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Existence of Positive Solutions for Non-Local Magnetic Fractional Systems

Tahar Bouali [1], Rafik Guefaifia [2], Salah Boulaaras [2] and Taha Radwan [3,4,*]

[1] Laboratory of Mathematics, Informatics and Systems (LAMIS), Echahid Cheikh Larbi Tebessi University, Tebessa 12000, Algeria; tahar.bouali@univ-tebessa.dz
[2] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; r.guefaifia@qu.edu.sa (R.G.); s.boularas@qu.edu.sa (S.B.)
[3] Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 51452, Saudi Arabia
[4] Department of Mathematics and Statistics, Faculty of Management Technology and Information Systems, Port Said University, Port Said 42511, Egypt
* Correspondence: t.radwan@qu.edu.sa

Abstract: In this paper, the existence of a weak positive solution for non-local magnetic fractional systems is studied in the fractional magnetic Sobolev space through a sub-supersolution method combined with iterative techniques.

Keywords: positive solutions; non-local system; fractional derivatives; sub-supersolution; magnetic Sobolev space; mathematical operators

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1. Introduction and Motivation

Fractional calculus, the branch of mathematical analysis dealing with derivatives and integrals of arbitrary non-integer order, has found a wide array of applications in various real-world problems [1,2]. This extension of classical calculus has proven particularly useful in fields where systems exhibit memory and hereditary properties [3,4]. On the other hand, the analysis of the solutions for nonlinear systems is a fundamental concept in mathematical analysis and is crucial for ensuring that models accurately describe real-world phenomena [5,6]. The existence of positive solutions in fractional systems is important because it ensures that the models we use are sound, practical, and stable. This reliability is crucial for accurately describing complex systems with non-local interactions and memory effects. As a result, it leads to better predictions, improved control strategies, and more effective optimization solutions in various scientific and engineering fields.

The study of positive solutions in non-local fractional systems contributes to the advancement of mathematical theory [7]. Establishing the existence of positive solutions often requires the development of new analytical techniques and methods, which can then be applied to a broader class of problems in fractional calculus and differential equations. The existence of positive solutions to non-local magnetic fractional systems involves a blend of advanced mathematical techniques from fractional calculus, variational methods, and topological arguments. The interplay between the fractional nature of the operators and the magnetic effects makes this a rich and ongoing area of research.

In this paper, we study the existence of positive solutions to the following non-local magnetic fractional systems:

$$\begin{cases} \mathcal{K}\left(\|\xi\|^2_{X_{0,\varphi}}\right)(-\triangle)^{\gamma}_{\varphi}\xi = \rho_1\alpha(x)\psi(\eta) + \mu_1\beta(x)\Upsilon(\xi) \text{ in } \Lambda, \\ \mathcal{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)(-\triangle)^{\gamma}_{\varphi}\eta = \rho_2\nu(x)\chi(\xi) + \mu_2\tau(x)\Phi(\eta) \text{ in } \Lambda, \\ \xi = \eta = 0 \text{ in } \mathbb{R}^N\backslash\Lambda, \end{cases} \tag{1}$$

where ρ_1, ρ_2, μ_1, and μ_2 are positive parameters and $\partial\Lambda$ is the smooth boundary of Λ, $\Lambda \subset \mathbb{R}^N$ $(N \geq 3)$ with $C^{1,1}$ for $N \geq 3$, $\mathcal{K}, \mathcal{R} : \mathbb{R}^+ \to \mathbb{R}^+$. The nonlinear source terms $\chi, \psi, \Upsilon, \Phi : \Lambda \times \mathbb{R} \to \mathbb{R}$ are positives, and $\alpha(.), \beta(.), \nu(.), \tau(.) \in C(\overline{\Lambda})$.

Take $\gamma \in (0,1)$, $(-\triangle)^{\gamma}_{\varphi}$ as the non-local fractional magnetic operator, defined in [8]

$$(-\triangle)^{\gamma}_{\varphi}u(x) := C_{N,\gamma} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\backslash B_{\varepsilon}(x)} \frac{u(x) - e^{i(x-y)\cdot\varphi\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2\gamma}}dy,\ x \in \mathbb{R}^N$$

with $u \in C_0^{\infty}(\mathbb{R}^N, \mathbb{C})$, where $B_{\varepsilon}(x)$ denotes the open ball in $\mathbb{R}^N$ of center x and radius ε, and $C_{N,\gamma} := \gamma 2^{2\gamma}\pi^{\frac{-N}{2}}\Gamma\left(\frac{N+2\gamma}{2}\right)/\Gamma(1-\gamma) > 0$. It is a constant mentioned in [9].

In recent years, mathematical modeling has developed using fractional operators and has appeared in many physical applications (for example, see [10,11]). We mention, but are not limited to, material transfer processes in fractured media and unusual diffusion processes in turbulent fluid movements. The fractional Laplacian also appears, in particular, in modern physics and in vibrational motions [12].

We can consider $(-\triangle)^{\gamma}_{\varphi}$ as a fractional counterpart of the magnetic Laplacian $(\nabla - i\varphi)^2$, with $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ being an L^{∞}_{loc}–vector potential; see, e.g., [13], Chapter 7. It is the Schrodinger factor for a particle in the presence of an external magnetic field, and plays a fundamental role in quantum mechanics in describing the dynamics of a particle in a non-relativistic environment.

In this context, when $N = 3$, the operator $\nabla \times \varphi$ represents a magnetic field acting on a charged particle. Obviously, when $\varphi \equiv 0$ and $u \in C_0^{\infty}(\mathbb{R}^N, \mathbb{R})$, the operator $(-\triangle)^{\gamma}_{\varphi}$ corresponds to the fractional Laplacian norm $(-\triangle)^{\gamma}$, defined as as principal value integral

$$(-\triangle)^{\gamma}u(x) := C_{N,\gamma} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\backslash B_{\varepsilon}(x)} \frac{u(x) - u(y)}{|x-y|^{N+2\gamma}}dy, x \in \mathbb{R}^N.$$

For more detail, see [14,15] .

The systems in Equation (1) are a fractional version related to the hyperbolic equation

$$\rho\frac{\partial^2\xi}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial\xi}{\partial x}\right|^2 dx\right)\frac{\partial^2\xi}{\partial x^2} = 0, \tag{2}$$

and they are considered a generalization of the D'Alembert wave equation, which was first presented by Kirchhoff [16].

Recently, many authors have been interested in elliptic problems involving non-local operators by using variational and topological methods. See, for example, the works in [17–19] and the references therein.

