) - \sum_{k=0}^{n-1} \varsigma^{q-k-1} f^{(k)}(0), \quad n-1 < q < n,$$

where $F(\varsigma)$ *is the Laplace transform of* $f(\iota)$*, i.e.,* $F(\varsigma) = \int_0^\infty f(\iota) e^{-\varsigma\iota} d\iota$*.*

The following linear fractional-order time-delayed system is considered:

$$D_\iota^q X(\iota) = AX(\iota) + KX(\iota - \tau), \tag{1}$$

where $A = (\alpha_{ij})_{n\times n}, K = (\kappa_{ij})_{n\times n}, X(\iota) = (x_1(\iota), x_2(\iota), \cdots, x_n(\iota))^T, X(\iota-\tau) = (x_1(\iota-\tau_1), x_2(\iota-\tau_2), \cdots, x_n(\iota-\tau_n))^T$.

The Laplace transform is applied to both sides of Equation (1), which yields the characteristic matrix [38]

$$\Delta(\varsigma) = \begin{pmatrix} \varsigma^q - \kappa_{11}e^{-\varsigma\tau_{11}} - \alpha_{11} & -\kappa_{12}e^{-\varsigma\tau_{12}} - \alpha_{12} & \cdots & -\kappa_{1n}e^{-\varsigma\tau_{1n}} - \alpha_{1n} \\ -\kappa_{21}e^{-\varsigma\tau_{21}} - \alpha_{21} & \varsigma^q - \kappa_{22}e^{-\varsigma\tau_{22}} - \alpha_{22} & \cdots & -\kappa_{2n}e^{-\varsigma\tau_{2n}} - \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_{n1}e^{-\varsigma\tau_{n1}} - \alpha_{n1} & -\kappa_{n2}e^{-\varsigma\tau_{n2}} - \alpha_{n2} & \cdots & \varsigma^q - \kappa_{nn}e^{-\varsigma\tau_{nn}} - \alpha_{nn} \end{pmatrix}. \tag{2}$$

According to the distribution of the eigenvalues of $\det(\Delta(\varsigma)) = 0$, the stability of system (1) is totally determined. If $\tau_{ij} = 0$, Equation (1) can be rewritten as

$$D_\iota^q X(\iota) = AX(\iota) + KX(\iota) = \Lambda X(\iota), \tag{3}$$

where the coefficient matrix $\Lambda = A + K$. Then, a useful Lemma about the stability of system (1) is introduced as:

Lemma 1 ([38]). *When $q \in (0,1)$, if all the eigenvalues of Λ satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $det(\Delta(\varsigma)) = 0$ has no pure imaginary roots for any $\tau_{ij} > 0$, $i, j = 1, \ldots, n$, then the zero solution of Equation (1) is asymptotically stable by Lyapunov.*

2.2. Model Description

In this subsection, a fractional-order time-delayed economic growth model with environmental pollution based on the Solow model is set up. According to [22,39], the following fractional-order generalization of the Solow-type equation with time delay is given as

$$\begin{cases} D_\iota^q L_\iota = r(1 - L_\iota/L_m)L_\iota, \\ D_\iota^q K_\iota = sA_\iota K_\iota^\alpha L_\iota^{1-\alpha} - (\delta + g + n)K(\iota - \tau), \\ D_\iota^q A_\iota = pA_\iota + \overline{w}L_\iota + hK(\iota - \tau), \end{cases} \tag{4}$$

where $q \in (0,1)$ is the fractional order; L_ι is the working population; $r \in (0,1)$ denotes the natural growth rate of the working population; K_ι is the capital stock; τ represents time delay; $s \in (0,1)$ is the constant saving rate; $\delta \in (0,1)$ is the capital depreciation rate; $g \in (0,1)$ is the growth rate of technology; $n \in (0,1)$ is the population growth rate; Y_ι, K_ι are defined as Cobb–Douglas production function (gross domestic product) and capital stock at time ι. A_ι is the index of total factor productivity (TFP) and $\alpha \in (0,1)$ is the output elasticity of capital stock.

The economic–environmental system consists of two subsystems: the economic subsystem and environmental subsystem. These two subsystems operate through the exchange of materials and energy. During the process of capital accumulation, the discharge of pollutants generated by the economic system into the environmental system increases the stock of environmental pollutants. By contrast, the accumulated pollutants in the environmental system cause economic losses for the economic system, thereby inhibiting the accumulation of capital. The economic losses caused by environmental pollution stimulate investors in the economic system to invest in pollution control, thereby removing a portion of the pollution and reducing the stock of environmental pollutants in the environmental system. Consequently, this reduces the economic losses caused by the environmental system on the economic system. An environmental purification factor is introduced into the classical Solow model.

Investment in pollution control, denoted by E, is directly responsive to the economic losses caused by environmental pollution, represented by G. The larger the economic losses from pollution, the greater the corresponding investment in pollution control. The relationship between the two can be expressed as

$$E = \rho G,$$

where ρ is the degree of pollution control investment. The amount of waste pollution, denoted by Z, is directly related to gross domestic product Y_ι. According to environmental Kuznets curve (EKC) theory [33,40], the relationship between waste pollution and gross domestic product is described as follows:

$$Z(\iota) = \epsilon Y_\iota e^{-\lambda Y_\iota},$$

where $\epsilon > 0$ is referred to as the environmental pollution index, which indicates the severity of environmental pollution and $\lambda > 0$ is a parameter. The pollution amount, denoted by R, is related to pollution control investment E and waste pollution amount Z.

To define a pollution control function $R(Z, E)$ that describes the amount of pollution removed, this process has a saturation effect, and its functional form is assumed as follows:

$$R(Z, E) = ZE/(E + \omega Z),$$

where ω is a control parameter.

The function $\psi(P_\iota)$ represents the natural purification capacity for pollution and exhibits strong nonlinear characteristics. The environmental purification capacity $\psi(P_\iota)$ is related to the stock of environmental pollutants P_ι and can be expressed as follows:

$$\psi(P_\iota) = \frac{\sigma P_\iota d^\beta}{d^\beta + P_\iota^\beta},$$

where σ, β, and d are parameters, and $\beta > 1$.

The economic losses caused by pollution, denoted by G, are related to the environmental purification capacity. It is assumed that G is proportional to the difference between the nonlinear environmental purification capacity and the linear purification capacity. The pollution economic loss function is expressed as

$$G(Y_\iota, P_\iota) = \frac{l Y_\iota P_t^{\beta+1}}{(d^\beta + P_\iota^\beta)},$$

where l is a parameter that represents the sensitivity of economic losses to the level of pollution, and β and d are parameters introduced in the environmental purification function.

Remark 1. *Note that the $\psi(P_\iota)$ and $G(Y_\iota, P_\iota)$ are the case of the Hill function [41,42]. The exponent β determines the steepness of the switch occurring around P_ι. Considering that the higher the β values are, the steeper the Hill functions, and stronger the hysteresis they create; then, $\beta > 1$ is chosen here.*

The change in the capital stock as K_ι includes an increase in investment $sY(\iota)$, economic losses caused by pollution G, capital depreciation. Based on system (4), the evolution equation for the capital stock K_ι is given by

$$D_\iota^q K_t = sA_\iota K_\iota^\alpha L_\iota^{1-\alpha} - G(Y_\iota, P_\iota) - (\delta + g + n)K(\iota - \tau).$$

The change in environmental pollutant stock $P(\iota)$ includes the input of waste pollution Z, the removal of pollution due to pollution control investment R, and the natural purification capacity $\psi(P_\iota)$. The evolution equation for the environmental system is given by

$$D_\iota^q P_\iota = Z_\iota - R(Z_\iota, P_\iota) - \psi(P_\iota).$$

From the above analysis, the following fractional-order time-delayed economic growth model with environmental purification can be provided:

$$\begin{cases} D_\iota^q L_\iota = r(1 - L_\iota/L_m)L_\iota, \\ D_\iota^q K_\iota = sA_\iota K_\iota^\alpha L_\iota^{1-\alpha} - G(Y_\iota, P_\iota) - (\delta + g + n)K(\iota - \tau), \\ D_\iota^q A_\iota = pA_\iota + \overline{w} L_\iota + hK(\iota - \tau), \\ D_\iota^q P_\iota = Z_\iota - R(Z_\iota, P_\iota) - \psi(P_\iota). \end{cases} \tag{5}$$

The functional relationship for system (5) is as follows:

$$\begin{cases} Y_\iota = A_\iota K_\iota^\alpha L_\iota^{1-\alpha}, \\ E_\iota = \rho G_\iota, \\ Z_\iota = \epsilon Y_\iota e^{-\lambda Y_\iota}, \\ R(Z_\iota, E_\iota) = Z_\iota E_\iota/(E_\iota + \omega Z_\iota), \\ \psi(P_\iota) = \frac{\sigma P_\iota d^\beta}{d^\beta + P_\iota^\beta}, \\ G(Y_\iota, P_\iota) = \frac{l Y_\iota P_\iota^{\beta+1}}{(d^\beta + P_\iota^\beta)}. \end{cases} \tag{6}$$

A summary of the definitions of the parameters and variables in system (5) and Equation (6) are shown in Table 1.

