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Fuzzy Differential Subordination for Classes of Admissible Functions Defined by a Class of Operators

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Abstract: This paper's findings are related to geometric function theory (GFT). We employ one of the most recent methods in this area, the fuzzy admissible functions methodology, which is based on fuzzy differential subordination, to produce them. To do this, the relevant fuzzy admissible function classes must first be defined. This work deals with fuzzy differential subordinations, ideas borrowed from fuzzy set theory and applied to complex analysis. This work examines the characteristics of analytic functions and presents a class of operators in the open unit disk $\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})$ for $\varsigma > -1, \eta > 0$, such that $\mathrm{a},\mathrm{e} \in \mathbb{R}, (\mathrm{e}-\mathrm{a}) \geq 0, \mathrm{a} > -\mathrm{x}$. The fuzzy differential subordination results are obtained using (GFT) concepts outside the field of complex analysis because of the operator's compositional structure, and some relevant classes of admissible functions are studied by utilizing fuzzy differential subordination.

Keywords: fuzzy set; fuzzy differential subordination; analytic functions; admissible functions; fuzzy best dominant

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1. Introduction

In 2011, a connection was made between the study of fuzzy sets theory and the area of complex analysis that examines analytic functions' geometric characteristics. [1]. The notion of unequal subordination was first investigated by Miller and Mocanu in [2,3]. Fuzzy subordination was investigated by Oros and Oros [1] in 2011, and they originally presented fuzzy differential subordination in 2012 [4]. A publication from 2017 [5] provides a good overview of the background of the concept of a fuzzy set and its connections to many scientific and technological fields. It also includes references to the research conducted up to that point on fuzzy differential subordination theory. Without the first findings, which adjusted the conventional differential subordination hypothesis to the unique characteristics of fuzzy differential subordination and offered strategies for analyzing fuzzy differential subordinations' dominants and best dominants, it would not have been possible for the study in this field to continue [6]. After that, Ref. [7] studied the specific form of Briot–Bouquet fuzzy differential subordinations. After embracing the idea, Haydar in [8] started investigating the recent discoveries of fuzzy differential subordinations. This subsequent research gave the investigation a new direction by associating fuzzy differential subordinations with various operators [9,10].

Fuzzy differential subordination, including fractional calculus, has advanced significantly in recent years, and it has been demonstrated to have applications in numerous study areas [11,12].

The following equation denotes the $\mathcal{H}(\mathbb{U})$ class of analytic functions in the open unit disk:

$$\mathbb{U} = \{\zeta : \zeta \in \mathbb{C} \quad \text{and} \quad |\zeta| < 1\}.$$

A notable subclass of $\mathcal{H}(\mathbb{U})$ is defined by $\mathcal{H}[a_0, n]$ and contains $\mathfrak{f} \in \mathcal{H}$, given by

$$\mathfrak{f}(\zeta) = a_0 + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \cdots \tag{1}$$

$$(a_0 \in \mathbb{C};\ n \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

Another remarkable subclass of $\mathcal{H}$ is denoted by $\mathcal{A}$ and consists of $\mathfrak{f} \in \mathcal{H}$ of the type

$$\mathfrak{f}(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n\zeta^n \quad (\zeta \in \mathbb{U}). \tag{2}$$

Suppose that $\kappa \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $\varsigma > -1, \eta > 0$, a linear operator $\mathcal{J}^{\kappa}_{\eta,\varsigma} : \mathcal{A} \to \mathcal{A}$ be defined by

$$\begin{aligned} \mathcal{J}^{\kappa}_{\eta,\varsigma}\mathfrak{f}(\zeta) &= \mathfrak{f}(\zeta), \quad \kappa = 0, \\ &= \tfrac{\varsigma+1}{\eta}\zeta^{1-\frac{\varsigma+1}{\eta}} \int_0^{\zeta} t^{\frac{\varsigma+1}{\eta}-2}\mathcal{J}^{\kappa+1}_{\eta,\varsigma}\mathfrak{f}(\zeta)dt, \quad \kappa = -1, -2, ... \\ &= \tfrac{\eta}{\varsigma+1}\zeta^{2-\frac{\varsigma+1}{\eta}} \tfrac{d}{d\zeta}\left(\zeta^{\frac{\varsigma+1}{\eta}-1}\mathcal{J}^{\kappa-1}_{\eta,\varsigma}\mathfrak{f}(\zeta)\right), \quad \kappa = 1, 2, ... \end{aligned}$$

Consider

$$\left(I^{\alpha}_{0+}\ \mathfrak{f}\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1}\ \mathfrak{f}(\tau)\ \mathrm{d}\tau \quad (x > 0;\ \Re(\alpha) > 0) \tag{3}$$

as a Riemann–Liouville fractional integral operator of order $\alpha \in \mathbb{C}$, $(\Re(\alpha) > 0)$ (see, for instance, [13,14], and see also [15,16]). Using the widely recognized Gamma function $\Gamma(\alpha)$ (Euler's), the Riemann–Liouville operator I^{α}_{0+} is interestingly replaced by the Erdelyi–Kober fractional integral operator of order $\alpha \in \mathbb{C}$, $(\Re(\alpha) > 0)$ given by

$$\left(I^{\alpha}_{0+;\sigma,\eta}\ f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x \tau^{\sigma(\eta+1)-1}\ (x^{\sigma} - \tau^{\sigma})^{\alpha-1}\ \mathfrak{f}(\tau)\ \mathrm{d}\tau \tag{4}$$

$$(x > 0;\ \Re(\alpha) > 0),$$

which basically matches with (3), where $\sigma - 1 = \eta = 0$, for

$$\left(I^{\alpha}_{0+;1,0}\ \mathfrak{f}\right)(x) = x^{-\alpha}\ \left(I^{\alpha}_{0+}\ f\right)(x) \quad (x > 0;\ \Re(\alpha) > 0).$$