In [19], the authors use a variational method to investigate the existence and multiplicity of weak solutions to non-local equations involving the magnetic fractional Laplacian when the nonlinearity is subcritical and asymptotically linear at infinity.

FiscellaI and Eugenio, in [17], performed a study with bifurcation phenomena and the existence of multiple solutions for a non-local boundary value problem driven by the magnetic fractional Laplacian $(-\triangle)^{\gamma}_{\varphi}$. In particular, they used the equation

$$(-\triangle)^{\gamma}_{\varphi}u = \lambda u + |u|^{2^*_\gamma-2}u, \text{ in } \Omega,$$
$$u = 0, \text{ in } \mathbb{R}^N\backslash\Omega.$$

where λ is a real parameter and $\Omega \subset \mathbb{R}^N$ is an open and bounded set with Lipschitz boundary.

In [18], Jiabin Zuo et al. present a study on the existence and asymptotic behavior of solutions to a fractional Kirchhoff-type problem involving electromagnetic fields and critical nonlinearities using classical critical point theory. An example is given to illustrate the application of the main result.

In [9], when $\varphi \equiv 0$, the authors discussed proving the existence of positive solutions theoretically using the supersolution method. They also presented branch diagrams based on famous theories in numerical analysis. The author of [20] studied a fractional population model with a homogeneous Dirichlet boundary condition. According to the positive coefficient, the authors of [21] studied a Dirichlet-type problem for an equation involving the fractional Laplacian and a reaction term subject to either subcritical or critical growth conditions, depending on a positive parameter. Applying a critical point result of Bonanno, they proved the existence of one or two positive solutions as soon as the parameter lies under a (explicitly determined) value. As an application, they found two positive solutions for a fractional Ambrosetti–Brezis–Cerami problem.

The authors of [22] proved the existence of at least one positive solution for a non-local semipositone problem of the type

$$\begin{aligned} (-\triangle)^{\gamma}u &= \lambda(u^q - 1) + \mu u^r, \text{ in } \Omega, \\ u &> 0, \text{ in } \Omega, \\ u &= 0, \text{ in } \mathbb{R}^N\backslash\Omega. \end{aligned} \tag{3}$$

Here, $\Omega \subset \mathbb{R}^N$ is assumed to be a bounded open set with a smooth boundary, $\gamma \in (0,1)$, $N > 2\gamma$, and $0 < q < 1 < r \leq \frac{N+2\gamma}{N-2\gamma}$. The proof was based on constructing a positive subsolution for (3) when $\lambda > \lambda_0$, and, in the case of $1 < r \leq \frac{N+2\gamma}{N-2\gamma}$, they show the existence of a second positive solution by using the mountain pass argument.

In this paper, we provide some new results for non-local fractional magnetic system (1). According to [17,19] and the references therein, we prove the validity of the comparison principal in the magnetic fractional Laplacian, and by using the sub-supersolution method combined with an iterative technic we prove the existence of a positive solution for magnetic fractional system (1). The results are obtained by considering the nonlinear source terms as sublinear at infinity. When then the nonlinearities $\chi$and ψ are written in a special case, we prove that magnetic fractional system (1) does not accept solutions.

This work is organized as follows: we give definitions and the properties of magnetic fractional spaces that are important for our paper in Section 2, and then demonstrate our results in Section 3. Finally, in Section 4 we provide the result of the non-existence of solutions for magnetic fractional system (1).

2. Preliminaries

Notations

- $\Re z$, $\overline{z}$, and $|z|$ are, respectively, the real part, the complex conjugate, and the modulus of a given $z \in \mathbb{C}$.
- $L^2(\Lambda, \mathbb{C})$ denotes the Lebesgue space of measurable functions $u : \Lambda \to \mathbb{C}$ such that

$$|u|_2^2 = \int\limits_{\Lambda} |u(x)|^2 dx < +\infty,$$

where $|.|$ is the Euclidean norm in $\mathbb{C}$, endowed with the real scalar product

$$\langle u,v\rangle_2 := \Re\int\limits_{\Lambda} u\overline{v}dx,\ \text{for all } u,v \in L^2(\Lambda,\mathbb{C}).$$

Based on [17,19], we present some basic definitions of spaces.

Let

$$\mathcal{H}_\varphi^\gamma(\Lambda) = \left\{\xi \in L^2(\Lambda,\mathbb{C}):\ \|\xi\|_{\mathcal{H}_\varphi^\gamma(\Lambda)} < +\infty\right\},$$

in which

$$\|\xi\|_{\mathcal{H}_\varphi^\gamma(\Lambda)} := \left(\|\xi\|^2_{L^2(\Lambda,\mathbb{C})} + [\xi]^2_{\mathcal{H}_\varphi^\gamma(\Lambda)}\right)^{\frac{1}{2}}, \tag{4}$$

and

$$[\xi]_{\mathcal{H}^\gamma(\Lambda)} := \left(\frac{C_{N,\gamma}}{2}\int_\Lambda\int_\Lambda \frac{|\xi(x) - e^{i(x-y).\varphi\left(\frac{x+y}{2}\right)}\xi(y)|^2}{|x-y|^{N+2\gamma}}dxdy\right)^{\frac{1}{2}},$$

where $[\xi]^2_{\mathcal{H}^\gamma(\Lambda)}$ is Gagliardo seminorm. The magnetic fractional Sobolev space $\mathcal{H}_\varphi^\gamma(\mathbb{R}^N)$ (see [19] or [17]) is defined by the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to norm (5).

