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Fractional Hermite–Hadamard–Mercer-Type Inequalities for Interval-Valued Convex Stochastic Processes with Center-Radius Order and Their Related Applications in Entropy and Information Theory

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Abstract: We propose a new definition of the γ-convex stochastic processes ($\mathfrak{C}\mathcal{SP}$) using center and radius ($\mathcal{CR}$) order with the notion of interval valued functions ($^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$). By utilizing this definition and Mean-Square Fractional Integrals, we generalize fractional Hermite–Hadamard–Mercer-type inclusions for generalized $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$ versions of convex, tgs-convex, P-convex, exponential-type convex, Godunova–Levin convex, s-convex, Godunova–Levin s-convex, h-convex, n-polynomial convex, and fractional n-polynomial ($\mathfrak{C}\mathcal{SP}$). Also, our work uses interesting examples of $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}(\mathfrak{C}\mathcal{SP})$ with Python-programmed graphs to validate our findings using an extension of Mercer's inclusions with applications related to entropy and information theory.

Keywords: Hermite–Hadamard; Jensen–Mercer inclusions; interval-valued functions; mean-square fractional integral; γ-convexity; interval-valued stochastic γ-convexity with center-radius order relation

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1. Introduction

Interval analysis is a useful tool for dealing with problems involving uncertainty. The credit it deserved did not come until Moore's [1] seminal application of interval analysis for automated error analysis (even though it had roots in Archimedes' p-calculation). It has been extended to interval-valued and fuzzy-valued functions by numerous researchers, such as Costa et al. [2], who established integral inequalities for fuzzy-interval-valued functions; Flores-Franuli et al. [3], who introduced integral inequalities for interval-valued functions; and Chalco-Cano et al. [4], with their Ostrowski-type inequalities and applications in numerical integration for interval-valued functions. The integral inequality was demonstrated by Zhao et al. [5] by using an interval-valued h-convex function and the interval inclusion relation. Since the same comparison of intervals may not be applicable in all situations, the intriguing and challenging milestone of determining a sensible order to investigate inequality problems involving interval-valued functions is hard to deal with. Using the ($\mathcal{CR}$) of the interval, Bhunia et al. [6] computed the $\mathcal{C}.\mathcal{R}$-order in 2014. This new ordering relationship is a combination of the mean and scaled difference of the end-points of an interval, respectively.

Stochastic processes ($\mathcal{SP}$) significantly escalate the training of neural networks, optimizing energy perspectives and modeling complex processes with each possible combination. Whether implemented via stochastic control, stochastic computing, or generative models, these connections increase the efficiency of both the entropy and neural networks. Modern research has linked the smoothing of energy landscapes in neural networks to classical work in stochastic control. Adding stochastic elements, such as randomness or noise, to the training process can modify and improve the optimization process and more effectively escape local minima [7]. Stochastic control theory allows us to identify complex energy surfaces, making it relevant for understanding the training dynamics of neural networks. $\mathcal{SP}s$ are also applicable in fault detection, as CNNs (Convolutional Neural Networks) can learn to create patterns for identifying anomalies or faults in industrial automation systems. Several neural networks, such as ANNs (Artificial Neural Networks) and SNNs (Stochastic Neural Networks), can also model stochastic variations in data visualizations, helping with tasks like image recognition and segmentation [8].

In optimization and information theory, convex and non-convex functions have a significant impact. Also, convexity and $\mathcal{SP}s$ are connected closely. The theory of convexity plays a foundational role in many fields of science, such as modern mathematics and analysis. The pivotal relationship between the theory of inequalities and the theory of convexity has forced many researchers to explore several classical inequalities, which were discussed for $(\mathfrak{CF})$s and have also been generalized for other extensions of $(\mathfrak{CF})$s.

Jensen's inequality [9], Mercer's inequality [10], and Hermite–Hadamard's inequality [11] using $(\mathfrak{CF})$s are some of the most praised and celebrated inequalities in different areas of mathematics and optimization. In [12], Fejér inequality is provided, which is the weighted extension of the Hermite–Hadamard inequality. Jensen and Mercer inequalities are essential for investigating bounds for entropies. In this paper, by generalizing Mercer's inequality, we explore some approaches to Shannon's entropy since entropy and $\mathcal{SP}s$ are used in finance, signal processing, and neuroscience.

The idea of convexity for $(\mathcal{SP})$s has recently attracted much attention because of its applications in numerical estimations, optimal designs, and optimization. In 1974, Nagy [13] applied a characterization of measurable $(\mathcal{SP})$s for solving a generalization of the (additive) Cauchy functional equation. In 1980, Nikodem [14] introduced convex $(\mathcal{SP})$s and explored their regularity properties. In 1992, Skowronski [15] provided some interesting remarks on convex $(\mathcal{SP})$s that extended some famous $(\mathfrak{CF})$s. Pales discussed more nonconvex mappings' characteristics and power means in [16]. Kotrys presented a modern extension of the Hermite–Hadamard inequality in [17] using convex $(\mathcal{SP})$s. In [18], Saleem explored h-convex $(\mathcal{SP})$s. In [19], Işcan investigated the p-convex $(\mathcal{SP})$s. In [20], Maden introduced s-convex $(\mathcal{SP})$s in the first sense. In [21], Set proposed s-convex $(\mathcal{SP})$ in the second sense. In [22], Fu discussed the n-polynomial convex $(\mathcal{SP})$.

Rahman et al. [23] were the first to introduce the idea of $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$-$(\mathfrak{CF})$s, which paved the way for studies of generalized inequality types such as Hermite–Hadamard's, Jensen's, Mercer's, Schur's, Fejér and Pachpatte's. Vivas-Cortez et al. [24] recently provided fractional inequalities that pertain to generalizations of $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}\gamma$-$(\mathfrak{CF})$s with interval values. Harmonical $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}(h_1, h_2)$-Godunova–Levin functions were the focus of Sen et al. [25], whereas Botmart et al. [26] expanded on this class by studying the $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$ order relation.

Merging the concepts mentioned above and especially given by [24,27,28], we explore the properties of γ-convex $(\mathcal{SP})$s, a new generalization of $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$ functions, and use them to find modern integral inclusions like Hermite–Hadamard's, Jensen's, and Mercer's extending over fractional integrals.

In the future, one can extend this field by using generalized harmonically $^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$-$(\mathfrak{C}\mathcal{SP})$s using stochastic integrals, quantum integrals, and post-quantum integrals for exciting applications. This paper is organized as follows: First, we will provide some background information about our research. Then, in Section 3, we will describe our main results, and in Section 4, we will explain our work's applications.

2. Preliminaries

First, we recall notions from $\mathfrak{CF}s$.

Definition 1 ([9]). *Let* $\mathsf{S} : [\mathsf{U}, \mathsf{V}] \to \Re$; *then,* S *is said to be* $(\mathfrak{CF})$ *if* $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ *being a convex subset of* $\Re$ *and* $\mathsf{U}_1 < \mathsf{V}_1$, $\mathsf{N} \in [0,1]$,

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1-\mathsf{N})\mathsf{V}_1) \le \mathsf{N}\mathsf{S}(\mathsf{U}_1) + (1-\mathsf{N})\mathsf{S}(\mathsf{V}_1). \tag{1}$$

Hermite and Hadamard's inequality is among the most famous and frequently utilized [11]. An example of a popular phrasing for this inequality is as follows.

Let $\mathsf{S} : [\mathsf{U}, \mathsf{V}] \subseteq \Re \to \Re$ be a $(\mathfrak{CF})$ with $\mathsf{U}_1 < \mathsf{V}_1$. Then, if $\forall\ \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$,

$$\mathsf{S}\left(\frac{\mathsf{U}_1 + \mathsf{V}_1}{2}\right) \le \frac{1}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{n}) dn \le \frac{\mathsf{S}(\mathsf{V}_1) + \mathsf{S}(\mathsf{U}_1)}{2}. \tag{2}$$

Given Jensen's inequality [9], which is based on the same assumption, which may be expressed in the same way as Hermite and Hadamard's inequality, for any $\Sigma_{i=1}^{\mathsf{w}} \mathsf{z}_i = 1$ where $\mathsf{z}_i \ge 0$,

$$\mathsf{S}\left(\Sigma_{i=1}^{\mathsf{w}} \mathsf{z}_i \mathsf{n}_i\right) \le \Sigma_{i=1}^{\mathsf{w}} \mathsf{z}_i \mathsf{S}\left(\mathsf{n}_i\right). \tag{3}$$

Definition 2 ([24]). *Consider* $\gamma : [0,1] \to \Re^+$. $\mathsf{S} : [\mathsf{U}, \mathsf{V}] \to \Re^+$ *is said to be* γ-$(\mathfrak{CF})$, *denoted as* $\mathsf{S} \in SX(\gamma, [\mathsf{U}, \mathsf{V}], \Re^+)$, *if* $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ *and* $\mathsf{N} \in [0,1]$, *then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1-\mathsf{N})\mathsf{V}_1) \le \mathsf{N}\gamma(\mathsf{N})\mathsf{S}(\mathsf{U}_1) + (1-\mathsf{N})\gamma(1-\mathsf{N})\mathsf{S}(\mathsf{V}_1). \tag{4}$$

In (4), if "$\le$" is interchanged with "$\ge$", then becomes a γ-concave function or $\mathsf{S} \in \Im_{\aleph}^{\pi}(\gamma, [\mathsf{U}, \mathsf{V}], \Re^+)$.

If the function $\gamma : (0,1) \to (0, \infty)$ *satisfies the following inequality,*

$$\gamma(\mathsf{m}\mathsf{z}) \ge \gamma(\mathsf{m})\gamma(\mathsf{z}). \tag{5}$$

for all $\mathsf{m}, \mathsf{z} \in [0,1]$, *then* γ *is said to be super-multiplicative. If the sign in inequality (5) is replaced by* $\le$, *then* γ *is considered sub-multiplicative.*

Definition 3 ([29]). $\mathsf{S} : [\mathsf{U}, \mathsf{V}] \to \Re^+$ *is said to be n-polynomial* $(\mathfrak{CF})$, *denoted as* $\mathsf{S} \in SX$ $(n, [\mathsf{U}, \mathsf{V}], \Re^+)$, *if* $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ *and* $\mathsf{N} \in [0,1]$, *then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1-\mathsf{N})\mathsf{V}_1) \le \frac{1}{\aleph}\left[\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{N})^{\Im})\mathsf{S}(\mathsf{U}_1) + \Sigma_{\Im=1}^{\aleph}(1-\mathsf{N}^{\Im})\mathsf{S}(\mathsf{V}_1)\right]. \tag{6}$$

In (6), if "$\le$" is interchanged with "$\ge$", then it becomes a n-polynomial concave function or $\mathsf{S} \in \Im_{\aleph}^{\pi}(n, [\mathsf{U}, \mathsf{V}], \Re^+)$.

Definition 4 ([10]). $\mathsf{S} : [\mathsf{U}, \mathsf{V}] \to \Re^+$ *is said to be fractional n-polynomial* $(\mathfrak{CF})$, *denoted as* $\mathsf{S} \in SX\left(n_q^p, [\mathsf{U}, \mathsf{V}], \Re^+\right)$, *if* $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ *and* $\mathsf{N} \in [0,1]$, *then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1-\mathsf{N})\mathsf{V}_1) \le \frac{1}{\aleph}\left[\Sigma_{\Im=1}^{\aleph}(\mathsf{N}^{\frac{1}{\Im}})\mathsf{S}(\mathsf{U}_1) + \Sigma_{\Im=1}^{\aleph}(1-\mathsf{N})^{\frac{1}{\Im}}\mathsf{S}(\mathsf{V}_1)\right]. \tag{7}$$

In (7), if "$\le$" is interchanged with "$\ge$", then it becomes a fractional n-polynomial concave function or $\mathsf{S} \in \Im_{\aleph}^{\pi}\left(n_q^p, [\mathsf{U}, \mathsf{V}], \Re^+\right)$.

To avoid mistakes that could lead to erroneous findings, interval analysis uses interval variables instead of point variables and displays computing results as intervals. Moore released his first book on interval analysis in 1966 [30]. Additionally, interval arithmetic is thoroughly covered in [1]. The set S of all real numbers with real values that are both closed and bounded is said to be an interval. The definition is,

$$\mathsf{S} = \left[\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}\right] = \left\{x \in \Re : \mathsf{S}_{\circledast} \leq x \leq \mathsf{S}^{\circledast}\right\}.$$

where $\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast} \in \Re$ and $\mathsf{S}_{\circledast} < \mathsf{S}^{\circledast}$.

On the left side of an interval S, we have $\mathsf{S}_{\circledast}$, and on the right side, we have $\mathsf{S}^{\circledast}$. If the absolute value of $\mathsf{S}_{\circledast}$ is greater than zero, then the interval $[\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}]$ is non-negative. We represent the sets of all closed intervals as $\Re_I$ and closed intervals that are positive of the real numbers as $\Re_I^+$, respectively.

