*fractal and fractional*

*Article*

# Derivation of Closed-Form Expressions in Apéry-like Series Using Fractional Calculus and Applications

**Ampol Duangpan [1], Ratinan Boonklurb [1,*,†], Udomsak Rakwongwan [2,3,*,†] and Phiraphat Sutthimat [2,3]**

[1] Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand; ampol.d@chula.ac.th
[2] Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand; phiraphat.sut@ku.th
[3] Financial Mathematics, Data Science and Computational Innovations Research Unit (FDC), Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand
* Correspondence: ratinan.b@chula.ac.th (R.B.); udomsak.ra@ku.th (U.R.)
† These authors contributed equally to this work.

**Abstract:** This paper explores the Apéry-like series and demonstrates the derivation of closed-form expressions using fractional calculus. We consider a variety of Apéry-like functions, which were categorized by their functional forms and coefficients by applying the Riemann–Liouville fractional integral and derivative to examine their properties across various domains. The study focuses on establishing rigorous mathematical frameworks that unveil new insights into the behaviors of these series, contributing to a deeper understanding of number theory and mathematical analysis. Key results include proofs of convergence and divergence within specified intervals and the derivation of closed-form solutions through fractional integration and differentiation. This paper also introduces a method aimed at conjecturing mathematical constants through continued fractions as an application of our results. Finally, we provide the proof of validation for three unproven conjectures of continued fractions obtained from the Ramanujan Machine.

**Keywords:** Apéry-like series; fractional calculus; closed-form solutions; summation techniques

**MSC:** 11A55; 11M32; 26A33; 33B15

check for updates

**Citation:** Duangpan, A.; Boonklurb, R.; Rakwongwan, U.; Sutthimat, P. Derivation of Closed-Form Expressions in Apéry-like Series Using Fractional Calculus and Applications. *Fractal Fract.* **2024**, *8*, 406. https://doi.org/10.3390/fractalfract8070406

Academic Editors: Andrey Zahariev and Hristo Kiskinov

Received: 10 June 2024
Revised: 7 July 2024
Accepted: 9 July 2024
Published: 11 July 2024



## 1. Introduction

In the fields of mathematics, Apéry-like series hold significant value in number theory and combinatorics, which is noted especially for the challenges they posed in deriving closed-form expressions. Historically, these series have been crucial and involved in various mathematical proofs, particularly in establishing the transcendence and irrationality of numbers. In addition, Apéry-like series have also been utilized with many applications such as stochastic process [1], continued fraction [2–4] and other applications; see [5–7]. The application of fractional calculus, which extends ordinary differentiation and integration to non-integer orders, introduces a novel approach to these complex series. The knowledge of fractional calculus offers a sophisticated toolkit for managing differential and integral equations, often surpassing the capabilities of traditional techniques. Its ability to precisely adjust differentiation and integration parameters makes it exceptionally adept at exploring the intricate behaviors of the Apéry-like series.

This paper explores the intersection of Apéry-like series and fractional calculus by using the fractional calculus idea to derive closed-form expressions for these series. Using the flexibility of fractional operators, we examine their convergence properties in different mathematical domains. This research not only connects two significant areas of mathematical study but also expands the theoretical framework, introducing innovative methods and results that enrich our understanding of advanced mathematical concepts.

We start by revisiting the fundamental definitions and properties of the Apéry-like series, highlighting key equations that underpin our study. The following series is typically expressed by

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} f_n(x) \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{f_n(x)}{\binom{2n}{n}}. \tag{1}$$

This formula and its variations have been extensively studied and documented in the mathematical literature [8–11]. In 2000, Sherman [11] identified numerous Apéry-like series, which are beautiful and interesting. Unfortunately, closed-form expressions for his Apéry-like series are only available in certain cases. Moreover, even advanced mathematical software such as MATLAB and Mathematica, which define the state of the art in technical computing and provide essential computations, cannot offer these closed-form expressions. This poses a significant inconvenience for researchers who need explicit solutions.

This paper aims to address this gap by analyzing the closed-form expressions of the Apéry-like series, focusing on three specific types of functions $f_n(x)$, which are described as follows:

$$\text{Type I:} \quad f_n(x) = \frac{(4x)^n}{2n+a} \quad \text{for a positive odd number } a, \tag{2}$$

$$\text{Type II:} \quad f_n(x) = \frac{(4x)^n}{n+b} \quad \text{for a non-negative integer } b, \tag{3}$$

$$\text{Type III:} \quad f_n(x) = \frac{(4x)^n Q(n)}{P_{A,B}(n)} := \frac{(4x)^n Q(n)}{\prod_{a\in A}(2n+a) \prod_{b\in B}(n+b)}, \tag{4}$$

where $x$ is a real number, $n$ is a non-negative integer, and $Q(n)$ and $P_{A,B}(n)$ are polynomials in variable $n$ with $\deg Q(n) < \deg P_{A,B}(n)$. Here, $A$ is a finite subset of positive odd numbers (denoted by $A \subset 2\mathbb{N} - 1$) and $B$ is a finite subset of natural numbers (denoted by $B \subset \mathbb{N}$). We note here that the Type III function results from the interaction between Types I and II.

One application of (2)–(4) is to provide rigorous proofs for conjectures generated by the Ramanujan Machine [12–14], which signifies a shift toward automated discovery processes using exhaustive search techniques. Despite notable successes, this approach is inherently limited by its computational expense and the finite scope of search parameters. Our results address and mitigate these limitations, as demonstrated in Section 4.

The rest of this paper is organized into five sections. In Section 2, the foundational concepts of fractional calculus, particularly the Riemann–Liouville definitions, are introduced for both the fractional integral and fractional derivative. In Section 3, we apply fractional calculus techniques to derive new closed-form expressions for these Apéry-like series (2)–(4) and also consider the convergence and divergence domains. Section 4 demonstrates applications of our results by proving and validating three unproven conjectures of continued fractions from the Ramanujan Machine based on the MITM-RF algorithm [13] to illustrate the versatility of our work. Finally, the conclusion is discussed in Section 5. Additionally, several closed-form results of Apéry-like series are proposed in Appendices A and B.

## 2. Fractional Calculus

In this section, we explore the fundamental concepts and properties of fractional calculus, as referenced in [15–19], which are crucial to formulating the summations of Apéry-like series as discussed in Section 3. Our discussion begins with definitions of the Gamma function and a pivotal element in the development of fractional calculus, together with its properties; see [20] for more details.

**Definition 1.** *The Gamma function is defined by the following improper integral:*

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx \quad \textit{for } \operatorname{Re} z > 0.$$

Integration by parts confirms that the Gamma function satisfies the recursive relation $\Gamma(z+1) = z\Gamma(z)$. This recursion with $\Gamma(1) = 1$ allows us to deduce by induction that $\Gamma(n+1) = n!$ for all non-negative integers $n$. Moreover, an interesting property of the Gamma function when $z = \frac{1}{2}$ is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. By utilizing the recursive relation $\Gamma(z+1) = z\Gamma(z)$ combined with the base case $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the evaluation of $\Gamma(n + \frac{1}{2})$ for a non-negative integer $n$ is demonstrated in Property 1. This property is essential for later discussions on half-integrals and half-derivatives, which rely on the Gamma characteristics.

**Definition 2.** *The double factorial of integer $n \geq -1$ is defined in the usual factorial $n!$ by*

$$n!! = \begin{cases} 1 & \text{for } n = -1, 0, \\ n(n-2)(n-4)\dots 5 \cdot 3 \cdot 1 & \text{for odd } n > 0, \\ n(n-2)(n-4)\dots 6 \cdot 4 \cdot 2 & \text{for even } n > 0. \end{cases}$$

**Property 1** ([20])**.** *For any non-negative integer $n$, the Gamma function satisfies the following:*

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}. \tag{5}$$

In addition to the specific properties of the Gamma function, we consider the Riemann–Liouville operator, which is the most common tool for fractional integration and differentiation. The concept of the Riemann–Liouville fractional integral operator arises from generalizing Cauchy's formula for an $n$-fold integral [15]:

$$\int_a^x \int_a^{x_1} \int_a^{x_2} \cdots \int_a^{x_{n-1}} u(x_n)\, dx_n \dots dx_3\, dx_2\, dx_1 = \frac{1}{(n-1)!} \int_a^x \frac{u(\tau)\, d\tau}{(x-\tau)^{1-n}}. \tag{6}$$

Since $(n-1)! = \Gamma(n)$, Riemann extrapolated that the formula (6) could be adapted for non-integer orders $n$, denoted by $\alpha$, motivating the definitions of fractional integration. Now, we consider the two definitions established by Krug [21] as presented below.

**Definition 3.** *Let $\alpha > 0$. The left-sided Riemann–Liouville fractional integral in $[a, b]$ is defined by*

$$\mathcal{I}_{a+}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{u(\tau)}{(x-\tau)^{1-\alpha}}\, d\tau \tag{7}$$

*and the right-sided Riemann–Liouville fractional integral in $[a, b]$ is defined by*

$$\mathcal{I}_{b-}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(\tau)}{(\tau-x)^{1-\alpha}}\, d\tau. \tag{8}$$

By employing the idea of the fundamental theorem of calculus for ordinary integration and differentiation, we can define the derivative of non-integer order from Definition 3, also known as the fractional derivative, as follows.

**Definition 4.** *The left-sided Riemann–Liouville fractional derivative in $[a, b]$ is defined by*

$$\mathcal{D}_{a+}^{\alpha} u(x) = (D_x)^n \left(\mathcal{I}_{a+}^{n-\alpha} u(x)\right) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{u(\tau)}{(x-\tau)^{1-n+\alpha}}\, d\tau \tag{9}$$

*and the right-sided Riemann–Liouville fractional derivative in $[a, b]$ is defined by*

$$\mathcal{D}_{b-}^{\alpha} u(x) = (-D_x)^n \left(\mathcal{I}_{b-}^{n-\alpha} u(x)\right) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{u(\tau)}{(\tau-x)^{1-n+\alpha}}\, d\tau, \tag{10}$$

*where $\alpha > 0$ and $n = [\alpha] + 1$ when $[\alpha]$ is the integer part of order $\alpha$.*

Note that the operators $I_c^n$ and $D_c^n$ for $c \in \{a+, b-\}$ agree with the ordinary integral and differential operations of order $n \in \mathbb{N}$. The fractional integral and differential operators in the Riemann–Liouville sense, being linear operations, exhibit several significant properties [15–17], such as if $u(x)$ is the power function, then its fractional integral and derivative are evaluated as follows.

**Property 2** ([17]). *Let $\beta \geq 0$. Then, we have*

$$\mathcal{I}_{a+}^{\alpha}\frac{(x-a)^{\beta-1}}{\Gamma(\beta)} = \frac{(x-a)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}, \quad \mathcal{D}_{a+}^{\alpha}\frac{(x-a)^{\beta-1}}{\Gamma(\beta)} = \frac{(x-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)},$$
$$\mathcal{I}_{b-}^{\alpha}\frac{(b-x)^{\beta-1}}{\Gamma(\beta)} = \frac{(b-x)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}, \quad \mathcal{D}_{b-}^{\alpha}\frac{(b-x)^{\beta-1}}{\Gamma(\beta)} = \frac{(b-x)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

**Example 1.** *Consider Property 2 in cases of $\beta = n+1$ for $n \in \mathbb{N} \cup \{0\} := \mathbb{N}_0$ and zero limit of integrations; then, we obtain*

$$\mathcal{I}_{0+}^{\alpha}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}x^{n+\alpha} \quad \textit{for } x > 0, \tag{11}$$

$$\mathcal{D}_{0+}^{\alpha}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} \quad \textit{for } x > 0,$$

$$\mathcal{I}_{0-}^{\alpha}(-x)^n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}(-x)^{n+\alpha} \quad \textit{for } x < 0, \tag{12}$$

$$\mathcal{D}_{0-}^{\alpha}(-x)^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}(-x)^{n-\alpha} \quad \textit{for } x < 0.$$

Next, we consider the special case where $\alpha = \frac{1}{2}$ for both the fractional integral and derivative (also known as the half integral and half derivative) of the function $u(x) = x^n$ for all $n \in \mathbb{N}_0$, according to Definitions 3 and 4. These properties play an important role in establishing closed-form expressions of the Apéry-like series.

**Corollary 1.** *Let $n$ be a non-negative integer. Then, we have the following relations.*

1. *The left-sided half integral is*

$$\mathcal{I}_{0+}^{\frac{1}{2}}(x^n) = \frac{(n!)^2(4x)^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}} \quad \textit{for } x > 0. \tag{13}$$

2. *The left-sided half derivative is*

$$\mathcal{D}_{0+}^{\frac{1}{2}}(x^n) = \frac{2(n!)^2(4x)^{n-\frac{1}{2}}}{(2n)!\sqrt{\pi}} \quad \textit{for } x > 0. \tag{14}$$

3. *The right-sided half integral is*

$$\mathcal{I}_{0-}^{\frac{1}{2}}(x^n) = \frac{(n!)^2(-4x)^{n+\frac{1}{2}}(-1)^n}{(2n+1)!\sqrt{\pi}} \quad \textit{for } x < 0. \tag{15}$$

4. *The right-sided half derivative is*

$$\mathcal{D}_{0-}^{\frac{1}{2}}(x^n) = \frac{2(n!)^2(-4x)^{n-\frac{1}{2}}(-1)^n}{(2n)!\sqrt{\pi}} \quad \textit{for } x < 0. \tag{16}$$

**Proof of Corollary 1.** Since $(2n)!! = 2^n n!$, we transform the formula of Gamma function $\Gamma(n+\frac{1}{2})$ in Property 1 into another form as

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n} = \frac{(2n-1)!!\sqrt{\pi}}{2^n}\cdot\frac{(2n)!!}{2^n n!} = \frac{(2n)!\sqrt{\pi}}{4^n n!}. \tag{17}$$

Let $n \in \mathbb{N}_0$. We can prove the Formulas (13)–(16) by using the relation (17) together with the consequences as expressed in Example 1 as follows.

