fractal and fractional

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On Higher-Order Nonlinear Fractional Elastic Equations with Dependence on Lower Order Derivatives in Nonlinearity

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Abstract: The paper studied high-order nonlinear fractional elastic equations that depend on low-order derivatives in nonlinearity and established the existence and uniqueness results by using the Leray–Schauder alternative theorem and Perov's fixed point theorem on an appropriate space under mild assumptions. Examples are given to illustrate the key results.

Keywords: Leray–Schauder alternative theorem; Perov's fixed point theorem; existence and uniqueness of solutions; Riemann–Liouville fractional derivatives

1. Introduction

Integer and fractional differential equations have the ability to model tremendous phenomena in physics, mechanics, control, and other fields of sciences and engineering (see [1–4] and references therein). Due to the advancement of the calculus and fractional calculus theory, boundary value problems (BVPs) for differential equations have attracted extensive interest. Among them, the fourth-order BVPs have been extensively studied via the techniques of nonlinear analysis (e.g., [5–13]). For example, by using the contraction principle and the iterative method, the authors [5] investigated the problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \ \ t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

and established the existence result of the solution. Equation (1) can be used to model the deformation of an elastic beam in equilibrium state, whose two ends clamped. In BVPs (1), the physical meaning of the derivatives u' is the slope. In [6], Ma and Tisdel studied (1) with $f = p(t)u^{\sigma}$, where continuous $p : (0,1) \to [0,+\infty)$ may be singular at $t = 0,1$ and $\sigma \in (0,1)$ and achieved the necessary and sufficient conditions for a regular positive solution using a lower and upper solution method. In [7], Alsaedi studied the same problem as in [6] but with $\sigma \in (-1,1)$ and p, satisfying Karamata regularly varying function-related hypotheses, and obtained a positive solution with precise global behaviors and the existence and uniqueness result.

In [8], Imed Bachar and Habib Mâagli considered the following problem:

$$\begin{cases} u^{(4)}(t) + u\varphi(t, u(t)) = 0, \ \ t \in (0,1), \\ u(0) = u(1) = 0, \ u'(0) = a, \ u'(1) = -b, \end{cases}$$

where constants $a, b \in [0,+\infty)$ with $a + b > 0$. Under some appropriate conditions imposed on φ, they achieved a uniqueness solution. In [9], Yao obtained several existence and multiplicity results to (1) with $f = \lambda g(t,u)$ and $\lambda > 0$ through the Krasnosel'skii fixed point theorem (FPT).

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The authors in [10] proved the existence of multiple positive solution to (1) using the Green's function and FPT on a cone. In [11], Xu et al. extended the result in [10] to the fractional setting and studied the following BVP:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t)), \ \ t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

where D_{0+}^{α} denotes the standard Riemann–Liouville fractional derivative with real number $\alpha \in (3,4]$. By using the Leray–Schauder nonlinear alternative theorem and FPT on cones, they proved that (2) has positive solutions and established the existence, multiplicity, and uniqueness results. They also reported the features of Green's function of (2). In [12], Karimov and Sadarangani studied (2) in which the function $f(t,u)$ is singular and demonstrated the existence of a unique positive solution with novel contractive mappings in complete metric spaces. Recently, the authors in [13] investigated the following BVP involving the fractional boundary derivative:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t)), \ \ t \in (0,1), \\ u(0) = u(1) = D_{0+}^{\alpha-3}u(0) = u'(1) = 0, \end{cases}$$

where D_{0+}^{α} is the same as in (2), and the nonlinearity f that satisfies a mild Lipschitz assumption is continuous on $(0,1) \times \mathbb{R}$. They proved the existence of a unique positive solution by using the Banach FPT on an appropriate space and Green's functions.

The aim of this paper is to establish the existence and uniqueness results with the Leray–Schauder alternative theorem [14] and Perov's FPT [15,16] for

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t),D_{0+}^{\alpha-3}u(t)), \ \ t \in (0,1), \\ u(0) = u(1) = D_{0+}^{\alpha-3}u(0) = D_{0+}^{\alpha-3}u(1) = 0, \end{cases} \tag{3}$$

where $f \in C((0,1) \times \mathbb{R}^2, \mathbb{R})$ is continuous and given. Therefore, Equation (3) is converted into an equivalent Fredholm integral equation form via Green's function. At the same time, several essential properties of Green's function are presented and their discrepancies for Green's functions for the integer and fractional order differential equations are analyzed. We note that the problem (3) is novel and its investigation will enhance the scope of the literature on fractional BVPs of fractional differential equations.

This work is structured to the following plan. Section 2 shows several definitions of fractional calculus and useful lemmas. Then, the existence of a unique solution for (3) are obtained in Section 3. Section 4 gives some examples. The last section shows the key conclusions of the present paper.

2. Preliminaries

This section gives several useful definitions, lemmas, and theorems.

Let σ be a function, a Riemann–Liouville type fractional order $\alpha > 0$, let $[\alpha]$ be the integer part of α, and let $\Gamma(\dot{})$ be a Euler gamma function. We have the following definitions:

Definition 1 ([4,17])**.** *The α order integral of Riemann–Liouville type can be defined as*

$$I_{0+}^{\alpha}\sigma(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \sigma(s)(t-s)^{\alpha-1} ds.$$

Definition 2 ([4,17])**.** *The α order derivative of Riemann–Liouville type can be defined as*

$$D_{0+}^{\alpha}\sigma(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{\sigma(s)}{(t-s)^{\alpha+1-n}} ds, \ \ n = [\alpha]+1.$$

Lemma 1 ([4,17])**.** *Let $u \in C(0,1) \cap L^1(0,1)$ and $\alpha > 0$. We have the following assertions:*

(i) For $0 < \beta < \alpha$, $D_{0+}^{\beta} I_{0+}^{\alpha} u = I_{0+}^{\alpha-\beta} u$ and $D_{0+}^{\alpha} I_{0+}^{\alpha} u = u$.

(ii) $D_{0+}^{\alpha} u = 0$ if and only if $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_j \in \mathbb{R}$, $j = 1,2,\ldots,n$, where n is the smallest integer greater than or equal to α.

(iii) Suppose that $D_{0+}^{\alpha} u \in C(0,1) \cap L^1(0,1)$. Then,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

$c_j \in \mathbb{R}$, $j = 1,2,\ldots,n$.