A $(\mathcal{CR})$ or total order was applied to an interval provided in the following form by Bhunia et al. [6]:

$$\mathsf{S} = \langle \mathsf{S}_{\mathfrak{C}}, \mathsf{S}_{\mathfrak{R}} \rangle = \left\langle \frac{\mathsf{S}_{\circledast} + \mathsf{S}^{\circledast}}{2}, \frac{\mathsf{S}^{\circledast} - \mathsf{S}_{\circledast}}{2} \right\rangle$$

This is the relation between two intervals that is known as the $(\mathcal{CR})$ order or total order:

Definition 5 ([26])**.** *For any two intervals,* $\mathsf{S} = [\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}] = \langle \mathsf{S}_{\mathfrak{C}}, \mathsf{S}_{\mathfrak{R}} \rangle$ *and* $\mathsf{T} = [\mathsf{T}_{\circledast}, \mathsf{T}^{\circledast}] = \langle \mathsf{T}_{\mathfrak{C}}, \mathsf{T}_{\mathfrak{R}} \rangle$*, we describe the $\mathcal{C}.\mathcal{R}.$-order relation as follows:*

$$\mathsf{S} \preceq_{\mathcal{C}.\mathcal{R}.} \mathsf{T} \Leftrightarrow \begin{cases} \mathsf{S}_{\mathfrak{C}} < \mathsf{T}_{\mathfrak{C}}, & \text{if} \quad \mathsf{S}_{\mathfrak{C}} \neq \mathsf{T}_{\mathfrak{C}} \\ \mathsf{S}_{\mathfrak{R}} \leq \mathsf{T}_{\mathfrak{R}}, & \text{if} \quad \mathsf{S}_{\mathfrak{C}} = \mathsf{T}_{\mathfrak{C}} \end{cases}$$

So, for any two given intervals $\mathsf{S}, \mathsf{T} \in \Re_I$*, either* $\mathsf{S} \preceq_{\mathcal{C}.\mathcal{R}.} \mathsf{T}$ *or* $\mathsf{T} \preceq_{\mathcal{C}.\mathcal{R}.} \mathsf{S}$.

Definition 6. *For* $\mathsf{L} \in \Re$*, Minkowski addition and scalar multiplication are defined by [26],*

$$\mathsf{S} + \mathsf{T} = [\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}] + [\mathsf{T}_{\circledast}, \mathsf{T}^{\circledast}] = [\mathsf{S}_{\circledast} + \mathsf{T}_{\circledast}, \mathsf{S}^{\circledast} + \mathsf{T}^{\circledast}];$$

$$\mathsf{L}.\mathsf{S} = \mathsf{L}.\left[\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}\right] = \begin{cases} [\mathsf{L}\mathsf{S}_{\circledast}, \mathsf{L}\mathsf{S}^{\circledast}], & \mathsf{L} > 0, \\ \{0\}, & \mathsf{L} = 0, \\ [\mathsf{L}\mathsf{S}^{\circledast}, \mathsf{L}\mathsf{S}_{\circledast}], & \mathsf{L} < 0. \end{cases}$$

Moore et al. [1] were the first to introduce the concept of the Riemann integral for functions in the $\frac{\mathcal{I}}{\mathcal{V}}$ domain. Let $\mathfrak{IR}([\mathsf{U}_1, \mathsf{V}_1])$ and $\mathfrak{R}([\mathsf{U}_1, \mathsf{V}_1])$ denote the sets of all Riemann integrable $\frac{\mathcal{I}}{\mathcal{V}}$ and real-valued functions on $[\mathsf{U}_1, \mathsf{V}_1]$, respectively. The following outcome clarifies the connection between Riemann integrable $(\mathfrak{R})$-integrable functions and $(\mathfrak{IR})$-integrable functions.

Theorem 1 ([1])**.** *Suppose* $\mathsf{S} : [\mathsf{U}_1, \mathsf{V}_1] \to \Re_I$ *be an* $\frac{\mathcal{I}}{\mathcal{V}}$ *function, where* $\mathsf{S}(\mathsf{m}) = [\mathsf{S}_{\circledast}(\mathsf{m}), \mathsf{S}^{\circledast}(\mathsf{m})]$, $\mathsf{S} \in \mathfrak{IR}([\mathsf{U}_1, \mathsf{V}_1])$ *iff* $\mathsf{S}_{\circledast}(\mathsf{m}), \mathsf{S}^{\circledast}(\mathsf{m}) \in \mathfrak{R}([\mathsf{U}_1, \mathsf{V}_1])$*,*

$$(\mathfrak{IR}) \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{m})d\mathsf{m} = \left[(\mathfrak{R}) \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}_{\circledast}(\mathsf{m})d\mathsf{m}, (\mathfrak{R}) \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}^{\circledast}(\mathsf{m})d\mathsf{m}\right],$$

In their discussion of the order preservation property of integrals incorporating $\mathcal{C}.\mathcal{R}.$ order, Shi et al. [5] provided the following outcome.

Theorem 2. *Suppose* $\mathsf{S}, \mathsf{M} : [\mathsf{U}_1, \mathsf{V}_1] \to \Re_I^+$ *are two* $\frac{\mathcal{I}}{\mathcal{V}}$ *functions, where* $\mathsf{S}(\mathsf{n}) = [\mathsf{S}_{\circledast}(\mathsf{n}), \mathsf{S}^{\circledast}(\mathsf{n})]$, $\mathsf{M}(\mathsf{n}) = [\mathsf{M}_{\circledast}(\mathsf{n}), \mathsf{M}^{\circledast}(\mathsf{n})]$*. If* $\mathsf{S}, \mathsf{M} \in \mathfrak{IR}[\mathsf{U}_1, \mathsf{V}_1]$ *and* $\mathsf{S}(\mathsf{n}) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \mathsf{M}(\mathsf{n})$*, then*

$$\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{n})d\mathsf{n} \preceq_{\mathfrak{r}}^{\mathfrak{c}} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{M}(\mathsf{n})d\mathsf{n}.$$

Now, we recall notions from $\mathcal{SP}$s.

Definition 7 ([31])**.** *Consider $(\Omega, \mathsf{B}, \mathsf{Q})$ to be any probability space. A mapping $\mathsf{S} : \mathsf{I} \times \Omega \to \Re$ is called a random variable when it is B-measurable. A mapping $\mathsf{S} : \mathsf{I} \times \Re$ is called a stochastic process $(\mathcal{SP})$ when each $\mathsf{V} \in \mathsf{I}$, the mapping $\mathsf{S}(\mathsf{V}, .)$ is a random variable, having $\mathsf{I} \subseteq \Re$ being an interval.*

The $(\mathcal{SP})$ S is referred to as follows:

- *Stochastically continuous on I, if*

$$\mathsf{N}_1 - \lim_{\mathsf{V} \to \mathsf{V}_0} \mathsf{S}(\mathsf{V}, .) = \mathsf{S}(\mathsf{V}_0, .),$$

for every $\mathsf{V}_0 \in \mathsf{I}$, where $\mathsf{N}_1 - \lim$ shows the limit of probability.

- *Mean square continuous on I, if*

$$\lim_{\mathsf{V} \to \mathsf{V}_0} \mathsf{W}\big[\mathsf{S}(\mathsf{V}, .) - \mathsf{S}(\mathsf{V}_0, .)\big]^2 = 0,$$

for every $\mathsf{V}_0 \in \mathsf{I}$, where $\mathsf{W}[\mathsf{S}(\mathsf{V}, .)]$ shows the value of the expectation related to the random variable $\mathsf{S}(\mathsf{V}, .)$.

Now, we define our main definition, motivated by the works of [24].

Definition 8. *Consider $\gamma : [0, 1] \to \Re^+$. $\mathsf{S} : \mathsf{I} \times \Omega \to \Re^+$ is said to be γ-convex $(\mathcal{SP})$, denoted as $\mathsf{S} \in SX(\gamma, I, \Re^+)$, if $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ and $\mathsf{N} \in [0, 1]$, then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1 - \mathsf{N})\mathsf{V}_1, .) \leq \mathsf{N}\gamma(\mathsf{N})\mathsf{S}(\mathsf{U}_1, .) + (1 - \mathsf{N})\gamma(1 - \mathsf{N})\mathsf{S}(\mathsf{V}_1, .). \tag{8}$$

In (8), if "$\leq$" is interchanged with "$\geq$", then becomes γ-concave $(\mathcal{SP})$ or $\mathsf{S} \in \Im^{\pi}_{\aleph}(\gamma, I, \Re^+)$.

Motivated by works from [10,29], we introduce the following.

Definition 9. *$\mathsf{S} : \mathsf{I} \times \Omega \to \Re^+$ is said to be an n-polynomial convex $(\mathcal{SP})$, denoted as $\mathsf{S} \in \Im^{\alpha}_{\aleph}(n, I, \Re^+)$, if $\forall \mathsf{U}_1, \mathsf{V}_1 \in \mathsf{I}$ and $\mathsf{N} \in [0, 1]$, then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1 - \mathsf{N})\mathsf{V}_1, .) \leq \frac{1}{\aleph}\left[\Sigma^{\aleph}_{\Im=1}(1 - (1 - \mathsf{N})^{\Im})\mathsf{S}(\mathsf{U}_1, .) + \Sigma^{\aleph}_{\Im=1}(1 - \mathsf{N}^{\Im})\mathsf{S}(\mathsf{V}_1, .)\right]. \tag{9}$$

In (9), if "$\leq$" is interchanged with "$\geq$", then becomes n-polynomial concave $(\mathcal{SP})$ or $\mathsf{S} \in \Im^{\pi}_{\aleph}(n, I, \Re^+)$.

Definition 10. *$\mathsf{S} : \mathsf{I} \times \Omega \to \Re^+$ is said to be a fractional n-polynomial convex $(\mathcal{SP})$, denoted as $\mathsf{S} \in \Im^{\alpha}_{\aleph}\left(n^p_q, I, \Re^+\right)$, if $\forall \mathsf{U}_1, \mathsf{V}_1 \in \mathsf{I}$ and $\mathsf{N} \in [0, 1]$, then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1 - \mathsf{N})\mathsf{V}_1, .) \leq \frac{1}{\aleph}\left[\Sigma^{\aleph}_{\Im=1}(\mathsf{N}^{\frac{1}{\Im}})\mathsf{S}(\mathsf{U}_1, .) + \Sigma^{\aleph}_{\Im=1}(1 - \mathsf{N})^{\frac{1}{\Im}}\mathsf{S}(\mathsf{V}_1, .)\right]. \tag{10}$$

In (10), if "$\leq$" is interchanged with "$\geq$", then it becomes fractional n-polynomial concave $(\mathcal{SP})$ or $\mathsf{S} \in \Im^{\pi}_{\aleph}\left(n^p_q, I, \Re^+\right)$.

Now, we recall notions from $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ $\mathfrak{CF}$s.

Definition 11 ([24])**.** *Consider $\gamma : [0, 1] \to \Re^+$. $\mathsf{S} = [\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}] : [\mathsf{U}, \mathsf{V}] \to \Re^+_{\mathbf{I}}$ is said to be $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma$-$(\mathfrak{CF})$, denoted as $\mathsf{S} \in SX(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$, if $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ and $\mathsf{N} \in [0, 1]$, then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1 - \mathsf{N})\mathsf{V}_1) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{N}\gamma(\mathsf{N})\mathsf{S}(\mathsf{U}_1) + (1 - \mathsf{N})\gamma(1 - \mathsf{N})\mathsf{S}(\mathsf{V}_1). \tag{11}$$

In (11), if "$\preceq^{\mathfrak{c}}_{\mathfrak{r}}$" is interchanged with "$\succeq^{\mathfrak{c}}_{\mathfrak{r}}$", then it becomes a $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma$ concave function or $\mathsf{S} \in \Im^{\pi}_{\aleph}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$.

Motivated by the works from [10,29], we introduce the following.