1. Proof (13): Apply (17) and the left-sided half integral (11) with $\alpha = \frac{1}{2}$:

$$\mathcal{I}_{0+}^{\frac{1}{2}}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}x^{n+\frac{1}{2}} = \frac{n!\,x^{n+\frac{1}{2}}}{(n+\frac{1}{2})\Gamma(n+\frac{1}{2})} = \frac{n!\,x^{n+\frac{1}{2}}}{(\frac{2n+1}{2})\frac{(2n)!\sqrt{\pi}}{4^n n!}} = \frac{(n!)^2(4x)^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}}.$$

2. Proof (14): Utilize Definition 4 to transform operators of fractional derivative into a fractional integral and employ the obtained result of left-sided half integral (13) to yield the following result:

$$\mathcal{D}_{0+}^{\frac{1}{2}}(x^n) = \frac{d}{dx}\left(\mathcal{I}_{0+}^{\frac{1}{2}}(x^n)\right) = 4\left(n+\frac{1}{2}\right)\frac{(n!)^2(4x)^{n-\frac{1}{2}}}{(2n+1)(2n)!\sqrt{\pi}} = \frac{2(n!)^2(4x)^{n-\frac{1}{2}}}{(2n)!\sqrt{\pi}}.$$

3. Proof (15): Similar to the proof of (13), apply the Formula (17) and the right-sided half integral (12) with $\alpha = \frac{1}{2}$:

$$\mathcal{I}_{0-}^{\frac{1}{2}}(x^n) = (-1)^n\left(\mathcal{I}_{0-}^{\frac{1}{2}}(-x)^n\right) = (-1)^n\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}(-x)^{n+\frac{1}{2}} = \frac{(n!)^2(-4x)^{n+\frac{1}{2}}(-1)^n}{(2n+1)!\sqrt{\pi}}.$$

4. Proof (16): Similar to the proof of (14), utilize Definition 4 to transform the operators and use the obtained result of the right-sided half integral (15) to obtain the following:

$$\mathcal{D}_{0-}^{\frac{1}{2}}(x^n) = -\frac{d}{dx}\left(\mathcal{I}_{0-}^{\frac{1}{2}}(x^n)\right) = -\frac{(n!)^2(-1)^n\frac{d}{dx}(-4x)^{n-\frac{1}{2}}}{(2n+1)!\sqrt{\pi}} = \frac{2(n!)^2(-4x)^{n-\frac{1}{2}}(-1)^n}{(2n)!\sqrt{\pi}}.$$

This completes the proof. □

### 3. Summation of Apéry-like Series

In this section, our goal is to transform the summations of Apéry-like series into closed-form expressions. We divide our analysis into three subsections; each is dedicated to a different type of function $f_n(x)$, as defined in (2)–(4). Initially, we substitute these functions into the general form of the Apéry-like series (1). This leads to new series representations, denoted by $F_m(x)$, $G_m(x)$ and $H_{A,B}(x)$, which are defined as follows:

$$F_m(x) := \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{(2n+2m-1)}, \tag{18}$$

$$G_m(x) := \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{(n+m)} \quad \text{and} \tag{19}$$

$$H_{A,B}(x) := \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n Q(n)}{P_{A,B}(n)}, \tag{20}$$

where $x$ is a real number, $m$ is a natural number and $n$ is a non-negative integer. Polynomials $Q(n)$ and $P_{A,B}(n)$ are functions of $n$ with $\deg Q(n) < \deg P_{A,B}(n)$.

### 3.1. Analysis of Type I: Apéry-like Series $F_m(x)$

This section derives the closed form of the Apéry-like series $F_m(x)$ as defined in (18). Our analysis is structured into four main parts. First, Theorem 1 identifies the convergence and divergence domains for $x$. Then, Theorem 2 explores the behavior of the initial series $F_1(x)$. Building on this, Theorem 3 introduces a recurrence relation that outlines the series progression. Finally, Theorem 4 uses this relation to establish a closed-form expression, enhancing our understanding of the series. Additionally, we examine how variations in $x$ and $m$ affect the results, revealing diverse outcomes from applying the established formulas.

**Theorem 1.** *For a natural number m and a nonzero real number x, the Apéry-like series*

$$F_m(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+2m-1)}$$

*converges if $x \in [-1, 1)$ and diverges if $x \in (-\infty, -1) \cup [1, \infty)$.*

**Proof of Theorem 1.** Let $m \in \mathbb{N}$ and $x \in \mathbb{R}$. We define $a_n := \frac{(n!)^2(4x)^n}{(2n)!(2n+2m-1)}$. To determine the convergence of $F_m(x)$, we apply the ratio test as follows:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2(4x)(2n+2m-1)}{(2n+2)(2n+1)(2n+2m+1)} \right| = |x|.$$

Thus, the series converges when $|x| < 1$, which implies that the convergence domain is $x \in (-1, 1)$. Furthermore, the ratio test indicates that the series diverges when $|x| > 1$, establishing the divergence domain as $x \in (-\infty, -1) \cup (1, \infty)$.

Next, we explore the behavior of the Apéry-like series where $x = -1$:

$$F_m(-1) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{(2n+2m-1)} = \sum_{n=0}^{\infty} (-1)^n b_n,$$

where $b_n := \frac{(n!)^2 4^n}{(2n)!(2n+2m-1)} > 0$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. To verify convergence, we apply the alternating series test, which requires analyzing the limit of $b_n$ as $n \to \infty$. To consider the boundary of $b_n$, we have

$$0 < b_n = \frac{(n!)^2 4^n}{(2n)!(2n+2m-1)} \leq \frac{(n!)2^n}{(2n-1)!!(2n+1)} = \frac{(2n)!!}{(2n+1)!!}.$$

Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{(2n)!!}{(2n+1)!!} = 0$ were obtained in [22], which is one of the Wallis integral properties, by squeezing theorem, $\lim_{n\to\infty} b_n = 0$. This result implies that the terms of the series decrease in magnitude and approach zero, satisfying the conditions for the alternating series test. Further, we examine the monotonicity of the sequence $b_n$:

$$\begin{aligned} b_{n+1} &= \frac{4(n+1)^2(n!)^2 4^n}{(2n)!(2n+2)(2n+1)(2n+2m+1)} \cdot \frac{2n+2m-1}{2n+2m-1} \\ &= \frac{(n!)^2 4^n}{(2n)!(2n+2m-1)} \cdot \frac{(2n+2)(2n+2m-1)}{(2n+1)(2n+2m+1)} \\ &< \frac{(n!)^2 4^n}{(2n)!(2n+2m-1)} = b_n \end{aligned}$$

This calculation shows that $b_n$ forms a decreasing sequence of non-negative real numbers, and thus $b_n \to 0$ as $n \to \infty$. Hence, the alternating series test confirms that the Apéry-like series $F_m(x)$ converges at $x = -1$.

Finally, we examine the behavior of the Apéry-like series at $x = 1$ as follows

$$F_m(1) = \sum_{n=0}^{\infty} b_n \geq \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n)!(2n+2m)} = \sum_{n=0}^{\infty} \frac{(2n)!!}{(2n-1)!!(2n+2m)} \geq \sum_{n=0}^{\infty} \frac{1}{2n+2m}.$$

Since $\lim_{n\to\infty} \frac{n}{2n+2m} = \frac{1}{2} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent $p$-series, then $\sum_{n=0}^{\infty} \frac{1}{2n+2m}$ diverges by the limit comparison test. Thus, by the comparison test, $F_m(1)$ diverges. Hence, $x = 1$ falls within the divergence domain for the Apéry-like series $F_m(x)$. The proof is complete. □

Now, our focus shifts to computing the specific convergence values for $F_m(x)$ on the interval $[-1, 1)$, starting with the particular case where $m = 1$. In this context, Theorem 2 provides the explicit formula for $F_1(x)$, utilizing principles of fractional calculus as discussed in Section 2.

**Theorem 2.** *The initial Apéry-like series $F_1(x)$ can be expressed by the following explicit formula:*

$$F_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{2n+1} = \begin{cases} \frac{\operatorname{arcsinh}\sqrt{-x}}{\sqrt{x(x-1)}} & ;\ x \in [-1, 0), \\ 1 & ;\ x = 0, \\ \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}} & ;\ x \in (0, 1). \end{cases} \tag{21}$$

**Proof of Theorem 2.** Let $x \in (0, 1)$. By using the left-sided half integral (13), we can transform the Apéry-like series $F_1(x)$ to be

$$F_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n+1)!} \frac{\sqrt{4\pi x}}{\sqrt{4\pi x}} = \frac{\sqrt{\pi}}{2\sqrt{x}} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}} = \frac{\sqrt{\pi}}{2\sqrt{x}} \sum_{n=0}^{\infty} \left( \mathcal{I}_{0+}^{\frac{1}{2}} (x^n) \right).$$

Based on Definition 3, we can see that the half-integral operator $\mathcal{I}_{0+}^{\frac{1}{2}}$ is a linear operation and $\int_0^x \frac{\tau^n \, d\tau}{\sqrt{x-\tau}}$ is convergent for $n \in \mathbb{N}_0$ and $x > 0$. Then, we can apply the infinite geometric series $\sum_{n=0}^{\infty} \tau^n = \frac{1}{1-\tau}$ for $|\tau| < 1$ to consider the above equation as follows

$$\sum_{n=0}^{\infty} \left( \mathcal{I}_{0+}^{\frac{1}{2}} (x^n) \right) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{\tau^n d\tau}{\sqrt{x-\tau}} = \frac{1}{\sqrt{\pi}} \int_0^x \frac{\sum_{n=0}^{\infty} \tau^n}{\sqrt{x-\tau}} \, d\tau = \frac{1}{\sqrt{\pi}} \int_0^x \frac{d\tau}{(1-\tau)\sqrt{x-\tau}}.$$

Next, to evaluate the above integral, we can use a substitution to simplify the integrand. Let $\tau = x(1-u)$. Then, $d\tau = -x \, du$ and $0 \leq u \leq 1$. Now, substituting $\tau = x(1-u)$ into the integrand and $u = 1$ with $u = 0$ into the limits of integration, we obtain

$$\int_0^x \frac{d\tau}{(1-\tau)\sqrt{x-\tau}} = \int_1^0 \frac{-x \, du}{(1-x+xu)\sqrt{xu}} = \int_0^1 \frac{\sqrt{x} \, du}{(1-x+xu)\sqrt{u}}.$$

We use the substitution $v = \sqrt{u}$, where $u = v^2$ and $du = 2v \, dv$,

$$\int_0^1 \frac{\sqrt{x} \, du}{(1-x+xu)\sqrt{u}} = \int_0^1 \frac{\sqrt{x} \cdot 2v \, dv}{(1-x+xv^2)v} = \int_0^1 \frac{2\sqrt{x} \, dv}{1-x+xv^2}.$$

This integral can now be identified as a standard form of the integral that results in the inverse trigonometric of tangent function: $\int \frac{dv}{a+bv^2} = \frac{1}{\sqrt{ab}} \arctan\left(\frac{v\sqrt{b}}{\sqrt{a}}\right) + C$. Substituting $a = 1-x$ and $b = x$, the above integral can be evaluated as

$$\int_0^1 \frac{2\sqrt{x} \, dv}{1-x+xv^2} = \frac{2\sqrt{x}}{\sqrt{x(1-x)}} \arctan\left(\frac{v\sqrt{x}}{\sqrt{1-x}}\right)\Bigg|_0^1 = \frac{2\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right)}{\sqrt{1-x}} = \frac{2\arcsin\sqrt{x}}{\sqrt{1-x}}.$$

For $x \in (0,1)$, we thus obtain

$$F_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{2n+1} = \frac{\sqrt{\pi}}{2\sqrt{x}} \cdot \frac{1}{\sqrt{\pi}} \left( \frac{2 \arcsin \sqrt{x}}{\sqrt{1-x}} \right) = \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}}.$$

Next, let $x \in [-1,0)$; then, the Apéry-like series $F_1(x)$ can be proved in a similar way to $x \in (0,1)$ by using the right-sided half integral (15) to reformulate being

$$\begin{aligned} F_1(x) &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4x)^n(-1)^n}{(2n+1)} \frac{\sqrt{-4\pi x}}{\sqrt{-4\pi x}} \\ &= \frac{\sqrt{\pi}}{2\sqrt{-x}} \sum_{n=0}^{\infty} \frac{(n!)^2(-4x)^{n+\frac{1}{2}}(-1)^n}{(2n+1)!\sqrt{\pi}} \\ &= \frac{\sqrt{\pi}}{2\sqrt{-x}} \sum_{n=0}^{\infty} \left( \mathcal{I}_{0-}^{\frac{1}{2}}(x^n) \right). \end{aligned}$$

Subsequently, we consider the summation of the right-sided half integral according to Definition 3. In addition, $\int_x^0 \frac{\tau^n \, d\tau}{\sqrt{\tau - x}}$ is convergent for $n \in \mathbb{N}_0$ and $x < 0$. This summation applies the infinite geometric series $\sum_{n=0}^{\infty} \tau^n = \frac{1}{1-\tau}$ for $|\tau| < 1$; then, we have

$$\sum_{n=0}^{\infty} \left( \mathcal{I}_{0-}^{\frac{1}{2}}(x^n) \right) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2})} \int_x^0 \frac{\tau^n d\tau}{\sqrt{\tau - x}} = \frac{1}{\sqrt{\pi}} \int_x^0 \frac{\sum_{n=0}^{\infty} \tau^n}{\sqrt{\tau - x}} \, d\tau = \frac{1}{\sqrt{\pi}} \int_x^0 \frac{d\tau}{(1-\tau)\sqrt{\tau - x}}.$$

This integral can be evaluated similarly to the case of $x \in (0,1)$. Thus, it becomes

$$\int_x^0 \frac{d\tau}{(1-\tau)\sqrt{\tau - x}} = \frac{2}{\sqrt{1-x}} \operatorname{arctanh}\left( \frac{\sqrt{\tau - x}}{\sqrt{1-x}} \right) \Bigg|_x^0 = \frac{2 \operatorname{arctanh}(\frac{\sqrt{-x}}{\sqrt{1-x}})}{\sqrt{1-x}} = \frac{2 \operatorname{arcsinh} \sqrt{-x}}{\sqrt{1-x}}.$$

Hence, for $x \in [-1,0)$, we have

$$F_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{2n+1} = \frac{\sqrt{\pi}}{2\sqrt{-x}} \cdot \frac{1}{\sqrt{\pi}} \left( \frac{2 \operatorname{arcsinh} \sqrt{-x}}{\sqrt{1-x}} \right) = \frac{\operatorname{arcsinh} \sqrt{-x}}{\sqrt{-x(1-x)}}.$$

Finally, for $x = 0$, it is clear that $F_1(0)$ remains only the first term of summation, resulting in $F_1(0) = 1$. Therefore, the proof is complete. □

It should be noted that using classical integration with respect to $x$ and substituting $x$ with $x^2$ in (21) yields the same result, namely, $y(x) := \arcsin^2 x = \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2 x^{2n+2}}{(2n+2)!}$, as stated in Proposition 15 of [23], p. 262. This result is derived by using classical calculus to find the explicit solution of the ordinary differential equation $(1-x^2)(y')^2 = 4y$. In other words, applying classical differential calculus to the result in [23], p. 262 yields (21).

