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Certain Properties and Characterizations of Two-Iterated Two-Dimensional Appell and Related Polynomials via Fractional Operators

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Abstract: This paper introduces the operational rule for 2-iterated 2D Appell polynomials and derives its generalized form using fractional operators. It also presents the generating relation and explicit forms that characterize the generalized 2-iterated 2D Appell polynomials. Additionally, it establishes the monomiality principle for these polynomials and obtains their recurrence relations. The paper also establishes corresponding results for the generalized 2-iterated 2D Bernoulli, 2-iterated 2D Euler, and 2-iterated 2D Genocchi polynomials.

Keywords: hybrid special polynomials; monomiality principle; explicit form; symmetric identities; summation formulae; operational connection

MSC: 33E20; 33C45; 33B10; 33E30; 11T23

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1. Introduction

In 1880 and 1939, Appell and Sheffer polynomials were introduced, as outlined in the works of Appell and Sheffer [1,2], and they have a wide range of applications in fields such as applied mathematics, theoretical physics, and approximation theory. A multitude of scholars have studied these polynomials, employing distinct methodologies to deepen our understanding. For example, Blasiak and colleagues established a connection between the principle of monomiality and umbral calculus, as outlined in their work [3]. This endeavour created explicit representations of the Heisenberg–Weyl algebra, shedding light on the intricate interplay between fundamental algebraic principles and polynomial structures. Meanwhile, Dattoli and fellow researchers have made notable contributions to the field. In their works [4,5], they provided valuable insights by offering series expansions and connection coefficients tailored to specific expressions of Appell polynomials. These contributions have enriched our comprehension of these polynomial families' intricate properties and applications. Through these studies, researchers have gained a deeper understanding of the complex structures underlying these polynomial frameworks and their far-reaching implications across a variety of fields.

The discovery and examination of the 2-iterated Appell polynomials in [6] defined by the following generating function:

$$\mathcal{J}_1(\theta)\mathcal{J}_2(\theta)e^{\sigma\theta} = \sum_{n=0}^{\infty} \mathcal{J}_n^{[2]}(\sigma)\frac{\theta^n}{n!}, \tag{1}$$

where

$$\mathcal{J}_1(\theta) = \sum_{k=0}^{\infty} r_k \frac{\theta^k}{k!}, \quad r_0 \neq 0, \tag{2}$$

and
$$\mathcal{J}_2(\theta) = \sum_{k=0}^{\infty} s_k \frac{\theta^k}{k!}, \quad s_0 \neq 0. \tag{3}$$

Furthermore, their multiplicative and derivative operators, as well as differential equations and operational principles. For certain special cases of $\mathcal{J}_1(\theta)$ and $\mathcal{J}_2(\theta)$, the 2-iterated Bernoulli, 2-iterated Euler and Bernoulli-Euler (or Euler-Bernoulli) polynomials are defined as follows [6]:
$$\left(\frac{\theta}{e^{\theta}-1}\right)^2 e^{\sigma\theta} = \sum_{n=0}^{\infty} B_n^{[2]}(\sigma)\frac{\theta^n}{n!}, \tag{4}$$
$$\left(\frac{2}{e^{\theta}+1}\right)^2 e^{\sigma\theta} = \sum_{n=0}^{\infty} E_n^{[2]}(\sigma)\frac{\theta^n}{n!} \tag{5}$$
and
$$\left(\frac{2\theta}{e^{2\theta}-1}\right) e^{\sigma\theta} = \sum_{n=0}^{\infty} {}_BE_n(\sigma)\frac{\theta^n}{n!}. \tag{6}$$

Recently, Shahid et al. introduced and examined the 2-iterated 2D Appell and related polynomials in [7] defined by the generating function:
$$\mathcal{J}_1(\theta)\mathcal{J}_2(\theta)e^{\sigma\theta+\mu\theta^j} = \sum_{n=0}^{\infty} \mathcal{J}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!} \tag{7}$$
and derived families of differential equations. For certain special cases of $\mathcal{J}_1(\theta)$ and $\mathcal{J}_2(\theta)$, the 2-iterated 2D Bernoulli, 2-iterated 2D Euler polynomials. The relationship between trigonometric and hyperbolic secant functions and 2-iterated 2D Euler numbers warrants attention. Their Taylor series expansions incorporate 2-iterated 2D Euler numbers and their derivatives, serving as essential tools in diverse domains such as signal processing and quantum field theory. As a result, 2-iterated 2D Euler numbers have become indispensable in mathematical circles and interdisciplinary research pursuits. Furthermore, trigonometric and hyperbolic secant functions share a profound connection with 2-iterated 2D Euler numbers, integrating them into their expansions and extending their utility across various scientific disciplines. They are defined as follows [7]:
$$\left(\frac{\theta}{e^{\theta}-1}\right)^2 e^{\sigma\theta+\mu\theta^j} = \sum_{n=0}^{\infty} \mathcal{B}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!} \tag{8}$$
and
$$\left(\frac{2}{e^{\theta}+1}\right)^2 e^{\sigma\theta+\mu\theta^j} = \sum_{n=0}^{\infty} \mathcal{E}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{9}$$
respectively.

The 2-variable Appell polynomials emerge when $\mathcal{J}_1(\theta)$ equals unity and $\mathcal{J}_2(\theta)$ equals $\mathcal{J}(\theta)$. Conversely, setting μ to zero transforms the "2-iterated 2D Appell polynomials into their 2-iterated counterparts", by opting for $\mathcal{J}_1(\theta)$ to match $\mathcal{J}(\theta)$ (of the Appell polynomials) and appropriately selecting $\mathcal{J}_2(\theta)$. Therefore, we deduce the generating expressions for the "2D Bernoulli-Appell, 2D Euler-Appell, and 2D Hermite-Appell polynomials":
$$\mathcal{J}(\theta)\frac{\theta}{e^{\theta}-1}e^{\sigma\theta+\mu\theta^j} = \sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{J}_n^{(j)}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{10}$$
$$\mathcal{J}(\theta)\frac{2}{e^{\theta}+1}e^{\sigma\theta+\mu\theta^j} = \sum_{n=0}^{\infty} {}_{\mathcal{E}}\mathcal{J}_n^{(j)}(\sigma,\mu)\frac{\theta^n}{n!} \tag{11}$$

and

$$\mathcal{J}(\theta)e^{\sigma\theta-\frac{\theta^2}{2}+\mu\theta^j} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{J}_n^{(j)}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{12}$$

respectively.