Definition 12. $S = [S_{\circledast}, S^{\circledast}] : [U, V] \to \Re_{\mathbf{I}}^{+}$ *is said to be* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-n-polynomial* $(\mathfrak{CF})$*, denoted as* $S \in SX(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n, [U, V], \Re_{\mathbf{I}}^{+})$*, if* $\forall U_1, V_1 \in [U, V]$ *and* $N \in [0, 1]$*, then*

$$S(NU_1 + (1 - N)V_1) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\aleph}\left[\Sigma_{\Im=1}^{\aleph}(1 - (1 - N)^{\Im})S(U_1) + \Sigma_{\Im=1}^{\aleph}(1 - N^{\Im})S(V_1)\right]. \quad (12)$$

In (12), if "$\preceq_{\mathfrak{r}}^{\mathfrak{c}}$*" is interchanged with "*$\succeq_{\mathfrak{r}}^{\mathfrak{c}}$*", then it becomes a* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-n-polynomial concave function or* $S \in \Im_{\aleph}^{\pi}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n, [U, V], \Re_{\mathbf{I}}^{+})$*.*

Definition 13. $S = [S_{\circledast}, S^{\circledast}] : [U, V] \to \Re_{\mathbf{I}}^{+}$ *is said to be* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$*, denoted as* $S \in SX\left(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n_q^p, [U, V], \Re_{\mathbf{I}}^{+}\right)$*, if* $\forall U_1, V_1 \in [U, V]$ *and* $N \in [0, 1]$*, then*

$$S(NU_1 + (1 - N)V_1) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\aleph}\left[\Sigma_{\Im=1}^{\aleph}(N^{\frac{1}{\Im}})S(U_1) + \Sigma_{\Im=1}^{\aleph}(1 - N)^{\frac{1}{\Im}}S(V_1)\right]. \quad (13)$$

In (13), if "$\preceq_{\mathfrak{r}}^{\mathfrak{c}}$*" is interchanged with "*$\succeq_{\mathfrak{r}}^{\mathfrak{c}}$*", then it becomes a* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial concave function or* $S \in \Im_{\aleph}^{\pi}\left(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n_q^p, [U, V], \Re_{\mathbf{I}}^{+}\right)$*.*

Now let us introduce the concept for γ-convex $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ function using the works of [24].

Definition 14. *Consider* $\gamma : [0, 1] \to \Re^{+}$*.* $(\mathcal{SP})$ $S = [S_{\circledast}, S^{\circledast}] : I \times \Omega \to \Re_{\mathbf{I}}^{+}$ *where* $[U, V] \subseteq I$ *is said to be* γ*-convex* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *denoted as* $S \in \Im_{\aleph}^{\alpha}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, [U, V], \Re_{\mathbf{I}}^{+})$*, if* $\forall U_1, V_1 \in [U, V]$ *and* $N \in [0, 1]$*, then*

$$S(NU_1 + (1 - N)V_1, .) \preceq_{\mathfrak{r}}^{\mathfrak{c}} N\gamma(N)S(U_1, .) + (1 - N)\gamma(1 - N)S(V_1, .) \quad (14)$$

In (14), if "$\preceq_{\mathfrak{r}}^{\mathfrak{c}}$*" is interchanged with "*$\succeq_{\mathfrak{r}}^{\mathfrak{c}}$*", then it becomes* γ*-concave* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *or* $S \in \Im_{\aleph}^{\pi}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, [U, V], \Re_{\mathbf{I}}^{+})$*.*

- If $\gamma(x) = 1$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-$(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = (1 - x)$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-tgs-$(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = \frac{1}{x}$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-P $(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = \frac{(e^x - 1)}{x}$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ exponential-type $(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = \frac{1}{x^2}$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-Godunova–Levin $(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = x^{s-1}$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-s-$(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = x^{-s-1}$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-Godunova–Levin s-$(\mathfrak{C}\mathcal{SP})$.
- If $h(x) = x\gamma(x)$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-h-$(\mathfrak{C}\mathcal{SP})$[27].
- If $\gamma(x) = \frac{1}{x\aleph}\Sigma_{\Im=1}^{\aleph}(1 - (1 - x)^{\Im})$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ n-polynomial $(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = \frac{1}{x\aleph}\Sigma_{\Im=1}^{\aleph}(x^{\frac{1}{\Im}})$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ fractional n-polynomial $(\mathfrak{C}\mathcal{SP})$.
- If $\gamma(x) = \frac{1}{x\aleph}\frac{\Sigma_{\Im=1}^{\aleph}M_{\Im}(1-(1-x)^{\Im})}{\Sigma_{\Im=1}^{\aleph}M_{\Im}}$, (14) gives an $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ generalized n-polynomial $(\mathfrak{C}\mathcal{SP})$.

Similarly, we introduce the following notions for $(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}\ \mathcal{SP}s)$, using the works of [10,29].

Definition 15. $(\mathcal{SP})$ $S = [S_{\circledast}, S^{\circledast}] : I \times \Omega \to \Re_{\mathbf{I}}^{+}$ *where* $[U, V] \subseteq I$ *is said to be n-polynomial convex* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *denoted as* $S \in \Im_{\aleph}^{\alpha}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n, [U, V], \Re_{\mathbf{I}}^{+})$*, if* $\forall U_1, V_1 \in [U, V]$ *and* $N \in [0, 1]$*, then*

$$S(NU_1 + (1 - N)V_1, .) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\aleph}\left[\Sigma_{\Im=1}^{\aleph}(1 - (1 - N)^{\Im})S(U_1, .) + \Sigma_{\Im=1}^{\aleph}(1 - N^{\Im})S(V_1, .)\right]. \quad (15)$$

In (15), if "$\preceq_{\mathfrak{r}}^{\mathfrak{c}}$*" is interchanged with "*$\succeq_{\mathfrak{r}}^{\mathfrak{c}}$*", then it becomes n-polynomial concave* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *or* $S \in \Im_{\aleph}^{\pi}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n, [U, V], \Re_{\mathbf{I}}^{+})$*.*

Definition 16. $(\mathcal{SP})$ $\mathsf{S} = [\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}] : \mathsf{I} \times \Omega \to \Re_{\mathbf{I}}^{+}$ *where* $[\mathsf{U}, \mathsf{V}] \subseteq \mathsf{I}$ *is said to be fractional n-polynomial convex* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *denoted as* $\mathsf{S} \in \Im_{\aleph}^{\alpha}\left(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n_q^p, [\mathsf{U}, \mathsf{V}], \Re_{\mathbf{I}}^{+}\right)$*, if* $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ *and* $\mathsf{N} \in [0, 1]$*, then*

$$\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1-\mathsf{N})\mathsf{V}_1, .) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\aleph}\left[\Sigma_{\Im=1}^{\aleph}(\mathsf{N}^{\frac{1}{\Im}})\mathsf{S}(\mathsf{U}_1, .) + \Sigma_{\Im=1}^{\aleph}(1-\mathsf{N})^{\frac{1}{\Im}}\mathsf{S}(\mathsf{V}_1, .)\right]. \tag{16}$$

In (16), if "$\preceq_{\mathfrak{r}}^{\mathfrak{c}}$" is interchanged with "$\succeq_{\mathfrak{r}}^{\mathfrak{c}}$", then it becomes fractional n-polynomial concave $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *or* $\mathsf{S} \in \Im_{\aleph}^{\pi}\left(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n_q^p, [\mathsf{U}, \mathsf{V}], \Re_{\mathbf{I}}^{+}\right)$.

Motivated by the works from [32], we introduce the following.

Definition 17. $(\mathcal{SP})$ $\mathsf{S} = [\mathsf{S}_{\circledast}, \mathsf{S}^{\circledast}] : \mathsf{I} \times \Omega \to \Re_{\mathbf{I}}^{+}$ *where* $[\mathsf{U}, \mathsf{V}] \subseteq \mathsf{I}$ *is said to be generalized n-polynomial convex* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *denoted as* $\mathsf{S} \in \Im_{\aleph}^{\alpha}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n_{\mathsf{M}}, [\mathsf{U}, \mathsf{V}], \Re_{\mathbf{I}}^{+})$*, if* $\forall \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}]$ *and* $\mathsf{N} \in [0, 1]$*, then* $\mathsf{M}_{\Im} \geq 0$, $\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im} > 0$,

$$\begin{aligned}&\mathsf{S}(\mathsf{N}\mathsf{U}_1 + (1-\mathsf{N})\mathsf{V}_1, .)\\ &\preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\aleph}\left[\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{N})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}\mathsf{S}(\mathsf{U}_1, .) + \frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-\mathsf{N}^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}\mathsf{S}(\mathsf{V}_1, .)\right].\end{aligned} \tag{17}$$

In (17), if "$\preceq_{\mathfrak{r}}^{\mathfrak{c}}$" is interchanged with "$\succeq_{\mathfrak{r}}^{\mathfrak{c}}$", then it becomes generalized n-polynomial concave $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *or* $\mathsf{S} \in \Im_{\aleph}^{\pi}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - n_{\mathsf{M}}, [\mathsf{U}, \mathsf{V}], \Re_{\mathbf{I}}^{+})$.

Readers can see some recent works related to the interval order relation [33], Kulisch and Miranker-type inclusions for generalized classes of stochastic processes [34], and the center radius order relation [35] for further study, respectively [36–57].

3. Main Results

In this section, we will prove the results related to Jensen, Mercer, Hermite–Hadamard, and a fractional variant of Hermite–Hadamard inclusion, respectively.

3.1. Jensen-Type Inclusion

Theorem 3. *Let* $n_{\Im} \in \Re^{+}$*. If* γ *is a super multiplicative non-negative function and* $\mathsf{S} : \mathsf{I} \times \Omega \to \Re$ *is non-negative* γ*-convex* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *or we term as* $\mathsf{S} \in \Im_{\aleph}^{\alpha}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, I, \Re_{\mathfrak{I}}^{+})$ *with* $z_{\Im} \in \mathsf{I}$*, then the following holds:*

$$\mathsf{S}\left(\frac{1}{\mathsf{M}_{\aleph}}\sum_{\Im=1}^{\aleph} n_{\Im} z_{\Im}, .\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \sum_{\Im=1}^{\aleph}\left[\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}(z_{\Im}, .)\right]. \tag{18}$$

where $\mathsf{M}_{\aleph} = \Sigma_{\Im=1}^{\aleph} n_{\Im}$.

Proof. By mathematical induction, when $\aleph = 2$, then (18) is true. Suppose that (18) holds for $\aleph - 1$, then,

$$\begin{aligned}\mathsf{S}\left(\frac{1}{\mathsf{M}_{\aleph}}\sum_{\Im=1}^{\aleph} n_{\Im} z_{\Im}, .\right) &= \mathsf{S}\left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}} z_{\aleph} + \sum_{\Im=1}^{\aleph-1}\frac{n_{\Im}}{\mathsf{M}_{\aleph}} z_{\Im}, .\right),\\ &= \mathsf{S}\left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}} z_{\aleph} + \frac{\mathsf{M}_{\aleph-1}}{\mathsf{M}_{\aleph}}\sum_{\Im=1}^{\aleph-1}\frac{n_{\Im}}{\mathsf{M}_{\aleph-1}} z_{\Im}, .\right),\end{aligned}$$

$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}(z_{\aleph}, .) + \left(\frac{\mathsf{M}_{\aleph-1}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{\mathsf{M}_{\aleph-1}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}\left(\sum_{\Im=1}^{\aleph-1}\frac{n_{\Im}}{\mathsf{M}_{\aleph-1}} z_{\Im}, .\right),$$

$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}(z_{\aleph}, .) + \left(\frac{\mathsf{M}_{\aleph-1}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{\mathsf{M}_{\aleph-1}}{\mathsf{M}_{\aleph}}\right)\sum_{\Im=1}^{\aleph-1}\left[\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph-1}}\right)\gamma\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph-1}}\right)\mathsf{S}(z_{\Im}, .)\right],$$

$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{n_{\aleph}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}(z_{\aleph},.) + \sum_{\Im=1}^{\aleph-1}\left[\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}(z_{\Im},.)\right],$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \sum_{\Im=1}^{\aleph}\left[\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph}}\right)\gamma\left(\frac{n_{\Im}}{\mathsf{M}_{\aleph}}\right)\mathsf{S}(z_{\Im},.)\right].$$

Hence, it is proved by mathematical induction. □

Corollary 1.

- *If* $\gamma(\mathsf{x}) = 1$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = (1-\mathsf{x})$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-tgs-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}}$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-P* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}^2}$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{s-1}$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-s-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{-s-1}$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$.
- *If* $h(\mathsf{x}) = \mathsf{x}\gamma(\mathsf{x})$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-h-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(18) gives a Jensen-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *generalized n-polynomial* $(\mathfrak{CF})$.

3.2. Mercer-Type Inclusion

An extension of the Jensen inequality, given by Sahoo [31], is as follows.