Let $x = \sigma = 1$, $\eta = \mathrm{a} - 1$, and $\alpha = \mathrm{e} - \mathrm{a}$. We consider for $\mathrm{x} > 0;\ \mathrm{a}, \mathrm{e} \in \mathbb{R}$; be such that $\Re(\mathrm{e} - \mathrm{a}) \geq 0$ integral operator $\mathcal{I}^{\mathrm{a,e}}_{\mathrm{x}} : \mathcal{A} \to \mathcal{A}$ be defined for $\Re(\mathrm{e} - \mathrm{a}) > 0$ and $\Re(\mathrm{a}) > -\mathrm{x}$:

$$\mathcal{I}^{\mathrm{a,e}}_{\mathrm{x}}\mathfrak{f}(\zeta) = \frac{\Gamma(\mathrm{e}+\mathrm{x})}{\Gamma(\mathrm{a}+\mathrm{x})\Gamma(\mathrm{e}-\mathrm{a})} \int_0^1 \tau^{\mathrm{a}-1}\ (1-\tau)^{\mathrm{e}-\mathrm{a}-1}\ \mathfrak{f}(\zeta\tau^{\mathrm{x}})\mathrm{d}\tau \tag{5}$$

$$(\mathrm{x} > 0;\ \mathrm{a}, \mathrm{e} \in \mathbb{R};\ \mathrm{e} > \mathrm{a}).$$

Applying the Eulerian Beta-function integral for evaluation:

$$\mathrm{B}(\alpha, \beta) := \begin{cases} \int_0^1 \tau^{\alpha-1}\ (1-\tau)^{\beta-1}\ \mathrm{d}\tau & (\min\{\Re(\alpha), \Re(\beta)\} > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha, \beta \in \mathbb{C}\backslash\mathbb{Z}^{-}_{0} = \{0, -1, -2, ...\}\right), \end{cases}$$

we readily find that

$$\mathcal{I}_{\mathrm{x}}^{\mathrm{a,e}}\mathfrak{f}(\zeta) = \begin{cases} \zeta + \dfrac{\Gamma(\mathrm{e}+\mathrm{x})}{\Gamma(\mathrm{a}+\mathrm{x})} \sum\limits_{n=2}^{\infty} \dfrac{\Gamma(\mathrm{a}+\mathrm{x}n)}{\Gamma(\mathrm{e}+\mathrm{x}n)} a_n \zeta^n & (\mathrm{e} > \mathrm{a}) \\ \\ \mathfrak{f}(\zeta) & (\mathrm{e} = \mathrm{a}), \end{cases}$$

Through iterations of the previously described, a class of operators $\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\colon \mathcal{A} \to \mathcal{A}$ is given by

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) = \mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathcal{I}_{\mathrm{x}}^{\mathrm{a,e}}\mathfrak{f}(\zeta)) = \mathcal{I}_{\mathrm{x}}^{\mathrm{a,e}}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}\mathfrak{f}(\zeta)\right),$$

for $\kappa \in \mathbb{Z}, \varsigma > -1, \eta > 0, \mathrm{x} > 0, \Re(e-a) \geq 0, \Re(a) > -\mathrm{x}$ moreover, the form (2) for $\mathfrak{f}$ is provided by

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) = \zeta + \frac{\Gamma(\mathrm{e}+\mathrm{x})}{\Gamma(\mathrm{a}+\mathrm{x})} \sum_{n=2}^{\infty} \left(1 + \frac{\eta(n-1)}{\varsigma+1}\right)^{\kappa} \frac{\Gamma(\mathrm{a}+\mathrm{x}n)}{\Gamma(\mathrm{e}+\mathrm{x}n)} a_n \zeta^n. \tag{6}$$

It is noteworthy that a class of operators $\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})$ was presented in [17].
From (6), it is clear that

$$\zeta\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right)' = \left(\frac{\varsigma+1}{\eta}\right) \mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) - \left(\frac{\varsigma+1}{\eta} - 1\right) \mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta). \tag{7}$$

$$\zeta\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right)' = \left(\frac{a}{\mathrm{x}} + 1\right) \mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a}+1,\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) - \frac{a}{\mathrm{x}} \mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta). \tag{8}$$

It is also noted that a large number of additional integral operators that were examined in previous publications are generalized by a class of operators $\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})$.

(i) $\mathcal{J}_{\eta,0}^{\kappa}(\mathrm{a},\mathrm{a},\mathrm{x})\mathfrak{f}(\zeta) = D_{\eta}^{\kappa}\mathfrak{f}(\zeta)$ ($\kappa \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, (Al-Oboudi [18]));
(ii) $\mathcal{J}_{1,0}^{\kappa}(\mathrm{a},\mathrm{a},\mathrm{x})\mathfrak{f}(\zeta) = D^{\kappa}\mathfrak{f}(\zeta)$ ($\kappa \in \mathbb{N}_0$, (Salagean [19]));
(iii) $\mathcal{J}_{\eta,0}^{\kappa}(\mathrm{a},\mathrm{e},1)\mathfrak{f}(\zeta) = D_{\eta}^{\kappa}(\mathrm{a}+1,\mathrm{e}+1)\mathfrak{f}(\zeta)$ ($\kappa \in \mathbb{N}_0$, (Selvaraj-Karthikeyan [20]));
(iv) $\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},0,1)\mathfrak{f}(\zeta) = I^{\kappa}(\eta,\mathrm{a},\mathrm{x})\mathfrak{f}(\zeta)$ ($\kappa \in \mathbb{N}_0$, (Catas [21]));
(v) $\mathcal{J}_{\eta,\varsigma}^{0}(\mathrm{a},\mathrm{a}+\alpha,1)\mathfrak{f}(\zeta) = Q_a^{\alpha}\mathfrak{f}(\zeta)(\alpha > 0, \mathrm{a} > -1)$ (Jung et al. [22]; see also [23]);
(vi) $\mathcal{J}_{1,a}^{-\kappa}(\mathrm{a},\mathrm{a},\mathrm{x})\mathfrak{f}(\zeta) = L_{a+1}^{\kappa}(\eta,\mathrm{a},\mathrm{x})\mathfrak{f}(\zeta)$ ($\kappa \in \mathbb{N}_0, \mathrm{a} \geq 0$ (Komatu [24])).