**Example 2.** *The Apéry-like series $F_1(x)$ for $x \in \{-\frac{1}{8}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\}$ converges to the following values:*

$$\begin{aligned} F_1\left(-\frac{1}{8}\right) &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+1)} = \frac{8}{3} \operatorname{arcsinh}\left(\frac{1}{\sqrt{8}}\right) = \frac{4\ln(2)}{3}, \\ F_1\left(-\frac{1}{4}\right) &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+1)} = \frac{4}{\sqrt{5}} \operatorname{arcsinh}\left(\frac{1}{2}\right) = \frac{4\sqrt{5}\ln(\phi)}{5}, \\ F_1\left(\frac{1}{4}\right) &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(2n+1)} = \frac{4}{\sqrt{3}} \arcsin\left(\frac{1}{2}\right) = \frac{2\pi\sqrt{3}}{9}, \\ F_1\left(\frac{1}{2}\right) &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(2n+1)} = 2 \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}, \end{aligned}$$

*where* $\phi = \frac{1+\sqrt{5}}{2}$ *denotes the golden ratio.*

Prior to establishing the closed-form formula for the Apéry-like series $F_m(x)$, we can derive the linear recurrence relation from $F_m(x)$, as delineated in (18). The formulation of this relation involves a strategy of re-indexing the series coupled with applying the partial fraction decomposition. The relation is explicated in Theorem 3.

**Theorem 3.** *The Apéry-like series $F_m(x)$ defined in (18) can be expressed in the following linear recurrence relation*

$$F_{m+1}(x) = \frac{2mF_m(x)}{(2m-1)x} - \frac{F_1(x)}{2m-1} - \frac{2m}{(2m-1)^2 x}, \tag{22}$$

*where $m \in \mathbb{N}$ and $x \in [-1,1)$, starting with $F_1(x)$, as defined in (21).*

**Proof of Theorem 3.** Consider

$$\begin{aligned} F_m(x) &= \frac{1}{2m-1} + \sum_{n=0}^{\infty} \frac{(n!)^2(n+1)^2(4x)^{n+1}}{(2n)!(2n+1)(2n+2)(2n+2m+1)} \\ &= \frac{1}{2m-1} + \frac{x}{2}\sum_{n=0}^{\infty} \frac{(n!)^2(4x)^n}{(2n)!} \frac{4(n+1)}{(2n+1)(2n+2m+1)} \\ &= \frac{1}{2m-1} + \frac{x}{2}\sum_{n=0}^{\infty} \frac{(n!)^2(4x)^n}{(2n)!} \left(\frac{1}{m}\cdot\frac{1}{2n+1} + \frac{2m-1}{m}\cdot\frac{1}{2n+2m+1}\right) \\ &= \frac{1}{2m-1} + \frac{x}{2m}\sum_{n=0}^{\infty} \frac{(n!)^2(4x)^n}{(2n)!(2n+1)} + \frac{(2m-1)x}{2m}\sum_{n=0}^{\infty} \frac{(n!)^2(4x)^n}{(2n)!(2n+2m+1)} \\ &= \frac{1}{2m-1} + \frac{x}{2m}F_1(x) + \frac{(2m-1)x}{2m}F_{m+1}(x). \end{aligned}$$

To manipulate the preceding equation, we arrive at the expression of $F_{m+1}(x)$ that follows as (22). The application of mathematical induction on $m$ allows us to substantiate the linear recurrence relation delineated in (22). □

The results elucidated from Theorem 3 are herein presented. Example 3 delineates the initial five instances of the Apéry-like series $F_m(x)$ for $m \in \{1,2,3,4,5\}$.

**Example 3.** *The first five Apéry-like series $F_m(x)$ for $m \in \{1,2,3,4,5\}$ are the following:*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+1} = F_1(x), \\ &\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+3} = \left(\frac{2}{x}-1\right)F_1(x) - \frac{2}{x}, \\ &\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+5} = \left(\frac{8}{3x^2}-\frac{4}{3x}-\frac{1}{3}\right)F_1(x) - \frac{8}{3x^2} - \frac{4}{9x}, \\ &\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+7} = \left(\frac{16}{5x^3}-\frac{8}{5x^2}-\frac{2}{5x}-\frac{1}{5}\right)F_1(x) - \frac{16}{5x^3} - \frac{8}{15x^2} - \frac{6}{25x} \quad \textit{and} \\ &\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+9} = \left(\frac{128}{35x^4}-\frac{64}{35x^3}-\frac{16}{35x^2}-\frac{8}{35x}-\frac{1}{7}\right)F_1(x) - \frac{128}{35x^4} - \frac{64}{105x^3} - \frac{48}{175x^2} - \frac{8}{49x}. \end{aligned}$$

The results asserted in Example 3 are recursively derived from the linear recurrence relation in Theorem 3 with the initial function $F_1(x)$ provided in Theorem 2. Consequently, we elucidate the closed-form expression for the Apéry-like series $F_m(x)$ by employing the linear recurrence relation delineated in (22), as thoroughly expounded in Theorem 4.

**Theorem 4.** *For $m \in \mathbb{N}$, $x \in [-1,1)$ and $F_1(x)$ as defined in (21), we have*

$$F_{m+1}(x) = \frac{1}{2m+1} - \sum_{n=0}^{m} \left( \frac{F_1(x)}{2n-1} + \frac{1}{2n+1} \right) \frac{\binom{2n}{n}}{\binom{2m}{m}} \left( \frac{x}{4} \right)^{n-m}.$$

**Proof of Theorem 4.** First, we rewrite the linear recurrence relation (22) expressed as

$$F_{m+1}(x) = a_m(x) F_m(x) - b_m(x), \tag{23}$$

where $a_m(x) := \frac{2m}{(2m-1)x}$ and $b_m(x) := \frac{F_1(x)}{2m-1} + \frac{2m}{(2m-1)^2 x}$. Given that $m \in \mathbb{N}$ and $x \neq 0$, it follows that $a_m(x) \neq 0$. Subsequently, we divide the linear recurrence relation (23) by the product $\prod_{r=1}^{m} a_r(x)$ to obtain $\frac{F_{m+1}(x)}{\prod_{r=1}^{m} a_r(x)} = \frac{a_m(x) F_m(x)}{\prod_{r=1}^{m} a_r(x)} - \frac{b_m(x)}{\prod_{r=1}^{m} a_r(x)}$ or it can be simplified as

$$\frac{F_{m+1}(x)}{\prod_{r=1}^{m} a_r(x)} - \frac{F_m(x)}{\prod_{r=1}^{m-1} a_r(x)} = -\frac{b_m(x)}{\prod_{r=1}^{m} a_r(x)}. \tag{24}$$

For convenience, let us define $Q_m(x) := \frac{F_m(x)}{\prod_{r=1}^{m-1} a_r(x)}$ for $m > 1$ and $Q_1(x) := F_1(x)$. Upon substituting $Q_m(x)$ into (24), we obtain the following relation:

$$Q_{m+1}(x) - Q_m(x) = -\frac{b_m(x)}{\prod_{r=1}^{m} a_r(x)}. \tag{25}$$

Next, we consider the summation of any consecutive differences $Q_n(x)$ and $Q_{n+1}(x)$ for all $n \in \{1, 2, 3, \ldots, m\}$ by using (25), which facilitates the derivation of an explicit formula $F_{m+1}(x)$ in terms of the initial function $F_1(x)$ and the coefficients $a_n(x)$ and $b_n(x)$,

$$Q_{m+1}(x) - Q_1(x) = \sum_{n=1}^{m} \Big( Q_{n+1}(x) - Q_n(x) \Big) = -\sum_{n=1}^{m} \frac{b_n(x)}{\prod_{r=1}^{n} a_r(x)}. \tag{26}$$

Since $Q_{m+1}(x) = \frac{F_{m+1}(x)}{\prod_{r=1}^{m} a_r(x)}$ and $Q_1(x) = F_1(x)$, we substitute them back into (26). Then, it can be expressed as the explicit solution:

$$F_{m+1}(x) = \left( \prod_{r=1}^{m} a_r(x) \right) \left( F_1(x) - \sum_{n=1}^{m} \frac{b_n(x)}{\prod_{r=1}^{n} a_r(x)} \right). \tag{27}$$

To simplify $\prod_{r=1}^{m} a_r(x)$ into an explicit form, we proceed as follows:

$$\prod_{r=1}^{m} a_r(x) = \prod_{r=1}^{m} \frac{2r}{(2r-1)x} = \left( \frac{(2m)!!}{(2m-1)!!} \cdot \frac{(2m)!!}{(2m)!!} \right) \left( \frac{1}{x} \right)^m = \frac{(m!)^2}{(2m)!} \left( \frac{4}{x} \right)^m. \tag{28}$$

Finally, we substitute (28) into (27) and manipulate it to obtain the closed-form formula of the Apéry-like series $F_{m+1}(x)$ as follows:

$$\begin{aligned}
F_{m+1}(x) &= \frac{(m!)^2}{(2m)!} \left( \frac{4}{x} \right)^m \left( F_1(x) - \sum_{n=1}^{m} \frac{\frac{F_1(x)}{2n-1} + \frac{2n}{(2n-1)^2 x}}{\frac{(n!)^2}{(2n)!} \left( \frac{4}{x} \right)^n} \right) \\
&= \frac{(m!)^2}{(2m)!} \left( \frac{4}{x} \right)^m \left( -\sum_{n=0}^{m} \frac{\frac{F_1(x)}{2n-1}}{\frac{(n!)^2}{(2n)!} \left( \frac{4}{x} \right)^n} - \sum_{n=0}^{m} \frac{\frac{1}{2n+1}}{\frac{(n!)^2}{(2n)!} \left( \frac{4}{x} \right)^n} + \frac{\frac{1}{2m+1}}{\frac{(m!)^2}{(2m)!} \left( \frac{4}{x} \right)^m} \right) \\
&= \frac{(m!)^2}{(2m)!} \left( \frac{4}{x} \right)^m \left( -\sum_{n=0}^{m} \frac{\frac{F_1(x)}{2n-1} + \frac{1}{2n+1}}{\frac{(n!)^2}{(2n)!} \left( \frac{4}{x} \right)^n} \right) + \frac{1}{2m+1}
\end{aligned}$$

$$= \frac{1}{2m+1} - \sum_{n=0}^{m} \left( \frac{F_1(x)}{2n-1} + \frac{1}{2n+1} \right) \frac{\binom{2n}{n}}{\binom{2m}{m}} \left( \frac{x}{4} \right)^{n-m}.$$

Hence, the proof is complete. □

**Remark 1.** *We provide several closed-form formulas of Type I Apéry-like series $F_m(x)$ with $m \in \{1,2,3,4,5\}$ and $x \in \{\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$ obtained from Theorem 4 in Appendix A. These formulas are utilized to be instrumental for analyzing the Type III Apéry-like series $H_{A,B}(x)$ later.*

### 3.2. Analysis of Type II: Apéry-like Series $G_m(x)$

This section delineates the closed-form derivation of the Apéry-like series $G_m(x)$, as initially defined in (19). Our exploration consists of four critical stages. First, Theorem 5 identifies the convergence domain for $G_m(x)$. Subsequently, Theorem 6 delves into the initial series $G_1(x)$. Theorem 7 introduces a linear recurrence relation characterizing $G_m(x)$, leading to Theorem 8, which establishes the closed form of the series. Additionally, the interaction between variables $x$ and $m$ is examined to uncover significant insights.

**Theorem 5.** *For a natural number $m$ and a nonzero real number $x$, the Apéry-like series*

$$G_m(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+m)}$$

*converges if $x \in [-1,1)$ and diverges if $x \in (-\infty,-1) \cup [1,\infty)$.*

**Proof of Theorem 5.** By similar argument as in the proof of Theorem 1 and the fact that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2(4x)(n+m)}{(2n+2)(2n+1)(n+m+1)} \right| = |x|,$$

where $a_n := \frac{(n!)^2(4x)^n}{(2n)!(n+m)}$, we can conclude that $G_m(x)$ converges on $x \in (-1,1)$ and diverges on $x \in (-\infty,-1) \cup (1,\infty)$. Next, we examine the convergence at $x = -1$ for the Apéry-like series $G_m(-1)$ using the comparison test. The series is defined as

$$G_m(-1) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{(n+m)} = \sum_{n=0}^{\infty} (-1)^n b_n,$$

where $b_n := \frac{(n!)^2 4^n}{(2n)!(n+m)} > 0$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. To verify convergence, we apply the alternating series test, which requires analyzing the limit of $b_n$ as $n \to \infty$. We have

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{(n!)^2 4^n}{(2n)!(n+m)} \cdot \frac{2n+1}{2n+1} \\ &= \lim_{n\to\infty} \frac{(2^n n!)^2(2n+1)}{(2n)!!(2n+1)!!(n+m)} \\ &= \lim_{n\to\infty} \frac{(2n)!!}{(2n+1)!!} \cdot \lim_{n\to\infty} \frac{2n+1}{n+m} = 0 \end{aligned}$$

because $\lim_{n\to\infty} \frac{2n+1}{n+m} = 2$ and $\lim_{n\to\infty} \frac{(2n)!!}{(2n+1)!!} = 0$ provided in [22], which is one of the Wallis integral properties. Further, we examine the monotonicity of the sequence $b_n$:

$$\begin{aligned} b_{n+1} &= \frac{4(n+1)^2(n!)^2 4^n}{(2n)!(2n+2)(2n+1)(n+m+1)} \\ &= \frac{(n!)^2 4^n}{(2n)!(n+m)} \cdot \frac{(2n+2)(n+m)}{(2n+1)(n+m+1)} \end{aligned}$$

$$\leq \frac{(n!)^2 4^n}{(2n)!(n+m)} = b_n.$$

Thus, $b_{n+1} \leq b_n$ for all $n \geq m-1$; in other words, $b_n$ is then an eventually decreasing sequence and also $\lim_{n\to\infty} b_n = 0$. Hence, the alternating series test confirms that the Apéry-like series $G_m(x)$ converges at $x = -1$.