Again for $\mathcal{J}_1(\theta)=\mathcal{J}_2(\theta)=\frac{\theta}{e^\theta-1}$, $\mathcal{J}_1(\theta)=\mathcal{J}_2(\theta)=\frac{2}{e^\theta+1}$ and $\mathcal{J}_1(\theta)=\mathcal{J}_2(\theta)=e^{\frac{-\theta^2}{2}}$, we possess the the generating functions of the 2-iterated 2D Bernoulli, Euler and Hermite polynomials as follows:

$$\left(\frac{\theta}{e^\theta-1}\right)^2 e^{xt+y\theta^j} = \sum_{n=0}^{\infty} \mathcal{B}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{13}$$

$$\left(\frac{2}{e^\theta+1}\right)^2 e^{xt+y\theta^j} = \sum_{n=0}^{\infty} \mathcal{E}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!} \tag{14}$$

and

$$e^{xt-\theta^2+y\theta^j} = \sum_{n=0}^{\infty} \mathcal{H}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{15}$$

respectively.

Certain combinations of $\mathcal{J}_1(\theta)$ and $\mathcal{J}_2(\theta)$ can be used to derive the generating functions of three types of 2D polynomials: "Bernoulli–Euler (also known as Euler–Bernoulli), Hermite–Bernoulli (also known as Bernoulli–Hermite), and Hermite–Euler (also known as Euler–Hermite)". These polynomials are essential mathematical tools used in various fields such as physics and engineering. The "2D Bernoulli–Euler polynomials" are commonly used in the study of beam deflection, while the "2D Hermite–Bernoulli and 2D Hermite–Euler polynomials" are often used in the study of quantum mechanics and statistical mechanics. They possess the following generating relations:

$$\left(\frac{2\theta}{e^{2\theta}-1}\right)e^{\sigma\theta+\mu\theta^j} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}_n^{(j)}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{16}$$

$$\left(\frac{\theta}{e^{\theta}-1}\right)e^{\sigma\theta-\frac{\theta^2}{2}+\mu\theta^j} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}_n^{(j)}(\sigma,\mu)\frac{\theta^n}{n!} \tag{17}$$

and

$$\left(\frac{2}{e^{\theta}+1}\right)e^{\sigma\theta-\frac{\theta^2}{2}+\mu\theta^j} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{E}_n^{(j)}(\sigma,\mu)\frac{\theta^n}{n!}, \tag{18}$$

respectively.

The manuscript is structured as follows: Section 2 of the article introduces the operational identity for the 2-iterated 2D Appell polynomials which is then further used to develop operational identity and the generating function for the generalized 2-iterated 2D Appell polynomials. Further, the explicit forms of these polynomials for these generalized polynomials are established. Moving on to Section 3, we verify the monomiality principle for this polynomial family. Section 4 establishes the operational identity, generating function, and explicit form for the "generalized 2-iterated 2D Bernoulli, generalized 2-iterated 2D Euler, and generalized 2-iterated 2D Genocchi polynomials". Finally, the article concludes with some remarks.

2. Main Results in View of Fractional Operators

Operational rules play a key role in mathematical analysis by providing structured procedures for manipulating mathematical expressions. In the domain of special polynomial theory, operational rules offer guidance for performing algebraic operations, differentiation, integration, and other transformations on polynomials. They empower researchers to discover new identities, solve equations, and efficiently investigate the properties of polynomial functions. Operational rules enable the application of mathematical concepts in various fields, such as physics, engineering, and computer science, thereby allowing for the development of practical solutions to intricate problems and the advancement of scientific understanding.

Moreover, we utilize the 2-iterated 2D Appell polynomials and establish their operational formalism in the following manner:

Differentiating expression (7) with respect to σ continuously, we find

$$\frac{\partial}{\partial \sigma}\Big[\mathcal{J}_1(\theta)\mathcal{J}_2(\theta)e^{\sigma\theta+\mu\theta^j}\Big] = \theta\Big[\mathcal{J}_1(\theta)\mathcal{J}_2(\theta)e^{\sigma\theta+\mu\theta^j}\Big] \tag{19}$$

which implies

$$\frac{\partial}{\partial \sigma}\left[\sum_{n=0}^{\infty}\mathcal{J}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!}\right] = \left[\sum_{n=0}^{\infty}\mathcal{J}_n^{[2,j]}(\sigma,\mu)\frac{\theta^{n+1}}{n!}\right]. \tag{20}$$

Further, upon replacing $n \to n-1$ in preceding expression and then comparing the coefficients of the same exponents $\frac{\theta^n}{n!}$ on both sides of the resultant expression, it follows that

$$\frac{\partial}{\partial \sigma}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big] = n\Big[\mathcal{J}_{n-1}^{[2,j]}(\sigma,\mu)\Big]. \tag{21}$$

Upon similar observations, we find that

$$\begin{aligned}
\frac{\partial^2}{\partial \sigma^2}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big] &= n(n-1)\Big[\mathcal{J}_{n-2}^{[2,j]}(\sigma,\mu)\Big]\\
\frac{\partial^3}{\partial \sigma^3}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big] &= n(n-1)(n-2)\Big[\mathcal{J}_{n-3}^{[2,j]}(\sigma,\mu)\Big]\\
\vdots \quad & \vdots\\
\frac{\partial^j}{\partial \sigma^j}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big] &= n(n-1)(n-2)\cdots(n-j+1)\Big[\mathcal{J}_{n-j}^{[2,j]}(\sigma,\mu)\Big].
\end{aligned} \tag{22}$$

Further, differentiating expression (7) with respect to μ, we have

$$\frac{\partial}{\partial \mu}\Big[\mathcal{J}_1(\theta)\mathcal{J}_2(\theta)e^{\sigma\theta+\mu\theta^j}\Big] = \theta^j\Big[\mathcal{J}_1(\theta)\mathcal{J}_2(\theta)e^{\sigma\theta+\mu\theta^j}\Big] \tag{23}$$

which implies

$$\frac{\partial}{\partial \mu}\left[\sum_{n=0}^{\infty}\mathcal{J}_n^{[2,j]}(\sigma,\mu)\frac{\theta^n}{n!}\right] = \left[\sum_{n=0}^{\infty}\mathcal{J}_n^{[2,j]}(\sigma,\mu)\frac{\theta^{n+j}}{n!}\right]. \tag{24}$$

Therefore, upon replacing $n \to n-j$ in preceding expression and then comparing the coefficients of the same exponents $\frac{\theta^n}{n!}$ on both sides of the resultant expression, it follows that

$$\frac{\partial}{\partial \mu}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big] = n(n-1)(n-2)\cdots(n-j+1)\Big[\mathcal{J}_{n-j}^{[2,j]}(\sigma,\mu)\Big]. \tag{25}$$

