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Existence, Uniqueness, and Stability of a Nonlinear Tripled Fractional Order Differential System

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Abstract: This manuscript investigates the existence, uniqueness, and different forms of Ulam stability for a system of three coupled differential equations involving the Riemann–Liouville (RL) fractional operator. The Leray–Schauder alternative is employed to confirm the existence of solutions, while the Banach contraction principle is used to establish their uniqueness. Stability conditions are derived utilizing classical nonlinear functional analysis techniques. Theoretical findings are illustrated with an example. The proposed system generalizes third-order ordinary differential equations (ODEs) with different boundary conditions (BCs).

Keywords: fractional derivatives; differential equations; nonlinear equations; nonlinear systems; existence theory; Ulam stability

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1. Introduction

Fractional differential equations (FDEs) have become essential tools for modeling real-world phenomena. These equations are particularly effective in capturing the memory and hereditary properties of various materials and processes, making them indispensable in several fields. Numerous fundamental phenomena in diverse areas such as physics and polymer technology [1], fitting of experimental data [2], blood flow [3], biology [4], capacitor theory [5], fluid mechanics [6], aerodynamics [7], viscoelasticity [8], thermodynamics [9], control theory [10], electrodynamics [11], electrochemistry [12], electrical circuits [13], etc., are well described by the aforementioned equations.

In recent times, there has been significant focus on investigating the existence and uniqueness of solutions to FDEs, considering a wide range of BCs, such as Dirichlet [14], nonlocal [15], integral [16], periodic [17], anti-periodic [18], and multi-point [19]. Many researchers have studied fractional differential systems due to their extensive applications in modeling diverse physical and engineering phenomena, including diffusion and reactions [20], chaos theory [21], fluid dynamics [22], heat equations [23], and Burgers equations [24]. For more details about applications, refer to [25].

The study of stability in functional and differential equations has become a key area in mathematical analysis. The literature covers various types of stability, including exponential [26] and Lyapunov [27] stability. A notable type is Ulam–Hyers (UH) stability, which links exact and numerical solutions. Ulam introduced this problem in 1940 [28], and Hyers provided a partial solution for linear functional equations in the following year using Banach spaces [29]. In 1978, Rassias extended these results to linear mappings [30]. Rassias's work has inspired many researchers to extend his results to ODEs and FDEs, such as functional equations in several variables [31], linear differential equations of the first order [32], advection-reaction diffusion system [33], biology and economics [34], impulsive switched coupled evolution equations [35].

In recent years, the existence theory and various forms of Ulam stability for coupled systems of FDEs with two equations have garnered significant attention, particularly with different fractional order operators. Examples include coupled systems utilizing the Caputo operator [36], RL operator [37], Hadamard-type operator [38], Atangana–Baleanu fractional derivative [39], Langevin equations using the Caputo operator [40], coupled p-Laplacian systems of FDEs [41], generalized Hilfer derivatives [42], sequential FDEs [43], Riesz–Caputo operator [44], and ψ-Caputo operator [45], among others.

Based on the literature on the existence and stability of FDEs, it is evident that there currently exists no similar model involving a system of three or tripled FDEs, as will be studied in this article. Furthermore, this system represents a generalization of third-order ODEs and includes various boundary conditions. Motivated by the above discussion, this manuscript aims to examine the existence, uniqueness, and stability, including UH and generalized UH stability, of the following three FDEs systems incorporating the RL operators:

$$\begin{cases} \mathcal{D}^{g}u(t)=f_1(t,u(t),w(t),y(t));\ t\in\mathcal{G},\\ \mathcal{D}^{\hbar}w(t)=f_2(t,u(t),w(t),y(t));\ t\in\mathcal{G},\\ \mathcal{D}^{\ell}y(t)=f_3(t,u(t),w(t),y(t));\ t\in\mathcal{G},\\ \mathcal{D}^{g-3}u(0)=\sigma_0\mathcal{D}^{g-3}u(\mathcal{T}),\ \mathcal{D}^{g-2}u(0)=\varsigma_0\mathcal{D}^{g-2}u(\mathcal{T}),\ \mathcal{D}^{g-1}u(0)=\tau_0\mathcal{D}^{g-1}u(\mathcal{T}),\\ \mathcal{D}^{\hbar-3}w(0)=\sigma_1\mathcal{D}^{\hbar-3}w(\mathcal{T}),\ \mathcal{D}^{\hbar-2}w(0)=\varsigma_1\mathcal{D}^{\hbar-2}w(\mathcal{T}),\ \mathcal{D}^{\hbar-1}w(0)=\tau_1\mathcal{D}^{\hbar-1}w(\mathcal{T}),\\ \mathcal{D}^{\ell-3}y(0)=\sigma_2\mathcal{D}^{\ell-3}y(\mathcal{T}),\ \mathcal{D}^{\ell-2}y(0)=\varsigma_2\mathcal{D}^{\ell-2}y(\mathcal{T}),\ \mathcal{D}^{\ell-1}y(0)=\tau_2\mathcal{D}^{\ell-1}y(\mathcal{T}), \end{cases} \tag{1}$$

where $g,\hbar,\ell\in(2,3]$, $\mathcal{G}=[0,\mathcal{T}]$, $\mathcal{T}>0$ and $\sigma_\ell,\varsigma_\ell,\tau_\ell\neq 1$ $(\ell=0,1,2)$. The functions f_i $(i=1,2,3):\mathcal{G}\times\mathcal{R}^3\to\mathcal{R}$ are continuous and $\mathcal{D}^{g},\mathcal{D}^{\hbar},\mathcal{D}^{\ell}$ are RL fractional derivatives.

This manuscript addresses FDEs as described in Problem 1, which generalizes third-order ODEs. These ODEs have numerous applications in various fields of applied sciences, including fluid mechanics [46], physics and engineering [47], pseudospherical surfaces [48], resonance [49], biology [50], optimal control problems [51], and nuclear spin generators [52], among others. One prominent application of the third-order ODEs is the jerk-type equation, which is widely used in various fields, including economic systems, electrical engineering, chaos theory, and secure communication, [53]. Furthermore, for $\sigma_\ell=\varsigma_\ell=\tau_\ell=-1$ $(\ell=0,1,2)$, we obtain anti-periodic BCs, which frequently arise in models of several physical processes, such as ordinary and partial differential equations, impulsive differential equations, anti-periodic wavelets [54], anti-periodic trigonometric polynomials [55], and shunting inhibitory cellular neural networks [56]. For more details, see [57].

This manuscript is organized as follows: Section 2 covers key preliminaries to prove the theoretical results. Section 3 focuses on demonstrating the existence and uniqueness of solutions to system (1). Section 4 outlines the necessary conditions for the Ulam stability of problem (1). An example illustrating the practical application of these results is provided in Section 5. The conclusion and special applications of the study are presented in Section 6.

2. Preliminaries

Foundational concepts and materials are presented in this section.

Consider $\mathcal{C}(\mathcal{G})$ as the Banach space with the norm specified as $\|u\|=\max_{t\in\mathcal{G}}|u(t)|$. For $t\in\mathcal{G}$, define $u_r(t)=t^r u(t)$ for $r\geq 0$. Let $\mathcal{S}_1=\mathcal{C}_r(\mathcal{G})$ be the space of all functions u such that $u_r\in\mathcal{S}_1$, which becomes a Banach space when endowed with the norm $\|u\|_{\mathcal{S}_1}=\max_{t\in\mathcal{G}} t^r|u(t)|$. Similarly, we can define Banach spaces $\mathcal{S}_2$ and $\mathcal{S}_3$ endowed with norms $\|w\|_{\mathcal{S}_2}=\max_{t\in\mathcal{G}} t^r|w(t)|$ and $\|y\|_{\mathcal{S}_3}=\max_{t\in\mathcal{G}} t^r|y(t)|$, respectively. Likewise, the norm specified on the product space is $\|(u,w,y)\|=\|u\|_{\mathcal{S}_1}+\|w\|_{\mathcal{S}_2}+\|y\|_{\mathcal{S}_3}$. Clearly, $(\mathcal{S}=\mathcal{S}_1\times\mathcal{S}_2\times\mathcal{S}_3,\|(u,w,y)\|)$ is a Banach space.