Table 1. Values of some parameters.

Variables	Representations	Parameters	Representations
Y_ι	Cobb–Douglas production function (gross domestic product)	s	Constant saving rate
A_ι	Total factor productivity(TFP)	δ	Capital depreciation rate
P_ι	Natural purification capacity	g	Growth rate of technology
R_ι	Economic losses caused	r	Natural growth rate
ψ_ι	Environmental purification capacity	ϵ	Environmental pollution index
L_ι	Working population	q	Fractional order
K_ι	Capital stock	L_m	Maximum number of labor force
E_ι	Investment in pollution control	n	population growth rate
Z_ι	Waste pollution	τ	time delay
G_ι	Pollution economic loss function	p	TFP growth rate
		ρ	Degree of pollution control investment
		α	Output elasticity $0 < \alpha < 1$
		$\overline{w}, l, h$	Scale factors
		β	Parameter and $\beta > 1$
		$d, \omega, \sigma, \lambda$	Parameters
		ι	time

3. Main Results

In this section, the focus is the local stability of system (5). Lemma 1 is a local stability theorem for system (5). Local properties can be analyzed for stability using the eigenvalue distribution of $\Delta(s)$. Thus, simplification (part of linearization) is performed in this step. Note that $\lambda > 0$ and $Y_\iota > 0$, and then $e^{-\lambda Y_\iota} \leq 1$; hence, $Z_\iota \approx \epsilon Y_\iota$ is considered. The equation Z_ι is approximated to first order using the Taylor expansion, which meets the approximation requirement. According to Equation (6), $R(Z_\iota, E_\iota) = Z_\iota E_\iota/(E_\iota + \omega Z_\iota)$. If $E_\iota \ll \omega Z_\iota$, then $R(\iota) \simeq E_\iota/\omega$. Therefore, $R(Z_\iota, E_\iota) \simeq \frac{1}{\omega} E_\iota$, that is, it is approximated by a linear function. The ability of environmental purification is related to the level of environmental pollutants. When the level of pollutants is low, the self-purification ability of the environmental system is strong. However, when the level of pollutants reaches a certain upper limit, the self-purification ability of the environment gradually weakens. This characteristic can be described using Hill functions. Hence,

$$\psi(P_\iota) = \frac{\sigma P_\iota d^\beta}{d^\beta + P_\iota^\beta} \qquad and \qquad G(Y_\iota, P_\iota) = \frac{l Y_\iota P_\iota^{\beta+1}}{(d^\beta + P_\iota^\beta)}.$$

To obtain the stability conditions of system (5), $P_\iota^\beta \gg d^\beta$ can be chosen. Then,

$$\psi(P_\iota) \simeq \sigma d^\beta P_\iota^{1-\beta} \qquad and \qquad G(Y_\iota, P_\iota) \simeq l Y_\iota P_\iota.$$

The simplification of system (5) can be obtained:

$$\begin{cases} D_\iota^q L_\iota = r(1 - L_\iota/L_m)L_\iota, \\ D_\iota^q K_\iota = sA_\iota K_\iota^\alpha L_\iota^{1-\alpha} - G(Y_\iota, P_\iota) - (\delta + g + n)K(\iota - \tau), \\ D_\iota^q A_\iota = pA_\iota + \overline{w}L_\iota + hK(\iota - \tau), \\ D_\iota^q P_\iota = Z_\iota - R(Z_\iota, P_\iota) - \psi(P_\iota). \end{cases} \tag{7}$$

The functional relationship in Equation (7) is as follows:

$$\begin{cases} Y_\iota = A_\iota K_\iota^\alpha L_\iota^{1-\alpha}, \\ E_\iota = \rho G_\iota, \\ Z_\iota = \epsilon Y_\iota \iota e^{-\lambda Y_\iota}, \\ R(Z_\iota, E_\iota) \simeq \frac{1}{\omega}E_\iota, \\ \psi(P_\iota) \simeq \sigma d^\beta P_\iota^{1-\beta}, \\ G(Y_\iota, P_\iota) \simeq lY_\iota P_\iota. \end{cases} \tag{8}$$

Remark 2. *Actually, the functional relationship in system (7) is*

$$\begin{cases} Y_\iota = A_\iota K_\iota^\alpha L_\iota^{1-\alpha}, \\ E_\iota = \rho G(Y_\iota, P_\iota), \\ Z_\iota = \epsilon Y_\iota e^{-\lambda Y_\iota}, \\ R(Z_\iota, E_\iota) = Z_\iota E_\iota/(E_\iota + \omega Z_\iota), \\ \psi(P_\iota) = \frac{\sigma P_\iota d^\beta}{d^\beta + P_\iota^\beta}, \\ G(Y_\iota, P_\iota) = \frac{lY_\iota P_\iota^{\beta+1}}{(d^\beta + P_\iota^\beta)}. \end{cases}$$

In order to obtain the local stability of system (5), the Equation (6) is reduced as Equation (8). If Equation (6) is used, the equilibrium point equation is difficult to solve, and the Jacobian determinant at the equilibrium point is also quite complex. Hence, it is very hard to obtain stability conditions and the parameter stability interval of system (5). However, according to the facts mentioned, in [38,43], the consideration here is locality, which means that linear equations conform to the linear form of the model. Therefore, when fractional-order systems have the same linear form, their stability can be studied through their linear equations, regardless of the complexity of the original equations. Thus, the simplification Equation (8) is reasonable. Based on this fact, to obtain stability conditions, the parameter $\sigma = 0$ is considered.

Theorem 1. *When $q \in (0,1)$ and $\sigma = 0$, if $\lambda = 0$, $r > 0$, $p + b_{22} + c_{22} < 0$, $p(b_{22} + c_{22}) - hb_{23} > 0$ and $b_{44} < 0$, $\overline{b}^2 - 4\chi - 2\overline{c} + \frac{\overline{b}^3 - 4\overline{b}\overline{c} + 8\overline{d}}{\sqrt{8\chi + \overline{b}^2 - 4\overline{c}}} < 0$ or $\overline{b}^2 - 4\chi - 2\overline{c} - \frac{\overline{b}^3 - 4\overline{b}\overline{c} + 8\overline{d}}{\sqrt{8\chi + \overline{b}^2 - 4\overline{c}}} < 0$, the positive equilibrium point of system (7) is locally asymptotically stable by Lyapunov, where χ is any real root of the equation $8\chi^3 - 4\overline{c}\chi^2 + 2(\overline{b}\overline{d} - 8\overline{e})\chi + \overline{e}(4\overline{c} - \overline{b}^2) - \overline{d}^2 = 0$, and the coefficients of the equation are given as*

$$\begin{cases} \overline{b} = -2(p + b_{22})(\cos q\pi \cos \frac{q\pi}{2} + \sin(\pm q\pi)\sin(\pm\frac{q\pi}{2})), \\ \overline{c} = (p + b_{22})^2 - c_{22}^2 + 2pb_{22}\cos q\pi, \\ \overline{d} = 2[p(c_{22}^2 - b_{22}^2 - b_{22}) - hc_{22}b_{23}]\cos\frac{q\pi}{2}, \\ \overline{e} = p^2 b_{22}^2 - (pc_{22} - hb_{23})^2, \end{cases} \tag{9}$$

where b_{22}, b_{23}, b_{44} and c_{22} satisfy

$$\begin{cases} b_{22} = \alpha(s - l\overline{P})\overline{AK}^{\alpha-1}\overline{L}^{1-\alpha}, \\ b_{23} = (s - l\overline{P})\overline{K}^\alpha \overline{L}^{1-\alpha}, \\ b_{44} = -\frac{\rho l}{\omega}\overline{AK}^\alpha \overline{L}^{1-\alpha}, \\ c_{22} = -(\delta + g + n), \end{cases} \tag{10}$$

and the $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ *is a solution of the equation*

$$\begin{cases} r(1-\overline{L}/L_m)\overline{L} = 0, \\ s\overline{AK}^{\alpha}\overline{L}^{1-\alpha} - l\overline{PAK}^{\alpha}\overline{L}^{1-\alpha} - (\delta+g+n)\overline{K} = 0, \\ p\overline{A} + \overline{w}\overline{L} + h\overline{K} = 0, \\ (\epsilon - \frac{\rho l}{\omega}\overline{P})\overline{AK}^{\alpha}\overline{L}^{1-\alpha} = 0. \end{cases} \tag{11}$$