2. Preliminaries

Let $\wp$ be the collection of injective and analytic functions on $\overline{\mathbb{U}} \backslash E(\chi)$, with $\chi'(\xi) \neq 0$, for $\xi \in \partial\mathbb{U} \backslash E(\chi)$, and

$$E(\chi) = \{\xi : \xi \in \partial\mathbb{U} \ \text{ and } \ \lim_{\zeta \to \xi} f(\zeta) = \infty\}.$$

Also, $\wp(\mathfrak{a})$ is the subclass of $\wp$ with $\chi(0) = \mathfrak{a}$, and let

$$\wp(0) = \wp_0 \ \text{ and } \ \wp(1) = \wp_1.$$

Definition 1 ([25]). *Let $\varrho \neq \phi$. A fuzzy subset of ϱ is defined as $F : \varrho \to [0,1]$.*

Definition 2 ([25]). *A fuzzy subset of ϱ is a pair $(\mathcal{L}, F_{\mathcal{L}})$, where $\mathcal{L} = \{x \in \varrho : 0 < F_{\mathcal{L}}(x) \leq 1\} = sup(\mathcal{L}, F_{\mathcal{L}})$ is referred to as a fuzzy subset and $F_{\mathcal{L}} : \varrho \to [0,1]$ is the membership function of the fuzzy set $(\mathcal{L}, F_{\mathcal{L}})$.*

Definition 3 ([1]). *Fuzzy subsets $(\varepsilon_1, F_{\varepsilon_1})$ and $(\varepsilon_2, F_{\varepsilon_2})$ of ϱ are equal iff $\varepsilon_1 = \varepsilon_2$, whereas $(\varepsilon_1, F_{\varepsilon_1}) \subseteq (\varepsilon_2, F_{\varepsilon_2})$ iff $F_{\varepsilon_1}(\eta) \leq F_{\varepsilon_2}(\eta)$, $\eta \in \varrho$.*

Definition 4 ([1]). *Let $\mathfrak{D} \subset \mathbb{C}$ and ζ_0 is a fixed point in $\mathfrak{D}$ and let $\mathfrak{f}, \mathfrak{h} \in \mathcal{H}(\mathbb{U})$ and we will say that $\mathfrak{f}$ fuzzy is subordinate to $\mathfrak{h}$, denoted by $\mathfrak{f} \prec_F \mathfrak{h}$ or $\mathfrak{f}(\zeta) \prec_F \mathfrak{h}(\zeta)$ if*

$$\mathfrak{f}(\zeta_0) = \mathfrak{h}(\zeta_0) \;\; and \;\; F_{\mathfrak{f}(\mathfrak{D})}(\mathfrak{f}(\zeta)) \leq F_{\mathfrak{h}(\mathfrak{D})}(\mathfrak{h}(\zeta)), \zeta \in \mathfrak{D},$$

where

$$\mathfrak{f}(\mathfrak{D}) = \sup\left(\mathfrak{f}(\mathfrak{D}), F_{\mathfrak{f}(\mathfrak{D})}\right) = \{\mathfrak{f}(\zeta) : 0 < F_{\mathfrak{f}(\mathfrak{D})}(\mathfrak{f}(\zeta)) \leq 1, \zeta \in \mathfrak{D}$$

and

$$\mathfrak{h}(\mathfrak{D}) = \sup\left(\mathfrak{h}(\mathfrak{D}), F_{\mathfrak{h}(\mathfrak{D})}\right) = \{\mathfrak{h}(\zeta) : 0 < F_{\mathfrak{h}(\mathfrak{D})}(\mathfrak{h}(\zeta)) \leq 1, \zeta \in \mathfrak{D}.$$

Definition 5 ([4]). *Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let $\hbar$ be univalent in $\mathbb{U}$. If ω is analytic in $\mathbb{U}$ and satisfies*

$$F_{\psi(\mathbb{C}^3 \times \mathbb{U})}\left(\psi\left(\omega(\zeta), \zeta\omega'(\zeta), \zeta^2\omega''(\zeta); \zeta\right)\right) \leq F_{\hbar(\mathbb{U})}(\mathfrak{h}(\zeta)), \tag{9}$$

i.e.,

$$\psi\left(\omega(\zeta), \zeta\omega'(\zeta), \zeta^2\omega''(\zeta); \zeta\right)) \leq_F (\mathfrak{h}(\zeta)), \;\; \zeta \in \mathbb{U},$$

then, ω is called a fuzzy solution of fuzzy differential subordination. The univalent function ω is called a fuzzy dominant if $\omega(\zeta) \prec_F \chi(\zeta))$, for all ω satisfying (9). A fuzzy dominant $\widetilde{\chi}$ that satisfies $\widetilde{\chi}(\zeta) \prec_F \chi(\zeta)$ for all fuzzy dominant χ of (9) is said to be the fuzzy best dominant of (9).