Lemma 2. *If $h \in C(0,1) \cap L^1(0,1)$, then there is a unique solution*

$$u(t) = \int_0^1 G_1(t,s) h(s) ds, \ t \in [0,1], \tag{4}$$

for fractional BVP

$$\begin{cases} D_{0+}^{\alpha} u(t) = h(t), \ \ t \in (0,1), \\ u(0) = u(1) = D_{0+}^{\alpha-3} u(0) = D_{0+}^{\alpha-3} u(1) = 0, \end{cases} \tag{5}$$

with

$$G_1(t,s) = G_{11}(t,s) - G_{12}(t,s) + G_{13}(t,s), \ \ t,s \in [0,1],$$

$$G_{11}(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1, \end{cases}$$

$$G_{12}(t,s) = \frac{t^{\alpha-1}}{(\alpha-3)\Gamma(\alpha)} [(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}], \ \ t,s \in [0,1],$$

$$G_{13}(t,s) = \frac{t^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)} [(1-s)^2 - (1-s)^{\alpha-1}], \ \ t,s \in [0,1].$$

Proof. By Lemma 1 there exists $c_i \in \mathbb{R}$ $(i = 1,2,3,4)$ such that

$$u(t) = I_{0+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4}.$$

Now, since $u(0) = 0$, we have $c_4 = 0$. Then,

$$u(t) = I_{0+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Applying operator $D_{0+}^{\alpha-3}$ on both sides of above equation yields

$$D_{0+}^{\alpha-3} u(t) = I_{0+}^{3} h(t) + \frac{c_1 \Gamma(\alpha)}{2} t^2 + c_2 \Gamma(\alpha-1) t + c_3 \Gamma(\alpha-2).$$

By using $D_{0+}^{\alpha-3} u(0) = 0$, we obtain $c_3 = 0$. Hence,

$$u(t) = I_{0+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

$$D_{0+}^{\alpha-3} u(t) = I_{0+}^{3} h(t) + \frac{c_1 \Gamma(\alpha)}{2} t^2 + c_2 \Gamma(\alpha-1) t.$$

Now, using the boundary conditions $u(1) = D_{0+}^{\alpha-3}u(1) = 0$ in the two equations above, we obtain

$$\begin{aligned} c_1 &= \frac{2}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left[-\Gamma(\alpha-1)I_{0+}^{\alpha}h(1)+I_{0+}^{3}h(1)\right] \\ &= \frac{1}{(\alpha-3)\Gamma(\alpha)}\int_0^1[2(1-s)^{\alpha-1}-(\alpha-1)(1-s)^2]h(s)ds, \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}[\Gamma(\alpha)I_{0+}^{\alpha}h(1)-I_{0+}^{3}h(1)] \\ &= \frac{1}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1[(1-s)^2-(1-s)^{\alpha-1}]h(s)ds. \end{aligned}$$

Then, we have the following unique solution for (5):

$$\begin{aligned} u(t) &= I_{0+}^{\alpha}h(t)+\frac{t^{\alpha-1}}{(\alpha-3)\Gamma(\alpha)}\int_0^1[2(1-s)^{\alpha-1}-(\alpha-1)(1-s)^2]h(s)ds \\ &\quad +\frac{t^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1[(1-s)^2-(1-s)^{\alpha-1}]h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

□

Applying operator $D_{0+}^{\alpha-3}$ on the integral Equation (4) and then using Lemma 1 yields

$$D_{0+}^{\alpha-3}u(t) = \int_0^1 G_2(t,s)h(s)ds,\ t\in[0,1],$$

where

$$G_2(t,s) = G_{21}(t,s)-G_{22}(t,s)+G_{23}(t,s),\ \ t,s\in[0,1],$$

$$G_{21}(t,s) = \begin{cases} \frac{1}{2}(t-s)^2, & 0\le s\le t\le 1, \\ 0, & 0\le t\le s\le 1, \end{cases}$$

$$G_{22}(t,s) = \frac{t^2}{2(\alpha-3)}[(\alpha-1)(1-s)^2-2(1-s)^{\alpha-1}],\ \ t,s\in[0,1],$$

and

$$G_{23}(t,s) = \frac{t}{2(\alpha-3)}[(1-s)^2-(1-s)^{\alpha-1}],\ \ t,s\in[0,1].$$

As stated in [5,10], Green's function G_1 with $\alpha = 4$ is nonnegative. However, it is invalid for $\alpha \in (3,4)$. In fact, $G_1(t,s)$ becomes $G_1(t,t) = \frac{t^{\alpha-2}(1-t)^{\alpha-1}}{(\alpha-3)\Gamma(\alpha)}[(\alpha-1)(1-t)^{4-\alpha} - (\alpha-1)+2t]$ along the diagonal, and $G_1(t,t)$ has a change of sign. Thus, Green's function $G_1(t,s)$ in this paper is split into three parts, each of which is either a nonnegative function or a nonpositive function as shown in the following results.

Lemma 3. $G_{ij}(t,s)$ $(i=1,2; j=1,2,3)$ *satisfies conditions:*

(i) $G_{i1}(t,s)\ge 0,\ t,s\in[0,1],\ i=1,2;$

(ii) $G_{ij}(t,s)\ge 0,\ t,s\in[0,1],\ \ i=1,2, j=2,3;$

(iii) $G_1(t,s)\in C([0,1]\times[0,1])$ *and* $|G_1(t,s)|\le h_1(t)g_1(s)$, *where* $h_1(t)=t^{\alpha-2}$, $g_1(s) = \frac{2}{(\alpha-3)\Gamma(\alpha-1)}(1-s)^2-\frac{4}{(\alpha-3)\Gamma(\alpha)}(1-s)^{\alpha-1};$

(iv) $G_2(t,s)\in C([0,1]\times[0,1])$ *and* $|G_2(t,s)|\le h_2(t)g_2(s)$, *where* $h_2(t)=t$, $g_2(s)=\frac{1}{2(\alpha-3)}[(2\alpha-3)(1-s)^2-3(1-s)^{\alpha-1}]$.

Proof. Obviously, (i) holds. For (ii), considering the definition of $G_{ij}(i = 1, 2, j = 2, 3)$, we only need to prove that

$$(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1} \geq 0,\ s \in [0,1],$$

and

$$(1-s)^2 - (1-s)^{\alpha-1} \geq 0,\ s \in [0,1].$$

Note that $\alpha \in (3,4)$. The simple calculation leads to the following:

$$\begin{aligned} & (\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1} \\ \geq\ & 2[(1-s)^2 - (1-s)^{\alpha-1}] \geq 2(1-s)^2[1-(1-s)^{\alpha-3}] \geq 0, s \in [0,1]. \end{aligned}$$