Definition 1 ([58])**.** *The RL integral of order $g > 0$ of a continuous function $u \in (\mathcal{R}^+, \mathcal{R})$ is expressed as:*

$$\mathbf{I}^g u(t) = \frac{1}{\Gamma(g)} \int_0^t \frac{u(\varrho)}{(t-\varrho)^{1-g}}\, d\varrho,$$

assuming the integral is well-defined.

Definition 2 ([58])**.** *The $g > 0$ order RL derivative of $u \in (\mathcal{R}^+, \mathcal{R})$, which is continuous, is given by:*

$$\mathcal{D}^g u(t) = \frac{1}{\Gamma(m-g)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{u(\varrho)}{(t-\varrho)^{g-m+1}}\, d\varrho,$$

where $m = [g] + 1$. We note that for $\varphi > -1$, $\varphi \neq g-1, g-2, \dots, g-m$, we have

$$\mathcal{D}^g t^\varphi = \frac{\Gamma(\varphi+1)}{\Gamma(\varphi - g + 1)}\, t^{\varphi - g} \quad \text{and} \quad \mathcal{D}^g t^{g-i} = 0, \quad i = 1, 2, \dots, m.$$

Lemma 1 ([58])**.** *The unique solution of the differential equation $\mathcal{D}^g u(t) = \varpi(t)$ is*

$$\mathbf{I}^g \mathcal{D}^g u(t) = \mathbf{I}^g \varpi(t) + k_0 t^{g-m} + k_1 t^{g-m-1} + \cdots + k_{m-2} t^{g-2} + k_{m-1} t^{g-1},$$

where $m = [g] + 1$ and $k_i \in \mathcal{R}$ for $i = 1, 2, \dots, m$.

Theorem 1 ([59])**.** *Consider an operator $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ that is completely continuous. Define the set*

$$\mathcal{B}(\mathcal{F}) = \{u \in \mathcal{S} : u = \lambda \mathcal{F}(u),\ \lambda \in [0,1]\}.$$

Then either the operator $\mathcal{F}$ possesses at least one fixed point, or the set $\mathcal{B}(\mathcal{F})$ exhibits unboundedness.

3. Existence Theory

Lemma 2. *Given $\mu_0 \in C(\mathcal{J})$ and $g \in (2, 3]$, the unique solution of FDE:*

$$\begin{cases} \mathcal{D}^g u(t) = \mu_0(t);\ t \in \mathcal{J}, \\ \mathcal{D}^{g-3} u(0) = \sigma_0 \mathcal{D}^{g-3} u(\mathcal{T}),\ \mathcal{D}^{g-2} u(0) = \varsigma_0 \mathcal{D}^{g-2} u(\mathcal{T}),\ \mathcal{D}^{g-1} u(0) = \tau_0 \mathcal{D}^{g-1} u(\mathcal{T}) \end{cases} \tag{2}$$

is expressed as

$$u(t) = \int_0^{\mathcal{T}} \mathbf{G}_g(t, \varrho) \mu_0(\varrho) d\varrho,$$

where

$$\mathbf{G}_g(t,\varrho) = \begin{cases} \frac{(t-\varrho)^{g-1}}{\Gamma(g)} + \frac{\sigma_0 t^{g-3} (\mathcal{T}-\varrho)^2}{2(1-\sigma_0)\Gamma(g-2)} + \frac{\varsigma_0 t^{g-3} [t(1-\sigma_0) + \sigma_0 \mathcal{T}(g-2)] (\mathcal{T}-\varrho)}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \\ + \frac{\tau_0 t^{g-2} [t(1-\varsigma_0) + \varsigma_0 \mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} + \frac{\sigma_0 \tau_0 t^{g-3} \mathcal{T}^2 (1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}, & 0 \le \varrho < t \le \mathcal{T}, \\ \\ \frac{\sigma_0 t^{g-3} (\mathcal{T}-\varrho)^2}{2(1-\sigma_0)\Gamma(g-2)} + \frac{\varsigma_0 t^{g-3} [t(1-\sigma_0) + \sigma_0 \mathcal{T}(g-2)] (\mathcal{T}-\varrho)}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \\ + \frac{\tau_0 t^{g-2} [t(1-\varsigma_0) + \varsigma_0 \mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} + \frac{\sigma_0 \tau_0 t^{g-3} \mathcal{T}^2 (1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}, & 0 \le t < \varrho \le \mathcal{T}. \end{cases} \tag{3}$$