Definition 6 ([4]). *Let Ω be a set in $\mathbb{C}, \chi \in \wp$ and $n \in \mathbb{N}$. The class $\Psi_n[\Omega, \chi]$ of admissible functions contain $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfies $F_\Omega(\psi(r, s, t; \zeta)) = 0$,*

$$r = \chi(\xi), \;\; s = k\xi\chi'(\xi) \;\; and \;\; \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(1 + \frac{\xi\chi''(\xi)}{\chi'(\xi)}\right),$$

where $\zeta \in \mathbb{U}, \xi \in \partial\mathbb{U} \backslash E(\chi)$ and $k \geqq n$. We can write $\Psi_1[\Omega, \chi]$ as $\Psi[\Omega, \chi]$.

Lemma 1 ([4]). *Let $\psi \in \Psi_n[\Omega, \chi]$ with $\chi(0) = \mathfrak{a}$. If $\omega \in \mathcal{H}[a_0, n]$ satisfies*

$$F_{\psi(\mathbb{C}^3 \times \mathbb{U})}\left(\psi\left(\omega(\zeta), \zeta\omega'(\zeta), \zeta^2\omega''(\zeta); \zeta\right)\right) \leq F_\Omega(\zeta), \;\; \zeta \in \mathbb{U},$$

then $F_{\omega(\mathbb{U})}(\omega(\zeta)) \leq F_{\chi(\mathbb{U})}(\chi(\zeta))$ i.e., $\omega(\zeta) \prec_F \chi(\zeta)$.

In this study, we establish suitable criteria for a class of operators $\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a}, \mathrm{e}, \mathrm{x})$ that corresponds to certain designated classes of admissible functions of analytic functions. The fuzzy best dominants are determined by obtaining fuzzy differential subordinations.

3. Main Results

Throughout this paper, unless otherwise mentioned, we set $\varsigma > -1, \eta > 0, \mathrm{a}, \mathrm{e} \in \mathbb{R}$, $(\mathrm{e} - \mathrm{a}) \geq 0$, and $\mathrm{a} > -\mathrm{x}$.

Definition 7. *Let Ω be a set in $\mathbb{C}$ and $\chi \in \wp_0 \cap \mathcal{H}$. The class $\Phi_A[\Omega, \chi]$ of admissible functions contains the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy*

$$F_\Omega(\varphi(u, v, w; \zeta)) = 0$$

when

$$u = \chi(\xi), \quad v = \frac{k\xi\chi'(\xi) + \left(\frac{\varsigma+1}{\eta} - 1\right)\chi(\xi)}{\frac{\varsigma+1}{\eta}}$$

and

$$\Re\left\{\frac{(\varsigma+1)^2 w-(\varsigma+1-\eta)^2 u}{(\varsigma+1)(v-u)+\eta v}+2\left(1-\frac{\varsigma+1}{\eta}\right)\right\} \geqq k\Re\left(\frac{\xi\chi''(\xi)}{\chi'(\xi)}+1\right) \quad (k>0), \tag{10}$$

where $\zeta \in \mathbb{U}$,$\xi \in \partial\mathbb{U}\backslash E(\chi)$ *and* $k \geqq 1$.

Here, we present and validate our initial finding, which we call Theorem 1.

Theorem 1. *Put* $\varphi \in \Phi_A[\Omega,\chi]$. *If* $\mathfrak{f} \in \mathcal{A}$ *satisfies*

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\left(\varphi\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta),\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta),\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta);\zeta\right)\right) \leq F_\Omega(\zeta), \tag{11}$$

then

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right) \leq F_{\chi(\mathbb{U})}(\chi(\zeta))$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \prec_F \chi(\zeta).$$

Proof. Let

$$\omega(\zeta) = \mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta). \tag{12}$$

Differentiating (12) and using (7), we obtain

$$\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) = \frac{\zeta\omega'(\zeta)+\left(\frac{\varsigma+1}{\eta}-1\right)\omega(\zeta)}{\left(\frac{\varsigma+1}{\eta}\right)}. \tag{13}$$

Further computations show that

$$\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) = \frac{\zeta^2\omega''(\zeta)+\left(\frac{2(\varsigma+1)}{\eta}-1\right)\zeta\omega'(\zeta)+\left(\frac{\varsigma+1}{\eta}-1\right)^2\omega(\zeta)}{\left(\frac{\varsigma+1}{\eta}\right)^2}. \tag{14}$$

The following transformations are now defined for $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$:

$$u(r,s,t) = r, \quad v(r,s,t) = \frac{s+\left(\frac{\varsigma+1}{\eta}-1\right)r}{\left(\frac{\varsigma+1}{\eta}\right)}$$

and

$$w(r,s,t) = \frac{t+\left(\frac{2(\varsigma+1)}{\eta}-1\right)s+\left(\frac{\varsigma+1}{\eta}-1\right)^2 r}{\left(\frac{\varsigma+1}{\eta}\right)^2}. \tag{15}$$

Additionally, we set

$$\psi(r,s,t;\zeta) = \varphi(u,v,w;\zeta) = \varphi\left(r,\frac{s+\left(\frac{\varsigma+1}{\eta}-1\right)r}{\left(\frac{\varsigma+1}{\eta}\right)},\frac{t+\left(\frac{2(\varsigma+1)}{\eta}-1\right)s+\left(\frac{\varsigma+1}{\eta}-1\right)^2 r}{\left(\frac{\varsigma+1}{\eta}\right)^2};\zeta\right). \tag{16}$$

Then, by using Equations (12)–(16), we obtain

$$\psi\Big(\omega(\zeta), \zeta\omega'(\zeta), \zeta^2\omega''(\zeta); \zeta\Big) = \varphi\Big(\Big(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta), \mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta), \mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta); \zeta\Big)\Big).$$