Thus, in view of expressions (22) and (25), it follows that

$$\frac{\partial}{\partial \mu}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big] = \frac{\partial^j}{\partial \mu^j}\Big[\mathcal{J}_n^{[2,j]}(\sigma,\mu)\Big]. \tag{26}$$

Thus, in view of expressions (21), (25) and (26) and under the initial condition,

$$\mathcal{J}_n^{[2,j]}(\sigma,0) = \mathcal{J}_n^{[2]}(\sigma), \tag{27}$$

the 2-iterated 2D Appell polynomials possess the succeeding operational rule:

$$\exp\left(\mu\,\frac{\partial^j}{\partial\sigma^j}\right)\left\{\mathcal{J}_n^{[2]}(\sigma)\right\}=\mathcal{J}_n^{[2,j]}(\sigma,\mu). \tag{28}$$

Note-1. For $\mathcal{J}_1(\theta)=1,\ \ j=2$, the 2-iterated 2D Appell polynomials $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ reduces to the the 2D Appell polynomials $\mathcal{J}_n^{[2]}(\sigma,\mu)$ and thus satisfying the operational rule:

$$\exp\left(\mu\,\frac{\partial^j}{\partial\sigma^j}\right)\left\{\mathcal{J}_n(\sigma)\right\}=\mathcal{J}_n^{[j]}(\sigma,\mu). \tag{29}$$

Note-2. For $j=2$, the 2-iterated 2D Appell polynomials $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ reduces to the the 2-iterated Hermite–Appell polynomials $\mathcal{J}_n^{[2,2]}(\sigma,\mu)$ and thus satisfying the operational rule:

$$\exp\left(\mu\,\frac{\partial^2}{\partial\sigma^2}\right)\left\{\mathcal{J}_n^{[2]}(\sigma)\right\}=\mathcal{J}_n^{[2,2]}(\sigma,\mu). \tag{30}$$

Note-3. For $\mathcal{J}_1(\theta)=1,\ j=2$, the 2-iterated 2D Appell polynomials $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ reduces to the the Hermite–Appell polynomials $\mathcal{J}_n(\sigma,\mu)$ and thus satisfying the operational rule:

$$\exp\left(\mu\,\frac{\partial^2}{\partial\sigma^2}\right)\left\{\mathcal{J}_n(\sigma)\right\}=\mathcal{J}_n(\sigma,\mu). \tag{31}$$

The application of integral transforms to fractional derivatives was initiated by the work of Riemann and Liouville [8,9]. An effective approach to fractional derivatives involves combining integral transformations with specialized polynomials, as demonstrated in works like [10,11]. This method effectively analyses and manipulates fractional derivatives, enhancing our ability to comprehend and solve problems in various scientific and engineering fields.

The Euler integral serves as a cornerstone for expanding and generalising special polynomials into more diverse forms. In a groundbreaking study referenced as [10], Dattoli and colleagues utilized the Euler integral to define operational relations and establish generating relations for novel versions of special polynomials. Employing the Euler integral as a fundamental tool effectively opened up new avenues for exploring special polynomial theory. This approach broadens the applicability of special polynomials and facilitates the investigation of their properties and relationships in various mathematical contexts. Through their meticulous analysis and utilization of the Euler integral, Dattoli et al. [10,12] contributed significantly to advancing the understanding and utility of special polynomials, offering valuable insights into their broader mathematical significance. Consequently, their work serves as a crucial reference point for researchers delving into the intricate interplay between special polynomials and mathematical analysis.

The integral associated with Euler is referenced in [13] (p. 218);

$$\delta^{-\tau}=\frac{1}{\Gamma(\tau)}\int_0^\infty e^{-\lambda\theta}\theta^{\tau-1}d\theta,\quad \min\{\mathrm{Re}(\tau),\mathrm{Re}(\lambda)\}>0, \tag{32}$$

this subsequently leads to the following [10]:

$$\left(\beta-\frac{\partial}{\partial\sigma}\right)^{-\tau}h(\sigma)=\frac{1}{\Gamma(\tau)}\int_0^\infty e^{-\beta\theta}\theta^{\tau-1}\,e^{\theta\frac{\partial}{\partial\sigma}}h(\sigma)d\theta=\frac{1}{\Gamma(\tau)}\int_0^\infty e^{-\beta\theta}\theta^{\tau-1}\,h(\sigma+\theta)d\theta. \tag{33}$$

The following formula applies to second-order derivatives:

$$\left(\beta - \frac{\partial^2}{\partial\sigma^2}\right)^{-\tau} h(\sigma) = \frac{1}{\Gamma(\tau)} \int_0^\infty e^{-\beta\theta}\theta^{\tau-1}\, e^{\theta \frac{\partial^2}{\partial\sigma^2}} h(\sigma) d\theta. \tag{34}$$

First, we derive the operational identity and generating expression by proving the following result:

Theorem 1. *The following operational definition applies to the generalized* 2*-iterated 2D Appell polynomials (2I2DAP)* $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$*:*

$$\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau} \{\mathcal{J}_n^{[2]}(\sigma)\} = \mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta). \tag{35}$$

Proof. Substituting δ with $\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)$ in integral (32) and applying it to $\mathcal{J}_n^{[2]}(\sigma)$, we have the following transformation:

$$\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau} \{\mathcal{J}_n^{[2]}(\sigma)\} = \frac{1}{\Gamma(\tau)} \int_0^\infty e^{-\beta\theta}\theta^{\tau-1} \exp\left(\theta\mu\,\frac{\partial^j}{\partial\sigma^j}\right) \mathcal{J}_n^{[2]}(\sigma) d\theta. \tag{36}$$

Considering Equation (28), the previous expression transforms to

$$\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau} \{\mathcal{J}_n^{[2]}(\sigma)\} = \frac{1}{\Gamma(\tau)} \int_0^\infty e^{-\beta\theta}\theta^{\tau-1} \mathcal{J}_n^{[2,j]}(\sigma,\mu\theta) d\theta. \tag{37}$$

The polynomial set $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ introduced on the right-hand side of Equation (37) is identified as the generalized 2-iterated 2D Appell polynomials and thus leads to the following:

$$\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta) = \frac{1}{\Gamma(\tau)} \int_0^\infty e^{-\beta\theta}\theta^{\tau-1} \mathcal{J}_n^{[2,j]}(\sigma,\mu\theta) d\theta. \tag{38}$$