Proof. By Lemma 1, we have [59]:

$$u(t) = \frac{1}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} \mu_0(\varrho) \mathrm{d}\varrho + k_2 t^{g-1} + k_1 t^{g-2} + k_0 t^{g-3}. \quad (4)$$

Using given boundary conditions on Equation (4), we will obtain

$$\begin{aligned} k_0 =& \frac{\sigma_0}{(1-\sigma_0)\Gamma(g-2)} \Big[\frac{1}{2} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 \mu_0(\varrho) \mathrm{d}\varrho + \frac{\varsigma_0 \mathcal{T}}{(1-\varsigma_0)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) \mu_0(\varrho) \mathrm{d}\varrho \\ &+ \Big(\frac{\tau_0 \mathcal{T}^2}{2(1-\tau_0)} + \frac{\varsigma_0 \tau_0 \mathcal{T}^2}{(1-\varsigma_0)(1-\tau_0)} \Big) \int_0^{\mathcal{T}} \mu_0(\varrho) \mathrm{d}\varrho \Big], \\ k_1 =& \frac{\varsigma_0}{(1-\varsigma_0)\Gamma(g-1)} \Big(\int_0^{\mathcal{T}} (\mathcal{T}-\varrho) \mu_0(\varrho) \mathrm{d}\varrho + \frac{\tau_0 \mathcal{T}}{(1-\tau_0)} \int_0^{\mathcal{T}} \mu_0(\varrho) \mathrm{d}\varrho \Big), \\ k_2 =& \frac{\tau_0}{(1-\tau_0)\Gamma(g)} \int_0^{\mathcal{T}} \mu_0(\varrho) \mathrm{d}\varrho. \end{aligned}$$

Put the values of k_0, k_1 and k_2 in Equation (4), we obtain

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} \mu_0(\varrho) \mathrm{d}\varrho \\ &+ \frac{\sigma_0 t^{g-3}}{2(1-\sigma_0)\Gamma(g-2)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 \mu_0(\varrho) \mathrm{d}\varrho \\ &+ \frac{\varsigma_0 t^{g-3} \big[t(1-\sigma_0) + \sigma_0 \mathcal{T}(g-2) \big]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) \mu_0(\varrho) \mathrm{d}\varrho \\ &+ \frac{\tau_0 t^{g-2} \big[t(1-\varsigma_0) + \varsigma_0 \mathcal{T}(g-1) \big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} \int_0^{\mathcal{T}} \mu_0(\varrho) \mathrm{d}\varrho \\ &+ \frac{\sigma_0 \tau_0 t^{g-3} \mathcal{T}^2 (1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)} \int_0^{\mathcal{T}} \mu_0(\varrho) \mathrm{d}\varrho \\ =& \int_0^{\mathcal{T}} \mathbf{G}_g(t,\varrho) \mu_0(\varrho) \mathrm{d}\varrho, \end{aligned} \quad (5)$$

where $\mathbf{G}_g(t,\varrho)$ is provided in (3). □

Lemma 3. *Assume $\mu_1 \in \mathcal{C}(\mathcal{J})$ and $\hbar \in (2,3]$, the unique solution of FDE:*

$$\begin{cases} \mathcal{D}^{\hbar} w(t) = \mu_1(t);\ t \in \mathcal{J}, \\ \mathcal{D}^{\hbar-3} w(0) = \sigma_1 \mathcal{D}^{\hbar-3} w(\mathcal{T}),\ \mathcal{D}^{\hbar-2} w(0) = \varsigma_1 \mathcal{D}^{\hbar-2} w(\mathcal{T}),\ \mathcal{D}^{\hbar-1} w(0) = \tau_1 \mathcal{D}^{\hbar-1} w(\mathcal{T}) \end{cases}$$

is represented by the integral formula:

$$w(t) = \int_0^{\mathcal{T}} \mathbf{G}_{\hbar}(t,\varrho) \mu_1(\varrho) \mathrm{d}\varrho,$$

where $\mathbf{G}_{\hbar}(t,\varrho)$ is:

$$\mathbf{G}_{\hbar}(t,\varrho) = \begin{cases} \frac{(t-\varrho)^{\hbar-1}}{\Gamma(\hbar)} + \frac{\sigma_1 t^{\hbar-3} (\mathcal{T}-\varrho)^2}{2(1-\sigma_1)\Gamma(\hbar-2)} + \frac{\varsigma_1 t^{\hbar-3} \left[t(1-\sigma_1) + \sigma_1 \mathcal{T}(\hbar-2) \right] (\mathcal{T}-\varrho)}{(1-\sigma_1)(1-\varsigma_1)\Gamma(\hbar-1)} \\ + \frac{\tau_1 t^{\hbar-2} \left[t(1-\varsigma_1) + \varsigma_1 \mathcal{T}(\hbar-1) \right]}{(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar)} + \frac{\sigma_1 \tau_1 t^{\hbar-3} \mathcal{T}^2 (1+\varsigma_1)}{2(1-\sigma_1)(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar-2)}, & 0 \le \varrho < t \le \mathcal{T}, \\ \\ \frac{\sigma_1 t^{\hbar-3} (\mathcal{T}-\varrho)^2}{2(1-\sigma_1)\Gamma(\hbar-2)} + \frac{\varsigma_1 t^{\hbar-3} \left[t(1-\sigma_1) + \sigma_1 \mathcal{T}(\hbar-2) \right] (\mathcal{T}-\varrho)}{(1-\sigma_1)(1-\varsigma_1)\Gamma(\hbar-1)} \\ + \frac{\tau_1 t^{\hbar-2} \left[t(1-\varsigma_1) + \varsigma_1 \mathcal{T}(\hbar-1) \right]}{(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar)} + \frac{\sigma_1 \tau_1 t^{\hbar-3} \mathcal{T}^2 (1+\varsigma_1)}{2(1-\sigma_1)(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar-2)}, & 0 \le t < \varrho \le \mathcal{T}. \end{cases}$$

Proof. The proof reflects the strategy used in Lemma 2. □

Lemma 4. *Assume $\mu_2 \in \mathcal{C}(\mathcal{J})$ and $\ell \in (2,3]$, the unique solution of FDE:*

$$\begin{cases} \mathcal{D}^{\ell} w(t) = \mu_2(t); \ t \in \mathcal{J}, \\ \mathcal{D}^{\ell-3} w(0) = \sigma_2 \mathcal{D}^{\ell-3} w(\mathcal{T}), \ \mathcal{D}^{\ell-2} w(0) = \varsigma_2 \mathcal{D}^{\ell-2} w(\mathcal{T}), \ \mathcal{D}^{\ell-1} w(0) = \tau_2 \mathcal{D}^{\ell-1} w(\mathcal{T}) \end{cases}$$

is represented by the integral equation:

$$w(t) = \int_0^{\mathcal{T}} \mathbf{G}_{\ell}(t,\varrho)\mu_2(\varrho)d\varrho,$$

where $\mathbf{G}_{\ell}(t,\varrho)$ is:

$$\mathbf{G}_{\ell}(t,\varrho) = \begin{cases} \frac{\left(t-\varrho\right)^{\ell-1}}{\Gamma(\ell)} + \frac{\sigma_2 t^{\ell-3}\left(\mathcal{T}-\varrho\right)^2}{2(1-\sigma_2)\Gamma(\ell-2)} + \frac{\varsigma_2 t^{\ell-3}\left[t(1-\sigma_2)+\sigma_2\mathcal{T}(\ell-2)\right]\left(\mathcal{T}-\varrho\right)}{(1-\sigma_2)(1-\varsigma_2)\Gamma(\ell-1)} \\ +\frac{\tau_2 t^{\ell-2}\left[t(1-\varsigma_2)+\varsigma_2\mathcal{T}(\ell-1)\right]}{(1-\varsigma_2)(1-\tau_2)\Gamma(\ell)} + \frac{\sigma_2\tau_2 t^{\ell-3}\mathcal{T}^2\left(1+\varsigma_2\right)}{2(1-\sigma_2)(1-\varsigma_2)(1-\tau_2)\Gamma(\ell-2)}, & 0 \le \varrho < t \le \mathcal{T}, \\ \\ \frac{\sigma_2 t^{\ell-3}\left(\mathcal{T}-\varrho\right)^2}{2(1-\sigma_2)\Gamma(\ell-2)} + \frac{\varsigma_2 t^{\ell-3}\left[t(1-\sigma_2)+\sigma_2\mathcal{T}(\ell-2)\right]\left(\mathcal{T}-\varrho\right)}{(1-\sigma_2)(1-\varsigma_2)\Gamma(\ell-1)} \\ +\frac{\tau_2 t^{\ell-2}\left[t(1-\varsigma_2)+\varsigma_2\mathcal{T}(\ell-1)\right]}{(1-\varsigma_2)(1-\tau_2)\Gamma(\ell)} + \frac{\sigma_2\tau_2 t^{\ell-3}\mathcal{T}^2\left(1+\varsigma_2\right)}{2(1-\sigma_2)(1-\varsigma_2)(1-\tau_2)\Gamma(\ell-2)}, & 0 \le t < \varrho \le \mathcal{T}. \end{cases}$$