Proof. If $\lambda = 0$, $\sigma = 0$ and $r > 0$, from the Equation (7), we can obtain the equilibrium solution equation as follows:

$$\begin{cases} r(1-\overline{L}/L_m)\overline{L} = 0, \\ s\overline{AK}^{\alpha}\overline{L}^{1-\alpha} - l\overline{PAK}^{\alpha}\overline{L}^{1-\alpha} - (\delta+g+n)\overline{K} = 0, \\ p\overline{A} + \overline{w}\overline{L} + h\overline{K} = 0, \\ (\epsilon - \frac{\rho l}{\omega}\overline{P})\overline{AK}^{\alpha}\overline{L}^{1-\alpha} = 0. \end{cases}$$

Assume a positive equilibrium point $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ is derived from Equation (11), then the linear centralized system (7) at the point $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ can be written as:

$$D_{\iota}^{q}\Phi(\iota) = A\Phi(\iota) + \overline{B}\Phi(\iota) + \overline{C}\Phi(\iota-\tau), \tag{12}$$

where

$$\Phi(\iota) = (L_{\iota}, K_{\iota}, A_{\iota}, P_{\iota})^T, A = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\overline{B}$ and $\overline{C}$ are the Jacobian matrices at equilibrium point $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ as

$$\overline{B} = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} \end{pmatrix}, \overline{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(\delta+g+n) & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{cases} b_{11} = -2r, \\ b_{21} = (1-\alpha)(s - l\overline{P})\overline{K}^{\alpha}\overline{L}^{-\alpha}, \\ b_{22} = \alpha(s - l\overline{P})\overline{AK}^{\alpha-1}\overline{L}^{1-\alpha}, \\ b_{23} = (s - l\overline{P})\overline{K}^{\alpha}\overline{L}^{1-\alpha}, \\ b_{24} = -l\overline{AK}^{\alpha}\overline{L}^{1-\alpha}, \\ b_{44} = -\frac{\rho l}{\omega}\overline{AK}^{\alpha}\overline{L}^{1-\alpha}. \end{cases}$$

The Laplace transform is applied to both sides of system (12), and the characteristic matrix is given as:

$$\Delta(s) = \begin{pmatrix} s^q + r & 0 & 0 & 0 \\ -b_{21} & s^q + (\delta+g+n)e^{-s\tau} - b_{22} & -b_{23} & -b_{23} \\ 0 & -he^{-s\tau} & s^q - p & 0 \\ 0 & 0 & 0 & s^q - b_{44} \end{pmatrix}. \tag{13}$$

The $det(\Delta(s)) = 0$ is calculated as

$$(s^q + r)(s^q - b_{44})(s^{2q} - as^q + (\delta+g+n)s^q e^{-s\tau} + be^{-s\tau} + pb_{22}) = 0, \tag{14}$$

where

$$\begin{cases} a = p + b_{22}, \\ b = -p(\delta+g+n) - hb_{23}. \end{cases}$$

The Equation (14) has no pure imaginary roots for any $\tau > 0$ which is verified for the next part. We testify the fact by contradiction. When $r > 0$, there are obviously no pure imaginary roots in the equation $s^q + r = 0$. Hence, the following equation is considered:

$$s^{2q} - as^q + (\delta + g + n)s^q e^{-s\tau} + be^{-s\tau} + pb_{22} = 0. \tag{15}$$

Assume that $s^* = \varpi i = |\varpi|(\cos(\frac{\pi}{2}) + i\sin(\pm\frac{\pi}{2}))$ is one of the pure imaginary roots of Equation (15), where ϖ is a positive real number. Taking s^* into Equation (15) yields

$$\begin{aligned} &|\varpi|^{2q}(\cos q\pi + i\sin(\pm q\pi)) - c_{22}|\varpi|^q(cos\tfrac{q\pi}{2} + i\sin(\pm\tfrac{q\pi}{2}))(\cos\varpi\tau - i\sin\varpi\tau) \\ &\quad - a|\varpi|^q(\cos\tfrac{q\pi}{2} + i\sin(\pm\tfrac{q\pi}{2})) + b(\cos\varpi\tau - i\sin\varpi\tau) + pb_{22} = 0. \end{aligned} \tag{16}$$

Both the real part and imaginary part of Equation (16) are zero, so we have

$$|\varpi|^{2q}\cos q\pi - a|\varpi|^q\cos\frac{q\pi}{2} + pb_{22} = c_{22}|\varpi|^q\sin(\pm\frac{q\pi}{2})\sin\varpi\tau + (c_{22}|\varpi|^q\cos\frac{q\pi}{2} - b)\cos\varpi\tau \tag{17}$$

and

$$|\varpi|^{2q}\sin(\pm q\pi) - a|\varpi|^q\sin(\pm\frac{q\pi}{2}) = (b - c_{22}|\varpi|^q\cos\frac{q\pi}{2})\sin\varpi\tau + c_{22}|\varpi|^q\sin(\pm\frac{q\pi}{2})\cos\varpi\tau. \tag{18}$$

The following is considered:

$$\begin{cases} \beta_1 = |\varpi|^{2q}\cos q\pi - a|\varpi|^q\cos\frac{q\pi}{2} + pb_{22}, \\ \beta_2 = |\varpi|^{2q}\sin(\pm q\pi) - a|\varpi|^q\sin(\pm\frac{q\pi}{2}), \\ \rho_1 = c_{22}|\varpi|^q\cos\frac{q\pi}{2} - b, \\ \rho_2 = c_{22}|\varpi|^q\sin(\pm\frac{q\pi}{2}). \end{cases} \tag{19}$$

Thus, from Equations (17) and (18), we can obtain

$$\begin{cases} \rho_1\cos\varpi\tau + \rho_2\sin\varpi\tau = \beta_1, \\ -\rho_1\sin\varpi\tau + \rho_2\cos\varpi\tau = \beta_2. \end{cases} \tag{20}$$

Sum the squares of two equations in Equation (20) on both sides:

$$\rho_1^2 + \rho_2^2 = \beta_1^2 + \beta_2^2. \tag{21}$$

Expressions β_1, β_2, ρ_1 and ρ_2 are substituted, which yields

$$|\varpi|^{4q} + \bar{b}|\varpi|^{3q} + \bar{c}|\varpi|^{2q} + \bar{d}|\varpi|^q + \bar{e} = 0, \tag{22}$$

where

$$\begin{cases} \bar{b} = -2a(\cos q\pi\cos\frac{q\pi}{2} + \sin(\pm q\pi)\sin(\pm\frac{q\pi}{2})), \\ \bar{c} = a^2 - c_{22}^2 + 2pb_{22}\cos q\pi, \\ \bar{d} = 2(-apb_{22} + c_{22}b)\cos\frac{q\pi}{2}, \\ \bar{e} = p^2b_{22}^2 - b^2. \end{cases} \tag{23}$$

If $0 < q < 1$ and χ satisfy the inequality $\bar{b}^2 - 4\chi - 2\bar{c} + \frac{\bar{b}^3 - 4\bar{b}\bar{c} + 8\bar{d}}{\sqrt{8\chi + \bar{b}^2 - 4\bar{c}}} < 0$ or $\bar{b}^2 - 4\chi - 2\bar{c} - \frac{\bar{b}^3 - 4\bar{b}\bar{c} + 8\bar{d}}{\sqrt{8\chi + \bar{b}^2 - 4\bar{c}}} < 0$, there exists no real roots in Equation (22), where χ is any real root of the equation $8\chi^3 - 4\bar{c}\chi^2 + 2(\overline{bd} - 8\bar{e})\chi + \bar{e}(4\bar{c} - \bar{b}^2) - \bar{d}^2 = 0$; so, for any $\tau > 0$, the characteristic equation $det(\Delta(s)) = 0$ has no pure imaginary roots.