Considering Equations (37) and (38), statement (35) is thereby demonstrated. □

Remark 1. *For,* $\mathcal{J}_1(\theta) = 1$*, the generalized 2I2DAP* $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ *reduces to the generalized 2DAP* $\mathcal{J}_n^{[j]}(\sigma,\mu)$ *and thus satisfying the operational rule:*

$$\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau} \{\mathcal{J}_n(\sigma)\} = \mathcal{J}_{n,\tau}^{[j]}(\sigma,\mu;\beta).$$

Remark 2. *For,* $\mathcal{J}_1(\theta) = 1$ *and* $j = 2$*, the generalized 2I2DAP* $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ *reduces to the generalized 2VHAP* $\mathcal{J}_n(\sigma,\mu)$ *and thus satisfying the operational rule:*

$$\left(\beta - \mu\,\frac{\partial^2}{\partial\sigma^2}\right)^{-\tau} \{\mathcal{J}_n(\sigma)\} = \mathcal{J}_{n,\tau}(\sigma,\mu;\beta).$$

Remark 3. *For,* $j = 2$*, the generalized 2I2DAP* $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ *reduces to the the generalized 2IAP* $\mathcal{J}_n^{[2]}(\sigma,\mu)$ *and thus satisfying the operational rule:*

$$\left(\beta - \mu\,\frac{\partial^2}{\partial\sigma^2}\right)^{-\tau} \{\mathcal{J}_n(\sigma)\} = \mathcal{J}_{n,\tau}^{[2]}(\sigma,\mu;\beta).$$

Theorem 2. *The following generating expression applies to the generalized 2I2DAP* $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$*:*

$$\frac{\mathcal{J}_1(w)\ \mathcal{J}_2(w)\exp(\sigma w)}{(\beta-\mu w^j)^\tau}=\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{w^n}{n!}. \tag{39}$$

Proof. By multiplying Equation (38) with $\frac{w^n}{n!}$ on both sides and then summing over n, it can be inferred that

$$\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{w^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\tau)}\int_0^\infty e^{-\beta\theta}\theta^{\tau-1}\mathcal{J}_n^{[2,j]}(\sigma,\mu\theta)\frac{w^n}{n!}d\theta. \tag{40}$$

Utilizing the corresponding generating functions from (7) on the right-hand side of Equation (40), it can be deduced that

$$\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{w^n}{n!}=\frac{\mathcal{J}_1(w)\ \mathcal{J}_2(w)\exp(\sigma w^j)}{\Gamma(\tau)}\int_0^\infty e^{-\left(\beta-\mu w\right)\theta}\theta^{\nu-1}d\theta, \tag{41}$$

thus, upon applying integral (32) on the right-hand side, assertion (39) is derived. □

Remark 4. *For* $\mathcal{J}_1(\theta)=1$*, the generalized 2I2DAP* $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ *reduces to the the generalized 2DAP* $\mathcal{J}_n^{[j]}(\sigma,\mu)$ *and thus satisfying the generating function:*

$$\frac{\mathcal{J}_2(w)\exp(\sigma w)}{(\beta-\mu w^j)^\tau}=\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[j]}(\sigma,\mu;\beta)\frac{w^n}{n!}.$$

Remark 5. *For* $\mathcal{J}_1(\theta)=1$ *and* $j=2$*, the generalized 2I2DAP* $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ *reduces to the the generalized 2VHAP* $\mathcal{J}_n(\sigma,\mu)$ *and thus satisfying the operational rule:*

$$\frac{\mathcal{J}_2(w)\exp(\sigma w)}{(\beta-\mu w^2)^\tau}=\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}(\sigma,\mu;\beta)\frac{w^n}{n!}.$$

Remark 6. *For* $j=2$*, the generalized 2I2DAP* $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ *reduces to the the generalized 2IAP* $\mathcal{J}_n^{[2]}(\sigma,\mu)$ *and thus satisfying the operational rule:*

$$\frac{\mathcal{J}_1(w)\ \mathcal{J}_2(w)\exp(\sigma w)}{(\beta-\mu w^2)^\tau}=\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,2]}(\sigma,\mu;\beta)\frac{w^n}{n!}.$$

Next, we derive the explicit forms for the generalized 2-iterated 2D Appell polynomials by demonstrating the succeeding results:

Theorem 3. *The generalized 2I2DAP* $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ *satisfies the succeeding explicit form:*

$$\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)=\sum_{k=0}^{n}\binom{n}{k}r_k\mathcal{J}_{n-k,\tau}^{[j]}(\sigma,\mu;\beta). \tag{42}$$

Proof. Inserting expression (2) in the l.h.s. of the expression (39), we find

$$\sum_{k=0}^{\infty}r_k\frac{\theta^k}{k!}\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}=\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}. \tag{43}$$

Therefore, upon replacing $n\to n-k$ in the preceeding expression and then comparing the coefficients of the same exponents $\frac{\theta^n}{n!}$ on both sides of the resultant expression, assertion (42) is obtained. □

Theorem 4. *The generalized 2I2DAP* $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ *satisfies the succeeding explicit form:*

$$\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta) = n! \sum_{k=0}^{n} \sum_{p=0}^{[\frac{k}{j}]} \frac{(\tau)_p \, s_{k-jp} \, \mu^p}{(n-k)! \, p! \, (k-jp)! \, \beta^p} \mathcal{J}_{n-k}(\sigma). \tag{44}$$

Proof. Taking $\mathcal{J}_1(w) = \mathcal{J}(\theta)$ in the l.h.s. of the expression (39), it becomes

$$\frac{\mathcal{J}(\theta)\exp(\sigma\theta)\ \mathcal{J}_2(\theta)\exp(\sigma\theta)}{(\beta-\mu\theta^j)^\tau} = \sum_{n=0}^{\infty} \mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}. \tag{45}$$

Inserting expression (1) with $\mathcal{J}_2(\theta) = 1$, (3) and the expansion

$$(1-\theta)^{-c} = \sum_{k=0}^{\infty} (c)_k \frac{\theta^k}{k!}, \quad |\theta| < 1,$$

in preceding expression, we have

$$\sum_{n=0}^{\infty} \mathcal{J}_n(\sigma)\frac{\theta^n}{n!} \sum_{k=0}^{\infty} s_k \frac{\theta^k}{k!} \sum_{p=0}^{\infty} (\tau)_p \frac{\mu^p \theta^{jp}}{\beta^p \ p!} = \sum_{n=0}^{\infty} \mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}, \tag{46}$$

thus, upon replacing $k \to k - jp$ in the l.h.s. of the preceding expression, it follows that

$$\sum_{n=0}^{\infty} \mathcal{J}_n(\sigma) \sum_{k=0}^{\infty} \sum_{p=0}^{[\frac{k}{j}]} (\tau)_p \, s_{k-jp} \frac{\mu^p \theta^{n+k}}{\beta^p \ (k-jp)! \ n! \ p!} = \sum_{n=0}^{\infty} \mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}. \tag{47}$$