Theorem 4 (Lemma 2.3)**.** *[31] If* S *being a convex* $(\mathcal{SP})$ *on interval* $\mathsf{I}\times\Omega := [\mathsf{U},\mathsf{V}]\times\Omega$, $n_{\Im}\in\mathsf{I}, 1\le\Im\le\aleph$ *and* $\sum_{\Im=1}^{\aleph} z_{\Im} = 1$, *then*

$$\mathsf{M}_{\mathsf{S}}(n,z) = \mathsf{S}\left(\mathsf{U}+\mathsf{V}-\sum_{\Im=1}^{\aleph} z_{\Im}n_{\Im},.\right) + \sum_{\Im=1}^{\aleph} z_{\Im}\mathsf{S}(n_{\Im},.) \le \mathsf{S}(\mathsf{U},.)+\mathsf{S}(\mathsf{V},.). \tag{19}$$

Theorem 5. *Let* S *be a* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ γ*-convex* $(\mathcal{SP})$ *on* $[\mathsf{U},\mathsf{V}]\times\Omega, \mathsf{N}_1,\ldots,\mathsf{N}_{\aleph}\in[\mathsf{U},\mathsf{V}]$ *and* $\frac{\mathsf{N}_1+\ldots+\mathsf{N}_{\aleph}}{\aleph} = \frac{\mathsf{U}+\mathsf{V}}{2}$, *Thus, the following holds:*

$$\frac{1}{\gamma\left(\frac{1}{\aleph}\right)}\mathsf{S}\left(\frac{\mathsf{U}+\mathsf{V}}{2},.\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{\sum_{\Im=1}^{\aleph}\mathsf{S}(\mathsf{N}_{\Im},.)}{\aleph} \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{\gamma(\frac{1}{2})}{2}[\mathsf{S}(\mathsf{U},.)+\mathsf{S}(\mathsf{V},.)]. \tag{20}$$

Proof. Since $\frac{\mathsf{N}_1+\ldots+\mathsf{N}_{\aleph}}{\aleph} = \frac{\mathsf{U}+\mathsf{V}}{2}$, the first inclusion is easy to achieve from Jensen's inclusion:

$$\mathsf{S}\left(\frac{\mathsf{U}+\mathsf{V}}{2},.\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{S}\left(\frac{\mathsf{N}_1+\ldots+\mathsf{N}_{\aleph}}{\aleph},.\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{\gamma(\frac{1}{\aleph})}{\aleph}\sum_{\Im=1}^{\aleph}\mathsf{S}(\mathsf{N}_{\Im},.),$$

The second inclusion is achieved as given in the following. Since $\mathsf{N}_{\Im}\in[\mathsf{U},\mathsf{V}]$, there is a sequence $\{\mathsf{A}_{\Im}\}_{\Im=1}^{\aleph}, \mathsf{A}_{\Im}\in[0,1]$ such that $\mathsf{N}_{\Im} = \mathsf{A}_{\Im}\mathsf{U}+(1-\mathsf{A}_{\Im})\mathsf{V}$. On the other hand, since $\frac{\mathsf{N}_1+\ldots+\mathsf{N}_{\aleph}}{\aleph} = \frac{\mathsf{U}+\mathsf{V}}{2}$, we have

$$\sum_{\Im=1}^{\aleph}\frac{\mathsf{A}_{\Im}\gamma(\mathsf{A}_{\Im})}{\aleph} = \sum_{\Im=1}^{\aleph}\frac{(1-\mathsf{A}_{\Im})\gamma(1-\mathsf{A}_{\Im})}{\aleph} = \frac{1}{2}\gamma\left(\frac{1}{2}\right),$$

So,

$$\frac{\sum_{\Im=1}^{\aleph} \mathsf{S}(\mathsf{N}_{\Im},.)}{\aleph} = \sum_{\Im=1}^{\aleph} \frac{\mathsf{S}(\mathsf{A}_{\Im}\mathsf{U} + (1-\mathsf{A}_{\Im})\mathsf{V},.)}{\aleph}$$
$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \sum_{\Im=1}^{\aleph} \frac{\mathsf{A}_{\Im}\gamma(\mathsf{A}_{\Im})\mathsf{S}(\mathsf{U},.) + (1-\mathsf{A}_{\Im})\gamma(1-\mathsf{A}_{\Im})\mathsf{S}(\mathsf{V},.)}{\aleph}$$
$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{\gamma(\frac{1}{2})}{2}[\mathsf{S}(\mathsf{U},.) + \mathsf{S}(\mathsf{V},.)].$$

□

Remark 1. *For* $\aleph = 2$ *in (20), then*

$$\mathsf{S}(\mathsf{N}_1,.) + \mathsf{S}(\mathsf{N}_2,.) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \gamma\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{U},.) + \mathsf{S}(\mathsf{V},.)]. \tag{21}$$

Remark 2. *Let* $\mathsf{S} : (0,\infty)$ *be a* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *γ-convex* $(\mathcal{SP})$ *and* $\frac{1+2+...+\aleph}{\aleph} = \frac{1+\aleph}{2}$. *Thus, (20) implies,*

$$\frac{1}{\gamma\left(\frac{1}{\aleph}\right)}\mathsf{S}\left(\frac{1+\aleph}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{\sum_{\Im=1}^{\aleph} \mathsf{S}(\Im,.)}{\aleph} \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{\gamma(\frac{1}{2})}{2}[\mathsf{S}(1,.) + \mathsf{S}(\aleph,.)]. \tag{22}$$

Remark 3. *Let* S *be an* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-convex* $(\mathcal{SP})$ *on* $[\mathsf{U},\mathsf{V}]$ *and let* $\mathsf{N} \in [\mathsf{U},\mathsf{V}]$, $\mathsf{N}_1 = \frac{\mathsf{U}+\mathsf{N}}{2}$, $\mathsf{N}_2 = \frac{\mathsf{V}+\mathsf{N}}{2}$ *and* $\mathsf{N}_3 = \mathsf{U} + \mathsf{V} - \mathsf{N}$. *Then,* $\frac{\mathsf{N}_1+\mathsf{N}_2+\mathsf{N}_3}{3} = \frac{\mathsf{U}+\mathsf{V}}{2}$*; hence, by the use of (20), we revive the Hermite–Hadamard inequality after integrating over* $[\mathsf{U},\mathsf{V}]$ *with respect to* N*:*

$$\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)d\mathsf{N} \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{[\mathsf{S}(\mathsf{U}_1,.) + \mathsf{S}(\mathsf{V}_1,.)]}{2}$$

Corollary 2.

- *If* $\gamma(\mathsf{x}) = 1$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = (1-\mathsf{x})$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-tgs-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}}$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-P* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}^2}$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{s-1}$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-s-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{-s-1}$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$.
- *If* $h(\mathsf{x}) = \mathsf{x}\gamma(\mathsf{x})$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-h-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(20) gives a Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *generalized n-polynomial* $(\mathfrak{CF})$.

3.3. Hermite–Hadamard-Type Inclusion

Theorem 6. *Let $\gamma : (0,1) \to \Re^+$ and $\gamma\left(\frac{1}{2}\right) \neq 0$. Suppose mapping $\mathsf{S} : \mathsf{I} \times \Omega \to \Re^+_{\mathbf{I}}$ is γ-convex $(\mathcal{SP})$ and also mean square integrable for $\frac{\mathcal{I.V}}{\mathcal{C.R}}$. $\forall$ $\mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}] \subseteq \mathsf{I}$, if $\mathsf{S} \in \Im^{\alpha}_{\aleph}(\frac{\mathcal{I.V}}{\mathcal{C.R}} - \gamma, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$ and $\mathsf{S} \in \Re^+_{\mathbf{I}}$. Then, we obtain the following result:*

$$\frac{1}{\gamma\left(\frac{1}{2}\right)} \mathsf{S}\left(\frac{\mathsf{U}_1 + \mathsf{V}_1}{2}, .\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N}, .) d\mathsf{N}$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} [\mathsf{S}(\mathsf{U}_1, .) + \mathsf{S}(\mathsf{V}_1, .)] \int_0^1 s\gamma(s) ds. \tag{23}$$

Proof. Since $\mathsf{S} \in \Im^{\alpha}_{\aleph}(\frac{\mathcal{I.V}}{\mathcal{C.R}} - \gamma, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$, and integrating over $(0,1)$, we have

$$\frac{2}{\left[\gamma\left(\frac{1}{2}\right)\right]} \mathsf{S}\left(\frac{\mathsf{U}_1 + \mathsf{V}_1}{2}, .\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{S}(s\mathsf{U}_1 + (1-s)\mathsf{V}_1, .) + \mathsf{S}((1-s)\mathsf{U}_1 + s\mathsf{V}_1, .).$$

$$\begin{aligned}
&\frac{2}{\left[\gamma\left(\frac{1}{2}\right)\right]} \mathsf{S}\left(\frac{\mathsf{U}_1 + \mathsf{V}_1}{2}, .\right) \\
&\preceq^{\mathfrak{c}}_{\mathfrak{r}} \left[\int_0^1 \mathsf{S}(s\mathsf{U}_1 + (1-s)\mathsf{V}_1, .) ds + \int_0^1 \mathsf{S}((1-s)\mathsf{U}_1 + s\mathsf{V}_1, .) ds\right] \\
&= [\int_0^1 \mathsf{S}_{\circledast}(s\mathsf{U}_1 + (1-s)\mathsf{V}_1, .) ds + \int_0^1 \mathsf{S}_{\circledast}((1-s)\mathsf{U}_1 + s\mathsf{V}_1, .) ds \\
&\quad , \int_0^1 \mathsf{S}^{\circledast}(s\mathsf{U}_1 + (1-s)\mathsf{V}_1, .) ds + \int_0^1 \mathsf{S}^{\circledast}((1-s)\mathsf{U}_1 + s\mathsf{V}_1, .) ds\Big] \\
&\quad = \left[\frac{2}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}_{\circledast}(\mathsf{N}, .) d\mathsf{N}, \frac{2}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}^{\circledast}(\mathsf{N}, .) d\mathsf{N}\right] \\
&\quad = \frac{2}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N}, .) d\mathsf{N}.
\end{aligned}$$

$$\frac{1}{\left[\gamma\left(\frac{1}{2}\right)\right]} \mathsf{S}\left(\frac{\mathsf{U}_1 + \mathsf{V}_1}{2}, .\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N}, .) d\mathsf{N}. \tag{24}$$

By (14), we can obtain,

$$\mathsf{S}(s\mathsf{U}_1 + (1-s)\mathsf{V}_1, .) \preceq^{\mathfrak{c}}_{\mathfrak{r}} s\gamma(s)\mathsf{S}(\mathsf{U}_1, .) + (1-s)\gamma(1-s)\mathsf{S}(\mathsf{V}_1, .).$$

Integrating over $(0,1)$, we can obtain,

$$\int_0^1 \mathsf{S}(s\mathsf{U}_1 + (1-s)\mathsf{V}_1, .) ds \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{S}(\mathsf{U}_1, .) \int_0^1 s\gamma(s) ds + \mathsf{S}(\mathsf{V}_1, .) \int_0^1 (1-s)\gamma(1-s) ds.$$

Accordingly,

$$\frac{1}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N}, .) d\mathsf{N} \preceq^{\mathfrak{c}}_{\mathfrak{r}} [\mathsf{S}(\mathsf{U}_1, .) + \mathsf{S}(\mathsf{V}_1, .)] \int_0^1 s\gamma(s) ds. \tag{25}$$

Now, combining (24) and (25), we obtain the required (23). □

Corollary 3.

- *If $\gamma(\mathsf{x}) = 1$, (23) gives a Hermite–Hadamard-type inclusion for $(\mathcal{SP})$ of $\frac{\mathcal{I.V}}{\mathcal{C.R}}$-$(\mathfrak{CF})$.*
- *If $\gamma(\mathsf{x}) = (1 - \mathsf{x})$, (23) gives a Hermite–Hadamard-type inclusion for $(\mathcal{SP})$ of $\frac{\mathcal{I.V}}{\mathcal{C.R}}$-tgs-$(\mathfrak{CF})$.*
- *If $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}}$, (23) gives a Hermite–Hadamard-type inclusion for $(\mathcal{SP})$ of $\frac{\mathcal{I.V}}{\mathcal{C.R}}$-P $(\mathfrak{CF})$.*

- *If* $\gamma(\mathsf{x}) = \frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}^2}$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{s-1}$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-s-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{-s-1}$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$.
- *If* $h(\mathsf{x}) = \mathsf{x}\gamma(\mathsf{x})$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-h-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(23) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *generalized n-polynomial* $(\mathfrak{CF})$.