If $\tau = 0$, the matrix Λ of Equation (12) is

$$\Lambda = \begin{pmatrix} -r & 0 & 0 & 0 \\ b_{12} & b_{22}+c_{22} & b_{23} & b_{24} \\ 0 & h & p & 0 \\ 0 & 0 & 0 & b_{44} \end{pmatrix}$$

and its characteristic equation is

$$f(\widehat{\lambda}) = (\widehat{\lambda}+r)(\widehat{\lambda}-b_{44})[\widehat{\lambda}^2 - (b_{22}+c_{22}+p)\widehat{\lambda} + p(b_{22}+c_{22}) - hb_{23}]. \tag{24}$$

Its eigenvalues $\widehat{\lambda}_{1,2,3,4}$ can be obtained. When $0 < p < (1-\alpha)(\delta+g+n)$, then

$$\begin{cases} \widehat{\lambda}_1 = -r < 0, \\ \widehat{\lambda}_2 = b_{44} < 0, \\ \widehat{\lambda}_3 + \widehat{\lambda}_4 = p + b_{22} + c_{22} < 0, \\ \widehat{\lambda}_3\widehat{\lambda}_4 = p(b_{22}+c_{22}) - hb_{23} > 0. \end{cases} \tag{25}$$

Based on (25), the four eigenvalues of the matrix Λ have negative real parts, and all the eigenvalues of Λ satisfy $|\arg(\widehat{\lambda})| > \frac{\pi}{2}$. Based on Lemma 1, the positive equilibrium point $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ of system (7) is asymptotically stable by Lyapunov. This completes the proof. $\square$

Remark 3. *In this paper, a fractional-order time-delayed economic growth model with environmental purification is proposed. The established model considers not only the environment and economic production but also the labor force population and total factor productivity. The delayed fractional-order economic growth model without pollution is given in [22]. So, our results obtained in this paper are further extended results than [22] on the analysis between the environment and economic production.*

According to Equation (11),

$$\begin{cases} \overline{L} = L_m, \\ \overline{P} = \frac{\epsilon\omega}{\rho l}, \\ s\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha} - (\delta+g+n)\overline{K} - l\overline{P}\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha} = 0, \\ p\overline{A} + h\overline{K} + \overline{w}\overline{L} = 0. \end{cases} \tag{26}$$

Based on Equation (26), it is noteworthy that the equilibrium points $\overline{K}$ and $\overline{A}$ are not easy to obtain. Hence, certain conditions are provided to facilitate the determination of the equilibrium points. The corresponding stability conditions are provided as two corollaries.

Corollary 1. *If $q \in (0,1)$, when $\overline{w} = 0, \lambda = 0, \sigma = 0, h < 0, r > 0, 0 < p < (1-\alpha)(\delta+g+n)$, $\frac{l}{\omega}\frac{1}{\epsilon\omega - s\rho} < 0$, and $\overline{b}^2 - 4\chi - 2\overline{c} + \frac{\overline{b}^3 - 4\overline{b}\overline{c} + 8\overline{d}}{\sqrt{8\chi + \overline{b}^2 - 4\overline{c}}} < 0$ or $\overline{b}^2 - 4\chi - 2\overline{c} - \frac{\overline{b}^3 - 4\overline{b}\overline{c} + 8\overline{d}}{\sqrt{8\chi + \overline{b}^2 - 4\overline{c}}} < 0$, then the positive equilibrium point $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ of system (7) is locally asymptotically stable by Lyapunov, where χ is any real root of the equation $8\chi^3 - 4\overline{c}\chi^2 + 2(\overline{b}\overline{d} - 8\overline{e})\chi + \overline{e}(4\overline{c} - \overline{b}^2) - \overline{d}^2 = 0$, and the coefficients of the equation are given as*

$$\begin{cases} \overline{b} = -2(p + \alpha(\delta+g+n))(\cos q\pi \cos \frac{q\pi}{2} + \sin(\pm q\pi)\sin(\pm\frac{q\pi}{2})), \\ \overline{c} = p^2 + 2\alpha p(\delta+g+n)(1+\cos q\pi) + (\alpha^2-1)(\delta+g+n)^2, \\ \overline{d} = -2\alpha p(\delta+g+n)[\alpha(\delta+g+n) - 1]\cos\frac{q\pi}{2}, \\ \overline{e} = p^2\alpha^2(\delta+g+n)^2. \end{cases} \tag{27}$$

Proof. If $\overline{w}=0$, as a result of taking $\tau=0$, then

$$\begin{cases} \overline{L}=L_m, \\ \overline{P}=\frac{\epsilon\omega}{\rho l}, \\ s\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha}-l\overline{P}\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha}-(\delta+g+n)\overline{K}=0, \\ p\overline{A}+h\overline{K}=0. \end{cases} \tag{28}$$

Solving the Equation (28), we can obtain the positive solution $(\overline{L},\overline{K},\overline{A},\overline{P})$ of system (7) as

$$\overline{L}=L_m, \overline{K}=\left(\frac{\rho p(\delta+g+n)}{(\epsilon\omega-s\rho)hL_m^{(1-\alpha)}}\right)^{\frac{1}{\alpha}}, \overline{A}=-\frac{h}{p}\overline{K}, \overline{P}=\frac{\epsilon\omega}{\rho l}. \tag{29}$$

Based on Equation (29), the coefficients can be computed as

$$\begin{cases} b_{11}=-2r, \\ b_{21}=(1-\alpha)(\delta+g+n)\left(\frac{\rho p(\delta+g+n)}{(\epsilon\omega-s\rho)hL_m^{(1-\alpha)}}\right)^{\frac{1}{\alpha}}, \\ b_{22}=(\delta+g+n)\alpha, \\ b_{23}=-\frac{p}{h}(\delta+g+n), \\ b_{24}=\frac{\rho l(\delta+g+n)}{(\epsilon\omega-s\rho)}\left(\frac{\rho p(\delta+g+n)}{(\epsilon\omega-s\rho)hL_m^{(1-\alpha)}}\right)^{\frac{1}{\alpha}}, \\ b_{44}=\frac{\rho^2 l(\delta+g+n)}{\omega(\epsilon\omega-s\rho)}\left(\frac{\rho p(\delta+g+n)}{(\epsilon\omega-s\rho)hL_m^{(1-\alpha)}}\right)^{\frac{1}{\alpha}}. \end{cases}$$

According to the stability conditions in Theorem 1, if $b_{44}<0$, it can be concluded that $\frac{l}{\omega}\frac{1}{\epsilon\omega-s\rho}<0$. According to $p+b_{22}+c_{22}<0$ and $p(b_{22}+c_{22})-hb_{23}>0$, it can be concluded that $0<p<(1-\alpha)(\delta+g+n)$.