Proof. The proof reflects the strategy used in Lemma 2. □

For clarity and ease of understanding, the following notations are introduced:

$$\begin{aligned} \mathfrak{C}_{g} = \max\Big\{ & \frac{\mathcal{T}^3}{\Gamma(g+1)} + \Big|\frac{\sigma_0\mathcal{T}^3}{6(1-\sigma_0)\Gamma(g-2)}\Big| + \Big|\frac{\varsigma_0\mathcal{T}^3\left[1+|\sigma_0|(g-3)\right]}{2(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big| \\ & + \Big|\frac{\tau_0\mathcal{T}^3\left[1+|\varsigma_0|(g-2)\right]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big| + \Big|\frac{\sigma_0\tau_0\mathcal{T}^3(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big|\Big\}, \end{aligned} \tag{6}$$

$$\begin{aligned} \mathfrak{C}_{\hbar} = \max\Big\{ & \frac{\mathcal{T}^3}{\Gamma(\hbar+1)} + \Big|\frac{\sigma_1\mathcal{T}^3}{6(1-\sigma_1)\Gamma(\hbar-2)}\Big| + \Big|\frac{\varsigma_1\mathcal{T}^3\left[1+|\sigma_1|(\hbar-3)\right]}{2(1-\sigma_1)(1-\varsigma_1)\Gamma(\hbar-1)}\Big| \\ & + \Big|\frac{\tau_1\mathcal{T}^3\left[1+|\varsigma_1|(\hbar-2)\right]}{(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar)}\Big| + \Big|\frac{\sigma_1\tau_1\mathcal{T}^3(1+\varsigma_1)}{2(1-\sigma_1)(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar-2)}\Big|\Big\}, \end{aligned} \tag{7}$$

$$\begin{aligned} \mathfrak{C}_{\ell} = \max\Big\{ & \frac{\mathcal{T}^3}{\Gamma(\ell+1)} + \Big|\frac{\sigma_2\mathcal{T}^3}{6(1-\sigma_2)\Gamma(\ell-2)}\Big| + \Big|\frac{\varsigma_2\mathcal{T}^3\left[1+|\sigma_2|(\ell-3)\right]}{2(1-\sigma_2)(1-\varsigma_2)\Gamma(\ell-1)}\Big| \\ & + \Big|\frac{\tau_2\mathcal{T}^3\left[1+|\varsigma_2|(\ell-2)\right]}{(1-\varsigma_2)(1-\tau_2)\Gamma(\ell)}\Big| + \Big|\frac{\sigma_2\tau_2\mathcal{T}^3(1+\varsigma_2)}{2(1-\sigma_2)(1-\varsigma_2)(1-\tau_2)\Gamma(\ell-2)}\Big|\Big\} \end{aligned} \tag{8}$$

and

$$\mathfrak{C}_0 = \min\left\{1-\mathfrak{Q}_{\psi}, 1-\mathfrak{Q}_{\theta}, 1-\mathfrak{Q}_{\Upsilon}\right\}, \tag{9}$$

where

$$\begin{aligned} \mathfrak{Q}_{\psi} &= \mathfrak{C}_{g}\psi^*_{f_1} + \mathfrak{C}_{\hbar}\psi^*_{f_2} + \mathfrak{C}_{\ell}\psi^*_{f_3}, \\ \mathfrak{Q}_{\theta} &= \mathfrak{C}_{g}\theta^*_{f_1} + \mathfrak{C}_{\hbar}\theta^*_{f_2} + \mathfrak{C}_{\ell}\theta^*_{f_3}, \\ \mathfrak{Q}_{\Upsilon} &= \mathfrak{C}_{g}\Upsilon^*_{f_1} + \mathfrak{C}_{\hbar}\Upsilon^*_{f_2} + \mathfrak{C}_{\ell}\Upsilon^*_{f_3}. \end{aligned}$$