Thus, clearly, Equation (11) becomes

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\Big(\psi\Big(\omega(\zeta), \zeta\omega'(\zeta), \zeta^2\omega''(\zeta); \zeta\Big)\Big) \leq F_{\Omega}(\zeta).$$

Using (15)

$$\frac{t}{s} + 1 = \frac{(\varsigma+1)^2 w - (\varsigma+1-\eta)^2 u}{(\varsigma+1)(v-u) + \eta u} + 2\Big(1 - \frac{\varsigma+1}{\eta}\Big),$$

$\varphi \in \Phi_{\mathcal{A}}[\Omega, \chi]$ is equivalent to the admissibility condition for ψ given in Definition 6. So, $\psi \in \Psi[\Omega, \chi]$ and by Lemma 1,

$$F_{\omega(\mathbb{U})}\omega(\zeta) \leq F_{\chi(\mathbb{U})}(\chi(\zeta)),$$

or equivalent

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\Big(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\Big) \leq F_{\chi(\mathbb{U})}(\chi(\zeta)),$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \prec_F \chi(\zeta),$$

which proves Theorem 1. □

A simply connected domain $\Omega = \mathfrak{h}(\mathbb{U})$ for every conformal mapping $\mathfrak{h}(\zeta)$ of $\mathbb{U}$ onto Ω exists when $\Omega \neq \mathbb{C}$. The class $\Phi_{\mathcal{A}}[\mathfrak{h}(\mathbb{U}), \chi]$ is represented as $\Phi_{\mathcal{A}}[\mathfrak{h}, \chi]$ in this instance.
Theorem 1 immediately leads to the following outcome:

Theorem 2. *Set $\varphi \in \Phi_{\mathcal{A}}[\mathfrak{h}, \chi]$. If $\mathfrak{f} \in \mathcal{A}$,*

$$\varphi\Big(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta), \mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta), \mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta); \zeta\Big) \tag{17}$$

is analytic in $\mathbb{U}$ and

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\Big(\varphi\Big(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta), \mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta), \mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta); \zeta\Big)\Big) \leq F_{\mathfrak{h}(\mathbb{U})}(\mathfrak{h}(\zeta)), \tag{18}$$

then

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\Big(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\Big) \leq F_{\chi(\mathbb{U})}(\chi(\zeta)),$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \prec_F \chi(\zeta).$$

By taking $(\varphi(u, v, w; \zeta)) = 1 + \frac{v}{u}$ in Theorem 2, we obtain

Corollary 1. *Let $\varphi \in \Phi_{\mathcal{A}}[\mathfrak{h}, \chi]$. If $\mathfrak{f} \in \mathcal{A}$,*

$$2 + \frac{\eta}{\varsigma+1}\left(\frac{\zeta\Big(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\Big)'}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} - 1\right)$$

is analytic in $\mathbb{U}$ *and*

$$2+\frac{\eta}{\varsigma+1}\left(\frac{\zeta\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right)'}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}-1\right)\prec_F \mathfrak{h}(\zeta), \tag{19}$$

then

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\prec_F \chi(\zeta).$$

Our finding extends Theorem 1 to the situation where χ based on $\mathbb{U}$'s boundary is uncertain.

Corollary 2. *Assume that* $\Omega\subset\mathbb{C}$ *and* $\chi(\zeta)$ *are univalent in* $\mathbb{U}$ *with* $\chi(0)=0$. *Also suppose that* $\varphi\in\Phi_{\mathcal{A}}[\Omega,\chi_\rho]$ *for some* $\rho\in(0,1)$, *where*

$$\chi_\rho(\zeta)=\chi(\rho\zeta).$$

If $\mathfrak{f}\in\mathcal{A}$ *satisfies*

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\left(\varphi\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta),\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta),\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta);\zeta\right)\right)\le F_\Omega(\zeta), \tag{20}$$

then

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right)\le F_{\chi(\mathbb{U})}(\chi(\zeta)),$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\prec_F \chi(\zeta).$$

Proof. By Theorem 1, we obtain

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right)\le F_{\chi_\rho(\mathbb{U})}(\chi_\rho(\zeta)).$$

Since

$$\chi_\rho(\zeta)\prec\chi(\rho\zeta),$$

we have

$$F_{\chi_\rho(\mathbb{U})}(\chi_\rho(\zeta))=F_{\chi(\rho\mathbb{U})}(\chi(\rho\zeta))\quad and\quad \chi_\rho(0)=\chi(0).$$

Hence,

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right)\le F_{\chi(\rho\mathbb{U})}(\chi(\rho\zeta)).$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\prec_F \chi(\rho\zeta).$$

By letting $\rho\to 1$, we obtain

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\prec_F \chi(\zeta).$$

□

Theorem 3. *Let* $\mathfrak{h}$ *and* χ *be univalent in* $\mathbb{U}$ *with* $\chi(0)=0$. *Put*

$$\chi_\rho(\zeta)=\chi(\rho\zeta)\quad and\quad \mathfrak{h}_\rho(\zeta)=\mathfrak{h}(\rho\zeta).$$

Let $\varphi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ *satisfy*

(1) $\varphi\in\Phi_{\mathcal{A}}[\mathfrak{h},\chi_\rho]$ *for some* $\rho\in(0,1)$.
(2) *For* $\rho_0\in(0,1)$ *such that* $\varphi\in\Phi_{\mathcal{A}}[\mathfrak{h}_\rho,\chi_\rho]$, $\rho\in(\rho_0,1)$.

If $\mathfrak{f} \in \mathcal{A}$ *satisfies (18), then*

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right) \leq F_{\chi(\mathbb{U})}(\chi(\zeta)).$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \prec_F \chi(\zeta).$$

Proof. Case (1): Since the proof is similar to Theorem 2, we will not include it.