Therefore, (ii) is true. For (iii) and (iv), by (i), (ii), and the expression of functions G_{ij}, we obtain

$$\begin{aligned} |G_1(t,s)| &\leq G_{11}(t,s) + G_{12}(t,s) + G_{13}(t,s) \\ &\leq \frac{1}{\Gamma(\alpha)}(t-ts)^{\alpha-1} + \frac{1}{(\alpha-3)\Gamma(\alpha)}t^{\alpha-1}[(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}] \\ &\quad + \frac{t^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}[(1-s)^2 - (1-s)^{\alpha-1}] \\ &= \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1} + \frac{1}{(\alpha-3)\Gamma(\alpha)}t^{\alpha-1}[(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}] \\ &\quad + \frac{t^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}[(1-s)^2 - (1-s)^{\alpha-1}] \\ &\leq \frac{1}{\Gamma(\alpha)}t^{\alpha-2}(1-s)^{\alpha-1} + \frac{1}{(\alpha-3)\Gamma(\alpha)}t^{\alpha-2}[(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}] \\ &\quad + \frac{t^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}[(1-s)^2 - (1-s)^{\alpha-1}] \\ &= h_1(t)g_1(s), \end{aligned}$$

and

$$\begin{aligned} |G_2(t,s)| &\leq \frac{1}{2}(t-ts)^2 + \frac{t^2}{2(\alpha-3)}[(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}] \\ &\quad + \frac{t}{2(\alpha-3)}[(1-s)^2 - (1-s)^{\alpha-1}] \\ &= \frac{1}{2}t^2(1-s)^2 + \frac{t^2}{2(\alpha-3)}[(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}] \\ &\quad + \frac{t}{2(\alpha-3)}[(1-s)^2 - (1-s)^{\alpha-1}] \\ &\leq \frac{1}{2}t(1-s)^2 + \frac{t}{2(\alpha-3)}[(\alpha-1)(1-s)^2 - 2(1-s)^{\alpha-1}] \\ &\quad + \frac{t}{2(\alpha-3)}[(1-s)^2 - (1-s)^{\alpha-1}] \\ &= h_2(t)g_2(s), \end{aligned}$$

which completes the proof of (iii) and (iv). □

According to Green's function G_1, the existence results for linear fractional BVP (5) can be obtained under weaker conditions.

Lemma 4. *Let h be a function, $3 < \alpha < 4$, and let the map $t \to (1-t)^2 h(t)$ be continuous and integrable on $(0,1)$. The unique continuous solution for (5) can expressed as*

$$u(t) = Th(t) = \int_0^1 G_1(t,s)h(s)ds,\ t \in [0,1].$$

There are two nonnegative constants M_1, M_2 such that

$$|u(t)| \le M_1 h_1(t),\ \ |D_{0+}^{\alpha-3}u(t)| \le M_2 h_2(t),\ \ t \in [0,1], \tag{6}$$

where $h_1(t), h_2(t)$ are given in Lemma 3.

Proof. For a given function h, let $(1-t)^2 h(t)$ belong to $C(0,1) \cap L^1(0,1)$. Since by Lemma 3 (iii), $G_1(t,s) \in C([0,1] \times [0,1])$ with

$$|G_1(t,s)| \le h_1(t)g_1(s) \le \frac{3\alpha - 5}{(\alpha-3)\Gamma(\alpha-1)} t^{\alpha-2}(1-s)^2, \tag{7}$$

we conclude that $Th \in C[0,1]$ and $Th(0) = Th(1) = 0$ by virtue of the dominated convergence theorem. Therefore, by Fubini's theorem, we have

$$I_{0+}^{4-\alpha}(Th)(t) = \frac{1}{\Gamma(4-\alpha)} \int_0^t (t-s)^{3-\alpha}(Th)(s)ds = \int_0^1 \widetilde{G}(t,\tau)h(\tau)d\tau,$$

with

$$\begin{aligned}
\widetilde{G}(t,\tau) &= \frac{1}{\Gamma(4-\alpha)} \int_0^t (t-s)^{3-\alpha} G_1(s,\tau)ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(4-\alpha)} \int_\tau^t (t-s)^{3-\alpha}(s-\tau)^{\alpha-1}ds \\
&\quad - \frac{(\alpha-1)(1-\tau)^2 - 2(1-\tau)^{\alpha-1}}{(\alpha-3)\Gamma(\alpha)\Gamma(4-\alpha)} \int_0^t (t-s)^{3-\alpha}s^{\alpha-1}ds \\
&\quad + \frac{(1-\tau)^2 - (1-\tau)^{\alpha-1}}{(\alpha-3)\Gamma(\alpha-1)\Gamma(4-\alpha)} \int_0^t (t-s)^{3-\alpha}s^{\alpha-2}ds \\
&= \frac{(\max\{t-\tau, 0\})^3}{6} - \frac{t^3}{6(\alpha-3)}[(\alpha-1)(1-\tau)^2 - 2(1-\tau)^{\alpha-1}] \\
&\quad + \frac{t^2}{2(\alpha-3)}[(1-\tau)^2 - (1-\tau)^{\alpha-1}],
\end{aligned}$$

implying that

$$\begin{aligned}
I_{0+}^{4-\alpha}(Th)(t) &= \int_0^t \frac{(t-\tau)^3}{6} h(\tau)d\tau \\
&\quad - \frac{t^3}{6(\alpha-3)} \int_0^1 [(\alpha-1)(1-\tau)^2 - 2(1-\tau)^{\alpha-1}]h(\tau)d\tau \\
&\quad + \frac{t^2}{2(\alpha-3)} \int_0^1 [(1-\tau)^2 - (1-\tau)^{\alpha-1}]h(\tau)d\tau.
\end{aligned}$$

Hence, for $t \in (0,1)$, we obtain

$$D_{0+}^{\alpha}(Th)(t) = \frac{d^4}{dt^4} I_{0+}^{4-\alpha}(Th)(t) = h(t),$$

and

$$\begin{aligned} D_{0+}^{\alpha-3}(Th)(t) &= \frac{d}{dt} I_{0+}^{4-\alpha}(Th)(t) = \int_0^t \frac{(t-\tau)^2}{2} h(\tau) d\tau \\ &-\frac{t^2}{2(\alpha-3)} \int_0^1 [(\alpha-1)(1-\tau)^2 - 2(1-\tau)^{\alpha-1}] h(\tau) d\tau \\ &+\frac{t}{(\alpha-3)} \int_0^1 [(1-\tau)^2 - (1-\tau)^{\alpha-1}] h(\tau) d\tau. \end{aligned} \tag{8}$$

Thus, it follows that $D_{0+}^{\alpha-3}(Th)(0) = 0$ and

$$\begin{aligned} D_{0+}^{\alpha-3}(Th)(1) &= \int_0^1 \frac{(1-\tau)^2}{2} h(\tau) d\tau \\ &-\frac{1}{2(\alpha-3)} \int_0^1 [(\alpha-1)(1-\tau)^2 - 2(1-\tau)^{\alpha-1}] h(\tau) d\tau \\ &+\frac{1}{(\alpha-3)} \int_0^1 [(1-\tau)^2 - (1-\tau)^{\alpha-1}] h(\tau) d\tau = 0. \end{aligned}$$

Therefore, $Th(t)$ is a solution for (5).