Furthermore, according to Equation (9),

$$\begin{cases} \overline{b}=-2(p+\alpha(\delta+g+n))(\cos q\pi\cos\frac{q\pi}{2}+\sin(\pm q\pi)\sin(\pm\frac{q\pi}{2})), \\ \overline{c}=p^2+2\alpha p(\delta+g+n)[(1+\cos q\pi)+(\alpha^2-1)(\delta+g+n)], \\ \overline{d}=-2\alpha p[\alpha(\delta+g+n)^2-(\delta+g+n)]\cos\frac{q\pi}{2}, \\ \overline{e}=p^2\alpha^2(\delta+g+n)^2. \end{cases}$$

This completes the proof. □

Corollary 2. *If $q\in(0,1)$, when $\lambda=0$, $p=0$, $\sigma=0$, $h<0$, $\overline{w}>0$, $r>0$, $\frac{l}{\omega(s\rho-\epsilon\omega)}\frac{\overline{w}}{h}<0$, $\frac{s\rho-\epsilon\omega}{\rho}>0$, and $\overline{b}^2-4\chi-2\overline{c}+\frac{\overline{b}^3-4\overline{b}\overline{c}+8\overline{d}}{\sqrt{8\chi+\overline{b}^2-4\overline{c}}}<0$ or $\overline{b}^2-4\chi-2\overline{c}-\frac{\overline{b}^3-4\overline{b}\overline{c}+8\overline{d}}{\sqrt{8\chi+\overline{b}^2-4\overline{c}}}<0$, then the positive equilibrium point $(\overline{L},\overline{K},\overline{A},\overline{P})$ of system (7) is locally asymptotically stable by Lyapunov, where χ is any real root of the equation $8\chi^3-4\overline{c}\chi^2+2(\overline{b}\overline{d}-8\overline{e})\chi+\overline{e}(4\overline{c}-\overline{b}^2)-\overline{d}^2=0$, and the coefficients of the equation are given by*

$$\begin{cases} \overline{b}=-2\alpha(\delta+g+n)(\cos q\pi\cos\frac{q\pi}{2}+\sin(\pm q\pi)\sin(\pm\frac{q\pi}{2})), \\ \overline{c}=(\alpha^2-1)(\delta+g+n)^2, \\ \overline{d}=-2h\frac{s\rho-\epsilon\omega}{\rho}(-\frac{\overline{w}}{h})^{\alpha}(\delta+g+n)L_m\cos\frac{q\pi}{2}, \\ \overline{e}=-[-h\frac{(s\rho-\epsilon\omega)}{\rho}(-\frac{\overline{w}}{h})^{\alpha}L_m]^2. \end{cases} \tag{30}$$

Proof. If $p = 0, h < 0$ and $\overline{w} > 0$, taking $\tau = 0$, then we can obtain

$$\begin{cases} \overline{L} = L_m, \\ \overline{P} = \frac{\epsilon\omega}{\rho l}, \\ s\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha} - l\overline{P}\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha} - (\delta + g + n)\overline{K} = 0, \\ \overline{w}\overline{L} + h\overline{K} = 0. \end{cases} \tag{31}$$

Solving the equations, and getting the unique positive solution $(\overline{L}, \overline{K}, \overline{A}, \overline{P})$ of system (7)

$$\overline{L} = L_m, \overline{K} = -\frac{\overline{w}}{h}L_m, \overline{A} = \frac{\rho(\delta + g + n)}{(s\rho - \epsilon\omega)}(-\frac{\overline{w}}{h})^{1-\alpha}, \overline{P} = \frac{\epsilon\omega}{\rho l}. \tag{32}$$

Based on Equation (32), the coefficients can be computed as

$$\begin{cases} b_{11} = -2r, \\ b_{21} = -(1-\alpha)\frac{\overline{w}}{h}(\delta + g + n), \\ b_{22} = \alpha(\delta + g + n), \\ b_{23} = -\frac{(\epsilon\omega - s\rho)}{\rho}(-\frac{\overline{w}}{h})^{\alpha}L_m, \\ b_{24} = -\frac{\rho l(\delta+g+n)}{(\epsilon\omega - s\rho)}\frac{\overline{w}}{h}L_m, \\ b_{44} = \frac{\rho^2 l(\delta+g+n)}{\omega(s\rho - \epsilon\omega)}\frac{\overline{w}}{h}L_m. \end{cases}$$

According to the stability conditions in Theorem 1, when $b_{44} < 0$, it can be concluded that $\frac{l}{\omega(s\rho - \epsilon\omega)}\frac{\overline{\omega}}{h} < 0$, if $p = 0, h < 0, \overline{w} > 0, \frac{s\rho - \epsilon\omega}{\rho} > 0$,

$$b_{22} + c_{22} = (\delta + g + n)(\alpha - 1) < 0,$$

$$-hb_{23} = -h\frac{(s\rho - \epsilon\omega)}{\rho}(-\frac{\overline{w}}{h})^{\alpha}L_m > 0$$

are obtained.

According to Equation (9),

$$\begin{cases} \overline{b} = -2\alpha(\delta + g + n)(\cos q\pi \cos\frac{q\pi}{2} + \sin(\pm q\pi)\sin(\pm\frac{q\pi}{2})), \\ \overline{c} = (\alpha^2 - 1)(\delta + g + n)^2, \\ \overline{d} = -2h\frac{s\rho - \epsilon\omega}{\rho}(-\frac{\overline{w}}{h})^{\alpha}(\delta + g + n)L_m\cos\frac{q\pi}{2}, \\ \overline{e} = -[-h\frac{(s\rho - \epsilon\omega)}{\rho}(-\frac{\overline{w}}{h})^{\alpha}L_m]^2. \end{cases}$$

This completes the proof. □

Remark 4. *Note that, if coefficient $l = 0$, this means that economic losses and production capital caused by pollution are not considered in the economic growth model. In [22], the prediction of China's economic growth based on the delayed fractional-order economic growth model without pollution is discussed and potential economic growth factors are explored. Based on this model, by considering the fractional orders as parameters and optimizing them, an appropriate fractional order based on the economic data of China from 1978 to 2020 is found and China's GDP in the next 30 years is predicted using the fractional-order delayed economic growth model. The factors that drive short-term high-speed economic growth are also found. The results indicate that China has a declining population dividend and capital accumulation deceleration. Therefore, the TFP is increasing along with technological progress and innovation. Based on the fractional-order economic growth model in [22], the environmental purification factor is considered in this paper.*

4. Numerical Analysis

In this section, the effectiveness of the theoretical results is demonstrated through three numerical examples, and the impact of system parameters is further investigated.

The ABM predictor–corrector algorithm [44] and the computed step $h = 0.01$ are used to solve the fractional-order time-delayed economic growth model with environmental purification. The specific values of some parameters of system (7) are shown in Table 2.

Table 2. Values of some parameters.

Parameters	Values	Parameters	Values
Constant saving rate s	$s = 0.4$	Maximum number of labor force L_m	$L_m = 8$
Capital depreciation rate δ	$\delta = 0.05$	Population growth rate n	$n = 0.05$
Growth rate of technology g	$g = 0.1$	Time delay τ	$\tau = 2$
Natural growth rate r	$r = 0.05$	Degree of pollution control investment ρ	$\rho = 18$
Parameter β	$\beta = 1.6$	Parameter d	$d = 1$

Remark 5. *The time delay τ is chosen as $\tau = 2$. According to [22], the fitting result with time delay $\tau = 2$ matches the original K_t best. Analysis shows that when $\tau = 0$, which represents the case without time delay, capital stock will be slightly overestimated because historical states are ignored. By contrast, when $\tau = 4$, because of the excessive emphasis on the role and impact of economic variables in the capital accumulation that results from considering previous historical values, the historical data of K_t will be underestimated. Therefore, the time delay $\tau = 2$ is chosen properly.*

Inputting these coefficients into model (7) yields a more specific model as follows:

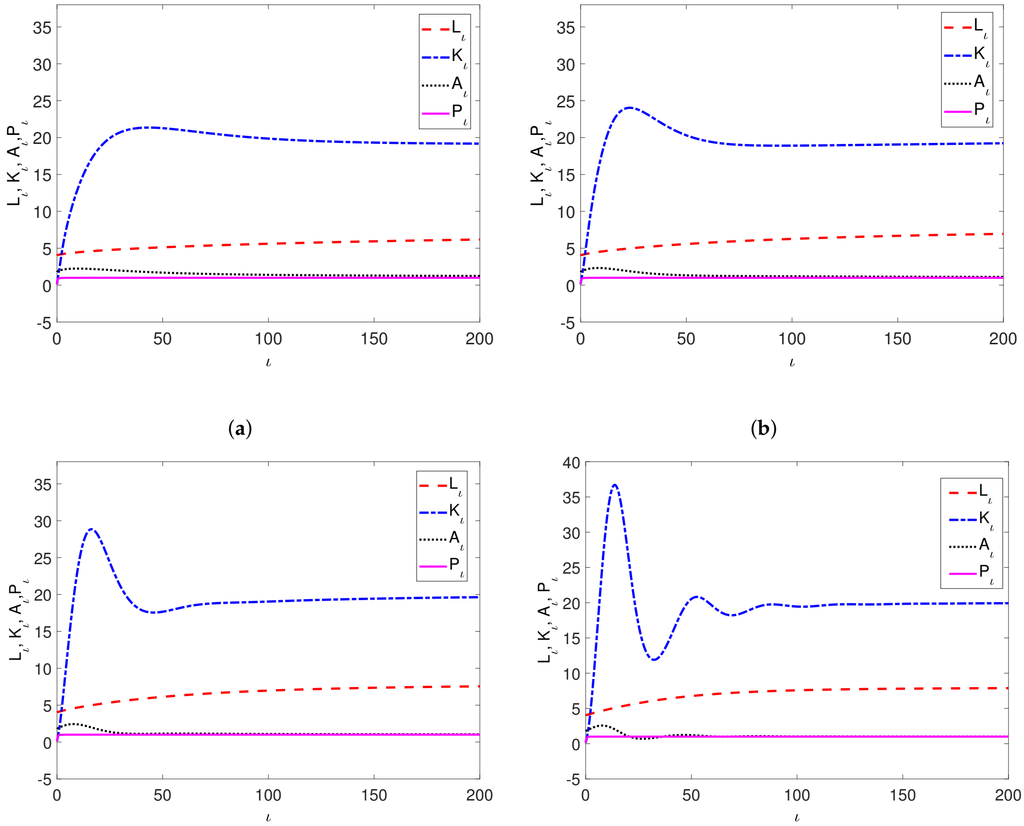
$$\begin{cases} D_\iota^q L_\iota = 0.05(1 - L_\iota/8)L_\iota, \\ D_\iota^q K_\iota = 0.4Y_\iota - lY_\iota P_\iota - 0.2K(\iota - 2), \\ D_\iota^q A_\iota = pA_\iota + \overline{w}L_\iota + hK(\iota - 2), \\ D_\iota^q P_\iota = \epsilon Y_\iota e^{-\lambda Y_\iota} - \frac{18l}{\omega} Y_\iota P_\iota - \sigma d^\beta P_\iota^{1-\beta}, \\ \ Y_\iota = A_\iota K_\iota^\alpha L_\iota^{1-\alpha}. \end{cases} \tag{33}$$

Example 1. $p = 0.04$, $\alpha = 0.4$, $\overline{w} = 0.02$, $h = -0.01, \epsilon = 1, \omega = 1, \lambda = 0, \sigma = 0, l = 1/18$ *are used to verify Theorem 1.*

The initial value is chosen as $E_0 = (4.0152, 0.1383, 1.7788, 0.1)$. From Equation (11), we can calculate the positive equilibrium point of system (33) and obtain $\overline{E} = (\overline{L}, \overline{K}, \overline{A}, \overline{P})$ $= (8.000, 19.978, 0.994, 1)$.Those can be obtained through further calculation:

$$\begin{cases} b_{22} = \alpha(s - l\overline{P})\overline{AK}^{\alpha-1}\overline{L}^{1-\alpha} = 0.079, \\ b_{23} = (s - l\overline{P})\overline{K}^\alpha\overline{L}^{1-\alpha} = 3.973, \\ b_{44} = -\frac{\rho l}{\omega}\overline{AK}^\alpha\overline{L}^{1-\alpha} = -11.467, \\ c_{22} = -(\delta + g + n) = -0.2. \end{cases}$$

It is verified that the corresponding conditions in Theorem 1 are fulfilled. Based on Theorem 1, the positive equilibrium point $\overline{E}$ is asymptotically stable. The convergence behaviors of the solution curve of system (33) about fractional order q are shown in Figure 1. From Figure 1, the smaller the fractional order, the slower the convergence speed.



(**a**) (**b**)

(**c**) (**d**)

Figure 1. Convergent behavior of system (33) about fractional order q. (**a**) Convergent behavior about fractional order $q = 0.6$. (**b**) Convergent behavior about fractional order $q = 0.7$. (**c**) Convergent behavior about fractional order $q = 0.8$. (**d**) Convergent behavior about fractional order $q = 0.9$.

Example 2. *$p = 0.02$, $\alpha = 0.5$, $h = -0.01, \overline{w} = 0, \epsilon = 1, \omega = 1, \lambda = 0, \sigma = 0, l = 1/18$ are used to verify Corollary 1. The initial value $E_0 = (4.0152, 0.1383, 1.7788, 0.1)$ is chosen. From Equation (11), the positive equilibrium point of system (33) is obtained as $\overline{E} = (\overline{L}, \overline{K}, \overline{A}, \overline{P}) = (8.000, 0.1686, 0.0843, 1)$.*

The corresponding conditions are satisfied in Corollary 1. Then, the positive equilibrium point $\overline{E}$ is asymptotically stable. The convergence behaviors of the solution curve of system (33) about the fractional order q are shown in Figure 2. From Figure 2, the smaller the fractional order, the slower the convergence speed. Similar numerical results for Corollary 2 can be obtained. Hence, verification is omitted.

Remark 6. *According to Theorem 1, the condition $s\rho - \epsilon\omega > 0$ is satisfied. This condition also can be given by Corollaries 1 and 2. Furthermore, $s - l\overline{P} > 0$ is obtained. According to [16,45], countries that have high savings/investment rates tend to be richer. If stable economic growth is to be maintained, saving rates need to be higher than the pollution rate, otherwise it will be difficult to achieve economic growth. Hence, this condition is perfectly logical and reasonable. Next, if this condition is not satisfied, is the system (33) still stable?*

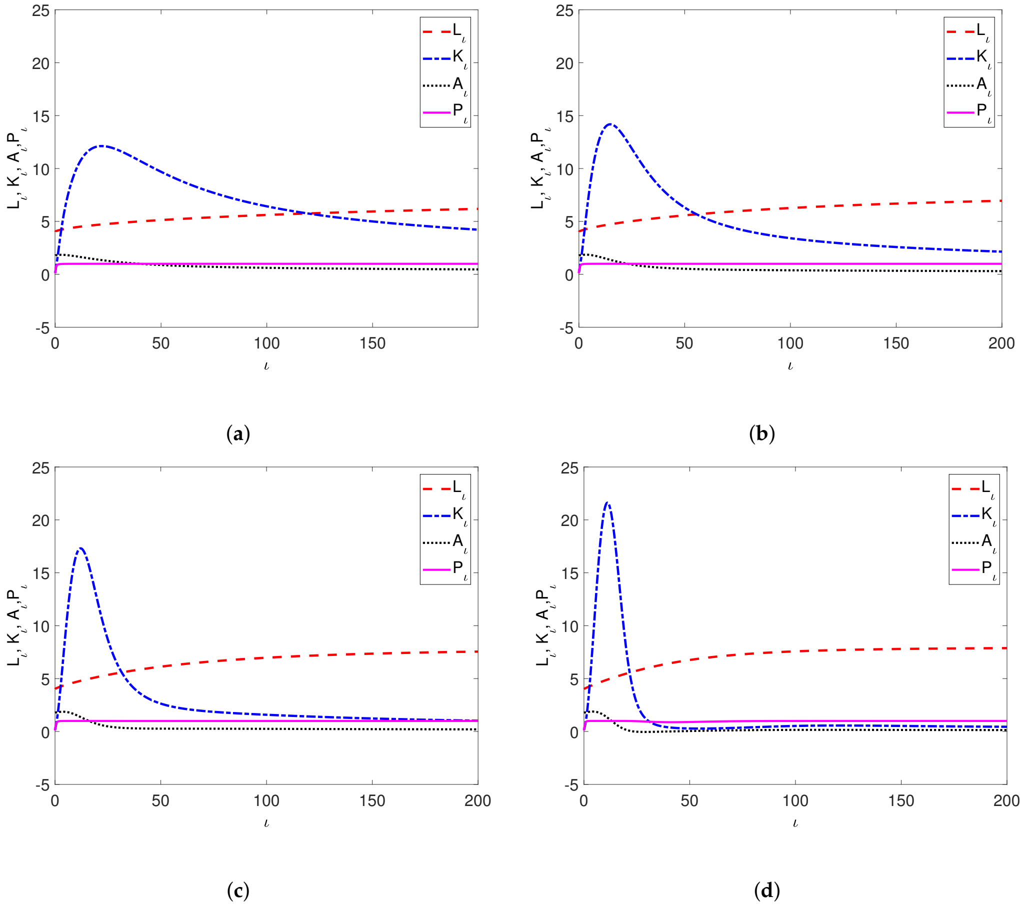


Figure 2. Convergent behavior of the system (33) about the fractional order q. (**a**) Convergent behavior about the fractional order $q = 0.6$. (**b**) Convergent behavior about fractional order $q = 0.7$. (**c**) Convergent behavior about the fractional order $q = 0.8$. (**d**) Convergent behavior about the fractional order $q = 0.9$.