Finally, we analyze the behavior of the Apéry-like series at the domain point $x = 1$, specifically to test the divergence of the series $G_m(1)$. Consider the series expressed as shown below:

$$G_m(1) = \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n)!(n+m)} \geq \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n)!(n+2m)} \geq \sum_{n=0}^{\infty} \frac{1}{n+2m}.$$

Since $\lim\limits_{n\to\infty} \frac{n}{n+2m} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent $p$-series, then $\sum_{n=0}^{\infty} \frac{1}{n+2m}$ diverges by the limit comparison test. Thus, by the comparison test, $G_m(1)$ diverges. Hence, we conclude that the point $x = 1$ lies within the divergence domain of the Apéry-like series, thus completing our proof. □

In the preceding analysis, we confirmed that the Apéry-like series $G_m(x)$ converges within the domain $x \in [-1, 0) \cup (0, 1)$, as demonstrated by Theorem 5. Building on this foundation, our next objective is to determine the convergence values for $G_m(x)$, beginning with the case where $m = 1$. To this end, we derive the explicit expression for $G_1(x)$ as outlined in Theorem 6, utilizing the principles of fractional calculus discussed in Section 2.

**Theorem 6.** *The initial Apéry-like series $G_1(x)$ is expressed in the explicit formula:*

$$G_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+1} = \begin{cases} \frac{2\,\mathrm{arcsinh}\sqrt{-x}}{\sqrt{x(x-1)}} + \frac{\mathrm{arcsinh}^2\sqrt{-x}}{x} & ;\ x \in [-1, 0), \\ 1 & ;\ x = 0, \\ \frac{2\arcsin\sqrt{x}}{\sqrt{x(1-x)}} - \frac{\arcsin^2\sqrt{x}}{x} & ;\ x \in (0, 1). \end{cases} \tag{29}$$

**Proof of Theorem 6.** Let $x \in (0, 1)$. By utilizing (14), we can transform $G_1(x)$ to become

$$G_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)!\,(n+1)} \frac{\sqrt{4\pi x}}{\sqrt{4\pi x}} = \sqrt{\pi x} \sum_{n=0}^{\infty} \frac{2(n!)^2 (4x)^{n-\frac{1}{2}}}{(2n)!\sqrt{\pi}\,(n+1)} = \sqrt{\pi x} \sum_{n=0}^{\infty} \left( \frac{\mathcal{D}_{0+}^{\frac{1}{2}}(x^n)}{n+1} \right).$$

From Definition 4, we can interchange the summation into half-derivative operator $\mathcal{D}_{0+}^{\frac{1}{2}}$, because $\int_0^x \frac{\tau^n\, d\tau}{(n+1)\sqrt{x-\tau}}$ is absolutely convergent for $n \in \mathbb{N}_0$ and $x > 0$. We further apply the infinite Mercator series [24], $\sum_{n=1}^{\infty} \frac{\tau^n}{n} = -\ln(1-\tau)$ for $|\tau| < 1$ to evaluate the above summation. Then, the result obtained is as follows.

$$\begin{aligned} \sqrt{\pi x} \sum_{n=0}^{\infty} \left( \frac{\mathcal{D}_{0+}^{\frac{1}{2}}(x^n)}{n+1} \right) &= \sqrt{\pi x} \sum_{n=0}^{\infty} \left( \frac{1}{\Gamma(\frac{1}{2})} \frac{d}{dx} \int_0^x \frac{\tau^n\, d\tau}{(n+1)\sqrt{x-\tau}} \right) \\ &= \frac{\sqrt{\pi x}}{\sqrt{\pi}} \frac{d}{dx} \int_0^x \left( \sum_{n=0}^{\infty} \frac{\tau^{n+1}}{n+1} \cdot \frac{1}{\tau\sqrt{x-\tau}} \right) d\tau \\ &= \sqrt{x} \frac{d}{dx} \int_0^x \left( \sum_{n=1}^{\infty} \frac{\tau^n}{n} \cdot \frac{1}{\tau\sqrt{x-\tau}} \right) d\tau \\ &= \sqrt{x} \frac{d}{dx} \int_0^x \frac{-\ln(1-\tau)}{\tau\sqrt{x-\tau}}\, d\tau. \end{aligned} \tag{30}$$

Using the integral substitution, let $\tau = xu$; then, $d\tau = x\,du$. The limits of integration change accordingly, when $\tau = 0$, $u = 0$ and when $\tau = x$, $u = 1$. Thus, the above integral becomes

$$\int_0^x \frac{-\ln(1-\tau)}{\tau\sqrt{x-\tau}}\,d\tau = \int_0^1 \frac{-\ln(1-xu)}{xu\sqrt{x(1-u)}}\,x\,du = \frac{-1}{\sqrt{x}}\int_0^1 \frac{\ln(1-xu)}{u\sqrt{1-u}}\,du.$$

According to the standard integral tables of Gradshteyn and Ryzhik [25], the result of this specific integral can be expressed in terms of the arcsin function,

$$\frac{-1}{\sqrt{x}}\int_0^1 \frac{\ln(1-xu)}{u\sqrt{1-u}}\,du = \frac{-1}{\sqrt{x}}\left(-2\arcsin^2\sqrt{x}\right) = \frac{2\arcsin^2\sqrt{x}}{\sqrt{x}}.$$

After that, we derivative the above expression with respect to the variable $x$. Thus, for $x \in (0,1)$, $G_1(x)$ can be evaluated in a simplified form as

$$G_1(x) = \sqrt{x}\,\frac{d}{dx}\left(\frac{2\arcsin^2\sqrt{x}}{\sqrt{x}}\right) = \frac{2\arcsin\sqrt{x}}{\sqrt{x(1-x)}} - \frac{\arcsin^2\sqrt{x}}{x}.$$

Similarly, for $x \in [-1,0)$, we consider $G_1(x)$ by using the relation of the right-sided half derivative (16) to obtain

$$G_1(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-4x)^n(-1)^n}{(n+1)}\frac{\sqrt{-4\pi x}}{\sqrt{-4\pi x}} = \sqrt{-\pi x}\sum_{n=0}^{\infty}\left(\frac{\mathcal{D}_{0-}^{\frac{1}{2}}(x^n)}{n+1}\right).$$

Then, we reformulate the above summation similarly to the process to receive (30) based on Definition 4 and the infinite Mercator series [24], that is $\sum_{n=1}^{\infty}\frac{\tau^n}{n} = -\ln(1-\tau)$ for $|\tau| < 1$. Therefore, we have

$$\begin{aligned}
\sqrt{-\pi x}\sum_{n=0}^{\infty}\left(\frac{\mathcal{D}_{0-}^{\frac{1}{2}}(x^n)}{n+1}\right) &= \sqrt{-\pi x}\sum_{n=0}^{\infty}\left(\frac{1}{\Gamma(\frac{1}{2})}\left(-\frac{d}{dx}\right)\int_x^0 \frac{\tau^n\,d\tau}{(n+1)\sqrt{\tau-x}}\right)\\
&= -\frac{\sqrt{-\pi x}}{\sqrt{\pi}}\frac{d}{dx}\int_x^0\left(\sum_{n=0}^{\infty}\frac{\tau^{n+1}}{n+1}\cdot\frac{1}{\tau\sqrt{\tau-x}}\right)d\tau\\
&= -\sqrt{-x}\frac{d}{dx}\int_x^0\left(\sum_{n=1}^{\infty}\frac{\tau^n}{n}\cdot\frac{1}{\tau\sqrt{\tau-x}}\right)d\tau\\
&= \sqrt{-x}\,\frac{d}{dx}\int_x^0 \frac{\ln(1-\tau)}{\tau\sqrt{\tau-x}}\,d\tau.
\end{aligned}$$

Evaluating the above integral according to the standard integral tables in [25], we obtain

$$G_1(x) = \sqrt{-x}\,\frac{d}{dx}\left(-\frac{2\operatorname{arcsinh}^2\sqrt{-x}}{\sqrt{-x}}\right) = \frac{2\operatorname{arcsinh}\sqrt{-x}}{\sqrt{x(x-1)}} + \frac{\operatorname{arcsinh}^2\sqrt{-x}}{x}.$$

Finally, for $x = 0$, it is clear that $G_1(0)$ consists only of the first term in the summation, giving $G_1(0) = 1$. Thus, the proof is complete. □

**Remark 2.** *We found that the Apéry-like series $G_1(x)$ can be written in terms of $F_1(x)$ as follows: $G_1(x) = 2F_1(x) - (1-x)F_1^2(x)$, where $F_1(x)$ and $G_1(x)$ are defined in (21) and (29), respectively.*

Before deducing the explicit formula for the Apéry-like series $G_m(x)$, we establish a linear recurrence relation as defined in (19). This process involves a deliberate re-indexing of the series and applying partial fraction decomposition, which is thoroughly explained in Theorem 7.

**Theorem 7.** *The Apéry-like series $G_m(x)$ defined in (19) can be expressed in the following linear recurrence relation:*

$$G_{m+1}(x) = \frac{(2m+1)G_m(x)}{2mx} - \frac{F_1(x)}{m} - \frac{2m+1}{2m^2x}, \tag{31}$$

*where $m \in \mathbb{N}$, $x \in [-1,1)$, and $F_1(x)$ and $G_1(x)$ defined in (21) and (29), respectively.*

**Proof of Theorem 7.** We derive the linear recurrence relation (31) starting from the definition of $G_m(x)$ in (19) with the following steps:

$$\begin{aligned} G_m(x) &= \frac{1}{m} + \sum_{n=1}^{\infty} \frac{((n-1)!)^2 n^2 (4x)^n}{(2n-2)!(2n-1)(2n)(n+m)} \\ &= \frac{1}{m} + \sum_{n=0}^{\infty} \frac{(n!)^2 (n+1)^2 (4x)^{n+1}}{(2n)!(2n+1)(2n+2)(n+m+1)} \\ &= \frac{1}{m} + 2x \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)!} \left( \frac{1}{2m+1} \cdot \frac{1}{2n+1} + \frac{m}{2m+1} \cdot \frac{1}{n+m+1} \right) \\ &= \frac{1}{m} + \frac{2x}{2m+1} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)!(2n+1)} + \frac{2mx}{2m+1} \sum_{n=0}^{\infty} \frac{(n!)^2 (4x)^n}{(2n)!(n+m+1)} \\ &= \frac{1}{m} + \frac{2x}{2m+1} F_1(x) + \frac{2mx}{2m+1} G_{m+1}(x). \end{aligned}$$

In addition, the mathematical induction technique applied to $m$ enables us to validate the linear recurrence relation (31). Thus, the proof is complete. □

The findings derived from Theorem 7 are hereby explained. Example 4 specifies the first five cases of the Apéry-like series $G_m(x)$ for $m$ that belong to the set $\{1,2,3,4,5\}$.

**Example 4.** *The first five Apéry-like series $G_m(x)$ for $m \in \{1,2,3,4,5\}$ hold the following:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+1} = G_1(x),$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+2} = \frac{3G_1(x)}{2x} - F_1(x) - \frac{3}{2x},$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+3} = \frac{15G_1(x)}{8x^2} - \left( \frac{5}{4x} + \frac{1}{2} \right) F_1(x) - \frac{15}{8x^2} - \frac{5}{8x},$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+4} = \frac{35G_1(x)}{16x^3} - \left( \frac{35}{24x^2} + \frac{7}{12x} + \frac{1}{3} \right) F_1(x) - \frac{35}{16x^3} - \frac{35}{48x^2} - \frac{7}{18x},$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+5} = \frac{315G_1(x)}{128x^4} - \left( \frac{105}{64x^3} + \frac{21}{32x^2} + \frac{3}{8x} + \frac{1}{4} \right) F_1(x) - \frac{315}{128x^4} - \frac{105}{128x^3} - \frac{7}{16x^2} - \frac{9}{32x}.$$

The results asserted in Example 4 are recursively derived from the linear recurrence relation in Theorem 7 with the initial functions $F_1(x)$ and $G_1(x)$ provided in Theorems 2 and 6, respectively. Accordingly, the explicit formula for the Apéry-like series denoted by $G_m(x)$ is articulated by utilizing (31), as comprehensively described in Theorem 8, maintaining the notation introduced in Theorem 7.

**Theorem 8.** *Keeping the notation in Theorem 7, we have the following closed-form formula*

$$G_{m+1}(x) = \frac{1}{m+1} + \frac{(2m+1)!(G_1(x)-1)}{(m!)^2(4x)^m} - \sum_{n=1}^{m} \left( \frac{F_1(x)}{n} + \frac{1}{n+1} \right) \frac{\binom{2m}{m} \frac{2m+1}{(4x)^m}}{\binom{2n}{n} \frac{2n+1}{(4x)^n}},$$

*where $m \in \mathbb{N}$; and $x \in [-1,1)$, $F_1(x)$ and $G_1(x)$ defined in (21) and (29), respectively.*

**Proof of Theorem 8.** From the linear recurrence relation (31), we denote its coefficients by $c_m(x)$ and $d_m(x)$. Thus, the recurrence relation (31) can be written as

$$G_{m+1}(x) = c_m(x)G_m(x) - d_m(x), \tag{32}$$

where $c_m(x) = \frac{2m+1}{2mx}$ and $d_m(x) = \frac{F_1(x)}{m} + \frac{2m+1}{2m^2x}$. Given that $m \in \mathbb{N}$ and $x \neq 0$, it follows that $c_m(x) \neq 0$. Next, we divide the recurrence relation (23) by the product $\prod_{r=1}^{m} c_r(x)$ and simplify it to become

$$\frac{G_{m+1}(x)}{\prod_{r=1}^{m} c_r(x)} - \frac{G_m(x)}{\prod_{r=1}^{m-1} c_r(x)} = -\frac{d_m(x)}{\prod_{r=1}^{m} c_r(x)}. \tag{33}$$

Then, we can solve the linear recurrence relation (33) using the same process as in the proof of Theorem 4. Hence, the explicit solution $G_{m+1}(x)$ of (33) can be expressed as follows

$$G_{m+1}(x) = \left(\prod_{r=1}^{m} c_r(x)\right)\left(G_1(x) - \sum_{n=1}^{m} \frac{d_n(x)}{\prod_{r=1}^{n} c_r(x)}\right). \tag{34}$$

To simplify $\prod_{r=1}^{m} c_r(x)$ into an explicit form, we proceed as follows:

$$\prod_{r=1}^{m} c_r = \prod_{r=1}^{m} \frac{2r+1}{2rx} = \left(\frac{(2m+1)!!}{(2m)!!} \cdot \frac{(2m)!!}{(2m)!!}\right)\left(\frac{1}{x}\right)^m = \frac{(2m+1)!}{(m!)^2(4x)^m}. \tag{35}$$