Again, upon replacing $n \to n - k$ in the l.h.s. of the preceding expression and then comparing the coefficients of the same exponents $\frac{\theta^n}{n!}$ on both sides of the resultant expression, the assertion (44) is obtained. □

Theorem 5. *The generalized 2I2DAP* $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ *satisfies the succeeding explicit form:*

$$\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta) = n! \sum_{p=0}^{[\frac{n}{j}]} \frac{(\tau)_p \, \mu^p}{p! \, (n-jp)! \, \beta^p} \mathcal{J}_{n-jp}^{[2]}(\sigma). \tag{48}$$

Proof. Inserting expression (1) and the expansion

$$(1-\theta)^{-c} = \sum_{k=0}^{\infty} (c)_k \frac{\theta^k}{k!}, \quad |\theta| < 1,$$

in expression (39), we have

$$\sum_{n=0}^{\infty} \mathcal{J}_n^{[2]}(\sigma)\frac{\theta^n}{n!} \sum_{p=0}^{\infty} (\tau)_p \frac{\mu^p \theta^{jp}}{\beta^p \ p!} = \sum_{n=0}^{\infty} \mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}. \tag{49}$$

Thus, upon replacing $n \to n - jp$ in the l.h.s. of the preceding expression, it follows that

$$\sum_{n=0}^{\infty} \sum_{p=0}^{[\frac{n}{j}]} n! \ \frac{(\tau)_p \ \mu^p}{p! \ (n-jp)! \ \beta^p} \mathcal{J}_{n-jp}^{[2]}(\sigma)\frac{\theta^n}{n!} = \sum_{n=0}^{\infty} \mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}. \tag{50}$$

Therefore, comparing the coefficients of the same exponents $\frac{\theta^n}{n!}$ on both sides of the resultant expression, assertion (48) is obtained. □

In the next section, the monomiality principle and recurrence relations for the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ are derived.

3. Monomiality Principle and Recurrence Relations

The idea of monomiality can be traced back to 1941 when Steffenson introduced the poweroid notion [14]. It was later refined by Dattoli [12]. The principle of monomiality is crucial in the theory of polynomials, especially in the field of special functions. It is a fundamental criterion that verifies the orthogonality and completeness of polynomial sets, which helps in the rigorous analysis of their properties and applications. By ensuring that polynomials satisfy the principle of monomiality, researchers can use them for various mathematical and computational tasks, such as function approximation, numerical analysis, and solving differential equations. Additionally, the principle enables the establishment of recurrence relations and explicit formulas, which enhances our understanding and utilization of these polynomial families across diverse scientific disciplines. The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ can be used as both multiplicative and derivative operators for a set of polynomials $\{\Phi_m(\sigma)\}_{m\in\mathbb{N}}$. These polynomials satisfy the following expressions:

$$\Phi_{m+1}(\sigma) = \hat{\mathcal{M}}\{\Phi_m(\sigma)\} \tag{51}$$

and

$$m\,\Phi_{m-1}(\sigma) = \hat{\mathcal{D}}\{\Phi_m(\sigma)\}. \tag{52}$$

The set of operators manipulating the quasi-monomial $\{\Phi_m(\sigma)\}_{m\in\mathbb{N}}$ must adhere to the following formula:

$$[\hat{\mathcal{D}},\hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{53}$$

thus displaying a Weyl group structure.

The characteristics of the quasi-monomial set $\{\Phi_m(\sigma)\}_{m\in\mathbb{N}}$ depend on the properties of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$, satisfying the following axioms:

(i) $\Phi_m(\sigma)$ satisfies the differential equation:

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{\Phi_m(\sigma)\} = m\,\Phi_m(\sigma), \tag{54}$$

provided $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ possesses differential recognitions.

(ii) The explicit representation of $\Phi_m(\sigma)$ is presented by

$$\Phi_m(\sigma) = \hat{\mathcal{M}}^m\,\{1\}, \tag{55}$$

with $\Phi_0(\sigma) = 1$.

(iii) The exponential form of the generating relation for $\Phi_m(\sigma)$ can be expressed as

$$e^{\theta\hat{\mathcal{M}}}\{1\} = \sum_{m=0}^{\infty}\Phi_m(\sigma)\frac{\theta^m}{m!}, \quad |\theta| < \infty\ , \tag{56}$$

on utilizing identity expression (55).

The monomiality principle is used to define raising and lowering operators. Additionally, we define the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ within this framework by demonstrating the succeeding results:

Theorem 6. *For the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$, the succeeding multiplicative and derivative operators hold true:*

$$\hat{\mathcal{M}}_{\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)} = \sigma + \frac{\mathcal{J}_1'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_1(\frac{\partial}{\partial\sigma})} + \frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})} + \frac{j\tau\mu\,(\frac{\partial}{\partial\sigma})^{j-1}}{\left(\beta - \mu\frac{\partial}{\partial\sigma}^{j}\right)} \tag{57}$$

and

$$\hat{\mathcal{D}}_{\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)} = \frac{\partial}{\partial\sigma}. \tag{58}$$

Proof. By differentiating expression (39) with respect to θ, it follows that

$$\frac{\partial}{\partial\sigma}\left[\frac{\mathcal{J}_1(\theta)\;\mathcal{J}_2(\theta)\exp(\sigma\theta)}{(\beta-\mu\theta^j)^\tau}\right] = \frac{\partial}{\partial\sigma}\left[\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}\right] \tag{59}$$

which further gives

$$\left(\sigma+\frac{\mathcal{J}_1'(\theta)}{\mathcal{J}_1(\theta)}+\frac{\mathcal{J}_2'(\theta)}{\mathcal{J}_2(\theta)}+\frac{j\,\tau\,\mu\,\theta^{j-1}}{(\beta-\mu\sigma^j)}\right)\left[\frac{\mathcal{J}_1(\theta)\;\mathcal{J}_2(\theta)\exp(\sigma\theta)}{(\beta-\mu\theta^j)^\tau}\right] = \sum_{n=0}^{\infty} n\,\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^{n-1}}{n!}. \tag{60}$$

Also, differentiating (39) with respect to σ, it follows that

$$\frac{\partial}{\partial\sigma}\left[\frac{\mathcal{J}_1(\theta)\;\mathcal{J}_2(\theta)\exp(\sigma\theta)}{(\beta-\mu\theta^j)^\tau}\right] = \theta\left[\frac{\mathcal{J}_1(\theta)\;\mathcal{J}_2(\theta)\exp(\sigma\theta)}{(\beta-\mu\theta^j)^\tau}\right] \tag{61}$$

which further can be written as

$$\frac{\partial}{\partial\sigma}\left[\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}\right] = \left[\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^{n+1}}{n!}\right]. \tag{62}$$

Using expression (61) in (60), assertion (57) is proven.