Example 1. *Consider* $[\mathsf{U}_1, \mathsf{V}_1] = [0,1], \gamma(\mathsf{n}) = 1, \forall \mathsf{n} \in [0,1]$. *If* $\mathsf{S} : [\mathsf{U}_1, \mathsf{V}_1] \times \Omega \to \Re_{\mathbf{I}}^{+}$ *is defined below and plotted using Python programmed graphs (Figure 1),*

$$\mathsf{S}(\mathsf{N}, .) = [-12\mathsf{N}^2 - 1, 4\mathsf{N}^2 + 9] = \langle -4\mathsf{N}^2 + 4, 8\mathsf{N}^2 + 5 \rangle, \quad \mathsf{N} \in [0,1].$$

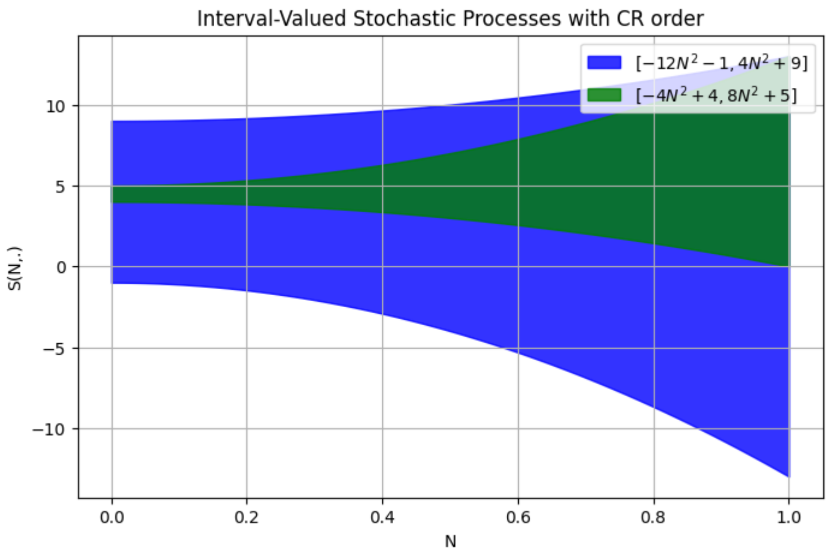

Figure 1. The plot above shows $\frac{\mathcal{I}}{\mathcal{V}}$ $\mathcal{SP}$ with concave and convex ends (blue).

Nevertheless, the newly constructed $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ $\mathcal{SP}$*s demonstrate that the* $\mathcal{SP}$*s at the left and right endpoints (green) are convex when the center and radius order is applied.*

Then,

$$\mathsf{S}\left(\frac{\mathsf{U}_1 + \mathsf{V}_1}{2}, .\right) = \mathsf{S}\left(\frac{1}{2}, .\right) = [3,7],$$

$$\frac{1}{\mathsf{V}_1 - \mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N}, .)d\mathsf{N} = \left[\int_0^1 \left(-4\mathsf{N}^2 + 4\right)d\mathsf{N}, \int_0^1 \left(8\mathsf{N}^2 + 5\right)d\mathsf{N}\right] = \left[\frac{8}{3}, \frac{23}{3}\right],$$

$$[\mathsf{S}(\mathsf{U}_1, .) + \mathsf{S}(\mathsf{V}_1, .)]\int_0^1 n\gamma(n)dn = [2,9].$$

As a result,

$$[3,7] \preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\frac{8}{3}, \frac{23}{3}\right] \preceq_{\mathfrak{r}}^{\mathfrak{c}} [2,9].$$

(Figure 2) demonstrates the newly constructed left (red dotted), middle (blue dotted), and right (green dotted) parts of (23) when substitutions are applied.

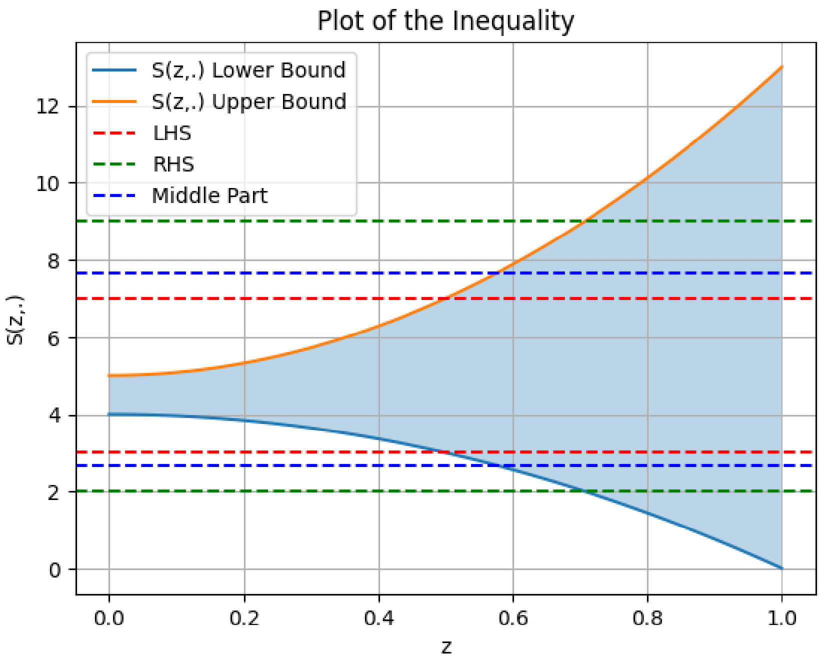

Figure 2. The plot above shows $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ $\mathcal{SP}$ with concave (blue) and convex (yellow) ends.

This verifies the (23).

Theorem 7. *Suppose $\gamma : (0,1) \to \Re^+$ and $\gamma\left(\frac{1}{2}\right) \neq 0$. Suppose mapping $\mathsf{S} : \mathsf{I} \times \Omega \to \Re^+_{\mathbf{I}}$ is γ-convex $(\mathcal{SP})$ and also mean square integrable for $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$. $\forall\ \mathsf{U}_1, \mathsf{V}_1 \in [\mathsf{U}, \mathsf{V}] \subseteq \mathsf{I}$, if $\mathsf{S} \in \Im^{\alpha}_{\aleph}\left(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}}\right)$ and $\mathsf{S} \in \Re^+_{\mathbf{I}}$. Then, we obtain the following result:*

$$\frac{1}{\left[\gamma\left(\frac{1}{2}\right)\right]^2}\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{S}_1 \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)d\mathsf{N}$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{S}_2 \preceq^{\mathfrak{c}}_{\mathfrak{r}} \left\{[\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)]\left[\frac{1}{2}+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right]\right\}\int_0^1 s\gamma(s)ds, \quad (26)$$

where,

$$\mathsf{S}_1 = \frac{1}{2\gamma\left(\frac{1}{2}\right)}\left[\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{4},.\right)+\mathsf{S}\left(\frac{3\mathsf{V}_1+\mathsf{U}_1}{4},.\right)\right],$$
$$\mathsf{S}_2 = \left[\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)+\frac{\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)}{2}\right]\int_0^1 s\gamma(s)ds.$$

Proof. Taking $\left[\mathsf{U}_1, \frac{\mathsf{U}_1+\mathsf{V}_1}{2}\right]$, we have

$$\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{4},.\right)$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{2}\gamma\left(\frac{1}{2}\right)\mathsf{S}\left(s\mathsf{U}_1+(1-s)\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\mathsf{S}\left((1-s)\mathsf{U}_1+s\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right),$$

Integrating over $(0,1)$, we have

$$\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{2}\gamma\left(\frac{1}{2}\right)[\int_0^1 \mathsf{S}\left(s\mathsf{U}_1+(1-s)\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)ds + \int_0^1 \mathsf{S}\left(s\frac{\mathsf{U}_1+\mathsf{V}_1}{2}+(1-s)\mathsf{V}_1,.\right)ds\Big]$$

$$= \frac{1}{2}\gamma\left(\frac{1}{2}\right)\left[\frac{2}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\frac{\mathsf{U}_1+\mathsf{V}_1}{2}} \mathsf{S}(\mathsf{N},.)d\mathsf{N} + \frac{2}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\frac{\mathsf{U}_1+\mathsf{V}_1}{2}} \mathsf{S}(\mathsf{N},.)d\mathsf{N}\right]$$
$$= \frac{1}{2}\gamma\left(\frac{1}{2}\right)\left[\frac{4}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\frac{\mathsf{U}_1+\mathsf{V}_1}{2}} \mathsf{S}(\mathsf{N},.)d\mathsf{N}\right].$$

Accordingly,

$$\frac{1}{2\gamma\left(\frac{1}{2}\right)}\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\frac{\mathsf{U}_1+\mathsf{V}_1}{2}} \mathsf{S}(\mathsf{N},.)d\mathsf{N}, \tag{27}$$

Similarly, for interval $\left[\frac{\mathsf{U}_1+\mathsf{V}_1}{2},\mathsf{V}_1\right]$, we have

$$\frac{1}{2\gamma\left(\frac{1}{2}\right)}\mathsf{S}\left(\frac{3\mathsf{V}_1+\mathsf{U}_1}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\frac{\mathsf{U}_1+\mathsf{V}_1}{2}}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)d\mathsf{N}, \tag{28}$$

Adding inclusions (27) and (28), we obtain

$$\mathsf{S}_1 = \frac{1}{2\gamma\left(\frac{1}{2}\right)}\left[\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{4},.\right)+\mathsf{S}\left(\frac{3\mathsf{V}_1+\mathsf{U}_1}{4},.\right)\right] \preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)d\mathsf{N}\right].$$

Now,

$$\frac{1}{\left[\gamma\left(\frac{1}{2}\right)\right]^2}\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)$$
$$= \frac{1}{\left[\gamma\left(\frac{1}{2}\right)\right]^2}\mathsf{S}\left(\frac{1}{2}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{4},.\right)+\frac{1}{2}\left(\frac{3\mathsf{V}_1+\mathsf{U}_1}{4},.\right)\right),$$
$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{\left[\gamma\left(\frac{1}{2}\right)\right]^2}\left[\frac{1}{2}\gamma\left(\frac{1}{2}\right)\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{4},.\right)+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\mathsf{S}\left(\frac{3\mathsf{V}_1+\mathsf{U}_1}{4},.\right)\right],$$
$$= \frac{1}{2\gamma\left(\frac{1}{2}\right)}\left[\mathsf{S}\left(\frac{3\mathsf{U}_1+\mathsf{V}_1}{4},.\right)+\mathsf{S}\left(\frac{3\mathsf{V}_1+\mathsf{U}_1}{4},.\right)\right],$$
$$= \mathsf{S}_1,$$

$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{2\gamma\left(\frac{1}{2}\right)}\{\frac{1}{2}\gamma\left(\frac{1}{2}\right)\left[\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\right], + \frac{1}{2}\gamma\left(\frac{1}{2}\right)\left[\mathsf{S}(\mathsf{V}_1,.)+\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\right]\},$$
$$= \frac{1}{2}\left[\frac{\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)}{2}+\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\right],$$
$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\frac{\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)}{2}+\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\right]\int_0^1 s\gamma(s)ds,$$

$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \left[\frac{\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)}{2}+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\mathsf{S}(\mathsf{U}_1,.)+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\mathsf{S}(\mathsf{V}_1,.)\right]\int_0^1 s\gamma(s)ds,$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \left[\frac{\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)}{2}+\frac{1}{2}\gamma\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)]\right]\int_0^1 s\gamma(s)ds,$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \left\{[\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)]\left[\frac{1}{2}+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right]\right\}\int_0^1 s\gamma(s)ds.$$

□

Corollary 4.

- *If* $\gamma(\mathsf{x})=1$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-*$(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=(1-\mathsf{x})$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-tgs-*$(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}}$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-P* $(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}^2}$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\mathsf{x}^{s-1}$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-s-*$(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\mathsf{x}^{-s-1}$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$*.*
- *If* $h(\mathsf{x})=\mathsf{x}\gamma(\mathsf{x})$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-h-*$(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$*.*
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(26) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *generalized n-polynomial* $(\mathfrak{CF})$*.*

Example 2. *Recall Example 1, we have*

$$\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)=\mathsf{S}\left(\frac{1}{2},.\right)=[3,7].$$

$$\mathsf{S}_1=\frac{1}{2}\left[\mathsf{S}\left(\frac{1}{4},.\right)+\mathsf{S}\left(\frac{3}{4},.\right)\right]=\left[\frac{11}{4},\frac{15}{2}\right].$$

$$\frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1}\mathsf{S}(\mathsf{N},.)d\mathsf{N}=\left[\frac{8}{3},\frac{23}{3}\right].$$

$$\mathsf{S}_2=\left[\frac{\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)}{2}+\mathsf{S}\left(\frac{1}{2},.\right)\right]\int_0^1 n\gamma(n)dn=\left[\frac{5}{2},8\right].$$

and,

$$\left\{[\mathsf{S}(\mathsf{U}_1,.)+\mathsf{S}(\mathsf{V}_1,.)]\left[\frac{1}{2}+\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right]\right\}\int_0^1 n\gamma(n)dn=[2,9].$$

Thus, we obtain

$$[3,7]\preceq^{\mathfrak{c}}_{\mathfrak{r}}\left[\frac{11}{4},\frac{15}{2}\right]\preceq^{\mathfrak{c}}_{\mathfrak{r}}\left[\frac{8}{3},\frac{23}{3}\right]\preceq^{\mathfrak{c}}_{\mathfrak{r}}\left[\frac{5}{2},8\right]\preceq^{\mathfrak{c}}_{\mathfrak{r}}[2,9].$$

(Figure 3) demonstrates the newly constructed left (red), middle (blue), and right (green) parts of (26) when substitutions are applied.