Next, for proving the uniqueness, assume that the fractional BVP has two solutions $\varphi, \psi \in C[0,1]$ and set $w = \varphi - \psi$. Then, $w \in C[0,1]$ and $D_{0+}^{\alpha} w(t) = 0$. By Lemma 2 (ii), there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$w(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4}.$$

$w = 0$ can be determined from $w(0) = w(1) = D_{0+}^{\alpha-3} w(0) = D_{0+}^{\alpha-3} w(1) = 0$. Therefore, $\varphi = \psi$.

At last, making use of Lemma 3, (7) and (8), we obtain

$$|u(t)| \le \int_0^1 |G_1(t,s)||h(s)| ds \le h_1(t) \int_0^1 g_1(s)|h(s)| ds := M_1 h_1(t)$$

and

$$\begin{aligned} &|D_{0+}^{\alpha-3}(u)(t)| \\ \le\ & \frac{t^2}{2} \int_0^t |h(\tau)| d\tau + \frac{t^2}{2(\alpha-3)} \int_0^1 [(\alpha-1)(1-\tau)^2 - 2(1-\tau)^{\alpha-1}] |h(\tau)| d\tau \\ &+\frac{t}{(\alpha-3)} \int_0^1 [(1-\tau)^2 - (1-\tau)^{\alpha-1}] |h(\tau)| d\tau \\ \le\ & \frac{t}{2} \int_0^1 |h(\tau)| d\tau + \frac{t}{2(\alpha-3)} \int_0^1 (\alpha-1)(1-\tau)^2 |h(\tau)| d\tau \\ &+\frac{t}{(\alpha-3)} \int_0^1 (1-\tau)^2 |h(\tau)| d\tau := M_2 h_2(t) \end{aligned}$$

Hence, (6) holds. □

Let $E = C[0,1]$ be a Banach space having a standard norm $\|u\| = \max_{0 \le t \le 1} |u(t)|$. Then, $E \times E$ is a Banach space with a norm $\|(u,v)\|_{E \times E} = \max\{\|u\|, \|v\|\}$.

Let

$$E_1 = \{u \in E,\ \exists\, \lambda > 0,\ s.t.\ |u(t)| \le \lambda h_1(t),\ t \in [0,1]\},$$
$$E_2 = \{u \in E,\ \exists\, \lambda > 0,\ s.t.\ |u(t)| \le \lambda h_2(t),\ t \in [0,1]\}.$$

Then, E_1, E_2 are two Banach spaces having norm $\|u\|_1 = \inf\{\lambda : |u(t)| \le \lambda h_1(t), t \in [0,1]\}$ and $\|u\|_2 = \inf\{\lambda : |u(t)| \le \lambda h_2(t), t \in [0,1]\}$, respectively.

Let $F = \{u \in E : D_{0+}^{\alpha-3}(u) \in E\}$, and the norm $\|u\|_F = \max\{\|u\|, \|D_{0+}^{\alpha-3}u\|\}$. Then, F is a Banach space [18]. In a similar manner, let $F_1 = \{u \in E_1 : D_{0+}^{\alpha-3}(u) \in E_2\}$. Then, F_1 is a Banach space with the norm $\|u\|_{F_1} = \max\{\|u\|_1, \|D_{0+}^{\alpha-3}u\|_2\}$.

Based on Lemma 4, the fractional BVP (3) has a solution $u \in F$ (in fact, $u \in F_1$ follows from Lemma 5 below) that can be written by

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s)f(s,u(s),v(s))ds, \\ v(t) = \displaystyle\int_0^1 G_2(t,s)f(s,u(s),v(s))ds, \end{cases}$$

where $v(t) = D_{0+}^{\alpha-3}u(t)$. Therefore, it is a fixed point problem in $E \times E$ (or $E_1 \times E_2$) for an operator

$$S = (S_1, S_2), \ \ S : E \times E \to E \times E, \tag{9}$$

with

$$S_1(u,v) = \int_0^1 G_1(t,s)f(s,u(s),v(s))ds, \tag{10}$$

$$S_2(u,v) = \int_0^1 G_2(t,s)f(s,u(s),v(s))ds, \tag{11}$$

respectively.

Here, we make assumptions as follows:

(H_1) $f \in C((0,1) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 (1-t)^2 |f(t,0,0)| dt < +\infty$;

(H_2) There exist $p_1, p_2 \in C((0,1), [0,+\infty))$ such that, for $t \in (0,1)$ and $x_i, y_i \in \mathbb{R} (i = 1,2)$,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le p_1(t)|x_1 - y_1| + p_2(t)|x_2 - y_2|,$$

and

$$\int_0^1 (1-t)^2 p_i(t) dt < \infty, \ \ i = 1,2. \tag{12}$$

(H_3) There exist $p_1, p_2, p_3 \in C((0,1), [0,+\infty))$ such that

$$|f(t,x,y)| \le p_1(t)|x| + p_2(t)|y| + p_3(t), \ t \in (0,1), \ x,y \in \mathbb{R},$$

and

$$\int_0^1 (1-t)^2 p_i(t) dt < \infty, \ \ i = 1,2,3. \tag{13}$$

Lemma 5. *Suppose that $(H_1), (H_2)$ or $(H_1), (H_3)$ hold. Then, the operator $S : E \times E \to E \times E$ is completely continuous.*

Proof. We only prove Lemma 5 in the case that $(H_1), (H_3)$ hold. Similar arguments apply when $(H_1), (H_2)$ hold.

Let us first show that $S_1(u,v)$ and $S_2(u,v)$ defined by (10) and (11) are continuous on $[0,1]$ for $u, v \in E$. By Lemma 3, $|G_1(t,s)| \le h_1(t)g_1(s)$ and $|G_2(t,s)| \le h_2(t)g_2(s)$, it follows that

$$\begin{aligned} & |G_1(t,s)f(s,u(s),v(s))| \le h_1(t)g_1(s)|f(s,u(s),v(s))| \\ \le \ & h_1(t)\frac{2(1-s)^2}{(\alpha-3)\Gamma(\alpha-1)}(p_1(s)|u(s)| + p_2(s)|v(s)| + p_3(s)) \\ \le \ & h_1(t)\frac{2(1-s)^2}{(\alpha-3)\Gamma(\alpha-1)}(p_1(s)\|u\| + p_2(s)\|v\| + p_3(s)) \end{aligned} \tag{14}$$

and

$$\begin{aligned} &|G_2(t,s)f(s,u(s),v(s))| \leq h_2(t)g_2(s)|f(s,u(s),v(s))| \\ \leq\ & h_2(t)\frac{2\alpha-3}{2(\alpha-3)}(1-s)^2(p_1(s)|u(s)|+p_2(s)|v(s)|+p_3(s)) \\ \leq\ & h_2(t)\frac{2\alpha-3}{2(\alpha-3)}(1-s)^2(p_1(s)\|u\|+p_2(s)\|v\|+p_3(s)). \end{aligned} \tag{15}$$

Since $G_i(t,s)(i=1,2)$ is continuous on $[0,1]\times[0,1]$, S is defined on $E\times E$ and $S(u,v)\in E\times E$ for $(u,v)\in E\times E$, according to (H_1), (H_2), and the dominated convergence theorem.