Further, replacing $n \to n-1$ in expression (62), we have

$$\frac{\partial}{\partial\sigma}\left[\sum_{n=0}^{\infty}\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!}\right] = \left[\sum_{n=0}^{\infty}\mathcal{J}_{n-1,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^{n}}{n!}\right] \tag{63}$$

which proves assertion (58) while comparing same powers of θ both sides. □

Remark 7. *For $\mathcal{J}_1(\theta)=1$, the generalized 2I2DAP $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ reduces to the generalized 2DAP $\mathcal{J}_n^{[j]}(\sigma,\mu)$ and thus satisfying the succeeding multiplicative and derivative operators:*

$$\hat{\mathcal{M}}_{\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)} = \sigma+\frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})}+\frac{j\tau\mu\,(\frac{\partial}{\partial\sigma})^{j-1}}{\left(\beta-\mu\frac{\partial}{\partial\sigma}^j\right)}$$

and

$$\hat{\mathcal{D}}_{\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)} = \frac{\partial}{\partial\sigma}.$$

Remark 8. *For $\mathcal{J}_1(\theta)=1$ and $j=2$, the generalized 2I2DAP $\mathcal{J}_n^{[2,j]}(\sigma,\mu)$ reduces to the generalized 2VHAP $\mathcal{J}_n(\sigma,\mu)$ and thus satisfying the succeeding multiplicative and derivative operators:*

$$\hat{\mathcal{M}}_{\mathcal{J}_{n,\tau}(\sigma,\mu;\beta)} = \sigma+\frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})}+\frac{2\tau\mu\,(\frac{\partial}{\partial\sigma})}{\left(\beta-\mu\frac{\partial}{\partial\sigma}^2\right)}$$

and

$$\hat{\mathcal{D}}_{\mathcal{J}_{n,\tau}(\sigma,\mu;\beta)} = \frac{\partial}{\partial\sigma}.$$

Remark 9. *For, $j = 2$, the generalized 2I2DAP $\mathcal{J}_n^{[2,j]}(\sigma, \mu)$ reduces to the the generalized 2IAP $\mathcal{J}_n^{[2]}(\sigma, \mu)$ and thus satisfying the succeeding multiplicative and derivative operators:*

$$\hat{\mathcal{M}}_{\mathcal{J}_{n,\tau}^{[2,2]}(\sigma,\mu;\beta)} = \sigma + \frac{\mathcal{J}_1'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_1(\frac{\partial}{\partial\sigma})} + \frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})} + \frac{2\tau\mu\,(\frac{\partial}{\partial\sigma})}{\left(\beta - \mu\frac{\partial}{\partial\sigma}^2\right)}$$

and

$$\hat{\mathcal{D}}_{\mathcal{J}_{n,\tau}^{[2,2]}(\sigma,\mu;\beta)} = \frac{\partial}{\partial\sigma}.$$

Theorem 7. *The generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ satisfy the succeeding differential equation:*

$$\left[\sigma\frac{\partial}{\partial\sigma} + \frac{\mathcal{J}_1'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_1(\frac{\partial}{\partial\sigma})}\frac{\partial}{\partial\sigma} + \frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})}\frac{\partial}{\partial\sigma} + \frac{j\,\tau\,\mu\,(\frac{\partial}{\partial\sigma})^j}{\left(\beta - \mu\frac{\partial}{\partial\sigma}^j\right)} - n\right]\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) = 0. \quad (64)$$

Proof. Inserting expressions (57) and (58) in expression (54), we obtain assertion (64). □

Remark 10. *For $\mathcal{J}_1(\theta) = 1$, the generalized 2I2DAP $\mathcal{J}_n^{[2,j]}(\sigma, \mu)$ reduces to the generalized 2DAP $\mathcal{J}_n^{[j]}(\sigma, \mu)$, thus satisfying the succeeding differential equation:*

$$\left[\sigma\frac{\partial}{\partial\sigma} + \frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})}\frac{\partial}{\partial\sigma} + \frac{j\,\tau\,\mu\,(\frac{\partial}{\partial\sigma})^j}{\left(\beta - \mu\frac{\partial}{\partial\sigma}^j\right)} - n\right]\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) = 0.$$

Remark 11. *For $\mathcal{J}_1(\theta) = 1$ and $j = 2$, the generalized 2I2DAP $\mathcal{J}_n^{[2,j]}(\sigma, \mu)$ reduces to the generalized 2VHAP $\mathcal{J}_n(\sigma, \mu)$, thus satisfying the succeeding differential equation:*

$$\left[\sigma\frac{\partial}{\partial\sigma} + \frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})}\frac{\partial}{\partial\sigma} + \frac{2\,\tau\,\mu\,(\frac{\partial}{\partial\sigma})^2}{\left(\beta - \mu\frac{\partial}{\partial\sigma}^2\right)} - n\right]\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) = 0.$$

Remark 12. *For $j = 2$, the generalized 2I2DAP $\mathcal{J}_n^{[2,j]}(\sigma, \mu)$ reduces to the the generalized 2IAP $\mathcal{J}_n^{[2]}(\sigma, \mu)$, thus satisfying the succeeding differential equation:*

$$\left[\sigma\frac{\partial}{\partial\sigma} + \frac{\mathcal{J}_1'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_1(\frac{\partial}{\partial\sigma})}\frac{\partial}{\partial\sigma} + \frac{\mathcal{J}_2'(\frac{\partial}{\partial\sigma})}{\mathcal{J}_2(\frac{\partial}{\partial\sigma})}\frac{\partial}{\partial\sigma} + \frac{2\,\tau\,\mu\,(\frac{\partial}{\partial\sigma})^2}{\left(\beta - \mu\frac{\partial}{\partial\sigma}^2\right)} - n\right]\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) = 0.$$

Next, we proceed to establish the recurrence relations for the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ by leveraging their generating relation (39). A recurrence relation defines a sequence or array of values recursively, where each subsequent term depends on the preceding ones.