Finally, we substitute (35) into (34) and manipulate it to obtain the closed-form formula of the Apéry-like series $G_{m+1}(x)$ as follows:

$$\begin{aligned}
G_{m+1}(x) &= \frac{(2m+1)!}{(m!)^2(4x)^m}\left(G_1(x) - \sum_{n=1}^{m} \frac{\frac{F_1(x)}{n} + \frac{2n+1}{2n^2x}}{\frac{(2n+1)!}{(n!)^2(4x)^n}}\right) \\
&= \frac{(2m+1)!}{(m!)^2(4x)^m}\left(G_1(x) - \sum_{n=1}^{m} \frac{\frac{F_1(x)}{n}}{\frac{(2n+1)!}{(n!)^2(4x)^n}} - \sum_{n=0}^{m} \frac{\frac{1}{n+1}}{\frac{(2n+1)!}{(n!)^2(4x)^n}} + \frac{\frac{1}{m+1}}{\frac{(2m+1)!}{(m!)^2(4x)^m}}\right) \\
&= \frac{(2m+1)!}{(m!)^2(4x)^m}\left(G_1(x) - \sum_{n=1}^{m} \frac{\frac{F_1(x)}{n}}{\frac{(2n+1)!}{(n!)^2(4x)^n}} - 1 - \sum_{n=1}^{m} \frac{\frac{1}{n+1}}{\frac{(2n+1)!}{(n!)^2(4x)^n}}\right) + \frac{1}{m+1} \\
&= \frac{1}{m+1} + \frac{(2m+1)!(G_1(x)-1)}{(m!)^2(4x)^m} - \sum_{n=1}^{m}\left(\frac{F_1(x)}{n} + \frac{1}{n+1}\right)\frac{\binom{2m}{m}\frac{2m+1}{(4x)^m}}{\binom{2n}{n}\frac{2n+1}{(4x)^n}}.
\end{aligned}$$

Hence, the proof is complete. □

**Remark 3.** *We also propose eight instances in Appendix B for the Type II familiar of Apéry-like series $G_m(x)$ for $m$ within the set $\{1,2,3,4,5\}$ where each instance, originating from Theorem 8, is defined for $x$ in the set $\{\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$. In the next section, these instances will serve a crucial role in the subsequent analysis of the Type III Apéry-like series $H_{A,B}(x)$.*

### *3.3. Analysis of Type III: Apéry-like Series $H_{A,B}(x)$*

This section delves into the combination of Type I and Type II Apéry-like series, referred to as Type III, which is represented by the series formula in (20). By synthesizing the methodologies applied in the series $F_m(x)$ and $G_m(x)$, defined in (18) and (19), respectively, we unify these expressions under a new framework. Let $k \in \mathbb{N}$. We define

$$H_k(x) := \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{(2n+k)} = \begin{cases} F_m(x) & ;\ k = 2m-1, \\ \frac{1}{2}G_m(x) & ;\ k = 2m. \end{cases} \tag{36}$$

Further extending this approach, we generalize the expression for Apéry-like series as in (20), which allows for a more inclusive representation:

$$H_{A,B}(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n Q(n)}{P_{A,B}(n)}, \tag{37}$$

where $x$ is a nonzero real number, $n$ is a non-negative integer, and $Q(n)$ and $P_{A,B}(n)$ are polynomials with $\deg Q(n) < \deg P_{A,B}(n)$. Here, $A$ denotes a finite set of positive odd numbers ($A \subset 2\mathbb{N} - 1$), and $B$ includes a finite set of natural numbers ($B \subset \mathbb{N}$).

Let $P_{A,B}(n)$ be a polynomial in term of $n$. Then, we factorize it into two products of linear functions $2n + a_i$ and $n + b_j$, where $a_i \in A := \{a_1, a_2, a_3, \ldots, a_{r_1}\}$, with $A$ being a subset of odd numbers and $b_j \in B := \{b_1, b_2, b_3, \ldots, b_{r_2}\}$, with $B$ being a subset of natural numbers. The degree of the polynomial $P_{A,B}(n)$, denoted by $\deg P_{A,B}(n) = r_1 + r_2 =: r$, encompasses the combined count of elements in sets $A$ and $B$. The polynomial $P_{A,B}(n)$ can thus be expressed as

$$P_{A,B}(n) = \prod_{i=1}^{r_1}(2n + a_i)\prod_{j=1}^{r_2}(n + b_j) = 2^{r_1}\prod_{i=1}^{r_1}\left(n + \frac{a_i}{2}\right)\prod_{j=1}^{r_2}(n + b_j) = 2^{r_1}\prod_{i=1}^{r}(n - \lambda_i), \tag{38}$$

where $r := r_1 + r_2 = n(A \cup B)$ and $\lambda_i \in \Lambda := \{-\frac{a_1}{2}, -\frac{a_2}{2}, \ldots, -\frac{a_{r_1}}{2}, -b_1, -b_2, \ldots, -b_{r_2}\}$.

Note that elements in the set $\Lambda$ represent the distinct roots of the polynomial $P_{A,B}(n)$. For convenience, we let the polynomial $R(n) := \prod_{i=1}^{r}(n - \lambda_i)$; then, $P_{A,B}(n) = 2^{r_1}R(n)$. This formulation allows us to analyze the rational function $\frac{Q(n)}{P_{A,B}(n)}$ in (37), with $Q(n)$ typically chosen to ensure non-repeated linear factors, facilitating the decomposition into partial fractions using the Lagrange interpolation method [26].

**Theorem 9** ([26])**.** *Let $Q(x)$ be a polynomial of degree less than n that interpolates the distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ such that $Q(x_k) = y_k$ for each k. Suppose $R(x)$ is an nth order polynomial with zeros at $x_1, x_2, \ldots, x_n$. The expressions for $R(x)$ and its derivative at these points are given by*

$$R(x) = \prod_{i=1}^{n}(x - x_i) \quad \textit{and} \quad R'(x_k) = \prod_{\substack{i=1 \\ i \neq k}}^{n}(x_k - x_i).$$

*The Lagrange interpolating polynomial for $Q(x)$ is then defined as*

$$Q(x) = \sum_{k=1}^{n} \frac{R(x)\, y_k}{(x - x_k)R'(x_k)}. \tag{39}$$

Considering the rational function in (37), assume unique constants $c_k \in \mathbb{R}$ exist such that the rational function $\frac{Q(n)}{P_{A,B}(n)}$ can be expressed as a sum of partial fractions:

$$\frac{Q(n)}{P_{A,B}(n)} = \frac{Q(n)}{2^{r_1}R(n)} = \frac{c_1}{n - \lambda_1} + \frac{c_2}{n - \lambda_2} + \frac{c_3}{n - \lambda_3} + \cdots + \frac{c_r}{n - \lambda_r} = \sum_{k=1}^{r} \frac{c_k}{n - \lambda_k}. \tag{40}$$

where $P_{A,B}(n)$ is factored as $2^{r_1}R(n)$. Evaluating the first-order derivative of $R(n)$ at $x = \lambda_k$ using the product rule, we find

$$R'(\lambda_k) = \frac{d}{dx}\left(\prod_{i=1}^{r}(x - \lambda_i)\right)\Bigg|_{x=\lambda_k} = \sum_{j=1}^{r}\left(\frac{d}{dx}(x - \lambda_j)\prod_{\substack{i=1 \\ i \neq j}}^{r}(x - \lambda_i)\right)\Bigg|_{x=\lambda_k} = \prod_{\substack{i=1 \\ i \neq k}}^{r}(\lambda_k - \lambda_i).$$

From the definition of the Lagrange interpolating polynomial in Theorem 9, the coefficients of the partial fraction decomposition are derived:

$$\frac{Q(n)}{P_{A,B}(n)} = \frac{Q(n)}{2^{r_1} R(n)} = \frac{1}{2^{r_1} R(n)} \sum_{k=1}^{r} \frac{R(n)\, Q(\lambda_k)}{(n-\lambda_k) R'(\lambda_k)} = \sum_{k=1}^{r} \frac{Q(\lambda_k)}{(n-\lambda_k)\, 2^{r_1} R'(\lambda_k)} \tag{41}$$

To compare the summations of (40) and (41), we thus yield the coefficients $c_k$ of decomposition into the partial fraction as follows:

$$c_k = \frac{Q(\lambda_k)}{2^{r_1} R'(\lambda_k)} = \frac{Q(\lambda_k)}{2^{r_1} \prod_{\substack{i=1 \\ i\neq k}}^{r} (\lambda_k - \lambda_i)}, \quad k \in \{1,2,3,\ldots,r\}, \tag{42}$$

where $\lambda_k$ represents the zeros of $P_{A,B}(n)$, $r = n(A \cup B)$ is the degree of $P_{A,B}(n)$ and $r_1 = n(A)$ is the count of terms involving the power of two in $P_{A,B}(n)$.

**Remark 4.** *It should be noted that this section is closely aligned with the results presented in [27]. However, it primarily offers a conceptual methodology for further exploration rather than delivering comprehensive proof.*

Ultimately, the generalized Apéry-like series $H_{A,B}(x)$ can be straightforwardly derived to the explicit formula by directly using the relations of $F_m(x)$ and $G_m(x)$. To showcase the usage of this section, we present the following example.

**Example 5.** *Consider the decomposition into partial fractions of the Apéry-like series*

$$H_{A,B}(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n (3n^2+5n-5)}{(2n+1)(2n+5)(n+2)(n+3)}.$$

*Let the polynomials $Q(n) = 3n^2+5n-5$ and $P_{A,B}(n) = (2n+1)(2n+5)(n+2)(n+3)$, in which sets $A = \{1,5\}$ and $B = \{2,3\}$. Then, we decompose the proper rational function $\frac{Q(n)}{P_{A,B}(n)}$ into the partial fractions as in (40), that is,*

$$\frac{Q(n)}{P_{A,B}(n)} = \frac{3n^2+5n-5}{4(n+\frac{1}{2})(n+\frac{5}{2})(n+2)(n+3)} = \frac{c_1}{n+\frac{1}{2}} + \frac{c_2}{n+\frac{5}{2}} + \frac{c_3}{n+2} + \frac{c_4}{n+3},$$

*where $c_k$ for $k \in \{1,2,3,4\}$ are unknown constants. Let $\Lambda := \{-\frac{1}{2}, -\frac{5}{2}, -2, -3\}$ be the set of zeros of the polynomial $P_{A,B}(n)$. Using (42), we can find the unknown constants $c_k$ when given that $\lambda_k \in \Lambda$ as the following:*

$$c_1 = \frac{Q(\lambda_1)}{2^2 \prod_{\substack{i=1 \\ i\neq 1}}^{4} (\lambda_1 - \lambda_i)} = \frac{3\lambda_1^2 + 5\lambda_1 - 5}{4(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} = -\frac{9}{40},$$

$$c_2 = \frac{Q(\lambda_2)}{2^2 \prod_{\substack{i=1 \\ i\neq 2}}^{4} (\lambda_2 - \lambda_i)} = \frac{3\lambda_2^2 + 5\lambda_2 - 5}{4(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} = \frac{5}{8},$$

$$c_3 = \frac{Q(\lambda_3)}{2^2 \prod_{\substack{i=1 \\ i\neq 3}}^{4} (\lambda_3 - \lambda_i)} = \frac{3\lambda_3^2 + 5\lambda_3 - 5}{4(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} = 1,$$

$$c_4 = \frac{Q(\lambda_4)}{2^2 \prod_{\substack{i=1 \\ i\neq 4}}^{4} (\lambda_4 - \lambda_i)} = \frac{3\lambda_4^2 + 5\lambda_4 - 5}{4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} = -\frac{7}{5}.$$

*Hence, after substituting the proper rational function* $\frac{Q(n)}{P_{A,B}(n)}$ *into the Apéry-like series* $H_{A,B}(x)$*, it can be expressed as the following partial fraction:*

$$\begin{aligned}H_{A,B}(x) &= \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\left(-\frac{9(4x)^n}{20(2n+1)}+\frac{5(4x)^n}{4(2n+5)}+\frac{(4x)^n}{n+2}-\frac{7(4x)^n}{5(n+3)}\right)\\ &= -\frac{9}{20}H_1(x)+\frac{5}{4}H_5(x)+2H_4(x)-\frac{14}{5}H_6(x),\end{aligned}$$

*where* $H_k(x)$ *for* $k \in \{1,4,5,6\}$ *is defined by (36), which can be actually transformed into both Apéry-like series* $F_m(x)$ *and* $G_m(x)$*, i.e.,* $H_{A,B}(x) = -\frac{9}{20}F_1(x)+\frac{5}{4}F_3(x)+G_2(x)-\frac{7}{5}G_3(x)$*.*

Furthermore, from Example 5, we can evaluate its value at different $x \in [-1,1)$ into the fundamental constants. For instance, if $x = \frac{1}{2}$, we can directly evaluate $HA, B(\frac{1}{2})$ using the Apéry-like series relations $F_m(\frac{1}{2})$ and $G_m(\frac{1}{2})$ in Propositions A6 and A15, respectively,

$$\begin{aligned}H_{A,B}\left(\frac{1}{2}\right) &= -\frac{9}{20}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{2n+1}+\frac{5}{4}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{2n+5}+\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{n+2}-\frac{7}{5}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{n+3}\\ &= -\frac{9}{20}\left(\frac{\pi}{2}\right)+\frac{5}{4}\left(\frac{23\pi}{6}-\frac{104}{9}\right)+\left(-3+\frac{5\pi}{2}-\frac{3\pi^2}{8}\right)-\frac{7}{5}\left(-\frac{35}{4}+6\pi-\frac{15\pi^2}{16}\right)\\ &= -\frac{187}{36}-\frac{4\pi}{3}+\frac{15\pi^2}{16}.\end{aligned}$$

However, the applications of this particular case in continued fractions will be discussed in detail in the next section.

## 4. Applications

This section applies the Apéry-like series in Section 3 to derive continued fractions. We also validate the Ramanujan Machine's unproven conjectures (identities) of continued fractions based on our proposed close-form formulas associated with the Apéry-like series. Typically, a generalized continued fraction is representable in various forms.