Furthermore, by (14) and (15), we obtain

$$\begin{aligned} |S_1(u,v)(t)| \leq\ & \frac{2h_1(t)}{(\alpha-3)\Gamma(\alpha-1)}\bigg(\|u\|\int_0^1(1-s)^2p_1(s)ds \\ & +\|v\|\int_0^1(1-s)^2p_2(s)ds+\int_0^1(1-s)^2p_3(s)ds\bigg), \end{aligned} \tag{16}$$

$$\begin{aligned} |S_2(u,v)(t)| \leq\ & h_2(t)\frac{2\alpha-3}{2(\alpha-3)}\bigg(\|u\|\int_0^1(1-s)^2p_1(s)ds \\ & +\|v\|\int_0^1(1-s)^2p_2(s)ds+\int_0^1(1-s)^2p_3(s)ds\bigg). \end{aligned} \tag{17}$$

From the definition of norms $\|\cdot\|_1$ and $\|\cdot\|_2$, we obtain that

$$S_1(u,v)\in E_1,\ \ S_2(u,v)\in E_2.$$

Thus, $S(E\times E)\subset E_1\times E_2$.

Next, we show that, for all bounded sets $\Omega\subset E\times E$, $S(\Omega)$ is relatively compact. For this end, let $\Omega=\{(u,v)\in E\times E:\|u\|\leq M,\ \|v\|\leq M\}\subset E\times E$ be a bounded set. Then, by (16) and (17), we obtain

$$\|S_1(u,v)\|,\|S_2(u,v)\|\leq\frac{2\max\{M,1\}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1(1-s)^2(p_1(s)+p_2(s)+p_3(s))ds.$$

Thus, $S(\Omega)\subset E\times E$ is bounded. For $s\in[0,1]$, $\rho(s)=(1-s)^2|f(s,u(s),v(s))\in L[0,1]$. Let $t_1<t_2$, $t_1,t_2\in[0,1]$. Then,

$$\begin{aligned} & |S_1(u,v)(t_2)-S_1(u,v)(t_1)| \\ \leq\ & \int_0^1|G_1(t_2,s)-G_1(t_1,s)||f(s,u(s),v(s))|ds \\ \leq\ & \int_0^1|G_{11}(t_2,s)-G_{11}(t_1,s)||f(s,u(s),v(s))|ds \\ & +\int_0^1|G_{12}(t_2,s)-G_{12}(t_1,s)||f(s,u(s),v(s))ds \\ & +\int_0^1|G_{13}(t_2,s)-G_{13}(t_1,s)||f(s,u(s),v(s))ds \\ =\ & \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}|f(s,u(s),v(s))|ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}|f(s,u(s),v(s))|ds \\ & +\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{(\alpha-3)\Gamma(\alpha)}\int_0^1[(\alpha-1)(1-s)^2-2(1-s)^{\alpha-1}]|f(s,u(s),v(s))|ds \\ & +\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1[(1-s)^2-(1-s)^{\alpha-1}]|f(s,u(s),v(s))|ds \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,u(s),v(s))|ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})|f(s,u(s),v(s))|ds \\
&\quad+\frac{\alpha-1}{(\alpha-3)\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^1\rho(s)ds+\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1\rho(s)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\rho(s)ds+\frac{\alpha-1}{2\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-2}(t_2+t_1-2s)(t_2-t_1)|f(s,u(s),v(s))|ds \\
&\quad+\frac{\alpha-1}{(\alpha-3)\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^1\rho(s)ds+\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1\rho(s)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\rho(s)ds+\frac{\alpha-1}{\Gamma(\alpha)}(t_2-t_1)\int_0^{t_1}(1-s)^{\alpha-1}|f(s,u(s),v(s))|ds \\
&\quad+\frac{\alpha-1}{(\alpha-3)\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^1\rho(s)ds+\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1\rho(s)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\rho(s)ds+\frac{\alpha-1}{\Gamma(\alpha)}(t_2-t_1)\int_0^{t_1}\rho(s)ds \\
&\quad+\frac{\alpha-1}{(\alpha-3)\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^1\rho(s)ds+\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1\rho(s)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\rho(s)ds+\frac{\alpha-1}{\Gamma(\alpha)}(t_2-t_1)\int_0^{1}\rho(s)ds \\
&\quad+\frac{\alpha-1}{(\alpha-3)\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^1\rho(s)ds+\frac{t_2^{\alpha-2}-t_1^{\alpha-2}}{(\alpha-3)\Gamma(\alpha-1)}\int_0^1\rho(s)ds,
\end{aligned}
$$

approaches 0 as $t_2-t_1\to 0$, independent of $(u,v)\in\Omega$. Hence, $S_1(\Omega)$ is equicontinuous. In an analogous manner, the equicontinuity of the operator S_2 can be established. In consequence, we deduce that $S(\Omega)$ is relatively compact.

Finally, we prove that the continuity of operator S. Let $\{(u_n,v_n)\}\subset E\times E$ be a convergent sequence and $\lim_{n\to\infty}\|(u_n,v_n)-(u,v)\|_{E\times E}=0$. Then, $u,v\in E$ and $\|u_n\|\le D$, $\|v_n\|\le D$ for $n\in\mathbb{N}$, where D is a positive constant. Note that $f\in C((0,1)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, we have

$$\lim_{n\to\infty}f(t,u_n(t),v_n(t))=f(t,u(t),v(t)) \text{ for a.e. } t\in[0,1].$$