Let us consider the functional space

$$X_{0,\varphi} = \left\{\xi \in \mathcal{H}_\varphi^\gamma\left(\mathbb{R}^N\right) : \xi = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Lambda\right\}$$

As in [8], we define the following real scalar product on $X_{0,\varphi}$:

$$\langle \xi,\eta\rangle_{X_{0,\varphi}} := \frac{C_{N,\gamma}}{2}\Re\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left(\xi(x) - e^{i(x-y).\varphi\left(\frac{x+y}{2}\right)}\xi(y)\right)\overline{\left(\eta(x) - e^{i(x-y).\varphi\left(\frac{x+y}{2}\right)}\eta(y)\right)}}{|x-y|^{N+2\gamma}}dxdy\right). \tag{5}$$

The norm $\|\xi\|^2_{X_{0,\varphi}} := \langle\xi,\xi\rangle_{X_{0,\varphi}}$ is equivalent to (5) in $\mathcal{H}_\varphi^\gamma(\mathbb{R}^N)$ (see [23], Lemma 2.1) and $(X_{0,\varphi}, \langle .,.\rangle_{X_{0,\varphi}})$ is a real separable Hilbert space (see [24], Lemma 7)

3. Main Results

Lemma 1. *Let* $\mathfrak{R}:\mathbb{R}^+\to\mathbb{R}^+$ *be a continuous function satisfy the following conditions:*

$$\mathfrak{R}(b) \le \mathfrak{R}(a) \text{ for all } b > a, \text{ for all } a,b \in \mathbb{R},\ \text{i.e., } \mathfrak{R} \text{ nonincreasing}$$

$$\mathfrak{R}(t) > r_0, \text{ for all } t \ge t_0,\ r_0 > 0 \tag{6}$$

For non-negative functions, ξ and η are satisfied

$$\begin{cases} \mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi\xi \le \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi\eta \text{ in } \Lambda, \\ \xi = \eta = 0 \text{ in } \mathbb{R}^N\backslash\Lambda. \end{cases} \tag{7}$$

Then, $\xi \le \eta$ in Λ.

Proof. To prove the result, we consider the function $H(t) = t\mathfrak{R}(t^2), t \ge 0$ on $\mathbb{R}^+$ such that

$$H(b) \ge H(a) \text{ for all a,} b \in \mathbb{R}, \text{i.e.,} H \text{ increasing}$$

Multiplying both sides of the inequality by ξ and η and integrating, we obtain

$$\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\|\xi\|^2_{X_{0,\varphi}} \le \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\langle\eta,\xi\rangle_{X_{0,\varphi}}$$

and

$$\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\langle\eta,\xi\rangle_{X_{0,\varphi}} \le \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\|\eta\|^2_{X_{0,\varphi}}.$$

Through mathematical skills, we have

$$\frac{\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\|\xi\|^2_{X_{0,\varphi}}}{\mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)} \leq \frac{\mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\|\eta\|^2_{X_{0,\varphi}}}{\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)}$$

This means that

$$\left(\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\right)^2\|\xi\|^2_{X_{0,\varphi}} \leq \left(\mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\right)^2\|\eta\|^2_{X_{0,\varphi}}$$

and so

$$\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\|\xi\|_{X_{0,\varphi}} \leq \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\|\eta\|_{X_{0,\varphi}}$$

i.e.,

$$H\left(\|\xi\|^2_{X_{0,\varphi}}\right) \leq H\left(\|\eta\|^2_{X_{0,\varphi}}\right).$$

With the function H increasing, that is,

$$\|\xi\|_{X_{0,\varphi}} \leq \|\eta\|_{X_{0,\varphi}},$$

then

$$\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right) \geq \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right). \tag{8}$$

This is because the function $\mathfrak{R}$ on $\mathbb{R}^+$ is nonincreasing, by (7):

$$\begin{cases} (-\triangle)^\gamma_\varphi\left(\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\xi - \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\eta\right) \leq 0 \text{ in } \Lambda, \\ \xi = \eta = 0 \text{ in } \mathbb{R}^N \backslash \Lambda, \end{cases}$$

The maximum principle of Laplace's partial non-local magnetic operator brings us to see [25]

$$\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\xi - \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\eta \leq 0$$

which implies

$$\mathfrak{R}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\xi \leq \mathfrak{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\eta.$$

From (8), we conclude that $\xi \leq \eta$. Hence, it was proven. □

Lemma 2 ([26], Proposition 2.1)**.** *The embedding $X_{0,\varphi}(\Lambda) \hookrightarrow L^\nu(\Lambda)$ is continuous and, for all $\xi \in X_{0,\varphi}(\Lambda)$ and $\nu \in \left[2, 2^*_\gamma\right]$, then*

$$\|\xi\|_\nu \leq S(N,\gamma)|\Lambda|^{\frac{2^*_\gamma - \nu}{2^*_\gamma . \nu}}\|\xi\|_{X_{0,\varphi}(\Lambda)},$$

where

$$\begin{aligned} S(N,\gamma) \quad : \quad &= \max_{\xi \in X_{0,\varphi}(\Lambda)/0} \frac{\|\xi\|_{2^*_\gamma}}{\|\xi\|_{X_{0,\varphi}(\Lambda)}} \\ : \quad &= \frac{\gamma^{\frac{1}{2}}\Gamma\left(\frac{N-2\gamma}{2}\right)^{\frac{1}{2}}\Gamma(N)^{\frac{\gamma}{N}}}{2^{\frac{1}{2}}\pi^{\frac{N+2\gamma}{4}}\Gamma(1-\gamma)^{\frac{1}{2}}\Gamma\left(\frac{N}{2}\right)^{\frac{\gamma}{N}}} > 0, \end{aligned}$$

is the best Sobolev constant of the magnetic Sobolev embedding, and $2^*_\gamma = \frac{2N}{N-2\gamma}$.

Considering the following hypotheses:

$(H1)$ Let $\mathcal{K}, \mathcal{R} : \mathbb{R}^+ \to \mathbb{R}^+$. The same condition applies to the function $\Re$, and so we have $k_i, r_i > 0, i = 1, 2$, in which

$$k_1 \leq \mathcal{K}(t) \leq k_2, \quad r_1 \leq \mathcal{R}(t) \leq r_2 \text{ for all } t \in \mathbb{R}^+;$$

$(H2)$ The functions $\alpha(.), \beta(.), \nu(.), \tau(.) \in C(\overline{\Lambda})$:

$$\alpha(x) \geq \alpha_0, \beta(x) \geq \beta_0, \upsilon(x) \geq \upsilon_0, \tau(x) \geq \tau_0 > 0,$$

$(H3)$ Let the functions ψ, χ, Υ, and Φ on $[0, +\infty[$ be increasing and continuous such that

$$\lim_{t \to +\infty} \psi(t) = \lim_{t \to +\infty} \chi(t) = \lim_{t \to +\infty} \Upsilon(t) = \lim_{t \to +\infty} \Phi(t) = +\infty$$

$(H4)$ The following is satisfied:

$$\lim_{t \to +\infty} \frac{\psi(K(\chi(t)))}{t} = 0, \text{ for all } R > 0,$$

$(H5)$

$$\lim_{t \to +\infty} \frac{\Upsilon(t)}{t} = \lim_{t \to +\infty} \frac{\Phi(t)}{t} = 0.$$