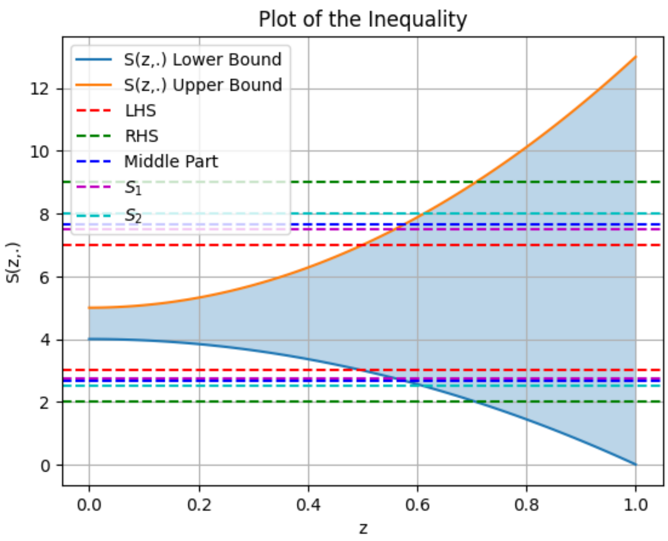

Figure 3. The plot above shows ${}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}\,\mathcal{SP}$ with concave (blue) and convex (yellow) ends.

This verifies (26) with the help of a Python-programmed graph.

Theorem 8. *Let* $\gamma_1, \gamma_2 : (0,1) \to \Re^+$ *and* $\gamma_1, \gamma_2 \neq 0$. *The functions* $\mathsf{S}, \mathsf{M} : \mathsf{I} \times \Omega \to \Re^+_{\mathbf{I}}$ *are* γ *convex* $(\mathcal{SP})$ *and also mean square integrable for* ${}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$. $\forall\, \mathsf{U}_1, \mathsf{V}_1 \in \mathsf{I}$, *if* $\mathsf{S} \in \Im^{\alpha}_{\aleph}({}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}} - \gamma_1, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$, $\mathsf{M} \in \Im^{\alpha}_{\aleph}({}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}} - \gamma_2, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$ *and* $\mathsf{S}, \mathsf{M} \in \mathbf{IR}_I$. *Then, we obtain the following result:*

$$\frac{1}{\mathsf{V}_1 - \mathsf{U}_1} \int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N}, .)\mathsf{M}(\mathsf{N}, .)d\mathsf{N}$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} P(\mathsf{U}_1, \mathsf{V}_1) \int_0^1 s^2 \gamma_1(s)\gamma_2(s)ds + Q(\mathsf{U}_1, \mathsf{V}_1) \int_0^1 s(1-s)\gamma_1(s)\gamma_2(1-s)ds. \quad (29)$$

where,

$$P(\mathsf{U}_1, \mathsf{V}_1) = \mathsf{S}(\mathsf{U}_1, .)\mathsf{M}(\mathsf{U}_1, .) + \mathsf{S}(\mathsf{V}_1, .)\mathsf{M}(\mathsf{V}_1, .),$$
$$Q(\mathsf{U}_1, \mathsf{V}_1) = \mathsf{S}(\mathsf{U}_1, .)\mathsf{M}(\mathsf{V}_1, .) + \mathsf{S}(\mathsf{V}_1, .)\mathsf{M}(\mathsf{U}_1, .).$$

Proof. Consider $\mathsf{S} \in \Im^{\alpha}_{\aleph}({}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}} - \gamma_1, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$, $\mathsf{M} \in \Im^{\alpha}_{\aleph}({}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}} - \gamma_2, [\mathsf{U}, \mathsf{V}], \Re^+_{\mathbf{I}})$, then we have

$$\mathsf{S}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .) \preceq^{\mathfrak{c}}_{\mathfrak{r}} s\gamma_1(s)\mathsf{S}(\mathsf{U}_1, .) + (1-s)\gamma_1(1-s)\mathsf{S}(\mathsf{V}_1, .),$$
$$\mathsf{M}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .) \preceq^{\mathfrak{c}}_{\mathfrak{r}} s\gamma_2(s)\mathsf{M}(\mathsf{U}_1, .) + (1-s)\gamma_2(1-s)\mathsf{M}(\mathsf{V}_1, .),$$

Then,

$$\mathsf{S}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)\mathsf{M}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} \left(s\gamma_1(s)\mathsf{S}(\mathsf{U}_1, .) + (1-s)\gamma_1(1-s)\mathsf{S}(\mathsf{V}_1, .)\right)$$
$$\times \left(s\gamma_2(s)\mathsf{M}(\mathsf{U}_1, .) + (1-s)\gamma_2(1-s)\mathsf{M}(\mathsf{V}_1, .)\right).$$

Integrating over $(0, 1)$, we have

$$\int_0^1 \mathsf{S}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)\mathsf{M}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)ds$$
$$= [\int_0^1 \mathsf{S}_{\circledast}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)\mathsf{M}_{\circledast}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)ds,$$
$$\int_0^1 \mathsf{S}^{\circledast}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)\mathsf{M}^{\circledast}(\mathsf{U}_1 s + (1-s)\mathsf{V}_1, .)ds]$$

$$= \left[\frac{1}{\mathsf{V}_1 - \mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}_{\circledast}(\mathsf{N},.)\mathsf{M}_{\circledast}(\mathsf{N},.)d\mathsf{N}, \frac{1}{\mathsf{V}_1 - \mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}^{\circledast}(\mathsf{N},.)\mathsf{M}^{\circledast}(\mathsf{N},.)d\mathsf{N}\right]$$
$$= \frac{1}{\mathsf{V}_1 - \mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)\mathsf{M}(\mathsf{N},.)d\mathsf{N}$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} P(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s^2\gamma_1(s)\gamma_2(s)ds + Q(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s(1-s)\gamma_1(s)\gamma_2(1-s)ds.$$

It follows that,

$$\frac{1}{\mathsf{V}_1 - \mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)\mathsf{M}(\mathsf{N},.)d\mathsf{N}$$
$$\preceq^{\mathfrak{c}}_{\mathfrak{r}} P(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s^2\gamma_1(s)\gamma_2(s)ds + Q(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s(1-s)\gamma_1(s)\gamma_2(1-s)ds.$$

The theorem is proved. □

Corollary 5.

- *If* $\gamma(\mathsf{x}) = 1$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$-$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = (1-\mathsf{x})$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$*-tgs-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}}$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$*-P* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}^2}$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{s-1}$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$*-s-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{-s-1}$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$.
- *If* $h(\mathsf{x}) = \mathsf{x}\gamma(\mathsf{x})$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$*-h-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(29) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{C.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.

Example 3. *Let* $[\mathsf{U}_1,\mathsf{V}_1] = [0,1], s\gamma_1(s) = s, s\gamma_2(s) = 1$ *for all* $s \in (0,1)$. *If* $\mathsf{S},\mathsf{M} : [\mathsf{U}_1,\mathsf{V}_1] \subseteq \mathsf{I}\times\Omega \to \Re_{\mathbf{I}}^{+}$ *are defined as*

$$\mathsf{S}(\mathsf{N},.) = [-12\mathsf{N}^2 - 1, 4\mathsf{N}^2 + 9] = \langle -4\mathsf{N}^2 + 4, 8\mathsf{N}^2 + 5\rangle,$$

and,

$$\mathsf{M}(\mathsf{N},.) = [-14\mathsf{N}^3 - 1, 4\mathsf{N}^3 + 11] = \langle -5\mathsf{N}^3 + 5, 9\mathsf{N}^3 + 6\rangle.$$

Then, we have

$$\frac{1}{\mathsf{V}_1 - \mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1} \mathsf{S}(\mathsf{N},.)\mathsf{M}(\mathsf{N},.)d\mathsf{N} = \left[\frac{35}{3}, \frac{277}{4}\right],$$
$$P(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s^2\gamma_1(s)\gamma_2(s)ds = P(0,1)\int_0^1 sds = \left[10, \frac{225}{2}\right],$$

and,

$$\mathcal{Q}(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s\gamma_1(s)(1-s)\gamma_2(1-s)ds = \mathcal{Q}(0,1)\int_0^1 sds = \left[0,\frac{153}{2}\right],$$

Since

$$\left[\frac{35}{3},\frac{277}{4}\right] \preceq^{\mathfrak{c}}_{\mathfrak{r}} \left[10,\frac{225}{2}\right] + \left[0,\frac{153}{2}\right] = [10,189].$$

The plot below (Figure 4) demonstrates the newly constructed left (red dotted) and right (green dotted) parts of (29) when substitutions are applied.

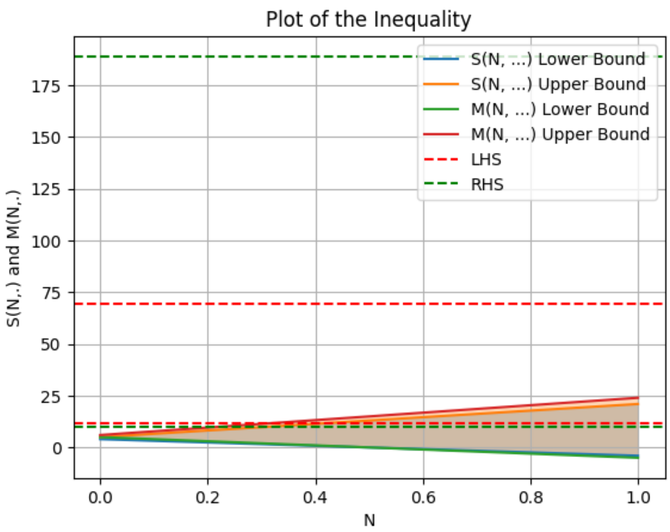

Figure 4. The plot above shows $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ $\mathcal{SP}s$ with concave and convex ends.

Thus, (29) is verified with the help of a Python-programmed graph.

Theorem 9. *Let $\gamma_1,\gamma_2:(0,1)\to\Re^+$ and $\gamma_1,\gamma_2\neq 0$. Suppose mappings $\mathsf{S},\mathsf{M}:\mathsf{I}\times\Omega\to\Re^+_{\mathbf{I}}$ are γ convex ($\mathcal{SP}$) and also mean square integrable for $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$. $\forall\ \mathsf{U}_1,\mathsf{V}_1 \in \mathsf{I}$, if $\mathsf{S}\in\Im^{\alpha}_{\aleph}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}-\gamma_1,[\mathsf{U},\mathsf{V}],\Re^+_{\mathbf{I}}),\mathsf{M}\in\Im^{\alpha}_{\aleph}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}-\gamma_2,[\mathsf{U},\mathsf{V}],\Re^+_{\mathbf{I}})$ and $\mathsf{S},\mathsf{M}\in\mathbf{IR}_I$. Then, we obtain the following result:*

$$\begin{aligned}\frac{1}{2\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)}&\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\\ &\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1}\mathsf{S}(\mathsf{N},.)\mathsf{M}(\mathsf{N},.)d\mathsf{N} + \mathcal{P}(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s(1-s)\gamma_1(s)\gamma_2(1-s)ds\\ &\quad + \mathcal{Q}(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s^2\gamma_1(s)\gamma_2(s)ds. \end{aligned} \tag{30}$$

Proof. Since $\mathsf{S}\in\Im^{\alpha}_{\aleph}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}-\gamma_1,[\mathsf{U},\mathsf{V}],\Re^+_{\mathbf{I}}),\mathsf{M}\in\Im^{\alpha}_{\aleph}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}-\gamma_2,[\mathsf{U},\mathsf{V}],\Re^+_{\mathbf{I}})$, we have

$$\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{2}\gamma_1\left(\frac{1}{2}\right)\mathsf{S}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.) + \frac{1}{2}\gamma_1\left(\frac{1}{2}\right)\mathsf{S}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.),$$

$$\mathsf{M}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{2}\gamma_2\left(\frac{1}{2}\right)\mathsf{M}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.) + \frac{1}{2}\gamma_2\left(\frac{1}{2}\right)\mathsf{M}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.).$$

$$
\begin{aligned}
&\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\\
&\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\\
&\qquad+\mathsf{S}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)]\\
&\quad+\frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)\\
&\qquad+\mathsf{S}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)]\\
&\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\\
&\qquad+\mathsf{S}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)]\\
&\quad+\frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[(s\gamma_1(s)\mathsf{S}(\mathsf{U}_1,.)+(1-s)\gamma_1(1-s)\mathsf{S}(\mathsf{V}_1,.))\\
&\qquad\times((1-s)\gamma_2(1-s)\mathsf{M}(\mathsf{U}_1,.)+s\gamma_2(s)\mathsf{M}(\mathsf{V}_1,.))]\\
&\quad+[((1-s)\gamma_1(1-s)\mathsf{S}(\mathsf{U}_1,.)+s\gamma_1(s)\mathsf{S}(\mathsf{V}_1,.))\\
&\qquad\times(s\gamma_2(s)\mathsf{M}(\mathsf{U}_1,.)+(1-s)\gamma_2(1-s)\mathsf{M}(\mathsf{V}_1,.))]\\
&\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1 s+(1-s)\mathsf{V}_1,.)\\
&\qquad+\mathsf{S}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)\mathsf{M}(\mathsf{U}_1(1-s)+s\mathsf{V}_1,.)]\\
&\quad+\frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[(s(1-s)\gamma_1(s)\gamma_2(1-s)+s(1-s)\gamma_1(1-s)\gamma_2(s))P(\mathsf{U}_1,\mathsf{V}_1)\\
&\qquad+\left(s^2\gamma_1(s)\gamma_2(s)+(1-s)^2\gamma_1(1-s)\gamma_2(1-s)\right)Q(\mathsf{U}_1,\mathsf{V}_1)\Big].
\end{aligned}
$$