**Definition 5.** *Let* $(a_m)_{m\geq 1}$ *and* $(b_m)_{m\geq 0}$ *be sequences of complex numbers. The continued fraction, denoted by x, can be expressed as*

$$x := b_0 + \cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots}}} = b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots. \tag{43}$$

*For any natural number m, the finite continued fraction of order* $m \geq 1$ *is defined by*

$$\frac{A_m}{B_m} := b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots+\frac{a_m}{b_m}. \tag{44}$$

*Furthermore, if the limit of (44) exists, the continued fraction (43) corresponds to* $x = \lim_{m\to\infty}\frac{A_m}{B_m}$*.*

Therefore, calculating $A_m$ and $B_m$ is essential to demonstrate the validity of the representation of the given continuation fraction. The following renowned result of the continued fraction theory establishes a connection between the convergence and the difference equations, as referenced in [28].

Laohakosol et al. [3] recently presented a specific closed-form solution for generalized second-order linear recurrence with coefficients represented by sequences of complex numbers, which is based on the idea of the counting set in [29,30]. The result can be applied with simple continued fractions to confirm the considerable conjectures discovered by the Ramanujan Machine.

**Theorem 10** ([3]). *Assume that there exist sequences $(c_m)_{m\geq 1}$ and $(d_m)_{m\geq 1}$ of complex numbers, such that $d_m \neq 0$ and $d_1 = b_0$, satisfying*

$$\begin{cases} a_m = -c_m d_m, \\ b_m = c_m + d_{m+1}, \end{cases} \tag{45}$$

*for all $m \geq 1$. Then, the multiplicative inverse of the sequence $\frac{A_m}{B_m}$, namely $\frac{B_m}{A_m}$, can be expressed by*

$$\frac{B_m}{A_m} = \frac{1}{b_0 +} \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 + \cdots +} \frac{a_m}{b_m} = \sum_{n=0}^{m} \frac{1}{d_{n+1}} \prod_{k=1}^{n} \frac{c_k}{d_k}. \tag{46}$$

*Furthermore, the generalized continued fraction can be computed by*

$$\lim_{m\to\infty} \frac{A_m}{B_m} = \left( \sum_{n=0}^{\infty} \frac{1}{d_{n+1}} \prod_{k=1}^{n} \frac{c_k}{d_k} \right)^{-1}. \tag{47}$$

The following Example 6 demonstrates applications of Theorem 10 to verify the unproven conjectures (identities) of the Ramanujan Machine discovered by Raayoni et al. [13] in 2021 and later proven by Sutthimat et al. [3] in 2024. Next, three unproven Ramanujan Machine conjectures listed in [13] are validated in the proofs of Conjectures 1–3, illustrating the versatility of our work. These conjectures are generated by the Ramanujan Machine using the MITM-RF algorithm provided in [14].

**Example 6** ([13]). *Verify the following Ramanujan Machine conjecture:*

$$\frac{2}{2-\pi} = -2 + \frac{-1}{-5 +} \frac{-6}{-8 +} \frac{-15}{-11 + \cdots +} \frac{-m(2m-1)}{-(3m+2)} + \cdots.$$

*First, we can establish that $a_m = -m(2m-1)$ and $b_m = -(3m+2)$, which allows us to set the sequences $c_m = -\frac{m^2}{m+1}$ and $d_m = -\frac{(2m-1)(m+1)}{m}$. By applying (46) from Theorem 10 and utilizing the partial fraction as the same process of (42), we obtain*

$$\begin{aligned} \frac{B_m}{A_m} &= \sum_{n=0}^{m} \frac{-(n+1)}{(2n+1)(n+2)} \prod_{k=1}^{n} \frac{k^3}{(2k-1)(k+1)^2} \\ &= \sum_{n=0}^{m} \frac{-1}{(2n+1)(n+2)(n+1)} \cdot \frac{n!}{(2n-1)!!} \frac{2^n n!}{(2n)!!} \\ &= \sum_{n=0}^{m} \frac{(n!)^2}{(2n)!} \frac{-2^n}{(2n+1)(n+1)(n+2)} \\ &= \sum_{n=0}^{m} \frac{(n!)^2 2^n}{(2n)!} \left( -\frac{4/3}{2n+1} + \frac{1}{n+1} - \frac{1/3}{n+2} \right). \end{aligned}$$

*It is evident that it converges when m approaches infinity. By using the relation (47) and the familiar identities of Propositions A6 and A15, we can therefore conclude that*

$$\lim_{m\to\infty} \frac{A_m}{B_m} = \left( -\frac{4}{3}\left(\frac{\pi}{2}\right) + \left(\pi - \frac{\pi^2}{8}\right) - \frac{1}{3}\left(-3 + \frac{5\pi}{2} - \frac{3\pi^2}{8}\right) \right)^{-1} = \frac{2}{2-\pi}.$$

Next, we proceed to apply the obtained results of the Apéry-like series in Section 3 with Theorem 10 to validate and prove the following unproven Conjectures 1–3 of continued fractions. The Ramanujan Machine obtains these conjectures with the MITM-RF technique provided by [14] and listed in [13].

**Conjecture 1** (unproven, listed in [13])**.** *The unproven conjecture (identity) of continued fractions generated by the Ramanujan Machine: MITM-RF algorithm is*

$$\frac{16}{\pi^2+4} = 1 + \frac{1}{7+}\,\frac{-8}{19+}\,\frac{-81}{37}\,+\cdots+\frac{-2m^4+3m^3}{3m(m+1)+1+}\cdots. \tag{48}$$

**Proof of Conjecture 1.** We take $a_m = -2m^4 + 3m^3$, $m \geq 1$ and $b_m = 3m(m+1)+1$, $m \geq 0$ so that the sequences satisfying (45) are $c_m = \frac{m^2(m^2+m-4)}{m^2+3m-2}$ and $d_m = \frac{(2m^2-3m)(m^2+3m-2)}{m^2+m-4}$ for $m \geq 1$. We denote the $m$th-order continued fraction (48) by

$$\frac{A_m}{B_m} := 1 + \frac{1}{7+}\,\frac{-8}{19+}\,\frac{-81}{37}\,+\cdots+\frac{-2m^4+3m^3}{3m(m+1)+1}.$$

Next, by using (46) from Theorem 10, we have

$$\begin{aligned}
\frac{B_m}{A_m} &= \sum_{n=0}^{m} \frac{(n^2+3n-2)}{(2n^2+n-1)(n^2+5n+2)} \prod_{k=1}^{n} \frac{k^2(k^2+k-4)^2}{(k^2+3k-2)^2(2k^2-3k)} \\
&= \sum_{n=0}^{m} \frac{(n^2+3n-2)}{(2n-1)(n+1)(n^2+5n+2)} \left( \prod_{k=1}^{n} \frac{(k^2+k-4)^2}{(k^2+3k-2)^2} \cdot \prod_{k=1}^{n} \frac{k}{2k-3} \right) \\
&= \sum_{n=0}^{m} \frac{(n^2+3n-2)}{(2n-1)(n+1)(n^2+5n+2)} \left( \frac{(-2)^2}{(n^2+3n-2)^2} \cdot \frac{-n!}{(2n-3)!!}\,\frac{2^n n!}{(2n)!!} \right) \\
&= \sum_{n=0}^{m} \frac{(n!)^2}{(2n)!}\,\frac{-2(2^n)}{(n+1)(n+2)} \left( \frac{2(n+2)}{(n^2+3n-2)(n^2+5n+2)} \right).
\end{aligned}$$

Through the telescoping series, considering the above expression $\frac{B_m}{A_m}$ as $m \to \infty$, we have

$$\begin{aligned}
\lim_{m\to\infty} \frac{B_m}{A_m} &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\,\frac{-2(2^n)}{(n+1)(n+2)} \left( \frac{1}{n^2+3n-2} - \frac{1}{n^2+5n+2} \right) \\
&= \frac{1}{2} - 2\sum_{n=1}^{\infty} \left( \frac{(n!)^2(2^n)}{(2n)!(n+1)(n+2)} - \frac{((n-1)!)^2(2^{n-1})}{(2n-2)!(n)(n+1)} \right) \left( \frac{1}{n^2+3n-2} \right) \\
&= \frac{1}{2} + 2\sum_{n=1}^{\infty} \left( \frac{((n-1)!)^2(2^n)}{(2n)!(n+1)(n+2)} \right) \\
&= \frac{1}{2} + 2\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \left( \frac{2^n}{(n+1)(n+2)(n+3)(2n+1)} \right) \\
&= \frac{1}{2} + 2\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)!} \left( -\frac{1/2}{n+1} + \frac{1/3}{n+2} - \frac{1/10}{n+3} + \frac{8/15}{2n+1} \right).
\end{aligned}$$

By using the familiar identities from Propositions A6 and A15, the last series converges to

$$\begin{aligned}
\lim_{m\to\infty} \frac{B_m}{A_m} &= \frac{1}{2} + 2\left( -\frac{1}{2}G_1\left(\frac{1}{2}\right) + \frac{1}{3}G_2\left(\frac{1}{2}\right) - \frac{1}{10}G_3\left(\frac{1}{2}\right) + \frac{8}{15}F_1\left(\frac{1}{2}\right) \right) \\
&= \frac{1}{2} - \left( \pi - \frac{\pi^2}{8} \right) + \frac{2}{3}\left( \frac{5\pi}{2} - \frac{3\pi^2}{8} - 3 \right) - \frac{1}{5}\left( 6\pi - \frac{15\pi^2}{16} - \frac{35}{4} \right) + \frac{16}{15}\left( \frac{\pi}{2} \right) \\
&= \frac{1}{2} + \frac{1}{16}\left( \pi^2 - 4 \right) = \frac{\pi^2+4}{16}.
\end{aligned}$$

Finally, the continued fraction (48) converges to $\lim\limits_{m\to\infty} \frac{A_m}{B_m} = \frac{16}{\pi^2+4}$, providing evident verification that Conjecture 1 is valid. Therefore, the proof is complete. □

**Conjecture 2** (unproven, listed in [13])**.** *Next, we address the unproven conjecture (identity) of continued fractions generated by the Ramanujan Machine. The MITM-RF algorithm is*

$$\frac{16}{\pi^2-8} = 9 + \frac{-9}{23\,+}\,\frac{-96}{43\,+}\,\frac{-375}{69}\,+\cdots+\,\frac{-m(m+2)^2(2m-1)}{m(3m+11)+9}\,+\cdots. \tag{49}$$

**Proof of Conjecture 2.** Take $a_m = -m(m+2)^2(2m-1)$, $m \geq 1$ and $b_m = m(3m+11)+9$, $m \geq 0$ so that sequences $c_m$ and $d_m$ which satisfy the relation (45) are $c_m = \frac{m(m+2)(m^2+5m-2)}{m^2+7m+4}$ and $d_m = \frac{(m+2)(2m-1)(m^2+7m+4)}{m^2+5m-2}$ for $m \geq 1$. We denote the continued fraction (48) by

$$\frac{A_m}{B_m} := 9 + \frac{-9}{23\,+}\,\frac{-96}{43\,+}\,\frac{-375}{69}\,+\cdots+\,\frac{-m(m+2)^2(2m-1)}{m(3m+11)+9}.$$

Next, we employ (46) from Theorem 10 to consider the inverse continued fraction:

$$\begin{aligned}
\frac{B_m}{A_m} &= \sum_{n=0}^{m} \frac{(n^2+7n+4)}{(n^2+9n+12)(n+3)(2n+1)} \prod_{k=1}^{n} \frac{(k^2+5k-2)^2(k)(k+2)}{(k^2+7k+4)^2(k+2)(2k-1)} \\
&= \sum_{n=0}^{m} \frac{(n^2+7n+4)}{(n^2+9n+12)(n+3)(2n+1)} \left( \prod_{k=1}^{n} \frac{(k^2+5k-2)^2}{(k^2+7k+4)^2} \cdot \prod_{k=1}^{n} \frac{k}{2k-1} \right) \\
&= \sum_{n=0}^{m} \frac{(n^2+7n+4)}{(n^2+9n+12)(n+3)(2n+1)} \left( \frac{4^2}{(n^2+7n+4)^2} \cdot \frac{n!}{(2n-1)!!}\frac{2^n n!}{(2n)!!} \right) \\
&= \sum_{n=0}^{m} \frac{8(n!)^2(2^n)}{(2n+1)!(n+3)(n+4)} \left( \frac{2(n+4)}{(n^2+7n+4)(n^2+9n+12)} \right)
\end{aligned}$$

Through the telescoping series, considering the above expression $\frac{B_m}{A_m}$ as $m \to \infty$, we have

$$\begin{aligned}
\lim_{m\to\infty} \frac{B_m}{A_m} &= \sum_{n=0}^{\infty} \frac{8(n!)^2(2^n)}{(2n+1)!(n+3)(n+4)} \left( \frac{1}{n^2+7n+4} - \frac{1}{n^2+9n+12} \right) \\
&= \frac{1}{6} + 8\sum_{n=1}^{\infty} \left( \frac{(n!)^2(2^n)}{(2n+1)!(n+4)} - \frac{((n-1)!)^2(2^{n-1})}{(2n-1)!(n+2)} \right) \left( \frac{1}{(n+3)(n^2+7n+4)} \right) \\
&= \frac{1}{6} - 8\sum_{n=1}^{\infty} \left( \frac{((n-1)!)^2(2^n)(n)}{(n+2)(n+3)(n+4)(2n+1)!} \right) \\
&= \frac{1}{6} - 8\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \left( \frac{2^n}{(n+3)(n+4)(n+5)(2n+1)(2n+3)} \right) \\
&= \frac{1}{6} - 8\sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)!} \left( \frac{1/30}{n+3} - \frac{1/35}{n+4} + \frac{1/126}{n+5} + \frac{4/315}{2n+1} - \frac{4/105}{2n+3} \right).
\end{aligned}$$

By using the familiar identities from Propositions A6 and A15, the last series converges to

$$\begin{aligned}
\lim_{m\to\infty} \frac{B_m}{A_m} &= \frac{1}{6} - 8\left( \frac{1}{30} G_3\left(\frac{1}{2}\right) - \frac{1}{35} G_4\left(\frac{1}{2}\right) + \frac{1}{126} G_5\left(\frac{1}{2}\right) + \frac{4}{315} F_1\left(\frac{1}{2}\right) - \frac{4}{105} F_2\left(\frac{1}{2}\right) \right) \\
&= \frac{1}{6} - \frac{4}{15}\left( -\frac{35}{4} + 6\pi - \frac{15\pi^2}{16} \right) + \frac{8}{35}\left( -\frac{763}{36} + \frac{83\pi}{6} - \frac{35\pi^2}{16} \right) \\
&\quad - \frac{4}{63}\left( -\frac{193}{4} + 31\pi - \frac{315\pi^2}{64} \right) - \frac{32}{315}\left( \frac{\pi}{2} \right) + \frac{32}{105}\left( \frac{3\pi}{2} - 4 \right) \\
&= \frac{1}{6} + \frac{1}{48}\left( 3\pi^2 - 32 \right) = \frac{\pi^2-8}{16}.
\end{aligned}$$