Definition 1. *We say that a function $(\xi, \eta) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$ is a weak solution of non-local magnetic fractional system (1) if it fulfills*

$$\begin{aligned} \mathcal{K}\left(\|\xi\|^2_{X_{0,\varphi}}\right)\langle \xi, u\rangle_{X_{0,\varphi}} &= \rho_1 \Re \int_\Lambda \alpha(x)\psi(\eta)\overline{u}dx + \mu_1 \Re \int_\Lambda \beta(x)\Upsilon(\xi)\overline{u}dx, \;\; in\ \Lambda, \\ \mathcal{R}\left(\|\eta\|^2_{X_{0,\varphi}}\right)\langle \eta, v\rangle_{X_{0,\varphi}} &= \rho_2 \Re \int_\Lambda \nu(x)\chi(\xi)\overline{v}dx + \mu_2 \Re \int_\Lambda \tau(x)\Phi(\eta)\overline{v}dx, \;\; in\ \Lambda, \end{aligned} \tag{9}$$

for all $(u, v) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$ with $(u, v) \geq (0, 0)$ in Λ.

Definition 2. *We say that a function $(\xi_*, \eta_*) \in (X_{0,\varphi}(\mathbb{R}^N) \times X_{0,\varphi}(\mathbb{R}^N))$ is a weak subsolution of non-local magnetic fractional system (1) if it fulfills*

$$\begin{aligned} \mathcal{K}\left(\|\xi_*\|^2_{X_{0,\varphi}}\right)\langle \xi_*, u\rangle_{X_{0,\varphi}} &\leq \rho_1 \Re \int_\Lambda \alpha(x)\psi(\eta_*)\overline{u}dx + \mu_1 \Re \int_\Lambda \beta(x)\Upsilon(\xi_*)\overline{u}dx, \quad in\ \Lambda, \\ \mathcal{R}\left(\|\eta_*\|^2_{X_{0,\varphi}}\right)\langle \eta_*, v\rangle_{X_{0,\varphi}} &\leq \rho_2 \Re \int_\Lambda \nu(x)\chi(\xi_*)\overline{v}dx + \mu_2 \Re \int_\Lambda \tau(x)\Phi(\eta_*)\overline{v}dx, \quad in\ \Lambda, \\ (\xi_*, \eta_*) &\geq 0 \text{ a.e. in } \mathbb{R}^N \backslash \Lambda. \end{aligned} \tag{10}$$

for all $(u, v) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$ with $(u, v) \geq (0, 0)$ in Λ.

Definition 3. *We say that a function $(\xi^*, \eta^*) \in (\mathcal{H}^\gamma(\mathbb{R}^N) \times \mathcal{H}^\gamma(\mathbb{R}^N))$ is a weak supersolution of magnetic fractional system (1) if it fulfills*

$$\begin{aligned} \mathcal{K}\left(\|\xi^*\|^2_{X_{0,\varphi}}\right)\langle \xi^*, u\rangle_{X_{0,\varphi}} &\geq \rho_1 \Re \int_\Lambda \alpha(x)\psi(\eta^*)\overline{u}dx + \mu_1 \Re \int_\Lambda \beta(x)\Upsilon(\xi^*)\overline{u}dx, \; in\ \Lambda, \\ \mathcal{R}\left(\|\eta^*\|^2_{X_{0,\varphi}}\right)\langle \eta^*, v\rangle_{X_{0,\varphi}} &\geq \rho_2 \Re \int_\Lambda \nu(x)\chi(\xi^*)\overline{v}dx + \mu_2 \Re \int_\Lambda \tau(x)\Phi(\eta^*)\overline{v}dx, \; in\ \Lambda, \\ (\xi^*, \eta^*) &\geq 0 \text{ a.e. in } \mathbb{R}^N \backslash \Lambda. \end{aligned} \tag{11}$$

for all $(u, v) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$ with $(u, v) \geq (0, 0)$ in Λ.

Remark 1. *If there exist subsolutions (ξ_*, η_*) and supersolutions (ξ^*, η^*) such that $(\xi_*, \eta_*) \leq (\xi^*, \eta^*)$ on Λ, non-local magnetic fractional system (1), then one can find a solution (ξ, η) fulfilling $(\xi_*, \eta_*) \leq (\varepsilon, \eta) \leq (\xi^*, \eta^*)$ on Λ.*

Theorem 1. *When $\rho_1\alpha_0 + \mu_1\beta_0$ and $\rho_2\nu_0 + \mu_2\tau_0$ are large, then non-local magnetic fractional system (1) has a positive weak solution if $(H1)$–$(H5)$ holds true.*

Proof. Let λ_1 be the first eigenvalue of $(-\Delta)^\gamma_\varphi$ in Λ and $\phi_1 > 0$ be the associated eigenfunction [17], that is,

$$\begin{cases} (-\triangle)^\gamma_\varphi \phi_1 = \lambda_1 \phi_1 \text{ in } \Lambda, \\ \phi_1 > 0 \qquad \text{in } \Lambda, \\ \phi_1 = 0 \qquad \text{in } \mathbb{R}^N \backslash \Lambda, \end{cases} \tag{12}$$

the variational characterization of λ_1 is given by

$$\lambda_1 := \inf_{\phi_1 \in X_{0,\varphi} \backslash \{0\}} \frac{\|\phi_1\|^2_{X_{0,\varphi}}}{|\phi_1|^2_2}.$$

Moreover, from [25], there are positive constants c_1, c_2 that satisfy

$$0 < c_1 \delta^\gamma < \phi_1(x) < c_2 \delta^\gamma, \text{ where } \delta := \delta(x) := d(x, \partial\Lambda). \tag{13}$$

Also, a positive constant, ν, can be obtained tha satisfies $\nu < h(x) < +\infty$ for all $x \in \Lambda$. It fulfills

$$h(x) := \int_{\mathbb{R}^N} \frac{|\phi_1(x) - e^{i(x-y).\varphi\left(\frac{x+y}{2}\right)} \phi_1(y)|^2}{|x-y|^{N+2\gamma}} dy. \tag{14}$$

Using a calculation similar to what was stated in ([22], Lemma 3.1), we have

$$(-\Delta)^\gamma_\varphi \phi_1^2(x) = 2\lambda_1 \phi_1^2(x) - 2C_{N,\gamma} h(x) \text{ in } \Lambda,$$

there is also $0 < \mu, m$ such that

$$m < h(x) - 2\lambda_1 \phi_1^2(x) \text{ in } \Lambda_\sigma, \tag{15}$$

and

$$\mu \leq \phi_1 \leq 1 \text{ in } \Lambda \backslash \overline{\Lambda}_\sigma, \tag{16}$$

where

$$\overline{\Lambda}_\sigma = \{x \in \Lambda : \delta(x) = d(x, \partial\Lambda) \leq \sigma\}.$$