Since, by (H_3),

$$|G_1(t,s)f(s,u_n(s),v_n(s))|\le\frac{2(1-s)^2}{(\alpha-3)\Gamma(\alpha-1)}(Dp_1(s)+Dp_2(s)+p_3(s)),$$

and

$$|G_2(t,s)f(s,u_n(s),v_n(s))|\le\frac{2\alpha-3}{2(\alpha-3)}(1-s)^2(Dp_1(s)+Dp_2(s)+p_3(s)),$$

we have

$$\lim_{n\to\infty}\int_0^1|G_1(t,s)|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds=0, \tag{18}$$

$$\lim_{n\to\infty}\int_0^1|G_2(t,s)|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds=0 \tag{19}$$

according to the Lebesgue dominated convergence theorem. Now, we conclude from (18) and (19),

$$|S_1(u_n,v_n)(t)-S_1(u,v)(t)|\le\int_0^1|G_1(t,s)|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds$$

and

$$|S_2(u_n,v_n)(t) - S_2(u,v)(t)| \leq \int_0^1 |G_2(t,s)|f(s,u_n(s),v_n(s)) - f(s,u(s),v(s))|ds$$

that $\lim_{n\to\infty} \|S_1(u_n,v_n) - S_1(u,v)\| = \lim_{n\to\infty} \|S_2(u_n,v_n) - S_2(u,v)\| = 0$ or $\lim_{n\to\infty} \|S(u_n,v_n) - S(u,v)\|_{E\times E} = 0$. Therefore, S is continuous. □

Lemma 6. *Let $p_i \in C((0,1),[0,+\infty))$ $(i=1,2)$ and assume that (9) holds. Set*

$$E_{ij} = \{a > 0 : \int_0^1 |G_i(t,s)|p_j(s)h_j(s)ds \leq ah_i(t)\} \ \ i,j = 1,2,$$

where $h_1(t), h_2(t)$ are given in Lemma 3. Then, $E_{ij} \neq \emptyset$ and $a_{ij} = \inf E_{ij} \leq M_{ij}$, where $M_{ij} = \int_0^1 g_i(s)p_j(s)h_j(s)ds$.

Proof. By Lemma 3 (iii) and (iv), we have the following conclusions:

$$\int_0^1 |G_1(t,s)|p_1(s)h_1(s)ds \leq h_1(t) \int_0^1 g_1(s)p_1(s)h_1(s)ds = M_{11}h_1(t),$$

$$\int_0^1 |G_1(t,s)|p_2(s)h_2(s)ds \leq h_1(t) \int_0^1 g_1(s)p_2(s)h_2(s)ds = M_{12}h_1(t),$$

$$\int_0^1 |G_2(t,s)|p_1(s)h_1(s)ds \leq h_2(t) \int_0^1 g_2(s)p_1(s)h_1(s)ds = M_{21}h_2(t),$$

and

$$\int_0^1 |G_2(t,s)|p_2(s)h_2(s)ds \leq h_2(t) \int_0^1 g_2(s)p_2(s)h_2(s)ds = M_{22}h_2(t).$$

It follows that $E_{ij} \neq \emptyset$ and $a_{ij} = \inf E_{ij} \leq M_{ij}$. This finishes the proof. □

For $i,j = 1,2$, let $N_{ij} = \int_0^1 g_i(s)p_j(s)ds$. With this, together with Lemma 6, we can introduce three nonnegative matrices, A, M, and N, as follows:

$$A = (a_{ij})_{2\times 2}, \ \ M = (M_{ij})_{2\times 2}, \ \ N = (N_{ij})_{2\times 2}. \tag{20}$$

For matrix $A = (a_{ij})$, we say $A \geq 0$ if $a_{ij} \geq 0$ for all i,j. For matrix A_1, A_2, we say $A_1 \geq A_2$ if $A_1 - A_2 \geq 0$. Clearly, $N \geq M \geq A \geq 0$ and matrices M and N are easier to acquire than the matrix A.

Let A be a nonnegative matrix with a spectral radius $\rho(A)$.

Lemma 7 ([15,16]). *If $\rho(A) < 1$, then $I - A$ is nonsingular, and $(I-A)^{-1}$ is nonnegative.*

Lemma 8 ([19,20]). *If $A_1 \geq A_2 \geq 0$, then $\rho(A_2) \leq \rho(A_1)$.*

Definition 3. *Let E be a vector space over $\mathbb{R}$. If a vector norm on E is a function $\|\cdot\| : E \to \mathbb{R}^n$ such that for all $x,y \in E$, $c \in \mathbb{R}$, then we have:*
(1) $\|x\| = 0$ *if and only if* $x = \theta$, *and* $\|x\| \geq 0$;
(2) $\|cx\| = |c|\|x\|$;
(3) $\|x+y\| \leq \|x\| + \|y\|$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, $\alpha, \beta \in \mathbb{R}^n$, and $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for $i = 1,2,\cdots,n$.

A vector space E equipped with a vector norm $\|\cdot\|$ is called a generalized norm space and represented with $(E, \|\cdot\|)$. For $u,v \in E$, $d(u,v) = \|u - v\|$ defines a vector metric on E. If there is a vector metric on a vector space E, then (E,d) is called a generalized metric space that shares key attributes with traditional norm spaces, including the space's completeness, as well as the Cauchy property and the convergence of sequences.

Theorem 1 (See [15,16])**.** *Let (E, d) be a complete generalized metric space and let $T : E \to E$ be such that*

$$d(T(u), T(v)) \leq Md(u, v), \quad u, v \in E \tag{21}$$

for some matrix M with $M \geq 0$. If $\rho(M) < 1$, then T has a unique fixed point.

3. Main Results

Firstly, the existence of a unique solution to fractional differential Equation (3) is proved by using Perov's FPT.

Theorem 2. *Problem (3) has a unique solution in F_1 provided that (H_1) and (H_2) hold and $\rho(A) < 1$ or $\rho(M) < 1$, where A, M are given in (20) and Lemma 6.*

Proof. For $(u_1, v_1), (u_2, v_2) \in E_1 \times E_2$, we define

$$\|(u_1, v_1)\| = (\|u_1\|_1, \|v_1\|_2)^T.$$

Obviously, $E_1 \times E_2$ is a complete generalized Banach space having a vector norm $\|(\cdot, \cdot)\|$.