Case (2): Let

$$\omega(\zeta) = \mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \quad \textit{and} \quad \omega_\rho(\zeta) = \omega(\rho\zeta).$$

Then

$$\begin{aligned} & F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\left(\varphi\left(\omega_\rho(\zeta), \zeta\omega_\rho'(\zeta), \zeta^2\omega_\rho''(\zeta); \rho\zeta\right)\right) \\ = \ & F_{\varphi(\mathbb{C}^3\times\mathbb{U})}(\varphi(\omega(\rho\zeta), \zeta\omega'(\rho\zeta), \zeta^2\omega''(\rho\zeta); \rho\zeta)) \\ \leq \ & F_{\mathfrak{h}_\rho(\mathbb{U})}(\mathfrak{h}_\rho(\zeta)). \end{aligned}$$

Applying Theorem 1 and the remark connected to

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\left(\varphi\left(\omega(\zeta), \zeta\omega'(\zeta), \zeta^2\omega''(\zeta); \varpi(\zeta)\right)\right) \leq F_\Omega(\zeta),$$

where $\varpi : \mathbb{U} \to \mathbb{U}$, with $\varpi(\zeta) = \rho\zeta$, we obtain $\omega_\rho(\zeta) \prec_F \chi_\rho(\zeta)$ for $\rho \in (0,1)$. Suppose that $\rho \to 1$, we obtain $\omega(\zeta) \prec_F \chi(\zeta)$. Then,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \prec_F \chi(\zeta).$$

□

The fuzzy differential subordination's best dominant (18) is obtained using the following theorem:

Theorem 4. *Let* $\mathfrak{h}$ *be univalent in* $\mathbb{U}$ *and let* $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. *Let*

$$\begin{aligned} & \varphi\left(\omega(\zeta), \frac{\zeta\omega'(\zeta) + \left(\frac{\varsigma+1}{\eta} - 1\right)\omega(\zeta)}{\left(\frac{\varsigma+1}{\eta}\right)}, \right. \\ & \left. \frac{\zeta^2\omega''(\zeta) + \left(\frac{2(\varsigma+1)}{\eta} - 1\right)\zeta\omega'(\zeta) + \left(\frac{\varsigma+1}{\eta} - 1\right)^2\omega(\zeta)}{\left(\frac{\varsigma+1}{\eta}\right)^2}; \zeta\right) \\ = \ & \mathfrak{h}(\zeta) \end{aligned} \tag{21}$$

has a solution $\chi(\zeta)$, *with* $\chi(0) = 0$, *satisfying one of the next conditions:*

(1) $\chi(\zeta) \in \wp_0$ *and* $\varphi \in \Phi_{\mathcal{A}}[\mathfrak{h}, \chi]$.
(2) $\chi(\zeta)$ *is univalent in* $\mathbb{U}$, *and* $\varphi \in \Phi_{\mathcal{A}}[\mathfrak{h}, \chi_\rho]$ *for some* $\rho \in (0,1)$.
(3) $\chi(\zeta)$ *is univalent in* $\mathbb{U}$ *and there exists* $\rho_0 \in (0,1)$ *such that* $\varphi \in \Phi_{\mathcal{A}}[\mathfrak{h}_\rho, \chi_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $\mathfrak{f} \in \mathcal{A}$ *satisfies (18), then*

$$F_{(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f})(\mathbb{U})}\left(\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)\right) \leq F_{\chi(\mathbb{U})}(\chi(\zeta)).$$

i.e.,

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta) \prec_F \chi(\zeta),$$

and $\chi(\zeta)$ *is the fuzzy best dominant.*

Proof. By Theorems 2 and 3, we conclude that $\chi(\zeta)$ is a fuzzy dominant of (18). Since $\chi(\zeta)$ satisfies (21), it is also a solution of (18) and $\chi(\zeta)$ will be dominated by all fuzzy dominants of (18). Thus, it is the fuzzy best dominant of (18). □

Definition 8. *Let Ω be a set in $\mathbb{C}$, and assume that $\chi(\zeta) \in \wp_0 \cap \mathcal{H}$. The class $\Phi_{A^*}[\Omega,\chi]$ of admissible functions contains the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy*

$$F_\Omega(\varphi(u,v,w;\zeta)) = 0$$

whenever

$$u = \chi(\xi), v = \chi(\xi) + \frac{\eta k \xi \chi'(\xi)}{(\varsigma+1)\chi(\xi)}$$

and

$$\Re\left(\frac{(\varsigma+1)(vw-u(3v-2u))}{\eta(v-u)}\right) \geqq k\Re\left(1+\frac{\xi\chi''(\xi)}{\chi'(\xi)}\right),$$

where $\zeta \in \mathbb{U}, \xi \in \partial\mathbb{U}\backslash E(\chi)$ and $k \geqq 1$.

Theorem 5. *Let $\varphi \in \Phi_{\mathcal{A}^*}[\Omega,\chi]$. For $\mathfrak{f} \in \mathcal{A}$,*

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\left\{\varphi\left(\frac{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}, \frac{\mathcal{J}^{\kappa+2}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}, \frac{\mathcal{J}^{\kappa+3}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa+2}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)};\zeta\right)\right\} \leq F_\Omega(\zeta), \quad (22)$$

then

$$F_{\left(\frac{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}\right)(\mathbb{U})}\left(\frac{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}\right) \leq F_{\chi(\mathbb{U})}(\chi(\zeta))$$

i.e.,

$$\frac{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} \prec_F \chi(\zeta).$$

Proof. Let

$$g(\zeta) = \frac{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}. \quad (23)$$