Starting with the differentiation of generating function (39) with respect to σ, μ, and β, we obtain the following recurrence relations for $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$:

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\left(\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)\right) &= n\,\mathcal{J}_{n-1,\tau}^{[2,j]}(\sigma, \mu; \beta), \\
\frac{\partial}{\partial\mu}\left(\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)\right) &= \tau\,\frac{n!}{(n-j+1)!}\mathcal{J}_{n-j+1,\tau}^{[2,j]}(\sigma, \mu; \beta), \\
\frac{\partial}{\partial\beta}\left(\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)\right) &= -\tau\,\mathcal{J}_{n,\tau+1}^{[2,j]}(\sigma, \mu; \beta).
\end{aligned} \quad (65)$$

Based on these relations, we deduce that

$$\frac{\partial}{\partial\mu}\left(\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\right)=-\frac{\partial^{j-1}}{\partial\sigma^{j-2}\partial\beta}\,\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta). \tag{66}$$

4. Examples

The Appell polynomial family comprises a set of polynomials that can be derived by selecting suitable functions, such as $\mathcal{J}_1(\theta)$ and $\mathcal{J}_2(\theta)$. A detailed compilation of diverse members of this family, featuring their names, generating functions, series definitions, and corresponding numerical values, is presented in Table 1. This valuable information can greatly assist those engaged in research or professional work involving polynomials.

Table 1. Expressions for select Appell family members.

S. No.	Name and Polynomials/Numbers	$\mathcal{J}_1(\theta)=\mathcal{J}_2(\theta)$	Generating Expression	Series Representation
I.	Bernoulli polynomials and numbers [15]	$\left(\frac{\theta}{e^\theta-1}\right)^2$	$\left(\frac{\theta}{e^\theta-1}\right)^2 e^{\theta\sigma}=\sum_{k=0}^{\infty}\mathcal{B}_k^{[2]}(\sigma)\frac{\theta^k}{k!}$ $\left(\frac{\theta}{e^\theta-1}\right)^2=\sum_{k=0}^{\infty}\mathcal{B}_k^{[2]}\frac{\theta^k}{k!}$ $\mathcal{B}_k^{[2]}:=\mathcal{B}_k^{[2]}(0)$	$\mathcal{B}_k^{[2]}(\sigma)=\sum_{m=0}^{k}\binom{k}{m}\mathcal{B}_m^{[2]}u^{k-m}$
II.	Euler polynomials and numbers [15]	$\left(\frac{2}{e^\theta+1}\right)^2$	$\left(\frac{2}{e^\theta+1}\right)^2 e^{\sigma u}=\sum_{k=0}^{\infty}\mathcal{E}_k^{[2]}(\sigma)\frac{u^k}{k!}$ $\left(\frac{2e^\theta}{e^{2\theta}+1}\right)^2=\sum_{k=0}^{\infty}\mathcal{E}_k^{[2]}\frac{\theta^s}{s!}$ $\mathcal{E}_k^{[2]}:=2^k\mathcal{E}_k^{[2]}\left(\frac{1}{2}\right)$	$\mathcal{E}_k^{[2]}(\sigma)=\sum_{m=0}^{k}\binom{k}{m}\frac{\mathcal{E}_m^{[2]}}{2^m}\left(\sigma-\frac{1}{2}\right)^{k-m}$
III.	Genocchi polynomials and numbers [16]	$\left(\frac{2\theta}{e^\theta+1}\right)^2$	$\left(\frac{2\theta}{e^\theta+1}\right)^2 e^{\sigma u}=\sum_{k=0}^{\infty}\mathcal{G}_k^{[2]}(\sigma)\frac{\theta^k}{k!}$ $\left(\frac{2\theta}{e^\theta+1}\right)^2=\sum_{k=0}^{\infty}\mathcal{G}_k^{[2]}\frac{\theta^s}{s!}$ $\mathcal{G}_k^{[2]}:=\mathbb{G}_k^{[2]}(0)$	$\mathcal{G}_k^{[2]}(\sigma)=\sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m^{[2]}\,\sigma^{k-m}$

The "Bernoulli, Euler, and Genocchi numbers" are of paramount importance in the field of mathematics, finding extensive applications across diverse domains. Similarly, Euler numbers, forming an integer sequence, are widely utilized in various mathematical areas such as algebraic topology, geometry, and number theory. They are pivotal in studying elliptic curves and modular forms, which are crucial for cryptography and coding theory. Conversely, Genocchi numbers, also integers, are valuable in combinatorial tasks like counting labelled rooted trees and up-down sequences, contributing significantly to graph and automata theory. Their collective significance in these fields is undeniable, making them indispensable tools for mathematicians and researchers alike.

Trigonometric and hyperbolic secant functions closely intertwine with Euler numbers through their Taylor series expansions, incorporating Euler numbers and their derivatives. This integration facilitates applications in signal processing and quantum field theory, rendering Euler numbers valuable beyond mathematics. Additionally, due to their rational nature, Bernoulli numbers play essential roles in mathematical formulas like Bernoulli polynomials and the Euler–Maclaurins formula, spanning number theory, numerical analysis, and combinatorics. Similarly, Euler numbers, as integer sequences, are indispensable in various mathematical fields such as algebraic topology, geometry, and number theory, especially in cryptography and coding theory. Conversely, Genocchi numbers, also integer sequences, contribute to combinatorial problems like counting labelled rooted trees and up-down sequences, impacting graph theory and automata theory.

By substituting $\mathcal{J}_1(\theta) = \mathcal{J}_2(\theta) = \left(\frac{\theta}{e^\theta - 1}\right)^2$, the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ reduces to the generalized 2-iterated 2D Bernoulli polynomials $\mathcal{B}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$. Thus, for $\mathcal{J}_1(\theta) = \mathcal{J}_2(\theta) = \left(\frac{\theta}{e^\theta - 1}\right)^2$ in the expressions (35), (39) and (48), the corresponding operational connection, generating function and explicit form for the generalized 2I2DBP $\mathcal{B}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ are as follows:

$$\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau}\{\mathcal{B}_n^{[2]}(\sigma)\} = \mathcal{B}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) \tag{67}$$

$$\frac{\left(\frac{\theta}{e^\theta - 1}\right)^2 \exp(\sigma\theta)}{(\beta - \mu\theta^j)^\tau} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)\frac{\theta^n}{n!} \tag{68}$$

$$\mathcal{B}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) = n!\sum_{p=0}^{[\frac{n}{j}]} \frac{(\tau)_p\,\mu^p}{p!\,(n - jp)!\,\beta^p}\mathcal{B}_{n-jp}^{[2]}(\sigma). \tag{69}$$