Example 3. *The initial value $E_0 = (4.0152, 0.1383, 1.7788, 0.1)$ is chosen. The parameters of the system (33) are chosen as*

$$q = 0.9, \alpha = 0.4, \ p = 0.04, \ h = -0.01, \overline{w} = 0.02, \ \lambda = 0, \sigma = 0, l = 1/18.$$

$\epsilon = 5$, $\omega = 3$ are chosen. $s\rho - \epsilon\omega = -7.8 < 0$ are obtained. Then, system (33) is unstable, which is shown as Figure 3a. A locally enlarged view of $t \in [0, 20]$ is also shown as Figure 3a. $\epsilon = 7.2, \omega = 1$ are chosen, $s\rho - \epsilon\omega = 0$ is obtained. Then, system (33) is unstable, as shown in Figure 3b. A locally enlarged view of $t \in [0, 20]$ is also shown in Figure 3b. However, when $\epsilon = 6.2, \omega = 1$ are chosen, $s\rho - \epsilon\omega = 1 > 0$ is obtained. Then, system (33) is also unstable, as shown in Figure 3c. A locally enlarged view of $t \in [0, 20]$ is also shown in Figure 3c. If $\epsilon = 3, \omega = 2$ are chosen, $s\rho - \epsilon\omega = 1.2 > 0$ is obtained. Then, system (33) is stable. The equilibrium point $\overline{E} = (\overline{L}, \overline{K}, \overline{A}, \overline{P}) = (8, 53.1003, 9.4197, 6.0000)$ is asymptotically stable by Lyapunov, as shown in Figure 3d. Hence, the condition $s\rho - \epsilon\omega > 0$ is a sufficient condition.

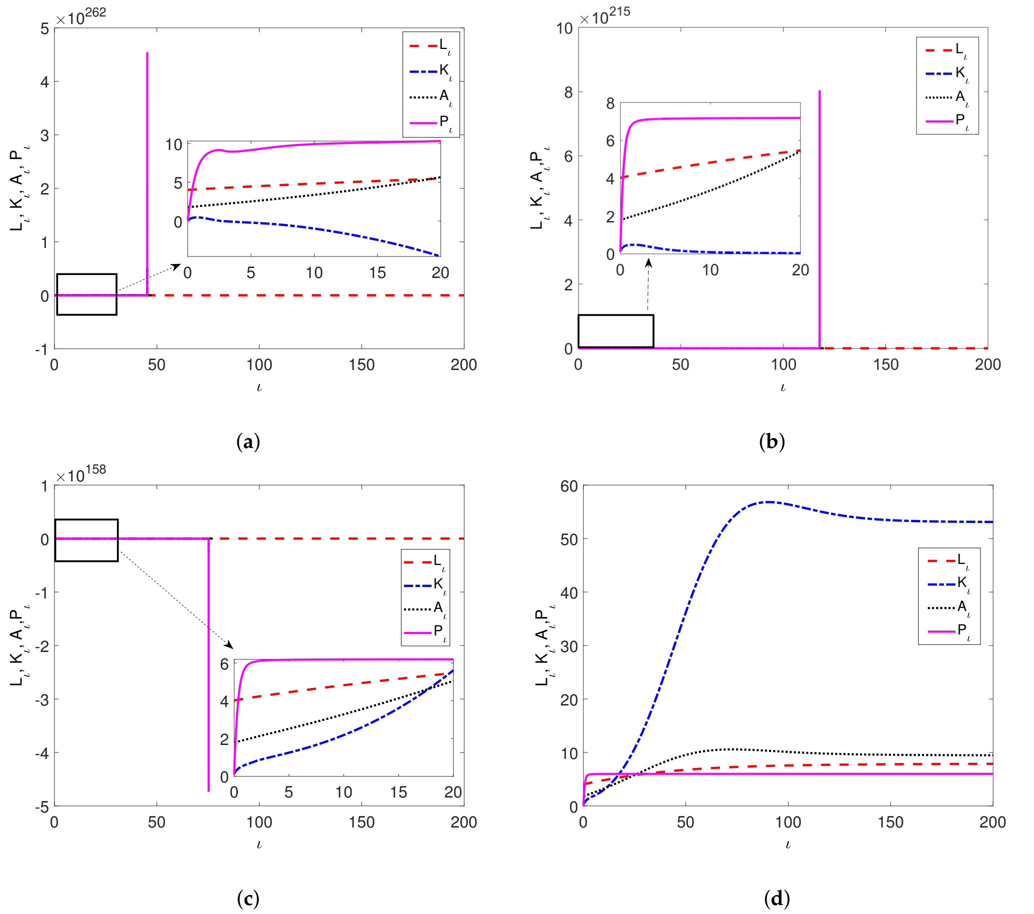


Figure 3. Convergent behavior of system (33) about the parameters ϵ, ω. (**a**) Convergent behavior about $\epsilon = 5, \omega = 3$. (**b**) Convergent behavior about $\epsilon = 7.2, \omega = 1$. (**c**) Convergent behavior about $\epsilon = 6.2, \omega = 1$. (**d**) Convergent behavior about $\epsilon = 3, \omega = 2$.

5. Discussion

Note that the conditions $\lambda = 0, \sigma = 0$ in the stability analysis of model (33), when $\lambda = 0$, approximate waste pollution $Z(\iota)$ as a linear function. When $\sigma = 0$, the function $\psi(P_\iota) = 0$; hence, the natural purification capacity is not considered in model (33). In this section, the effect of parameters λ, σ in the model (33) is mainly considered.

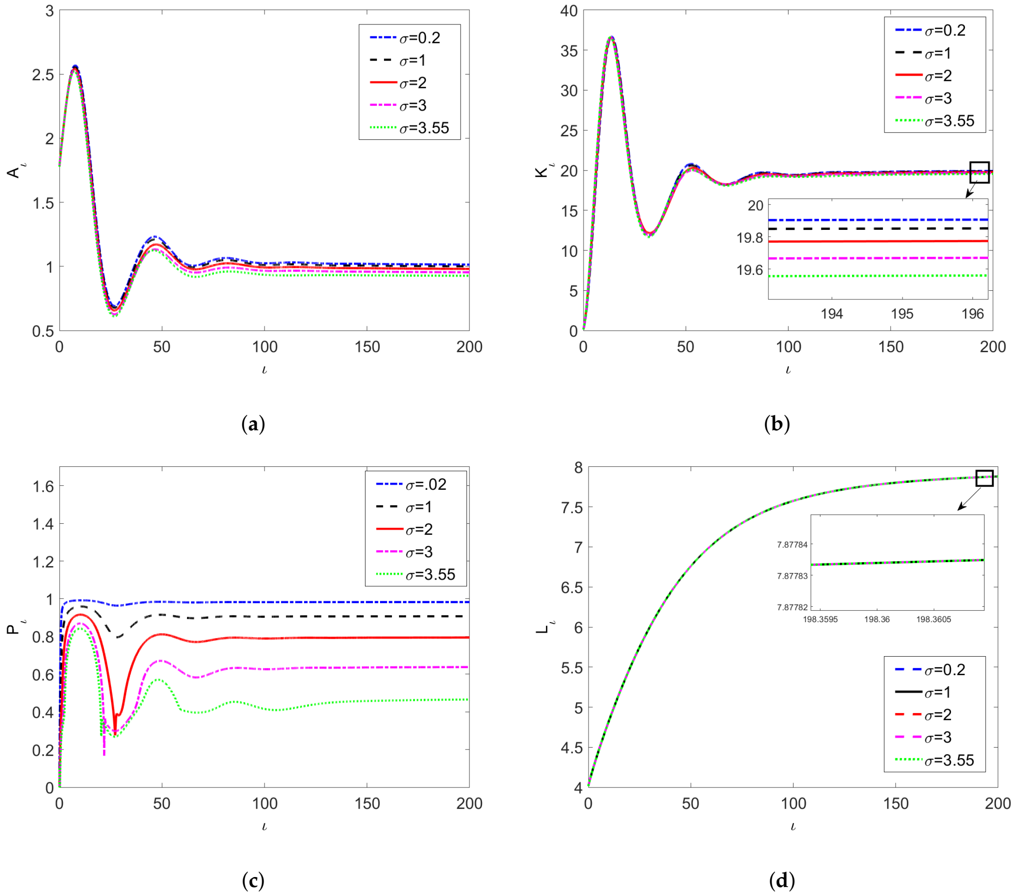
Parameters from Example 1 are considered:

$$p = 0.04,\ \alpha = 0.4,\ \overline{w} = 0.02,\ \epsilon = 1, h = -0.01, \omega = 1, l = 1/18.$$