As the functions ψ, Υ, Φ, and χ, are continuous on $[0, +\infty[$ and hold for $(H3)$–$(H5)$, then we have $\theta_0 > 0$ in a manner that

$$\psi(s) \geq -\theta_0, \ \chi(s) \geq -\theta_0, \ \Upsilon(s) \geq -\theta_0, \ \Phi(s) \geq -\theta_0 \text{ for all } s \geq 0. \tag{17}$$

Further, we define

$$\xi_* = \left(\frac{(\rho_1\alpha_0 + \mu_1\beta_0)\theta_0}{mk_1} \right) \phi_1^2,$$

and

$$\eta_* = \left(\frac{(\rho_2\nu_0 + \mu_2\tau_0)\theta_0}{mr_1} \right) \phi_1^2;$$

in which k_1 and r_1 are obtained from $(H1)$. Furthermore, it is obvious that $(\xi_*, \eta_*) \in (X_{0,\varphi}(\mathbb{R}^N) \times X_{0,\varphi}(\mathbb{R}^N))$.

When $\rho_1\alpha_0 + \mu_1\beta_0$ and $\rho_2\nu_0 + \mu_2\tau_0$ are large enough, we will confirm that (ξ_*, η_*) is a subsolution of non-local magnetic fractional system (1). Let $u \in X_{0,\varphi}(\Lambda)$ with $u \geq 0$ in Λ, and by direct calculation

$$\begin{aligned}\mathcal{K}\Big(\|\xi_*\|^2_{X_{0,\varphi}}\Big)\langle \xi_*, u\rangle_{X_{0,\varphi}} &= \mathcal{K}\Big(\|\xi_*\|^2_{X_{0,\varphi}}\Big)\frac{(\rho_1\alpha_0+\mu_1\beta_0)\theta_0}{mk_1}\langle \phi_1^2, u\rangle_{X_{0,\varphi}} \\ &= \frac{(\rho_1\alpha_0+\mu_1\beta_0)\theta_0}{mk_1}\mathcal{K}\Big(\|\xi_*\|^2_{X_{0,\varphi}}\Big)\Re\int_\Lambda\Big\{2\lambda_1\phi_1^2(x) - 2C_{N,\gamma}h(x)\Big\}\overline{u}(x)dx.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{R}\Big(\|\eta_*\|^2_{X_{0,\varphi}}\Big)\langle \eta_*, v\rangle_{X_{0,\varphi}} &= \mathcal{R}\Big(\|\eta_*\|^2_{X_{0,\varphi}}\Big)\frac{(\rho_2\nu_0+\mu_2\tau_0)\theta_0}{mr_1}\langle \phi_1^2, v\rangle_{X_{0,\varphi}} \\ &= \frac{(\rho_2\nu_0+\mu_2\tau_0)\theta_0}{mr_1}\mathcal{R}\Big(\|\eta_*\|^2_{X_{0,\varphi}}\Big)\Re\int_\Lambda\Big\{2\lambda_1\phi_1^2(x) - 2C_{N,\gamma}h(x)\Big\}\overline{v}(x)dx.\end{aligned}$$

Thus, (ξ_*, η_*) is a weak subsolution of non-local magnetic fractional system (1) if

$$\begin{aligned}&\frac{(\rho_1\alpha_0+\mu_1\beta_0)\theta_0}{mk_1}\mathcal{K}\Big(\|\xi_*\|^2_{X_{0,\varphi}}\Big)\Re\int_\Lambda\Big\{2\lambda_1\phi_1^2(x) - 2C_{N,\gamma}h(x)\Big\}\overline{u}(x)dx \\ \leq\ & \rho_1\Re\int_\Lambda \alpha(x)\psi(\eta_*)\overline{u}dx + \mu_1\Re\int_\Lambda \beta(x)\Upsilon(\xi_*)\overline{u}dx,\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\frac{(\rho_2\nu_0+\mu_2\tau_0)\theta_0}{mr_1}\mathcal{R}\Big(\|\eta_*\|^2_{X_{0,\varphi}}\Big)\Re\int_\Lambda\Big\{2\lambda_1\phi_1^2(x) - 2C_{N,\gamma}h(x)\Big\}\overline{v}(x)dx \\ \leq\ & \rho_2\Re\int_\Lambda \nu(x)\chi(\xi_*)\overline{v}dx + \mu_2\Re\int_\Lambda \tau(x)\Phi(\eta_*)\overline{v}dx,\end{aligned} \tag{19}$$

for all $(u,v) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$ with $(u,v) \geq (0,0)$ in Λ. Here, we distinguish two cases: $x \in \Lambda_\sigma$ and $x \in \Lambda\backslash\overline{\Lambda}_\sigma$. If $x \in \Lambda_\sigma$ using (18) and (19), it is

$$\begin{aligned}&\frac{(\rho_1\alpha_0+\mu_1\beta_0)\theta_0}{mk_1}\mathcal{K}\left(\frac{C_{N,\gamma}}{2}\int_{\Lambda_\sigma}\int_{\Lambda_\sigma}\frac{|\xi_*(x)-e^{i(x-y)\cdot\varphi\left(\frac{x+y}{2}\right)}\xi_*(y)|^2}{|x-y|^{N+2\gamma}}dxdy\right) \\ &\qquad\times\Re\int_{\Lambda_\sigma}\{2\lambda_1\phi_1^2(x) - 2C_{N,\gamma}h(x)\}\overline{u}(x)dx \\ =\ & -\frac{(\rho_1\alpha_0+\mu_1\beta_0)\theta_0}{mk_1}\mathcal{K}\left(\frac{C_{N,\gamma}}{2}\int_{\Lambda_\sigma}\int_{\Lambda_\sigma}\frac{|\xi_*(x)-e^{i(x-y)\cdot\varphi\left(\frac{x+y}{2}\right)}\xi_*(y)|^2}{|x-y|^{N+2\gamma}}dxdy\right) \\ &\qquad\times\Re\int_{\Lambda_\sigma}\{2C_{N,\gamma}h(x) - 2\lambda_1\phi_1^2(x))\}\overline{u}(x)dx \\ <\ & (\rho_1\alpha_0+\mu_1\beta_0)\Re\int_{\Lambda_\sigma} -\theta_0\overline{u}(x)dx \\ \leq\ & \rho_1\Re\int_{\Lambda_\sigma}\alpha(x)\psi(\eta_*)\overline{u}dx + \mu_1\Re\int_{\Lambda_\sigma}\beta(x)\Upsilon(\xi_*)\overline{u}dx,\end{aligned} \tag{20, 21}$$