Now, we show that the operator S given in (9) satisfies (21) for a nonnegative matrix. From the proof of Lemma 5, the operator S is defined on $E_1 \times E_2$ and $S(E_1 \times E_2) \subset E_1 \times E_2$. For any $(u_1, v_1), (u_2, v_2) \in E_1 \times E_2$, by using (H_2), we obtain that

$$\begin{aligned}
& |S_1(u_1, v_1)(t) - S_1(u_2, v_2)(t)| \\
\leq\ & \int_0^1 |G_1(t,s)||f(s, u_1(s), v_1(s)) - f(s, u_1(s), v_1(s))|ds \\
\leq\ & \int_0^1 |G_1(t,s)|p_1(s)|u_1(s) - u_2(s)|ds + \int_0^1 |G_1(t,s)|p_2(s)|v_1(s) - v_2(s)|ds \\
\leq\ & \|u_1 - u_2\|_1 \int_0^1 |G_1(t,s)|p_1(s)h_1(s)ds + \|v_1 - v_2\|_2 \int_0^1 |G_1(t,s)|p_2(s)h_2(s)ds \\
\leq\ & (a_{11}\|u_1 - u_2\|_1 + a_{12}\|v_1 - v_2\|_2)h_1(t).
\end{aligned}$$

Considering the definition of norm $\|\cdot\|_1$ on E_1, we obtain that

$$\|S_1(u_1, v_1) - S_1(u_2, v_2)\|_1 \leq a_{11}\|u_1 - u_2\|_1 + a_{12}\|v_1 - v_2\|_2.$$

In the same way, we can prove that

$$\|S_2(u_1, v_1) - S_2(u_2, v_2)\|_2 \leq a_{21}\|u_1 - u_2\|_1 + a_{22}\|v_1 - v_2\|_2.$$

Using the vector norm on $E_1 \times E_2$, we obtain a vector inequality:

$$\begin{pmatrix} \|S_1(u_1, v_1) - S_1(u_2, v_2)\|_1 \\ \|S_2(u_1, v_1) - S_2(u_2, v_2)\|_2 \end{pmatrix} \leq \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \|u_1 - u_2\|_1 \\ \|v_1 - v_2\|_2 \end{pmatrix}.$$

Therefore, for $\rho(A) < 1$ or $\rho(M) < 1$, S is a contraction, having a unique fixed point within $E_1 \times E_2$ (Theorem 1 [15,16]). Therefore, the problem (3) has a unique solution in F_1. □

To obtain Theorem 2, we use the basic complete generalized Banach space $E_1 \times E_2$ in the above proof. Theorem 2 remains true if we consider problem (3) in $E \times E$ with an appropriate vector norm on $E \times E$. Unlike Theorem 2, we remove the assumptions on $\rho(A)$ and $\rho(M)$ and replace them by $\rho(N)$.

Theorem 3. *Problem (3) has a unique solution in F provided that (H_1) and (H_2) hold and $\rho(N) < 1$.*

Proof. Let complete generalized Banach space $E \times E$ have a vector norm for $(u_1, v_1) \in E \times E$:

$$\|(u_1, v_1)\| = (\|u_1\|, \|v_1\|)^T.$$

In view of the assumption (H_2), for any $(u_1, v_1), (u_2, v_2) \in E \times E$, we have

$$\begin{aligned}
& |S_1(u_1, v_1)(t) - S_1(u_2, v_2)(t)| \\
\leq\ & \int_0^1 |G_1(t,s)||f(s, u_1(s), v_1(s)) - f(s, u_1(s), v_1(s))|ds \\
\leq\ & \int_0^1 |G_1(t,s)|p_1(s)|u_1(s) - u_2(s)|ds + \int_0^1 |G_1(t,s)|p_2(s)|v_1(s) - v_2(s)|ds \\
\leq\ & \|u_1 - u_2\| \int_0^1 h_1(t)g_1(s)p_1(s)ds + \|v_1 - v_2\| \int_0^1 h_1(t)g_1(s)p_2(s)ds \\
\leq\ & \|u_1 - u_2\| \int_0^1 g_1(s)p_1(s)ds + \|v_1 - v_2\| \int_0^1 g_1(s)p_2(s)ds \\
=\ & N_{11}\|u_1 - u_2\| + N_{12}\|v_1 - v_2\|.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
& |S_2(u_1, v_1)(t) - S_2(u_2, v_2)(t)| \\
\leq\ & \int_0^1 |G_2(t,s)||f(s, u_1(s), v_1(s)) - f(s, u_1(s), v_1(s))|ds \\
\leq\ & \int_0^1 |G_2(t,s)|p_1(s)|u_1(s) - u_2(s)|ds + \int_0^1 |G_2(t,s)|p_2(s)|v_1(s) - v_2(s)|ds \\
\leq\ & \|u_1 - u_2\| \int_0^1 h_2(t)g_2(s)p_1(s)ds + \|v_1 - v_2\| \int_0^1 h_2(t)g_2(s)p_2(s)ds \\
\leq\ & \|u_1 - u_2\| \int_0^1 g_2(s)p_1(s)ds + \|v_1 - v_2\| \int_0^1 g_2(s)p_2(s)ds \\
=\ & N_{21}\|u_1 - u_2\| + N_{22}\|v_1 - v_2\|.
\end{aligned}$$

Consequently, we have

$$\begin{pmatrix} \|S_1(u_1, v_1) - S_1(u_2, v_2)\| \\ \|S_2(u_1, v_1) - S_2(u_2, v_2)\| \end{pmatrix} \leq \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} \|u_1 - u_2\| \\ \|v_1 - v_2\| \end{pmatrix}.$$

Thus, Perov's FPT can be applied. □

Next, the existence result for the problem (3) is proved with the Leray–Schauder alternative theorem [14]. Unlike Theorem 3, we replace (H_2) by the weaker hypothesis (H_3) and prove the existence only, without uniqueness.

Theorem 4. *Problem (3) has at least one solution in F provided that (H_1) and (H_3) hold and $\rho(A) < 1$ or $\rho(M) < 1$.*

Proof. From Lemma 5, $S : E \times E \to E \times E$ is completely continuous.

We defined $\Omega = \{(u, v) \in E \times E : \ (u, v) = \lambda S(u, v),\ 0 \leq \lambda \leq 1\}$ and verified its boundedness. Let us take $(u, v) \in \Omega$. Then, $(u, v) = \lambda S(u, v)$ with $\lambda \in [0, 1]$, that is,

$u(t) = \lambda S_1(u(t), v(t))$, $v(t) = \lambda S_2(u(t), v(t))$, $\forall\, t \in [0,1]$. By Lemma 5, $(u, v) \in E_1 \times E_2$. Hence,

$$\begin{aligned} & |u(t)| \leq |S_1(u,v)(t)| \\ \leq\ & \int_0^1 |G_1(t,s)||f(s,u(s),v(s))|ds \\ \leq\ & \int_0^1 |G_1(t,s)|p_1(s)|u(s)|ds + \int_0^1 |G_1(t,s)|p_2(s)|v(s)|ds + \int_0^1 |G_1(t,s)|p_3(s)ds \\ \leq\ & \|u\|_1 \int_0^1 |G_1(t,s)|p_1(s)h_1(s)ds + \|v\|_2 \int_0^1 |G_1(t,s)|p_2(s)h_2(s)ds + h_1(t)\int_0^1 g_1(s)p_3(s)ds \\ \leq\ & (a_{11}\|u\|_1 + a_{12}\|v\|_2 + a_{13})h_1(t) \end{aligned}$$

and

$$\begin{aligned} & |v(t)| \leq |S_2(u,v)(t)| \\ \leq\ & \int_0^1 |G_2(t,s)||f(s,u(s),v(s))|ds \\ \leq\ & \int_0^1 |G_2(t,s)|p_1(s)|u(s)|ds + \int_0^1 |G_2(t,s)|p_2(s)|v(s)|ds + \int_0^1 |G_2(t,s)|p_3(s)ds \\ \leq\ & \|u\|_1 \int_0^1 |G_2(t,s)|p_1(s)h_1(s)ds + \|v\|_2 \int_0^1 |G_2(t,s)|p_2(s)h_2(s)ds + h_2(t)\int_0^1 g_2(s)p_3(s)ds \\ \leq\ & (a_{21}\|u\|_1 + a_{22}\|v\|_2 + a_{23})h_2(t). \end{aligned}$$