Integrating over $(0,1)$, we have

$$
\begin{aligned}
&\int_0^1 \mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)ds\\
&=\left[\int_0^1 \mathsf{S}_{\circledast}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}_{\circledast}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)ds,\int_0^1 \mathsf{S}^{\circledast}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}^{\circledast}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)ds\right]\\
&\quad=\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)ds\\
&\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{2}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)\left[\frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1}\mathsf{S}(\mathsf{N},.)\mathsf{M}(\mathsf{N},.)d\mathsf{N}\right]\\
&\quad+\frac{1}{2}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)\left[P(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s(1-s)\gamma_1(s)\gamma_2(1-s)ds\right.\\
&\qquad\left.+Q(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s^2\gamma_1(s)\gamma_2(s)ds\right].
\end{aligned}
$$

Multiplying both sides by $\frac{2}{\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)}$ in the above equation, we obtain the required result:

$$
\begin{aligned}
&\frac{1}{2\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)}\mathsf{S}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\mathsf{M}\left(\frac{\mathsf{U}_1+\mathsf{V}_1}{2},.\right)\\
&\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{\mathsf{V}_1-\mathsf{U}_1}\int_{\mathsf{U}_1}^{\mathsf{V}_1}\mathsf{S}(\mathsf{N},.)\mathsf{M}(\mathsf{N},.)d\mathsf{N}+P(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s(1-s)\gamma_1(s)\gamma_2(1-s)ds\\
&\qquad+Q(\mathsf{U}_1,\mathsf{V}_1)\int_0^1 s^2\gamma_1(s)\gamma_2(s)ds.
\end{aligned}
$$

It completes the proof. □

Corollary 6.

- *If* $\gamma(\mathsf{x}) = 1$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = (1-\mathsf{x})$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-tgs-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}}$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-P* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}^2}$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{s-1}$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-s-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \mathsf{x}^{-s-1}$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$.
- *If* $h(\mathsf{x}) = \mathsf{x}\gamma(\mathsf{x})$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-h-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(30) gives a Hermite–Hadamard-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *generalized n-polynomial* $(\mathfrak{CF})$.

3.4. Fractional Hermite–Hadamard–Mercer-Type Inclusion

Definition 18 ([31])**.** *Let* $\mathsf{S} : \mathsf{I} \times \Omega \to \Re$ *be a* $\mathcal{SP}$. *We recall the mean square continuous fractional integrals* $(\mathcal{MSCFI})$ $\mathfrak{I}^{\kappa}_{\mathsf{N}_1^+}$ *and* $\mathfrak{I}^{\kappa}_{\mathsf{N}_2^-}$ *having order* $\kappa > 0$ *are given as,*

$$\mathfrak{I}^{\kappa}_{\mathsf{N}_1^+}[\mathsf{S}](n) = \frac{1}{\kappa}\int_{\mathsf{N}_1}^{n} e^{-\frac{1-\kappa}{\kappa}(n-\mathsf{m})}\mathsf{S}(\mathsf{m},.)\,d\mathsf{m}, \quad (0 \le \mathsf{N}_1 < n < \mathsf{N}_2),$$

and,

$$\mathfrak{I}^{\kappa}_{\mathsf{N}_2^-}[\mathsf{S}](n) = \frac{1}{\kappa}\int_{n}^{\mathsf{N}_2} e^{-\frac{1-\kappa}{\kappa}(\mathsf{m}-n)}\mathsf{S}(\mathsf{m},.)\,d\mathsf{m}, \quad (0 \le \mathsf{N}_1 < n < \mathsf{N}_2),$$

respectively. Using Theorems 1 and 2, we can easily utilize $(\mathcal{MSCFI})$ *on* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *settings. For convenience, we use* $\alpha = \frac{1-\kappa}{\kappa}(\mathsf{n}-\mathsf{m})$.

Theorem 10. *Let* $\mathsf{S} : \mathsf{I} \times \Omega \to \Re$ *be a* γ*-convex* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *or we term as* $\mathsf{S} \in \Im^{\alpha}_{\aleph}(\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}} - \gamma, I, \Re^+_{\mathfrak{I}})$ *in the interval* I, *such that* $\mathsf{N}_1, \mathsf{N}_2 \in \mathsf{I}$, *with* $0 < \mathsf{N}_1 < \mathsf{N}_2$. *Then, for* $\mathsf{m}, \mathsf{n} > 0$, *the following fractional inclusion holds true:*

$$\begin{aligned}\frac{(1-e^{-\alpha})}{\gamma\left(\frac{1}{2}\right)}\mathsf{S}\left(\mathsf{N}_1+\mathsf{N}_2-\frac{\mathsf{m}+\mathsf{n}}{2},.\right) &\\ \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1-\kappa}{2}\left[\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}^-}[\mathsf{S}](\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}) + \mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}^+}[\mathsf{S}](\mathsf{N}_1+\mathsf{N}_2-\mathsf{m})\right] &\\ \preceq^{\mathfrak{c}}_{\mathfrak{r}} \left[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)-\frac{\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{n},.)}{2}\right]P_1. &\quad (31)\end{aligned}$$

where,

$$P_1 = \int_0^1 e^{-\frac{1-\kappa}{\kappa}(\mathsf{n}-\mathsf{m})\mathsf{s}}\,(\mathsf{s}\gamma(\mathsf{s}) + (1-\mathsf{s})\gamma(1-\mathsf{s}))d\mathsf{s}.$$

Proof. Suppose $\mathsf{S} : [\mathsf{N}_1, \mathsf{N}_2] \to \Re$ being an ${}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$ γ-convex $(\mathcal{SP})$. Thus, by assumption, we obtain

$$\mathsf{S}\left(\mathsf{N}_1 + \mathsf{N}_2 - \frac{\mathsf{U}+\mathsf{V}}{2}, .\right) = \mathsf{S}\left(\frac{\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{U} + \mathsf{N}_1 + \mathsf{N}_2 - \mathsf{V}}{2}, .\right) \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{2}\gamma\left(\frac{1}{2}\right)\big(\mathsf{S}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{U}, .) + \mathsf{S}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{V}, .)\big).$$

Subsequently, when we substitute the inputs as

$$\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{U} = \mathsf{s}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}) + (1-\mathsf{s})(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}),$$

$$\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{V} = \mathsf{s}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}) + (1-\mathsf{s})(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}),$$

Thus, we obtain

$$\begin{aligned}&\mathsf{S}\left(\mathsf{N}_1 + \mathsf{N}_2 - \frac{\mathsf{m}+\mathsf{n}}{2}, .\right)\\ &\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{1}{2}\gamma\left(\frac{1}{2}\right)[\mathsf{S}(\mathsf{s}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}) + (1-\mathsf{s})(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}), .)\\ &\qquad + \mathsf{S}(\mathsf{s}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}) + (1-\mathsf{s})(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}), .)].\end{aligned}$$

When multiplying on both sides of the above inclusion by $e^{-\frac{1-\kappa}{\kappa}(\mathsf{n}-\mathsf{m})\mathsf{s}}$ and also applying integration on the result over $[0,1]$, we obtain

$$\begin{aligned}&\left(\frac{1-e^{-\alpha}}{\alpha}\right)\mathsf{S}\left(\mathsf{N}_1 + \mathsf{N}_2 - \frac{\mathsf{m}+\mathsf{n}}{2}, .\right)\\ &\preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{\gamma\left(\frac{1}{2}\right)}{2(\mathsf{n}-\mathsf{m})}\left[\int_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}}^{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}} e^{-\frac{1-\kappa}{\kappa}(\mathsf{r}-(\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}))}\mathsf{S}(\mathsf{r}, .)\,d\mathsf{r}\right.\\ &\qquad \left. + \int_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}}^{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}} e^{-\frac{1-\kappa}{\kappa}((\mathsf{N}_1+\mathsf{N}_2-\mathsf{m})-\mathsf{r})}\mathsf{S}(\mathsf{r}, .)\,d\mathsf{r}\right]\\ &= \frac{\kappa\gamma\left(\frac{1}{2}\right)}{2(\mathsf{n}-\mathsf{m})}\left[\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}^-}[\mathsf{S}](\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}) + \mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}^+}[\mathsf{S}](\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m})\right],\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\frac{(1-e^{-\alpha})}{\gamma\left(\frac{1}{2}\right)}\mathsf{S}\left(\mathsf{N}_1 + \mathsf{N}_2 - \frac{\mathsf{m}+\mathsf{n}}{2}, .\right)\\ &\qquad \preceq^{\mathfrak{c}}_{\mathfrak{r}} \frac{(1-\kappa)}{2}\left[\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}^-}[\mathsf{S}](\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}) + \mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}^+}[\mathsf{S}](\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m})\right].\end{aligned}$$

It gives us the first part of the inclusion.
For, the second part of the inclusion, we utilize the ${}^{\mathcal{I}.\mathcal{V}}_{\mathcal{C}.\mathcal{R}}$ γ-convexity of S, applied as,

$$\begin{aligned}&\mathsf{S}\big(\mathsf{s}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}) + (1-\mathsf{s})(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}), .\big)\\ &\qquad \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{s}\gamma(\mathsf{s})\mathsf{S}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}, .) + (1-\mathsf{s})\gamma(1-\mathsf{s})\mathsf{S}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}, .),\end{aligned}$$

$$\begin{aligned}&\mathsf{S}\big(\mathsf{s}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}) + (1-\mathsf{s})(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}), .\big)\\ &\qquad \preceq^{\mathfrak{c}}_{\mathfrak{r}} \mathsf{s}\gamma(\mathsf{s})\mathsf{S}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{n}, .) + (1-\mathsf{s})\gamma(1-\mathsf{s})\mathsf{S}(\mathsf{N}_1 + \mathsf{N}_2 - \mathsf{m}, .).\end{aligned}$$

Summing both the above-given inclusions, we obtain that,

$$\begin{aligned}&\mathsf{S}\big(\mathsf{s}(\mathsf{N}_1+\mathsf{N}_2-\mathsf{m})+(1-\mathsf{s})(\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}),.\big)+\mathsf{S}\big(\mathsf{s}(\mathsf{N}_1+\mathsf{N}_2-\mathsf{n})+(1-\mathsf{s})(\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}),.\big)\\ &\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} (\mathsf{s}\gamma(\mathsf{s})+(1-\mathsf{s})\gamma(1-\mathsf{s}))[\mathsf{S}(\mathsf{N}_1+\mathsf{N}_2-\mathsf{m},.)+\mathsf{S}(\mathsf{N}_1+\mathsf{N}_2-\mathsf{n},.)]\\ &\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} (\mathsf{s}\gamma(\mathsf{s})+(1-\mathsf{s})\gamma(1-\mathsf{s}))[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)-\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)-\mathsf{S}(\mathsf{n},.)]\\ &\quad= (\mathsf{s}\gamma(\mathsf{s})+(1-\mathsf{s})\gamma(1-\mathsf{s}))\big(2\big[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)\big]-\big[\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{n},.)\big]\big).\end{aligned}$$

Taking the product on both sides of the above inclusion by $e^{-\frac{1-\kappa}{\kappa}(\mathsf{n}-\mathsf{m})\mathsf{s}}$ and thus taking the integration over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 e^{-\frac{1-\kappa}{\kappa}(\mathsf{n}-\mathsf{m})\mathsf{s}}\mathsf{S}\big(\mathsf{s}(\mathsf{N}_1+\mathsf{N}_2-\mathsf{m})+(1-\mathsf{s})(\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}),.\big)\,d\mathsf{s}\\ &\qquad+\int_0^1 e^{-\frac{1-\kappa}{\kappa}(\mathsf{n}-\mathsf{m})\mathsf{s}}\mathsf{S}\big(\mathsf{s}(\mathsf{N}_1+\mathsf{N}_2-\mathsf{n})+(1-\mathsf{s})(\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}),.\big)\,d\mathsf{s}\\ &\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}} 2\big[\big[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)\big]-\big[\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{n},.)\big]\big]\\ &\qquad\qquad\times\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathsf{n}-\mathsf{m})\mathsf{s}}\,(\mathsf{s}\gamma(\mathsf{s})+(1-\mathsf{s})\gamma(1-\mathsf{s}))d\mathsf{s}.\end{aligned}$$

This results in using the above inclusions,

$$\begin{aligned}&\frac{\kappa}{2(\mathsf{n}-\mathsf{m})}\big[\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}^-}[\mathsf{S}](\mathsf{N}_1+\mathsf{N}_2-\mathsf{n})+\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}^+}[\mathsf{S}](\mathsf{N}_1+\mathsf{N}_2-\mathsf{m})\big]\\ &\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}}\left[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)-\frac{\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{y},.)}{2}\right]P_1,\end{aligned}$$

This consequently implies,

$$\begin{aligned}&\frac{1-\kappa}{2}\big[\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{m}^-}[\mathsf{S}](\mathsf{N}_1+\mathsf{N}_2-\mathsf{n})+\mathfrak{I}^{\kappa}_{\mathsf{N}_1+\mathsf{N}_2-\mathsf{n}^+}[\mathsf{S}](\mathsf{N}_1+\mathsf{N}_2-\mathsf{m})\big]\\ &\quad\preceq^{\mathfrak{c}}_{\mathfrak{r}}\left[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)-\frac{\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{n},.)}{2}\right]P_1.\end{aligned}$$

This leads us to the proof of the desired fractional inclusion. □

Corollary 7.