for all $u \in X_{0,\varphi}(\Lambda)$ with $u \geq 0$ in Λ. In the same way, we have

$$\frac{(\rho_2\nu_0+\mu_2\tau_0)\theta_0}{mr_1}\mathcal{R}\Big(\|\eta_*\|^2_{X_{0,\varphi}}\Big)\Re\int_{\Lambda_\sigma}\Big\{2\lambda_1\phi_1^2(x) - 2C_{N,\gamma}h(x)\Big\}\overline{v}(x)dx \tag{22}$$

$$\begin{aligned}\leq\ & (\rho_2\nu_0+\mu_2\tau_0)\Re\int_{\Lambda_\sigma} -\theta_0(x)dx \\ \leq\ & \rho_2\Re\int_{\Lambda_\sigma}\nu(x)\chi(\xi_*)\overline{v}dx + \mu_2\Re\int_{\Lambda_\sigma}\tau(x)\Phi(\eta_*)\overline{v}dx.\end{aligned} \tag{23}$$

Now, let $x \in \Lambda \backslash \overline{\Lambda}_\sigma$. Next, conditions $(H3)$–(H) imply that for $\rho_1\alpha_0 + \mu_1\beta_0 >> 1$ and $\rho_2\nu_0 + \mu_2\tau_0 >> 1$ we have

$$\begin{aligned}\frac{\sigma_1\theta_0}{m} &\leq \psi(\eta_*), \ \frac{\sigma_1\theta_0}{m} \leq \Upsilon(\xi_*)\\ \frac{\sigma_1\theta_0}{m} &\leq \chi(\xi_*), \ \frac{\sigma_1\theta_0}{m} \leq \Phi(\eta_*)\end{aligned}$$

Hence, the inequality $h(x) > \nu > 0$ gives

$$\begin{aligned}&\rho_1\Re\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\psi(\eta_*)\overline{u}(x)dx + \mu_1\Re\int_\Lambda \beta(x)\Upsilon(\xi_*)\overline{u}dx \qquad (24)\\ \geq\ &(\rho_1\alpha_0+\mu_1\beta_0)\frac{\theta_0 r_2}{mk_1}\sigma_1\Re\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\overline{u}(x)dx\\ \geq\ &(\rho_1\alpha_0+\mu_1\beta_0)\frac{\theta_0}{mk_1}\mathcal{K}\left(\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\frac{|\xi_*(x)-e^{i(x-y)\cdot\varphi\left(\frac{x+y}{2}\right)}\xi_*(y)|^2}{|x-y|^{N+2\gamma}}dxdy\right)\lambda_1\Re\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\overline{u}(x)dx\\ \geq\ &(\rho_1\alpha_0+\mu_1\beta_0)\frac{\theta_0}{mk_1}\mathcal{K}\left(\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\frac{|\xi_*(x)-e^{i(x-y)\cdot\varphi\left(\frac{x+y}{2}\right)}\xi_*(y)|^2}{|x-y|^{N+2\gamma}}dxdy\right)\\ &\times\Re\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\left\{2\lambda_1\phi_1^2(x)-2C_{N,\gamma}h(x)\right\}\overline{u}(x)dx\\ =\ &\mathcal{K}\left(\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\int_{\Lambda\backslash\overline{\Lambda}_\sigma}\frac{|\xi_*(x)-e^{i(x-y)\cdot\varphi\left(\frac{x+y}{2}\right)}\xi_*(y)|^2}{|x-y|^{N+2\gamma}}dxdy\right)\langle\xi_*,u\rangle_{X_{0,\varphi}}\\ =\ &\mathcal{K}\left(\|\xi_*\|^2_{X_{0,\varphi}}\right)\langle\xi_*,u\rangle_{X_{0,\varphi}}\end{aligned}$$

for all $u \in X_{0,\varphi}(\Lambda)$ with $u \geq 0$ in Λ. Relationships (20) and (24) are met. Therefore, (18) is satisfied. By following similar steps, we obtain

$$\begin{aligned}\mathcal{R}\left(\|\eta_*\|^2_{X_{0,\varphi}}\right)\langle\eta_*,v\rangle_{X_{0,\varphi}} &= \frac{(\rho_2\nu_0+\mu_2\tau_0)r_2\theta_0}{mr_1}\mathcal{R}\left(\|\eta_*\|^2_{X_{0,\varphi}}\right)\\ &\times\Re\int_\Lambda\left\{2\lambda_1\phi_1^2(x)-2C_{N,\gamma}h(x)\right\}\overline{v}(x)dx\\ &\leq \rho_2\Re\int_{\Lambda_\sigma}\nu(x)\chi(\xi_*)\overline{v}dx + \mu_2\Re\int_{\Lambda_\sigma}\tau(x)\Phi(\eta_*)\overline{v}dx\end{aligned} \qquad (25)$$

Hence, (ξ_*,η_*) is a weak subsolution of non-local magnetic fractional system (1).

We are now interested in creating a supersolution of non-local magnetic fractional system (1). For this, let e be a solution to the problem

$$\begin{cases}(-\Delta)^\gamma_\varphi\omega = 1 \text{ in } \Lambda,\\ \omega = 0 \text{ on } \mathbb{R}^N\backslash\Lambda.\end{cases} \qquad (26)$$

In addition to this, one can find $\widehat{c}_1, \widehat{c}_2$ in a manner that [20]

$$0 < \widehat{c}_1\delta^\gamma < \omega(x) < \widehat{c}_2\delta^\gamma \text{ a.e in } \Lambda. \qquad (27)$$

Let

$$(\xi^*,\eta^*) = \left(K\omega, \frac{(\rho_2\|\nu\|_\infty+\mu_2\|\tau\|_\infty)}{r_2}[\chi(K\|\omega\|_\infty)]\omega\right),$$

where we choose $K > 0$ large. We prove that (ξ^*,η^*) is a supersolution of non-local magnetic fractional system (1).