Considering the definition of the norm $\|\cdot\|_i$ on E_1 and E_2, we obtain that

$$\|u\|_1 \leq a_{11}\|u\|_1 + a_{12}\|v\|_2 + a_{13}, \quad \|v\|_2 \leq a_{21}\|u\|_1 + a_{22}\|v\|_2 + a_{23}.$$

Then, we obtain a vector inequality with these two inequalities by using vector norm on $E_1 \times E_2$:

$$\begin{pmatrix} \|u\|_1 \\ \|v\|_2 \end{pmatrix} \leq \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \|u\|_1 \\ \|v\|_2 \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = A\begin{pmatrix} \|u\|_1 \\ \|v\|_2 \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}.$$

This, together with Lemma 7, yields

$$\begin{pmatrix} \|u\|_1 \\ \|v\|_2 \end{pmatrix} \leq (I-A)^{-1}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Thus,

$$\|u\|_1 \leq c_1, \quad \|v\|_2 \leq c_2. \tag{22}$$

According to the definition of norm $\|\cdot\|_i$ $(i = 1,2)$, for $u \in E_1$ and $v \in E_2$, we conclude that

$$\|u\| \leq \|u\|_1, \quad \|v\| \leq \|v\|_2.$$

Applying (22), we have

$$\|u\| \leq c_1, \quad \|v\| \leq c_2.$$

Therefore, we obtained boundedness of the set Ω. Therefore, the conclusion of the Leray–Schauder alternative theorem [14] holds. Hence, S has at least one fixed point in $E \times E$, indicating at least one solution for the problem (3). □

4. Application

Two concrete examples are presented in this section to illustrate the effectiveness of the acquired results.

4.1. Application 1

Consider BVP

$$\begin{cases} D_{0+}^{\frac{7}{2}} u(t) = \dfrac{1}{\sqrt{t(1-t)}} + 3\sin(u(t)) + \dfrac{|D_{0+}^{\frac{1}{2}} u(t)|}{1 + |D_{0+}^{\frac{1}{2}} u(t)|}, \ t \in (0,1), \\ u(0) = u(1) = D_{0+}^{\frac{1}{2}} u(0) = D_{0+}^{\frac{1}{2}} u(1) = 0, \end{cases} \tag{23}$$

where $\alpha = \frac{7}{2}$.

Let

$$f(t,u,v) = \frac{1}{\sqrt{t(1-t)}} + 3\sin u + \frac{|v|}{1+|v|}.$$

It is easy to see that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le 3|x_1 - y_1| + |x_2 - y_2|, \ t \in [0,1], x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

Thus, Hypothesis (H2) is fulfilled for $\alpha = \frac{7}{2}$, $p_1(t) = 3$, $p_2(t) = 1$. Also, we have $M_{11} \approx 0.192648$, $M_{12} \approx 0.097912$, $M_{21} \approx 0.278184$, $M_{22} \approx 0.142857$. Therefore, $\rho(M) \approx 0.33466 < 1$, and (23) has a unique solution in F_1 (Theorem 2).

4.2. Application 2

Consider the following fractional BVP

$$\begin{cases} D_{0+}^{\frac{15}{4}} u(t) = \dfrac{\sqrt{t} + \sqrt{1 + \sin D_{0+}^{\frac{3}{4}} u(t)}}{1+t^2} + \dfrac{2\arctan u(t)}{\sqrt{t}\sqrt{(1-t)^3}} + \dfrac{\ln\left(1 + \left(D_{0+}^{\frac{3}{4}} u(t)\right)^2\right)}{2\sqrt{t(1-t)}}, \\ u(0) = u(1) = D_{0+}^{\frac{3}{4}} u(0) = D_{0+}^{\frac{3}{4}} u(1) = 0, \end{cases} \tag{24}$$

where $\alpha = \frac{15}{4}$.

Let

$$f(t,u,v) = \frac{\sqrt{t} + \sqrt{1+\sin v}}{1+t^2} + \frac{2\arctan u}{\sqrt{t}\sqrt{(1-t)^3}} + \frac{\ln(1+v^2)}{2\sqrt{t(1-t)}}.$$

Obviously,

$$|f(t,u,v)| \le \frac{\sqrt{2}+\sqrt{t}}{1+t^2} + \frac{2}{\sqrt{t}\sqrt{(1-t)^3}}|u| + \frac{1}{\sqrt{t(1-t)}}|v|, \ t \in [0,1], u, v \in \mathbb{R}.$$

Thus, Hypothesis (H_2) does not hold, but the weaker hypothesis (H_3) has been met. For $\alpha = \frac{15}{4}$, $p_1(t) = \frac{2}{\sqrt{t}\sqrt{(1-t)^3}}$, and $p_2(t) = \frac{1}{\sqrt{t(1-t)}}$, $M_{11} \approx 0.4866$, $M_{12} \approx 0.1613$, $M_{21} \approx 0.9207$, and $M_{22} \approx 0.3166$. Therefore, $\rho(M) \approx 0.79626 < 1$. Thus, the problem (24) has at least one solution in F by Theorem 4.

5. Conclusions

We studied a new fractional differential equation with an order α, $3 < \alpha \le 4$, with the two-point BVP with nonlinearity depending on the lower-order derivatives of an unknown function. We first used Green's function to convert the given problem into the Fredholm integral equation form. Here, several properties and differences of Green's functions for integer and fractional order differential equations were explored. Since there are lower-order derivatives in nonlinearity, the fixed point problem for integral operators was treated as one for operator systems in which the functional spaces are equipped with a vector norm. The use of a vector norm enabled us to obtain some better results as shown in [15]. Under some suitable weaker assumptions, the uniqueness has been derived by means of Perov's FPT and matrix analysis, and the existence of solutions to the problem has been proved via the Leray–Schauder alternative theorem and matrix analysis. Our theoretical findings were verified with two examples. In the future, we intend to study the multivalued and impulsive cases of the problem with a fully nonlinear term.

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