Then, system (33) becomes

$$\begin{cases} D_\iota^q L_\iota = 0.05(1 - L_\iota/8)L_\iota, \\ D_\iota^q K_\iota = 0.4Y_\iota - 1/18 Y_\iota P_\iota - 0.2K(\iota - 2), \\ D_\iota^q A_\iota = 0.04A_\iota + 0.02L_\iota - 0.01K(\iota - 2), \\ D_\iota^q P_\iota = Y_\iota e^{-\lambda Y_\iota} - Y_\iota P_\iota - \sigma P_\iota^{1-\beta}, \\ \ Y_\iota = A_\iota K_\iota^\alpha L_\iota^{1-\alpha}. \end{cases} \tag{34}$$

When $\lambda = 0, \sigma = 0$, according to Example 1, system (34) is stable. If $\lambda = 0$ and $\sigma \in [0, 3.55]$, system (34) has a non-negative equilibrium point. When $\sigma = 1$ and $\lambda \in [0, 0.7]$, system (34) has a non-negative real equilibrium point. When $\lambda = 0$, $\sigma = 0.2, 1, 2, 3, 3.55$ are chosen and system (34) is asymptotically stable. Additionally, the convergence behavior of the solution curve of system (34) is given in Figure 4. The influence of σ on convergent behavior about A_ι is shown in Figure 4a. K_ι is shown in Figure 4b and P_ι is shown in Figure 4c. From Figure 4, A_ι, K_ι, and P_ι decrease along with σ increasing. This conclusion is drawn from system (34) ($0.4Y_\iota - 1/18Y_\iota P - 0.2K = 0$, $0.04A + 0.02L - 0.01K = 0$, $Y_\iota - Y_\iota P_\iota - \sigma P_\iota^{1-\beta} = 0$). However, L_ι remains unchanged along with σ increasing, as shown in Figure 4d.



(**a**) (**b**) (**c**) (**d**)

Figure 4. Convergent behavior of system (34) about parameters σ. (**a**) Influence of σ on the convergent behavior about A_ι. (**b**) Influence of σ on convergent behavior of K_ι. (**c**) Influence of σ on the convergent behavior of P_ι. (**d**) Influence of σ on the convergent behavior of L_ι.

When $\sigma = 0$ and $\lambda \in [0, 0.7]$, system (34) is asymptotically stable by Lyapunov. This numerical result is similar to $\lambda = 0$, $\sigma \in [0, 3.55]$. When $\lambda = 0, 0.1, 0.2, 0.4, 0.6, 0.7$ are chosen, Figure 5 shows the convergence behavior of the solution curve of the system (34). Taking $\lambda = 0$ and $\lambda > 0(\lambda = 0.1, 0.2, 0.4, 0.6, 0.7)$, the convergence behaviors of system (34) are similar. So, using $\lambda = 0$ is reasonable for getting the asymptotic stability conditions.

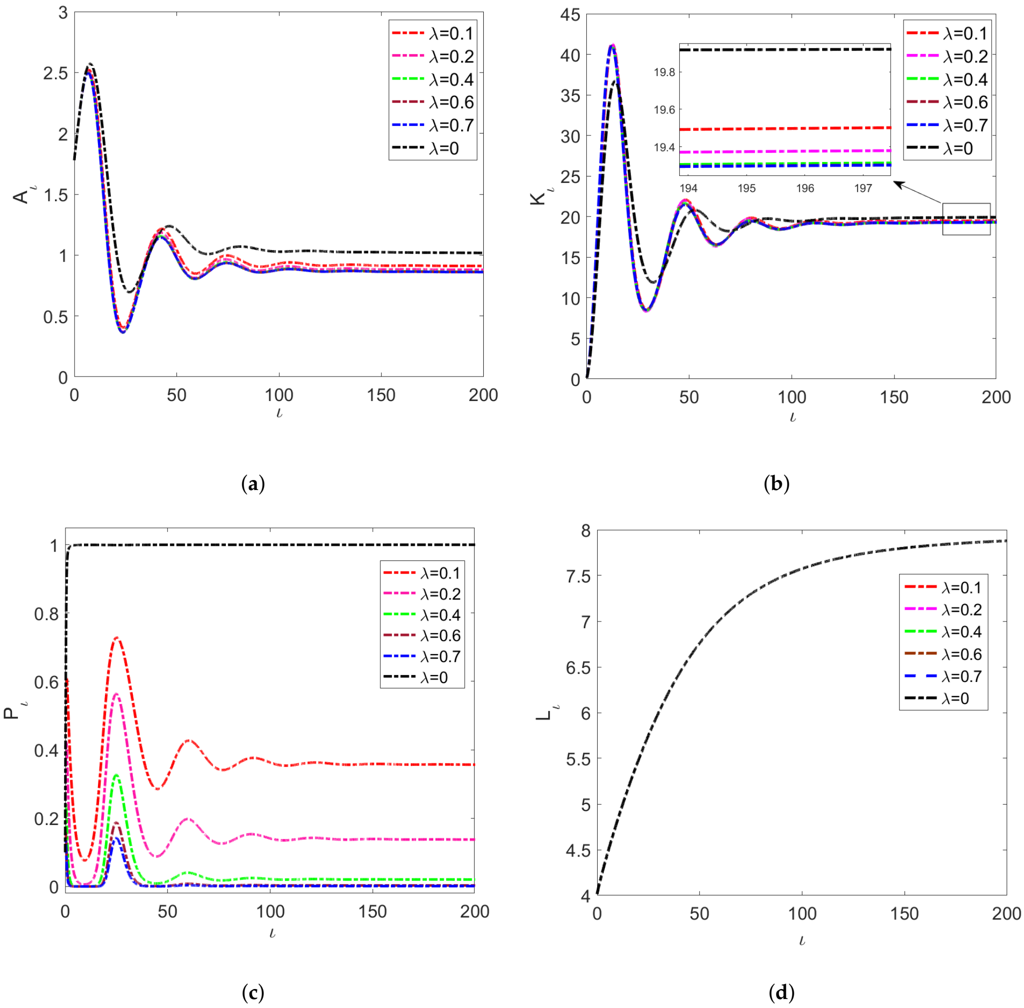


Figure 5. Convergent behavior of system (34) about parameters λ. (**a**) Influence of σ on the convergent behavior of A_ι. (**b**) Influence of σ on the convergent behavior of K_ι. (**c**) Influence of σ on the convergent behavior of P_ι. (**d**) Influence of σ on the convergent behavior of L_ι.

6. Conclusions

A fractional-order time-delayed economic growth model with environmental purification is proposed in this paper to analyze the interplay between economic growth and environmental pollution. Time delay is considered in capital stock to describe the lag effect and memory features in economic operations. The established model is proposed in the form of a fractional-order differential equation. The stability conditions of the established model are obtained, and the parameter stability interval are provided. The theoretical results are verified in the simulation. The convergence behaviors of the solution curve of the proposed model about the fractional order q are further discussed. The impacts of parameter variation on the stability of the proposed model are analyzed.

Some potential research directions of fractional-order economic growth model will be explored based on the proposed model. Note that the theoretical analysis of the proposed model is provided in this paper. In the future, the economic data and pollution data will be collected to analyze the relationship between economic growth and environmental pollution based on system (5) and Equation (6). Furthermore, a country's economic development cannot be separated from its energy consumption, and energy consumption directly leads to a large number of carbon emissions. It is of great significance to study the relationship between carbon dioxide emissions and economic growth for the implementation of energy conservation, emission reduction, and the development of low-carbon economy in cities. Based on the proposed model in this paper, some carbon dioxide emissions models (such as Kaya model [46], carbon emission simultaneous model [47]) can be used to study the relationship between carbon dioxide emissions and economic growth.

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