- *If* $\gamma(\mathsf{x})=1$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$-$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=(1-\mathsf{x})$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-tgs-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}}$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-P* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\frac{(e^{\mathsf{x}}-1)}{\mathsf{x}}$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *exponential-type* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}^2}$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\mathsf{x}^{s-1}$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-s-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\mathsf{x}^{-s-1}$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-Godunova–Levin s-*$(\mathfrak{CF})$.
- *If* $h(\mathsf{x})=\mathsf{x}\gamma(\mathsf{x})$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$*-h-*$(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(1-(1-\mathsf{x})^{\Im})$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *n-polynomial* $(\mathfrak{CF})$.
- *If* $\gamma(\mathsf{x})=\frac{1}{\mathsf{x}\aleph}\Sigma_{\Im=1}^{\aleph}(\mathsf{x}^{\frac{1}{\Im}})$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *fractional n-polynomial* $(\mathfrak{CF})$.

- *If* $\gamma(\mathsf{x}) = \frac{1}{\mathsf{x}\aleph}\frac{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}(1-(1-\mathsf{x})^{\Im})}{\Sigma_{\Im=1}^{\aleph}\mathsf{M}_{\Im}}$, *(31) gives a Hermite–Hadamard–Mercer-type inclusion for* $(\mathcal{SP})$ *of* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ *generalized n-polynomial* $(\mathfrak{CF})$*.*

Corollary 8. *By substituting* $\mathsf{N}_1 = \mathsf{m}$ *and* $\mathsf{N}_2 = \mathsf{n}$ *in (31), we obtain the following new fractional integral inclusion for* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ γ*-convex* $(\mathcal{SP})$*:*

$$\frac{(1-e^{-\alpha})}{\gamma\left(\frac{1}{2}\right)}\mathsf{S}\left(\frac{\mathsf{m}+\mathsf{n}}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1-\kappa}{2}\left[\mathfrak{I}_{\mathsf{m}^-}^{\kappa}[\mathsf{S}](\mathsf{n}) + \mathfrak{I}_{\mathsf{n}^+}^{\kappa}[\mathsf{S}](\mathsf{m})\right]$$
$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\frac{\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{n},.)}{2}\right]P_1.$$

The above inclusion becomes the one given by [31] for $\gamma(n) = 1$ *and* $\mathsf{S}_{\mathfrak{c}} = \mathsf{S}_{\mathfrak{r}}$*.*

Corollary 9. *When* $\kappa \to 1$*, we obtain* $\lim_{\kappa\to 1}\frac{1-\kappa}{2(1-e^{-\alpha})} = \frac{1}{2(\mathsf{n}-\mathsf{m})}$*. Thus, from (31), the given new Hermite–Hadamard–Mercer-type inclusion for* $\frac{\mathcal{I}.\mathcal{V}}{\mathcal{C}.\mathcal{R}}$ γ*-convex* $(\mathcal{SP})$ *is:*

$$\frac{1}{\gamma\left(\frac{1}{2}\right)}\mathsf{S}\left(\mathsf{N}_1+\mathsf{N}_2-\frac{\mathsf{m}+\mathsf{n}}{2},.\right) \preceq_{\mathfrak{r}}^{\mathfrak{c}} \frac{1}{2(\mathsf{n}-\mathsf{m})}\int_{\mathsf{m}}^{\mathsf{n}}\mathsf{S}(\mathsf{N}_1+\mathsf{N}_2-u,.)\,du$$
$$\preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\mathsf{S}(\mathsf{N}_1,.)+\mathsf{S}(\mathsf{N}_2,.)-\frac{\mathsf{S}(\mathsf{m},.)+\mathsf{S}(\mathsf{n},.)}{2}\right]\frac{P_1}{(1-e^{-\alpha})}.$$

The above inclusion becomes the one given by [31] for $\gamma(n) = 1$ *and* $\mathsf{S}_{\mathfrak{c}} = \mathsf{S}_{\mathfrak{r}}$*.*

Remark 4. *Substituting* $\mathsf{N}_1 = \mathsf{m}$*,* $\gamma(n) = 1$ *and* $\mathsf{N}_2 = \mathsf{n}$ *in (31), thus for* $\kappa \to 1$*, the Hermite–Hadamard-type inclusion for convex* $(\mathcal{SP})$ *presented by Kotrys [17] is reproduced for* $\mathsf{S}_{\mathfrak{c}} = \mathsf{S}_{\mathfrak{r}}$*.*

4. Applications

Entropy quantifies uncertainty in $(\mathcal{SP})$s. The greater the value of the entropy, the less predictable the next event will be. Shannon entropy, a vital idea in information theory, is usually used to calculate this. In this section, we discuss some valuable notations from the literature that provide applications in entropy and information theory related to our work using Python-programmed graphs.

Example 4. *Taking* $\mathsf{S}(n,.) = [(-1-n)\log(n), (-1+n)\log(n)] = \langle -\log(n), n\log(n)\rangle$*. See Figure 5.*

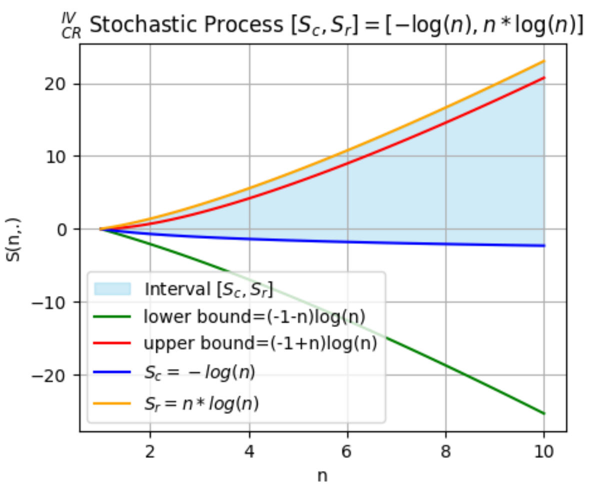

Figure 5. The plot above shows $\frac{\mathcal{I}}{\mathcal{V}}$ $\mathcal{SP}$ with concave (green) and convex (red) ends, respectively.

Nevertheless, the newly constructed $_{\mathcal{C}.\mathcal{R}}^{\mathcal{I}.\mathcal{V}}$ $\mathcal{SP}$s demonstrate that the $\mathcal{SP}$s at the left (blue) and right (yellow) endpoints are convex when the center and radius order is applied.

Definition 19. *The Shannon entropy of a positive probability distribution* $\mathsf{Q} = (q_1, \ldots, q_{\aleph})$ *is defined by,*

$$\mathsf{E}(q) := \sum_{\Im=1}^{\aleph} q_{\Im} \log \frac{1}{q_{\Im}}.$$

Let $\mathsf{L} = \{\mathsf{L}_{\Im}\}_{\Im=1}^{\aleph}$ *be a non-negative real sequence,*

$$\mathsf{A}_{\aleph}(n) = \frac{1}{\aleph} \sum_{\Im=1}^{\aleph} n_{\Im},$$

and

$$\mathsf{T}_{\aleph}(n) = \left(\eth_{\Im=1}^{\aleph} m_{\Im}\right)^{\frac{1}{\aleph}},$$

denote the usual arithmetic and geometric means of $\{n_{\Im}\}$*, respectively. From (20), we conclude with the following result.*

Proposition 1. *Let* $\mathsf{U} > 0, \gamma(n) = 1, \mathsf{N}_{\Im} \in [\mathsf{U}, \mathsf{V}]$, $\Im = 1, \ldots, \aleph$ *and* $\mathsf{A}_{\Im}(x) = \frac{\mathsf{U}+\mathsf{V}}{2}$*, then (20) implies*

$$\left[\sqrt{\mathsf{UV}}, \log\left(\frac{\aleph}{2}\right)\right] \preceq_{\mathfrak{r}}^{\mathfrak{c}} [\mathsf{T}_{\aleph}(x), \mathsf{E}(q)] \preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\frac{\mathsf{U}+\mathsf{V}}{2}, \frac{1}{\aleph}\log\left(\frac{2}{\aleph}\right)\right]. \tag{32}$$

Proof. Taking $\mathsf{S}_{\mathfrak{c}}(n,.) = -\log(n)$ in (20),

$$-\log\left(\frac{\mathsf{U}+\mathsf{V}}{2}\right) \leq \frac{-\log(\mathsf{N}_1) - \ldots - \log(\mathsf{N}_{\aleph})}{\aleph} \leq \frac{-\log(\mathsf{U}) - \log(\mathsf{V})}{2},$$

So,

$$\frac{\log(\mathsf{U}) + \log(\mathsf{V})}{2} \leq \frac{\log(\mathsf{N}_1) + \ldots + \log(\mathsf{N}_{\aleph})}{\aleph} \leq \log\left(\frac{\mathsf{U}+\mathsf{V}}{2}\right),$$

Subsequently,

$$\log(\sqrt{\mathsf{UV}}) \leq \log(\mathsf{T}_{\aleph}(n)) \leq \log\left(\frac{\mathsf{U}+\mathsf{V}}{2}\right),$$

Since $\mathsf{S}_{\mathfrak{c}}(n) = -\log(n)$ is nondecreasing, the result follows from above:

$$\sqrt{\mathsf{UV}} \leq \mathsf{T}_{\aleph}(x) \leq \frac{\mathsf{U}+\mathsf{V}}{2}. \tag{33}$$

Now, taking, $n_{\Im} = q_{\Im}$, $\mathsf{S}_{\mathfrak{r}}(n,.) = n\log(n)$, $n \neq 0$, $\mathsf{U} = 0$, $\mathsf{V} = \frac{2}{\aleph}$ in (20),

$$\frac{1}{\aleph}\log\left(\frac{1}{\aleph}\right) \leq \frac{q_1\log(q_1) + \ldots + q_{\aleph}\log(q_{\aleph})}{\aleph} \leq \frac{1}{\aleph}\log\left(\frac{2}{\aleph}\right),$$

$$\log\left(\frac{\aleph}{2}\right) \leq \mathsf{E}(q) \leq \log(\aleph), \tag{34}$$

(33) and (34) yield,

$$\left[\sqrt{\mathsf{UV}}, \log\left(\frac{\aleph}{2}\right)\right] \preceq_{\mathfrak{r}}^{\mathfrak{c}} [\mathsf{T}_{\aleph}(x), \mathsf{E}(q)] \preceq_{\mathfrak{r}}^{\mathfrak{c}} \left[\frac{\mathsf{U}+\mathsf{V}}{2}, \frac{1}{\aleph}\log\left(\frac{2}{\aleph}\right)\right].$$

□

5. Conclusions

Merging the concepts of interval analysis, stochastic processes, and generalized convexity, $\frac{\mathcal{I.V}}{\mathcal{C.R}}$ order relation is used in this manuscript for γ-convex $(\mathcal{SP})$s. Using these notions, we developed inclusions of the Jensen, Mercer, and Hermite–Hadamard types. A distinguishing feature of this notion is that the inclusion terms derived from it reproduce results for $\frac{\mathcal{I.V}}{\mathcal{C.R}}$ (convex, tgs-convex, P-convex, exponential-type convex, Godunova–Levin convex, s-convex, Godunova–Levin s-convex, h-convex, n-polynomial convex, and fractional n-polynomial convex $(\mathcal{SP})$s). Specific interesting examples related to Python-programmed graphs, entropy, and information theory applications make our work more advanced than the existing results in [17,27,31]. In the future, one can extend these results via stochastic integrals, quantum integrals, and post-quantum integrals in fuzzy interval settings.

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