Finally, the continued fraction (49) converges to $\lim_{m\to\infty} \frac{A_m}{B_m} = \frac{16}{\pi^2-8}$, providing evident verification that Conjecture 2 is valid. Therefore, the proof is complete. □

**Conjecture 3** (unproven, listed in [13])**.** *The unproven conjecture (identity) of continued fractions generated by the Ramanujan Machine. The MITM-RF algorithm is*

$$\frac{32}{\pi^2} = 3 + \frac{3}{13+} \frac{16}{29+} \frac{-135}{51} + \cdots + \frac{-m^2(m+2)(2m-3)}{m(3m+7)+3} + \cdots. \tag{50}$$

**Proof of Conjecture 3.** Take $a_m = -m^2(m+2)(2m-3)$, $m \geq 1$ and $b_m = m(3m+7)+3$, $m \geq 0$ so that the sequences $c_m = \frac{m^2(m^2+5m-10)}{m^2+7m-4}$ and $d_m = \frac{(2m-3)(m+2)(m^2+7m-4)}{m^2+5m-10}$ for $m \geq 1$ are corresponding to (45). We denote the continued fraction (48) by

$$\frac{A_m}{B_m} := 3 + \frac{3}{13+} \frac{16}{29+} \frac{-135}{51} + \cdots + \frac{-m^2(m+2)(2m-3)}{m(3m+7)+3}.$$

Next, we employ (46) from Theorem 10 to consider the inverse continued fraction:

$$\begin{aligned}
\frac{B_m}{A_m} &= \sum_{n=0}^{m} \frac{(n^2+7n-4)}{(2n-1)(n+3)(n^2+9n+4)} \prod_{k=1}^{n} \frac{k^2(k^2+5k-10)^2}{(2k-3)(k+2)(k^2+7k-4)^2} \\
&= \sum_{n=0}^{m} \frac{(n^2+7n-4)}{(2n-1)(n+3)(n^2+9n+4)} \left( \prod_{k=1}^{n} \frac{(k^2+5k-10)^2}{(k^2+7k-4)^2} \cdot \prod_{k=1}^{n} \frac{k}{k+2} \cdot \prod_{k=1}^{n} \frac{k}{2k-3} \right) \\
&= \sum_{n=0}^{m} \frac{(n^2+7n-4)}{(n+3)(n^2+9n+4)} \left( \frac{(-4)^2}{(n^2+7n-4)^2} \cdot \frac{2}{(n+1)(n+2)} \cdot \frac{-n!}{(2n-1)!!} \frac{2^n n!}{(2n)!!} \right) \\
&= \sum_{n=0}^{m} \frac{(n!)^2}{(2n)!} \frac{-16(2^n)}{(n+1)(n+2)(n+3)(n+4)} \left( \frac{2(n+4)}{(n^2+9n+4)(n^2+7n-4)} \right).
\end{aligned}$$

Let $\langle x \rangle_n = x(x+1)(x+2)\ldots(x+n-1)$ be the Pochhammer symbol used for simplifying the above expression. Through the telescoping series, the limit $\frac{B_m}{A_m}$ as $m \to \infty$ converges to

$$\begin{aligned}
\lim_{m\to\infty} \frac{B_m}{A_m} &= \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{-16(2^n)}{\langle n+1 \rangle_4} \left( \frac{1}{n^2+7n-4} - \frac{1}{n^2+9n+4} \right) \\
&= \frac{1}{6} - 16 \sum_{n=1}^{\infty} \left( \frac{(n!)^2(2^n)}{(2n)!\langle n+1 \rangle_4} - \frac{((n-1)!)^2(2^{n-1})}{(2n-2)!\langle n \rangle_4} \right) \left( \frac{1}{n^2+7n-4} \right) \\
&= \frac{1}{6} + 16 \sum_{n=1}^{\infty} \left( \frac{((n-1)!)^2(2^n)}{(2n)!(n+1)(n+2)(n+3)(n+4)} \right) \\
&= \frac{1}{6} + 16 \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \left( \frac{2^n}{(n+1)(n+2)(n+3)(n+4)(n+5)(2n+1)} \right) \\
&= \frac{1}{6} + 16 \sum_{n=0}^{\infty} \frac{(n!)^2 2^n}{(2n)!} \left( -\frac{1/24}{n+1} + \frac{1/18}{n+2} - \frac{1/20}{n+3} + \frac{1/42}{n+4} - \frac{1/216}{n+5} + \frac{32/945}{2n+1} \right).
\end{aligned}$$

By using the familiar identities from Propositions A6 and A15, the last series converges to

$$\begin{aligned}
\lim_{m\to\infty} \frac{B_m}{A_m} &= \frac{1}{6} + 16 \left( -\frac{G_1(\frac{1}{2})}{24} + \frac{G_2(\frac{1}{2})}{18} - \frac{G_3(\frac{1}{2})}{20} + \frac{G_4(\frac{1}{2})}{42} - \frac{G_5(\frac{1}{2})}{216} + \frac{32F_1(\frac{1}{2})}{945} \right) \\
&= \frac{1}{6} - \frac{2}{3} \left( \pi - \frac{\pi^2}{8} \right) + \frac{8}{9} \left( -3 + \frac{5\pi}{2} - \frac{3\pi^2}{8} \right) - \frac{4}{5} \left( -\frac{35}{4} + 6\pi - \frac{15\pi^2}{16} \right) \\
&\quad + \frac{8}{21} \left( \frac{83\pi}{6} - \frac{763}{36} - \frac{35\pi^2}{16} \right) - \frac{2}{27} \left( 31\pi - \frac{193}{4} - \frac{315\pi^2}{64} \right) + \frac{512}{945} \left( \frac{\pi}{2} \right) = \frac{\pi^2}{32}.
\end{aligned}$$

Finally, the continued fraction (50) converges to $\lim_{m\to\infty} \frac{A_m}{B_m} = \frac{32}{\pi^2}$, providing evident verification that Conjecture 3 is valid. Therefore, the proof is complete. □

## 5. Conclusions

This paper has explored the intersection of Apéry-like series and fractional calculus, leveraging the flexibility of fractional operators to derive closed-form expressions for these series rigorously. Our research addresses the gap in the availability of closed-form expressions for Apéry-like series by focusing on three specific types of functions and providing the conditions for their convergence. Despite the inherent limitations of computational approaches, our results mitigate these challenges and demonstrate their practical applicability. Additionally, we apply our findings to verify three unproven conjectures discovered by the Ramanujan Machine, illustrating the versatility and impact of our work. For future work, we plan to extend our study to more general classes of Apéry-like series involved with the following functions: $f_n(x) = \frac{(4x)^n}{(2n+a)^2}$, $f_n(x) = \frac{(4x)^n}{(n+b)^2}$ and $f_n(x) = \frac{(4x)^n}{an^2+bn+c}$. This will enable us to further explore the potential of our methods and to prove other conjectures such as

$$\frac{16+3\pi^2}{16-\pi^2} = 7 + \frac{8}{19+}\,\frac{-27}{37+}\,\frac{-192}{61}+\cdots+\frac{-n^3(2n-5)}{(n-1)(3n+6)+7+\cdots}$$

which needs the explicit solution of $f_n(x) = \frac{(4x)^n}{3n^2+23n+10}$ to complete the proof.

As a final remark, one can observe that a conjecture can be generated from a linear combination of the Apéry-like series. This raises a common question: which linear combination of the Apéry-like series satisfies the system (45) and ensures the convergence of (47)? For example, the following series is not provided by the Ramanujan Machine

$$\frac{4-\pi}{2} = \sum_{n=1}^{\infty} \frac{(n!)^2\,2^n}{(2n)!}\left(\frac{1}{n}-\frac{2}{2n+1}\right)$$

which satisfies Theorem 10 with $a_m = -(m+1)(2m+3)$ and $b_m = 3(m+1)$ for all $m \geq 1$. This observation may lead to improvements in the algorithm of the Ramanujan Machine.

**Author Contributions:** Conceptualization, A.D., R.B., U.R. and P.S.; methodology, A.D. and P.S.; software, A.D. and P.S.; validation, A.D., R.B., U.R. and P.S.; formal analysis, A.D., R.B., U.R. and P.S.; investigation, A.D., R.B., U.R. and P.S.; writing—original draft preparation, A.D. and P.S.; writing—review and editing, R.B. and U.R.; visualization, A.D. and P.S.; supervision, R.B. and U.R.; funding acquisition, R.B. and U.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** No new data were created or analyzed in this study. Data sharing is not applicable to this article.

**Acknowledgments:** We are grateful for valuable comments and suggestions from anonymous referees that have improved the quality of this paper. This research project is supported by the Second Century Fund (C2F), Chulalongkorn University.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Appendix A. Additional Results of Type I Apéry-like Series $F_m(x)$

In this appendix, we propose the closed-form expressions of the Apéry-like series $F_m(x)$ with $m \in \{1,2,3,4,5\}$ in Propositions A1–A9, where each proposition, derived from Theorems 2 and 4, is specified in $x \in \{-1, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$. These propositions are utilized to be elemental instruments for analyzing or evaluating other complicated infinite summations that are in terms of Apéry-like series $F_m(x)$.

**Proposition A1.** *For $x = -1$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m(-1) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{2n+1} &= \frac{\sqrt{2}\,\mathrm{arcsinh}(1)}{2}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{2n+3} &= 2 - \frac{3\sqrt{2}\,\mathrm{arcsinh}(1)}{2}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{2n+5} &= \frac{11\sqrt{2}\,\mathrm{arcsinh}(1)}{6} - \frac{20}{9}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{2n+7} &= \frac{218}{75} - \frac{23\sqrt{2}\,\mathrm{arcsinh}(1)}{10}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{2n+9} &= \frac{179\sqrt{2}\,\mathrm{arcsinh}(1)}{70} - \frac{11608}{3675}.
\end{aligned}$$

**Proposition A2.** *For $x = \frac{1}{8}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\left(\frac{1}{8}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(2n+1)} &= \frac{8\sqrt{7}}{7} \arcsin\left(\frac{\sqrt{2}}{4}\right), \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(2n+3)} &= \frac{120\sqrt{7}}{7} \arcsin\left(\frac{\sqrt{2}}{4}\right) - 16, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(2n+5)} &= \frac{3832\sqrt{7}}{21} \arcsin\left(\frac{\sqrt{2}}{4}\right) - \frac{1568}{9}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(2n+7)} &= \frac{61304\sqrt{7}}{35} \arcsin\left(\frac{\sqrt{2}}{4}\right) - \frac{125584}{75}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(2n+9)} &= \frac{560488\sqrt{7}}{35} \arcsin\left(\frac{\sqrt{2}}{4}\right) - \frac{56266432}{3675}.
\end{aligned}$$

**Proposition A3.** *For $x = -\frac{1}{8}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\left(-\frac{1}{8}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+1)} &= \frac{4\ln(2)}{3}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+3)} &= 16 - \frac{68\ln(2)}{3}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+5)} &= \frac{724\ln(2)}{3} - \frac{1504}{9}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+7)} &= \frac{120464}{75} - \frac{34756\ln(2)}{15}, \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+9)} &= \frac{2224364\ln(2)}{105} - \frac{53963072}{3675}.
\end{aligned}$$

**Proposition A4.** *For $x = \frac{1}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\left(\frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2n+1} = \frac{2\pi\sqrt{3}}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2n+3} = \frac{14\pi\sqrt{3}}{9} - 8,$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2n+5} = \frac{74\pi\sqrt{3}}{9} - \frac{400}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2n+7} = \frac{1774\pi\sqrt{3}}{45} - \frac{16072}{75},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2n+9} = \frac{56758\pi\sqrt{3}}{315} - \frac{3602528}{3675}.$$

**Proposition A5.** *For $x = -\frac{1}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\left(-\frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ as*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+1} = \frac{4\sqrt{5}\ln(\phi)}{5},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+3} = 8 - \frac{36\sqrt{5}\ln(\phi)}{5},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+5} = \frac{572\sqrt{5}\ln(\phi)}{15} - \frac{368}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+7} = \frac{14792}{75} - \frac{916\sqrt{5}\ln(\phi)}{5},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+9} = \frac{29308\sqrt{5}\ln(\phi)}{35} - \frac{3311008}{3675}.$$

**Proposition A6.** *For $x = \frac{1}{2}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+1} = \frac{\pi}{2},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+3} = \frac{3\pi}{2} - 4,$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+5} = \frac{23\pi}{6} - \frac{104}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+7} = \frac{91\pi}{10} - \frac{2116}{75},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+9} = \frac{1451\pi}{70} - \frac{238192}{3675}.$$

**Proposition A7.** *For $x = -\frac{1}{2}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\Big(-\frac{1}{2}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{2n+1} = \frac{2\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{2n+3} = 4 - \frac{10\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{2n+5} = \frac{26\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) - \frac{88}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{2n+7} = \frac{1796}{75} - \frac{314\sqrt{3}}{15} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{2n+9} = \frac{5014\sqrt{3}}{105} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) - \frac{199952}{3675}.$$

**Proposition A8.** *For $x = \frac{3}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\Big(\frac{3}{4}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{2n+1} = \frac{4\pi\sqrt{3}}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{2n+3} = \frac{20\pi\sqrt{3}}{27} - \frac{8}{3},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{2n+5} = \frac{284\pi\sqrt{3}}{243} - \frac{16}{3},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{2n+7} = \frac{2164\pi\sqrt{3}}{1215} - \frac{664}{75},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{2n+9} = \frac{67628\pi\sqrt{3}}{25515} - \frac{151136}{11025}.$$

**Proposition A9.** *For $x = -\frac{3}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$F_m\Big(-\frac{3}{4}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{(2n+2m-1)}$$

*using Theorems 2 and 4 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{2n+1} = \frac{4\sqrt{21}}{21} \operatorname{arcsinh}\Big(\frac{\sqrt{3}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{2n+3} = \frac{8}{3} - \frac{44\sqrt{21}}{63} \operatorname{arcsinh}\Big(\frac{\sqrt{3}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{2n+5} = \frac{668\sqrt{21}}{567} \operatorname{arcsinh}\Big(\frac{\sqrt{3}}{2}\Big) - \frac{112}{27},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{2n+7} = \frac{4696}{675} - \frac{5452\sqrt{21}}{2835} \operatorname{arcsinh}\Big(\frac{\sqrt{3}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{2n+9} = \frac{24692\sqrt{21}}{8505} \operatorname{arcsinh}\Big(\frac{\sqrt{3}}{2}\Big) - \frac{1030304}{99225}.$$

## Appendix B. Additional Results of Type II Apéry-like Series $G_m(x)$

In this appendix, we present closed-form expressions for the Apéry-like series of Type II, namely, $G_m(x)$ for $m \in \{1,2,3,4,5\}$ in Propositions A10–A18. The results in each proposition are derived from Theorems 6 and 8, which focus for $x$ in the set $\{-1, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}\}$. These propositions serve as fundamental tools for analyzing or evaluating other complex infinite summations involving the Apéry-like series $G_m(x)$.