Let $u \in X_{0,\varphi}(\Lambda)$ with $u \geq 0$ in Λ. By $(H3)$–$(H5)$, we can choose K large enough so that

$$k_1K \geq \rho_1\|\alpha\|_\infty\psi\left[\left(\frac{\rho_2\|\nu\|_\infty+\mu_2\|\tau\|_\infty}{r_2}\right)\chi(K\|\omega\|_\infty)\omega\right] + \mu_1\|\beta\|_\infty\Upsilon(K\|\omega\|_\infty)$$

From (27), we obtain $\xi^* = K\omega \in X_{0,\varphi}(\Lambda)$, which fulfills

$$\begin{aligned}\mathcal{K}\left(\|\xi^*\|^2_{X_{0,\varphi}}\right)\langle \xi^*, u\rangle_{X_{0,\varphi}} &= \mathcal{K}\left(\|\xi^*\|^2_{X_{0,\varphi}}\right)K\langle \omega, u\rangle_{X_{0,\varphi}} \\ &\geq k_1 K \Re \int_\Lambda \overline{u}(x)dx \\ &\quad \geq \left\{\rho_1\|\alpha\|_\infty \psi\left[\left(\frac{\rho_2\|\nu\|_\infty + \mu_2\|\tau\|_\infty}{r_2}\right)\chi(K\|\omega\|_\infty)\omega\right] + \mu_1\|\beta\|_\infty \Upsilon(K\|\omega\|_\infty)\right\} \\ &\qquad \times \Re\int_\Lambda \overline{u}(x)dx \\ &\geq \rho_1 \Re \int_\Lambda \alpha(x)\psi(\eta^*)\overline{u}dx + \mu_1 \Re \int_\Lambda \beta(x)\Upsilon(\xi^*)\overline{u}dx,\end{aligned} \tag{28}$$

Also,

$$\begin{aligned}\mathcal{R}\left(\|\eta^*\|^2_{X_{0,\varphi}}\right)\langle \eta^*, v\rangle_{X_{0,\varphi}} &\geq (\rho_2\|\nu\|_\infty + \mu_2\|\tau\|_\infty)[\chi(K\|\omega\|_\infty)]\omega\langle \omega, v\rangle_{X_{0,\varphi}} \\ &= (\rho_2\|\nu\|_\infty + \mu_2\|\tau\|_\infty)\Re\int_\Lambda [\chi(K\|\omega\|_\infty)]\omega\overline{v}(x)dx \\ &\geq \rho_2\Re\int_\Lambda \nu(x)\chi(\xi^*)\overline{v}(x)dx + \mu_2\Re\int_\Lambda \tau(x)\chi(K\|\omega\|_\infty)\overline{v}dx,\end{aligned}$$

Again, by $(H4)$ and $(H5)$, for a large enough K we have

$$\begin{aligned}\chi(K\|\omega\|_\infty) &\geq \Phi\left(\frac{(\rho_2\|\nu\|_\infty + \mu_2\|\tau\|_\infty)}{r_2}[\chi(K\|\omega\|_\infty)]\omega\right) \\ &\geq \Phi(\eta^*)\end{aligned}$$

Hence,

$$\mathcal{R}\left(\|\eta^*\|^2_{X_{0,\varphi}}\right)\langle \eta^*, v\rangle_{X_{0,\varphi}} \geq \rho_2\Re\int_\Lambda \nu(x)\chi(\xi^*)\overline{v}(x)dx + \mu_2\Re\int_\Lambda \tau(x)\Phi(\eta^*)\overline{v}dx, \tag{29}$$

From (28) and (29), the solution (ξ^*, η^*) is a supersolution of non-local magnetic fractional system (1), with $\xi_* \leq \xi^*$ and $\eta_* \leq \eta^*$ for large enough K values.

We will use the iterative method [27] to prove the existence of the weak solution to problem (1). For this, let the sequence be $\{(\xi_n, \eta_n)\} \subset (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$ as follows: $\xi_0 := \xi^*, \eta_0 = \eta^*$. (ξ_n, η_n) is the unique solution of

$$\begin{cases}\mathcal{K}\left(\|\xi_n\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi \xi_n = \rho_1\alpha(x)\psi(\eta_{n-1}) + \mu_1\beta(x)\Upsilon(\xi_{n-1}) \text{ in } \Lambda, \\ \mathcal{R}\left(\|\eta_n\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi \eta_n = \rho_2\nu(x)\chi(\xi_{n-1}) + \mu_2\tau(x)\Phi(\eta_{n-1}) \text{ in } \Lambda, \\ \xi_n = \eta_n = 0 \text{ in } \mathbb{R}^N\backslash\Lambda.\end{cases} \tag{30}$$

Here, system (30) is $(\mathcal{K}, \mathcal{R})$-linear if $(\xi_{n-1}, \eta_{n-1}) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$. Choose $\mathcal{K}(t) = t\mathcal{K}(t^2)$ and $\mathcal{R}(t) = t\mathcal{R}(t^2)$. Then, $\mathcal{K}(\mathbb{R}) = \mathbb{R}$, $\mathcal{R}(\mathbb{R}) = \mathbb{R}$, $\psi(\eta_{n-1})$, and $\chi(\xi_{n-1}) \in L^2(\Lambda, \mathbb{C})$ (in x).

We prove by iteration that non-local magnetic fractional system (30) has a unique solution $(\xi_n, \eta_n) \in (X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda))$. By using (30) and the fact that (ξ_0, η_0) is a supersolution of (1), we have

$$\begin{cases}\mathcal{K}\left(\|\xi_0\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi \xi_0 \geq \rho_1\alpha(x)\psi(\eta_0) + \mu_1\beta(x)\Upsilon(\xi_0) = \mathcal{K}\left(\|\xi_1\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi \xi_1, \\ \mathcal{R}\left(\|\eta_0\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi \eta_0 \geq \rho_2\nu(x)\chi(\xi_0) + \mu_2\tau(x)\Phi(\eta_0) = \mathcal{R}\left(\|\eta_1\|^2_{X_{0,\varphi}}\right)(-\triangle)^\gamma_\varphi \eta_1,\end{cases}$$

and, by the comparison principal in the magnetic fractional Laplacian (see Lemma 1), $\xi_0 \geq \xi_1$ and $\eta_0 \geq \eta_1$. Furthermore, since $\xi_0 \geq \xi_*$, $\eta_0 \geq \eta_*$ and the monotonicity of ψ, χ, Υ, and Φ has