Using (7) and (23), we obtain

$$\frac{\mathcal{J}^{\kappa+2}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa+1}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} = g(\zeta) + \frac{\eta}{\varsigma+1}\frac{\zeta g'(\zeta)}{g(\zeta)}. \quad (24)$$

Further computations show that

$$\frac{\mathcal{J}^{\kappa+3}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}^{\kappa+2}_{\eta,\varsigma}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} = g(\zeta) + \frac{\eta}{\varsigma+1}\left[\frac{\zeta g'(\zeta)}{g(\zeta)} + \frac{\frac{\varsigma+1}{\eta}\zeta g'(\zeta) + \frac{\zeta^2 g''(\zeta)}{g(\zeta)} + \frac{\zeta g'(\zeta)}{g(\zeta)} - \left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right)^2}{\frac{\varsigma+1}{\eta}g(\zeta) + \frac{\zeta g'(\zeta)}{g(\zeta)}}\right].$$

We next transformations are now defined for $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$

$$u = r, \quad v = r + \frac{\eta s}{(\varsigma+1)r},$$

and

$$w = r + \frac{\eta}{\varsigma+1}\left[\frac{s}{r} + \frac{\frac{\varsigma+1}{\eta}s + \frac{s+t}{r} - \left(\frac{s}{r}\right)^2}{\frac{\varsigma+1}{\eta}r + \frac{s}{r}}\right].$$

Also let

$$\begin{aligned}\psi(r,s,t;\zeta) &= \varphi(u,v,w;\zeta) \\ &= \varphi\Bigg(r, r+\frac{\eta s}{(\varsigma+1)r}, \\ & r+\frac{\eta}{\varsigma+1}\left[\frac{s}{r}+\frac{\frac{\varsigma+1}{\eta}s+\frac{s+t}{r}-\left(\frac{s}{r}\right)^2}{\frac{\varsigma+1}{\eta}r+\frac{s}{r}}\right];\zeta\Bigg).\end{aligned} \tag{25}$$

Thus, by using Equations (23)–(25), we obtain

$$\begin{aligned}&\psi\Big(g(\zeta),\zeta g'(\zeta),\zeta^2 g''(\zeta);\zeta\Big) \\ &= \varphi\left(\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}, \frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}, \frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+3}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a,e,x})\mathfrak{f}(\zeta)};\zeta\right).\end{aligned}$$

Hence, (22) implies that

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\Big(\psi\Big(g(\zeta),\zeta g'(\zeta),\zeta^2 g''(\zeta);\zeta\Big) \le F_{\Omega}(\zeta).$$

The proof of Theorem 5 is finished if it can be demonstrated that $\varphi \in \Phi_{\mathcal{A}^*}[\Omega,\chi]$ is equivalent to the admissibility condition for ψ given in Definition 6. In light of this, we observe that

$$\begin{aligned}\tfrac{s}{r} &= \tfrac{\varsigma+1}{\eta}(v-u) \\ \tfrac{t}{r} &= \left(\tfrac{\varsigma+1}{\eta}\right)^2 v(w-v) - \tfrac{s}{r}\left[\tfrac{\varsigma+1}{\eta}v - \tfrac{2s}{r}+1\right]\end{aligned}$$

and

$$\frac{t}{s}+1 = \frac{(\varsigma+1)(wv-u(3v-2u))}{\eta(v-u)}.$$

Thus, $\psi \in \Psi[\Omega,\chi]$. Consequently, we derive by Lemma 1 that

$$F_{g(\mathbb{U})}(g(\zeta)) \le F_{\chi(\mathbb{U})}(\chi(\zeta))$$

or equivalent

$$F_{\left(\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}\right)(\mathbb{U})}\left(\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}\right) \le F_{\chi(\mathbb{U})}(\chi(\zeta))$$

i.e.,

$$\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)} \prec_F \chi(\zeta),$$

which proves Theorem 5. □

Next, we take into account the situation where $\Omega = \mathfrak{h}(\mathbb{U})$ for some conformal mapping $\mathfrak{h}(\zeta) : \mathbb{U} \to \Omega$ and $\Omega \neq \mathbb{C}$ is a simply connected domain. $\Phi_{\mathcal{A}^*}[\mathfrak{h}(\mathbb{U}),\chi]$ is represented as $\Phi_{\mathcal{A}^*}[\mathfrak{h},\chi]$ in this instance.

Theorem 5 immediately leads to the next outcome.

Theorem 6. *Suppose* $\varphi \in \Phi_{\mathcal{A}^*}[\mathfrak{h},\chi]$. *If* $\mathfrak{f} \in \mathcal{A}$,

$$F_{\varphi(\mathbb{C}^3\times\mathbb{U})}\left\{\varphi\left(\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}, \frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}, \frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+3}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a,e,x})\mathfrak{f}(\zeta)};\zeta\right)\right\} \le F_{\mathfrak{h}(\mathbb{U})}(\mathfrak{h}(\zeta)),$$

then

$$F_{\left(\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}\right)(\mathbb{U})}\left(\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a,e,x})\mathfrak{f}(\zeta)}\right) \le F_{\chi(\mathbb{U})}(\chi(\zeta))$$

i.e.,

$$\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} \prec_F \chi(\zeta).$$

By taking $(\varphi(u,v,w;\zeta)) = uv$ in Theorem 6 we obtain

Corollary 3. *Let $\varphi \in \Phi_{\mathcal{A}^*}[\mathfrak{h},\chi]$. If $\mathfrak{f} \in \mathcal{A}$, $\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(a,e,x)\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(a,e,x)\mathfrak{f}(\zeta)}$ is analytic in $\mathbb{U}$ and*

$$\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+2}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} \prec_F \mathfrak{h}(\zeta), \tag{26}$$

then,

$$\frac{\mathcal{J}_{\eta,\varsigma}^{\kappa+1}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)}{\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})\mathfrak{f}(\zeta)} \prec_F \chi(\zeta).$$