The first few polynomials of the generalized 2I2DBP $\mathcal{B}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ are as follows:

$$\begin{aligned}
\mathcal{B}_{0,\tau}^{[2,j]}(\sigma, \mu; \beta) &= \beta^{-\tau},\\
\mathcal{B}_{1,\tau}^{[2,j]}(\sigma, \mu; \beta) &= (\sigma - 1)\beta^{-\tau},\\
\mathcal{B}_{2,\tau}^{[2,j]}(\sigma, \mu; \beta) &= \left(\frac{\sigma^2}{2} + \frac{1}{6} - 2\sigma\right)\beta^{-\tau}.
\end{aligned}$$

Further, substituting $\mathcal{J}_1(\theta) = \mathcal{J}_2(\theta) = \left(\frac{2}{e^\theta + 1}\right)^2$, the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ reduces to the generalized 2-iterated 2D Euler polynomials $\mathcal{E}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$. Thus, for $\mathcal{J}_1(\theta) = \mathcal{J}_2(\theta) = \left(\frac{2}{e^\theta + 1}\right)^2$ in the expressions (35), (39) and (48), the corresponding operational connection, generating function and explicit form for the generalized 2I2DEP $\mathcal{E}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ are as follows:

$$\left(\beta - \mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau}\{\mathcal{E}_n^{[2]}(\sigma)\} = \mathcal{E}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) \tag{70}$$

$$\frac{\left(\frac{2}{e^\theta + 1}\right)^2 \exp(\sigma\theta)}{(\beta - \mu\theta^j)^\tau} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)\frac{\theta^n}{n!} \tag{71}$$

$$\mathcal{E}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta) = n!\sum_{p=0}^{[\frac{n}{j}]} \frac{(\tau)_p\,\mu^p}{p!\,(n - jp)!\,\beta^p}\mathcal{E}_{n-jp}^{[2]}(\sigma). \tag{72}$$

The first few polynomials of the generalized 2I2DEP $\mathcal{E}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ are as follows:

$$\begin{aligned}
\mathcal{E}_{0,\tau}^{[2,j]}(\sigma, \mu; \beta) &= 4\beta^{-\tau},\\
\mathcal{E}_{1,\tau}^{[2,j]}(\sigma, \mu; \beta) &= (4\sigma - 8)\beta^{-\tau},\\
\mathcal{E}_{2,\tau}^{[2,j]}(\sigma, \mu; \beta) &= \left(4 + 4\sigma^2 - 8\right)\beta^{-\tau}.
\end{aligned}$$

Further, substituting $\mathcal{J}_1(\theta) = \mathcal{J}_2(\theta) = \left(\frac{2\theta}{e^\theta + 1}\right)^2$, the generalized 2I2DAP $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$ reduces to the generalized 2-iterated 2D Genocchi polynomials $\mathcal{G}_{n,\tau}^{[2,j]}(\sigma, \mu; \beta)$. Thus, for $\mathcal{J}_1(\theta) = \mathcal{J}_2(\theta) = \left(\frac{2\theta}{e^\theta + 1}\right)^2$ in the expressions (35), (39) and (48), the corresponding operational connection, generating function and explicit form for the generalized 2I2DGP $\mathcal{G}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ are as follows:

$$\left(\beta-\mu\,\frac{\partial^j}{\partial\sigma^j}\right)^{-\tau}\{\mathcal{G}_n^{[2]}(\sigma)\}=\mathcal{G}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta) \tag{73}$$

$$\frac{\left(\frac{2\theta}{e^\theta+1}\right)^2\exp(\sigma\theta)}{(\beta-\mu\theta^j)^\tau}=\sum_{n=0}^{\infty}\mathcal{G}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)\frac{\theta^n}{n!} \tag{74}$$

$$\mathcal{G}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)=n!\sum_{p=0}^{[\frac{n}{j}]}\frac{(\tau)_p\,\mu^p}{p!\,(n-jp)!\,\beta^p}\mathcal{G}_{n-jp}^{[2]}(\sigma). \tag{75}$$

The first few polynomials of the generalized 2I2DGP $\mathcal{G}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$ are as follows:

$$\mathcal{G}_{0,\tau}^{[2,j]}(\sigma,\mu;\beta)=0,$$
$$\mathcal{G}_{1,\tau}^{[2,j]}(\sigma,\mu;\beta)=0,$$
$$\mathcal{G}_{2,\tau}^{[2,j]}(\sigma,\mu;\beta)=4\beta^{-\tau},$$
$$\mathcal{G}_{2,\tau}^{[2,j]}(\sigma,\mu;\beta)=(4\sigma-8)\beta^{-\tau}.$$

Similarly, all other corresponding results are established for the other mixed polynomials coming under the umbrella of the generalized 2I2DAP family.

5. Conclusions

This research extends the groundwork laid by [7] and introduces a fresh set of the generalized 2I2DAP, denoted as $\mathcal{J}_{n,\tau}^{[2,j]}(\sigma,\mu;\beta)$, utilizing fractional operators. In Section 2, we delineate the operational guidelines according to Theorem 1, establish the generating expression per Theorem 2, and furnish explicit formulations as presented in Theorems 3–5, along with pertinent remarks detailing their special instances. Section 3 is devoted to validating the monomiality principle for these polynomials and deriving the recurrence relations for the generalized 2-iterated 2D Appell polynomials. In Section 4, we showcase the broad applicability of the outcomes outlined in Section 2 by providing analogous findings for generalized "2-iterated 2D Bernoulli, Euler, and Genocchi polynomials". These specialized polynomials, linked with Appell polynomials, hold significant sway in both mathematical and physical realms. Notably, their ties to quantum mechanics and probability theory, particularly to the normal distribution, underscore their importance in probability theory.

The use of operational methodologies is exceedingly important when it comes to creating new sets of functional equations and fully understanding their inherent qualities, which can include both standard and specialized functions. The remarkable contributions made by researchers like Dattoli et al. (as seen in works such as [10,12,17,18]) serve as a prime example of just how crucial operational methods are to the analysis of special functions and their practical implications.

Possible avenues for future research include exploring symmetric identities and determinant forms and investigating the Δ_h and degenerate forms associated with the polynomials mentioned above. Another potential direction is to consider implicit summation equations to gain additional insights. Moreover, validating the monomiality principle and examining the sequence of upward and downward operators more closely may be worthwhile, as these could provide fruitful opportunities for further observations.

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