$$\begin{aligned}
\mathcal{K}\Big(\|\xi_1\|^2_{X_{0,\varphi}}\Big)(-\triangle)^{\gamma}_{\varphi}\xi_1 &= \rho_1\alpha(x)\psi(\eta_0) + \mu_1\beta(x)\Upsilon(\xi_0) \\
&\geq \rho_1\alpha(x)\psi(\eta_*) + \mu_1\beta(x)\Upsilon(\xi_*) \\
&\geq \mathcal{K}\Big(\|\xi_*\|^2_{X_{0,\varphi}}\Big)(-\triangle)^{\gamma}_{\varphi}\xi_*,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}\Big(\|\eta_1\|^2_{X_{0,\varphi}}\Big)(-\triangle)^{\delta}_{\varphi}\eta_1 &= \rho_2\nu(x)\chi(\xi_0) + \mu_2\tau(x)\Phi(\eta_0) \\
&\geq \rho_2\nu(x)\chi(\xi_*) + \mu_2\tau(x)\Phi(\eta_*) \\
&\geq \mathcal{R}\Big(\|\eta_*\|^2_{X_{0,\varphi}}\Big)(-\triangle)^{\gamma}_{\varphi}\eta_*
\end{aligned}$$

then, according to Lemma 1, $\xi_1 \geq \xi_*$, $\eta_1 \geq \eta_*$. We repeat the process n times we obtain that $(\varepsilon_n, \eta_n) \in \big(X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda)\big)$ fulfills

$$\xi^* = \xi_0 \geq \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq \cdots \geq \xi_* > 0, \tag{31}$$

$$\eta^* = \eta_0 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_n \geq \cdots \geq \eta_* > 0. \tag{32}$$

We note that $(\varepsilon_n, \eta_n) \in \big(X_{0,\varphi}(\Lambda) \times X_{0,\varphi}(\Lambda)\big)$ is a bounded monotone sequence. From the function ψ, χ and the sequences $\{\xi_n\}, \{\eta_n\}$, we can find constants $C_i > 0$, $i = 1, \ldots, 4$ in which

$$|\psi(\eta_{n-1})| \leq C_1, \quad |\chi(\varepsilon_{n-1})| \leq C_2, |\Upsilon(\varepsilon_{n-1})| \leq C_3, |\Phi(\varepsilon_{n-1})| \leq C_4 \text{ for all } n. \tag{33}$$

From Lemma 2, and some estimates, we have

$$\begin{aligned}
k_1\|\xi_n\|^2_{X_{0,\varphi}} &\leq \mathcal{K}\Big(\|\xi_n\|^2_{X_{0,\varphi}}\Big)\|\xi_n\|^2_{X_{0,\varphi}} \\
&= \Re\int_{\Lambda}[\rho_1\alpha(x)\psi(\eta_{n-1}) + \mu_1\beta(x)\Upsilon(\xi_{n-1})]\overline{\xi}_n dx \\
&\leq \rho_1\|\alpha\|_\infty\Re\int_{\Lambda}|\psi(\eta_{n-1})|\,|\overline{\xi}_n|dx + \mu_1\|\beta\|_\infty\int_{\Lambda}\Upsilon(\xi_{n-1})\overline{\xi}_n dx \\
&\leq \rho_1 C_1 S(N,\gamma)|\Lambda|^{\frac{2^*_\gamma - 2}{2^*_\gamma . 2}}\|\xi_n\|_{X_{0,\varphi}} + \rho_1 C_3 S(N,\gamma)|\Lambda|^{\frac{2^*_\gamma - 2}{2^*_\gamma . 2}}\|\xi_n\|_{X_{0,\varphi}} \\
&\leq C_5\|\xi_n\|_{X_{0,\varphi}}
\end{aligned}$$

or

$$\|\xi_n\|_{X_{0,\varphi}} \leq C_3, \ \forall n, \tag{34}$$

in which $C_3 > 0$. Similarly,

$$\|\eta_n\|_{X_{0,\varphi}} \leq C_4, \ \ \forall n. \tag{35}$$

Equations (34) and (35) imply that $\{(\xi_n, \eta_n)\}$ has a subsequence which weakly converges in $X_{0,\varphi}(\Lambda)$ to a limit (ξ, η) with $\xi \geq \xi_* > 0$ and $\eta \geq \eta_* > 0$. Using monotone behavior and the standard regularity argument, we deduce the convergence of $\{(\xi_n, \eta_n)\}$ to (ξ, η); when $n \to +\infty$ in (30), the positive solution of (1) is (ξ, η). This concludes the proof. □

4. Practical Examples and Applications

Our study presents new existence results for positive solutions of magnetic fractional Laplacian equations. To further illustrate the practical relevance of our findings, we provide several examples and applications:

- Electromagnetic Fields:
  The magnetic fractional Laplacian can model electromagnetic fields in materials with complex conductivity properties. For instance, in plasmas or certain metamaterials, the fractional order of the Laplacian accounts for anomalous diffusion and non-local interactions, providing more accurate descriptions than classical models.
- Quantum Mechanics:
  In quantum mechanics, fractional Schrödinger equations with magnetic fields describe particles in potential fields with fractal-like properties. These models capture the effects of magnetic fields on the quantum states of particles, particularly in low-temperature physics and condensed matter systems.
- Biological Systems:
  Fractional differential equations, including magnetic fractional Laplacians, are used to model various biological processes, such as diffusion in heterogeneous media, population dynamics, and the spread of diseases. The fractional aspect helps account for memory effects and spatial heterogeneity.
- Engineering Applications:
  In engineering, fractional magnetic operators are applied to control systems and signal processing. They are used to design controllers and filters that can handle complex dynamical behaviors, such as those found in robotic systems and communications networks.

5. Conclusions

In this paper, we have established new existence results for positive solutions of magnetic fractional Laplacian equations. Our findings contribute to the theoretical understanding of these operators and open new avenues for their application in various scientific and engineering fields.

The practical examples provided demonstrate the broad relevance and utility of our results. By modeling phenomena in electromagnetic fields, quantum mechanics, biological systems, and engineering applications, the magnetic fractional Laplacian offers a powerful tool for addressing complex, real-world problems.

Despite the progress made, several open problems and future research directions remain. One significant area of future work is the numerical analysis and simulation of magnetic fractional Laplacian equations to further understand their behavior and potential applications. Additionally, extending the results to nonlinear and time-dependent problems could provide deeper insights into the dynamics governed by these operators.

Further investigation into the physical interpretation and experimental validation of magnetic fractional operators would also be valuable. This could involve collaborative efforts between mathematicians, physicists, and engineers to bridge the gap between theoretical models and practical implementations.

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