4. Conclusions

We have initially introduced the following linear integral operator by employing a somewhat specialized version of the Riemann–Liouville fractional integral operator and its varied form known as the Erdélyi–Kober fractional integral operator:

$$\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x}) \quad (\varsigma > -1, \eta > 0, \mathrm{a},\mathrm{e} \in \mathbb{R}, (\mathrm{e}-\mathrm{a}) \geq 0, \mathrm{a} > -\mathrm{x}).$$

Previous research on this class of operators was performed by Raina and Sharma [17]. Then, using the operator $\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})$ and the admissible classes $\Phi_{\mathcal{A}}[\Omega,\chi]$ and $\Phi_{\mathcal{A}^*}[\Omega,\chi]$ of analytic functions connected with the operator $\mathcal{J}_{\eta,\varsigma}^{\kappa}(\mathrm{a},\mathrm{e},\mathrm{x})$, several findings about the admissible fuzzy differential subordination have been obtained. The fact that there are differential subordinations and differential superordinations of the third and higher orders in the theory of differential subordinations and differential superordinations will lead to more research on this topic. We exclusively employed and examined second-order differential subordinations in this presentation. Since fuzzy differential subordination is still a relatively young theory, its potential uses in other scientific fields or in real life are unknown. Future research projects with a longer time frame should look into those topics.

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References

1. Oros, G.I.; Oros, G. The notion of subordination in fuzzy sets theory. *Gen. Math.* **2011**, *19*, 97–103.
2. Miller, S.S.; Mocanu, P.T. Second order-differential inequalities in the complex plane. *J. Math. Anal. Appl.* **1978**, *65*, 298–305. [CrossRef]
3. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Michig. Math. J.* **1981**, *28*, 157–171. [CrossRef]
4. Oros, G.I.; Oros, G. Fuzzy differential subordination. *Acta Univ. Apulensis* **2012**, *3*, 55–64.
5. Dzitac, I.; Filip, F.G.; Manolescu, M.J. Fuzzy logic is not fuzzy: World-renowned computer scientist Lotfi A. Zadeh. *Int. J. Comput. Commun. Control.* **2017**, *12*, 748–789. [CrossRef]

6. Oros, G.I.; Oros, G. Dominants and best dominants in fuzzy differential subordinations. *Stud. Univ. Babes-Bolyai Math.* **2012**, *57*, 239–248.
7. Oros, G.I. Briot-Bouquet fuzzy differential subordination. *Analele Univ. Oradea Fasc. Mat.* **2012**, *19*, 83–87.
8. Haydar, E.A. On fuzzy differential subordination. *Math. Moravica* **2015**, *19*, 123–129. [CrossRef]
9. Alb Lupas, A. A note on special fuzzy differential subordinations using generalized Salagean operator and Ruscheweyh derivative. *J. Comput. Anal. Appl.* **2013**, *15*, 1476–1483.
10. Rapeanu, E. Continuation method for boundary value problems with uniform elliptical operators. *J. Sci. Arts* **2011**, *3*, 273–277.
11. Ali, E.E.; Vivas-Cortez, M.; Shah, S.A.; Albalahi, A.M. Certain results on fuzzy p-valent functions involving the linear operator. *Mathematics* **2023**, *11*, 3968. [CrossRef]
12. Ali, E.E.; Vivas-Cortez, M.; El-Ashwah, R.M. New results about fuzzy γ-convex functions connected with the q-analogue multiplier-Noor integral operator. *AIMS Math.* **2024**, *9*, 5451–5465. [CrossRef]
13. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204.
14. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics; Longman Scientific and Technical: Harlow, UK, 1993; Volume 301.
15. Ali, E.E.; Srivastava, H.M.; El-Ashwah, R.M.; Albalahi, A.M. Differential subordination and differential superordination for classes of admissible multivalent functions associated with a linear operator. *Mathematics* **2022**, *10*, 4690. [CrossRef]
16. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Engrg. Comput.* **2021**, *5*, 135–166. [CrossRef]
17. Raina, R.K.; Sharma, P. Subordination properties of univalent functions involving a new class of operators. *Electron. J. Math. Anal. Appl.* **2014**, *2*, 37–52.
18. Al-Oboudi, F.M. On univalent functions defined by a generalized Sălăgean operator. *Int. J. Math. Math Sci.* **2004**, *27*, 1429–1436. [CrossRef]
19. Salagean, G.S. *Subclasses of Univalent Functions, in Complex Analysis: Fifth Romanian Finnish Seminar, Part I (Bucharest, 1981)*; Lecture Notes in Mathematics 1013; Springer: Berlin, Germany; New York, NY, USA, 1983.
20. Selvaraj, C.; Karthikeyan, K.R. Differential subordination and superordination for analytic functions defined using a family of generalized differential operators. *Analele Stiintifice Univ. Ovidus Constanta* **2009**, *17*, 201–210.
21. Catas, A. On a certain differential sandwich theorem associated with a new generalized derivative operator. *General Math.* **2009**, *17*, 83–95.
22. Jung, I.B.; Kim, Y.C.; Srivastava, H.M. The Hardy space of analytic functions associated with certain one-parameter families of integral operators. *J. Math. Anal. Appl.* **1993**, *176*, 138–147. [CrossRef]
23. Gao, C.-Y.; Yuan, S.M.; Srivastava, H.M. Some functional inequalities and inclusion relationships associated with certain families of integral operators. *Comput. Math. Appl.* **2005**, *49*, 1787–1795. [CrossRef]
24. Komatu, Y. On analytical prolongation of a family of operators. *Mathematica* **1990**, *32*, 141–145.
25. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]