**Proposition A10.** *For $x = -1$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m(-1) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{n+1} &= \sqrt{2}\,\text{arcsinh}(1) - \text{arcsinh}^2(1),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{n+2} &= \frac{3}{2} + \frac{3\,\text{arcsinh}^2(1)}{2} - 2\sqrt{2}\,\text{arcsinh}(1),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{n+3} &= -\frac{5}{4} - \frac{15\,\text{arcsinh}^2(1)}{8} + \frac{9\sqrt{2}\,\text{arcsinh}(1)}{4},\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{n+4} &= \frac{133}{72} + \frac{35\,\text{arcsinh}^2(1)}{16} - \frac{67\sqrt{2}\,\text{arcsinh}(1)}{24},\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-4)^n}{n+5} &= -\frac{115}{64} - \frac{315\,\text{arcsinh}^2(1)}{128} + \frac{193\sqrt{2}\,\text{arcsinh}(1)}{64}.
\end{aligned}$$

**Proposition A11.** *For $x = \frac{1}{8}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\left(\frac{1}{8}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(n+1)} &= \frac{16\sqrt{7}}{7} \arcsin\left(\frac{\sqrt{2}}{4}\right) - 8\arcsin^2\left(\frac{\sqrt{2}}{4}\right),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(n+2)} &= -12 + \frac{184\sqrt{7}}{7} \arcsin\left(\frac{\sqrt{2}}{4}\right) - 96\arcsin^2\left(\frac{\sqrt{2}}{4}\right),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(n+3)} &= -125 + \frac{1836\sqrt{7}}{7} \arcsin\left(\frac{\sqrt{2}}{4}\right) - 960\arcsin^2\left(\frac{\sqrt{2}}{4}\right),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(n+4)} &= -\frac{10528}{9} + \frac{51400\sqrt{7}}{21} \arcsin\left(\frac{\sqrt{2}}{4}\right) - 8960\arcsin^2\left(\frac{\sqrt{2}}{4}\right),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{2^n(n+5)} &= -\frac{42121}{4} + \frac{154198\sqrt{7}}{7} \arcsin\left(\frac{\sqrt{2}}{4}\right) - 80640\arcsin^2\left(\frac{\sqrt{2}}{4}\right).
\end{aligned}$$

**Proposition A12.** *For $x = -\frac{1}{8}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\left(-\frac{1}{8}\right) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+1)} &= \frac{8\ln(2)}{3} - 2\ln^2(2),\\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+2)} &= 12 - \frac{100\ln(2)}{3} + 24\ln^2(2),
\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+3)} = -115 + \frac{998\ln(2)}{3} - 240\ln^2(2),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+4)} = \frac{9688}{9} - \frac{9316\ln(2)}{3} + 2240\ln^2(2),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+5)} = -\frac{38743}{4} + \frac{83843\ln(2)}{3} - 20160\ln^2(2).$$

**Proposition A13.** *For $x = \frac{1}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\Big(\frac{1}{4}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+1} = \frac{4\pi\sqrt{3}}{9} - \frac{\pi^2}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+2} = \frac{22\pi\sqrt{3}}{9} - \frac{2\pi^2}{3} - 6,$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+3} = \frac{109\pi\sqrt{3}}{9} - \frac{10\pi^2}{3} - \frac{65}{2},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+4} = \frac{508\pi\sqrt{3}}{9} - \frac{140\pi^2}{9} - \frac{1379}{9},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+5} = \frac{4571\pi\sqrt{3}}{18} - 70\pi^2 - \frac{5525}{8}.$$

**Proposition A14.** *For $x = -\frac{1}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\Big(-\frac{1}{4}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ as*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+1} = \frac{8\sqrt{5}\ln(\phi)}{5} - 4\ln^2(\phi),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+2} = 6 - \frac{52\sqrt{5}\ln(\phi)}{5} + 24\ln^2(\phi),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+3} = -\frac{55}{2} + \frac{258\sqrt{5}\ln(\phi)}{5} - 120\ln^2(\phi),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+4} = \frac{1169}{9} - \frac{3616\sqrt{5}\ln(\phi)}{15} + 560\ln^2(\phi),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+5} = -\frac{4667}{8} + \frac{5423\sqrt{5}\ln(\phi)}{5} - 2520\ln^2(\phi).$$

**Proposition A15.** *For $x = \frac{1}{2}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\Big(\frac{1}{2}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+1} = \pi - \frac{\pi^2}{8},$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+2} = -3 + \frac{5\pi}{2} - \frac{3\pi^2}{8},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+3} = -\frac{35}{4} + 6\pi - \frac{15\pi^2}{16},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+4} = -\frac{763}{36} + \frac{83\pi}{6} - \frac{35\pi^2}{16},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+5} = -\frac{193}{4} + 31\pi - \frac{315\pi^2}{64}.$$

**Proposition A16.** *For $x = -\frac{1}{2}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\Big(-\frac{1}{2}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{n+1} = \frac{4\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) - 2\operatorname{arcsinh}^2\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{n+2} = 3 - \frac{14\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) + 6\operatorname{arcsinh}^2\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{n+3} = -\frac{25}{4} + \frac{34\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) - 15\operatorname{arcsinh}^2\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{n+4} = \frac{553}{36} - \frac{80\sqrt{3}}{3} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) + 35\operatorname{arcsinh}^2\Big(\frac{\sqrt{2}}{2}\Big),$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-2)^n}{n+5} = -34 + \frac{359\sqrt{3}}{6} \operatorname{arcsinh}\Big(\frac{\sqrt{2}}{2}\Big) - \frac{315}{4}\operatorname{arcsinh}^2\Big(\frac{\sqrt{2}}{2}\Big).$$

**Proposition A17.** *For $x = \frac{3}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\Big(\frac{3}{4}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{n+1} = \frac{8\pi\sqrt{3}}{9} - \frac{4\pi^2}{27},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{n+2} = \frac{4\pi\sqrt{3}}{3} - \frac{8\pi^2}{27} - 2,$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{n+3} = 2\pi\sqrt{3} - \frac{40\pi^2}{81} - \frac{25}{6},$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{n+4} = \frac{80\pi\sqrt{3}}{27} - \frac{560\pi^2}{729} - 7,$$
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{n+5} = \frac{13\pi\sqrt{3}}{3} - \frac{280\pi^2}{243} - \frac{87}{8}.$$

**Proposition A18.** *For $x = -\frac{3}{4}$ and $m \in \{1,2,3,4,5\}$, the Apéry-like series*

$$G_m\Big(-\frac{3}{4}\Big) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-3)^n}{(n+m)}$$

*using Theorems 6 and 8 can be expressed with the following closed-form formulas:*

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-3)^n}{n+1}=\frac{8\sqrt{21}}{21}\operatorname{arcsinh}\left(\frac{\sqrt{3}}{2}\right)-\frac{4}{3}\operatorname{arcsinh}^2\left(\frac{\sqrt{3}}{2}\right),$$
$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-3)^n}{n+2}=2-\frac{20\sqrt{21}}{21}\operatorname{arcsinh}\left(\frac{\sqrt{3}}{2}\right)+\frac{8}{3}\operatorname{arcsinh}^2\left(\frac{\sqrt{3}}{2}\right),$$
$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-3)^n}{n+3}=-\frac{5}{2}+\frac{94\sqrt{21}}{63}\operatorname{arcsinh}\left(\frac{\sqrt{3}}{2}\right)-\frac{40}{9}\operatorname{arcsinh}^2\left(\frac{\sqrt{3}}{2}\right),$$
$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-3)^n}{n+4}=\frac{119}{27}-\frac{1352\sqrt{21}}{567}\operatorname{arcsinh}\left(\frac{\sqrt{3}}{2}\right)+\frac{560}{81}\operatorname{arcsinh}^2\left(\frac{\sqrt{3}}{2}\right),$$
$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-3)^n}{n+5}=-\frac{449}{72}+\frac{667\sqrt{21}}{189}\operatorname{arcsinh}\left(\frac{\sqrt{3}}{2}\right)-\frac{280}{27}\operatorname{arcsinh}^2\left(\frac{\sqrt{3}}{2}\right).$$

## References


1. Borwein, J.M.; Straub, A.; Wan, J.; Zudilin, W.; Zagier, D. Densities of short uniform random walks. *Can. J. Math.* **2012**, *64*, 961–990. [CrossRef]
2. Laohakosol, V.; Meesa, R.; Sutthimat, P. Closed-form solutions of general second order linear recurrences and applications. *Discret. Math.* **2023**, *346*, 113206. [CrossRef]
3. Sutthimat, P.; Laohakosol, V.; Meesa, R. A unified approach to derive explicit solutions of generalized second-order linear recurrences and applications. *Discret. Math.* **2024**, *347*, 113757. [CrossRef]
4. Chaichana, T.; Laohakosol, V.; Meesa, R. Derivation of some identities and applications: Annual Meeting in Mathematics 2023. *Thai J. Math.* **2024**, *22*, 263–269.
5. Straub, A.; Zudilin, W. Positivity of rational functions and their diagonals. *J. Approx. Theory* **2015**, *195*, 57–69. [CrossRef]
6. Straub, A. Multivariate Apéry numbers and supercongruences of rational functions. *Algebra Number Theory* **2014**, *8*, 1985–2008. [CrossRef]
7. Rogers, M.D.; Straub, A. A solution of Sun's $520 challenge concerning. *Int. J. Number Theory* **2013**, *9*, 1273–1288. [CrossRef]
8. Berndt, B.C.; Lamphere, R.L.; Wilson, B. Chapter 12 of Ramanujan's second notebook: Continued fractions. *Rocky Mt. J. Math.* **1985**, *15*, 235–310. [CrossRef]
9. Van der Poorten, A.J. Some wonderful formulae... Footnotes to Apery's proof of the irrationality of $\zeta(3)$. *Séminaire-Delange-Pisot-Poitou. Théorie Nr.* **1978**, *20*, 7.
10. Zucker, I. On the series $\sum_{k=1}^{\infty}\binom{2k}{k}^{-1}k^{-n}$ and related sums. *J. Number Theory* **1985**, *20*, 92–102. [CrossRef]
11. Sherman, T. Summation of Glaisher- and Apéry-like Series. Available online: https://api.semanticscholar.org/CorpusID:146282789 (accessed on 25 February 2024).
12. The Ramanujan Machine. Available online: www.ramanujanmachine.com (accessed on 25 February 2024).
13. Raayoni, G.; Gottlieb, S.; Pisha, G.; Harris, Y.; Manor, Y.; Mendlovic, U.; Haviv, D.; Hadad, Y.; Kaminer, I. The Ramanujan Machine: Automatically generated conjectures on fundamental constants. *arXiv* **2019**, arXiv:1907.00205.
14. Raayoni, G.; Gottlieb, S.; Manor, Y.; Pisha, G.; Harris, Y.; Mendlovic, U.; Haviv, D.; Hadad, Y.; Kaminer, I. Generating conjectures on fundamental constants with the Ramanujan Machine. *Nature* **2021**, *590*, 67–73. [CrossRef] [PubMed]
15. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: London, UK, 1993.
16. Oldham, K.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*; Elsevier: Amsterdam, The Netherlands, 1974.
17. Yang, X.J.; Gao, F.; Ju, Y. Chapter 2—Fractional derivatives with singular kernels. In *General Fractional Derivatives with Applications in Viscoelasticity*; Academic Press: Cambridge, MA, USA, 2020; pp. 95–207.
18. Boonklurb, R.; Duangpan, A.; Rakwongwan, U.; Sutthimat, P. A novel analytical formula for the discounted moments of the ECIR process and interest rate swaps pricing. *Fractal Fract.* **2022**, *6*, 58. [CrossRef]
19. Duangpan, A.; Boonklurb, R.; Juytai, M. Numerical solutions for systems of fractional and classical integro-differential equations via Finite Integration Method based on shifted Chebyshev polynomials. *Fractal Fract.* **2021**, *5*, 103. [CrossRef]
20. Arfken, G.B.; Weber, H.J.; Harris, F.E. Chapter 13—Gamma Function. In *Mathematical Methods for Physicists*, 7th ed.; Academic Press: Boston, MA, USA, 2013; pp. 599–641.
21. Krug, A. *Theorie der Derivationen*; Kaiserlich-Königliche Hof-und Staatsdruckerei: Vienna, Austria, 1890.
22. Jungić, V.; Burazinb, A.; Lovrić, M. Wallis $m$-integrals and their properties. *SCIENTIA Ser. Math. Sci.* **2023**, *33*, 61–72.
23. Berndt, B.C. *Ramanujan's Notebooks: Part I*; Springer: New York, NY, USA, 1985.
24. Hofmann, J.E. On the discovery of the logarithmic series and its development in England up to Cotes. *Natl. Math. Mag.* **1939**, *14*, 37–45. [CrossRef]
25. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*; Academic Press: Boston, MA, USA, 2014.
26. Szeg, G. *Orthogonal Polynomials*; American Mathematical Society: Providence, RI, USA, 1939; Volume 23.

27. Jayarama, P.; Lim, D.; Rathie, A.K. On several results associated with the Apéry-like series. *Symmetry* **2023**, *15*, 1022. [CrossRef]
28. Jones, W.B.; Thron, W.J. *Continued Fractions: Analytic Theory and Applications*; Encyclopedia of Mathematics and Its Applications; Addison-Wesley Publishing Co.: Carrollton, TX, USA, 1980.
29. Sutthimat, P.; Mekchay, K. Closed-form formulas for conditional moments of inhomogeneous Pearson diffusion processes. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *106*, 106095. [CrossRef]
30. Sutthimat, P.; Mekchay, K.; Rujivan, S. Closed-form formula for conditional moments of generalized nonlinear drift CEV process. *Appl. Math. Comput.* **2022**, *428*, 127213. [CrossRef]