Suppose u, w, y represent solutions to problem (1), and $t \in \mathcal{G}$, then

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\sigma_0 t^{g-3}}{2(1-\sigma_0)\Gamma(g-2)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\varsigma_0 t^{g-3}\left[t(1-\sigma_0)+\sigma_0 \mathcal{T}(g-2)\right]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\tau_0 t^{g-2}\left[t(1-\varsigma_0)+\varsigma_0 \mathcal{T}(g-1)\right]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} \int_0^{\mathcal{T}} f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\sigma_0 \tau_0 t^{g-3} \mathcal{T}^2 (1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)} \int_0^{\mathcal{T}} f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho, \end{aligned}$$

$$\begin{aligned} w(t) = & \frac{1}{\Gamma(\hbar)} \int_0^t (t-\varrho)^{\hbar-1} f_2(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\sigma_1 t^{\hbar-3}}{2(1-\sigma_1)\Gamma(\hbar-2)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 f_2(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\varsigma_1 t^{\hbar-3}\left[t(1-\sigma_1)+\sigma_1 \mathcal{T}(\hbar-2)\right]}{(1-\sigma_1)(1-\varsigma_1)\Gamma(\hbar-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) f_2(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\tau_1 t^{\hbar-2}\left[t(1-\varsigma_1)+\varsigma_1 \mathcal{T}(\hbar-1)\right]}{(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar)} \int_0^{\mathcal{T}} f_2(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\sigma_1 \tau_1 t^{\hbar-3} \mathcal{T}^2 (1+\varsigma_1)}{2(1-\sigma_1)(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar-2)} \int_0^{\mathcal{T}} f_2(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \end{aligned}$$

and

$$\begin{aligned} y(t) = & \frac{1}{\Gamma(\ell)} \int_0^t (t-\varrho)^{\ell-1} f_3(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\sigma_2 t^{\ell-3}}{2(1-\sigma_2)\Gamma(\ell-2)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 f_3(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\varsigma_2 t^{\ell-3}\left[t(1-\sigma_2)+\sigma_2 \mathcal{T}(\ell-2)\right]}{(1-\sigma_2)(1-\varsigma_2)\Gamma(\ell-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) f_3(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\tau_2 t^{\ell-2}\left[t(1-\varsigma_2)+\varsigma_2 \mathcal{T}(\ell-1)\right]}{(1-\varsigma_2)(1-\tau_2)\Gamma(\ell)} \int_0^{\mathcal{T}} f_3(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ & + \frac{\sigma_2 \tau_2 t^{\ell-3} \mathcal{T}^2 (1+\varsigma_2)}{2(1-\sigma_2)(1-\varsigma_2)(1-\tau_2)\Gamma(\ell-2)} \int_0^{\mathcal{T}} f_3(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho. \end{aligned}$$

Now, to reformulate problem (1) as a fixed-point problem, introduce the operator $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ defined as follows:

$$\mathcal{F}(u, w, y)(t) = \begin{pmatrix} \int_0^{\mathcal{T}} \mathbf{G}_g(t, \varrho) f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ \int_0^{\mathcal{T}} \mathbf{G}_\hbar(t, \varrho) f_2(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \\ \int_0^{\mathcal{T}} \mathbf{G}_\ell(t, \varrho) f_3(\varrho, u(\varrho), w(\varrho), y(\varrho)) d\varrho \end{pmatrix} = \begin{pmatrix} \mathcal{F}_g(u, w, y)(t) \\ \mathcal{F}_\hbar(u, w, y)(t) \\ \mathcal{F}_\ell(u, w, y)(t) \end{pmatrix}. \quad (10)$$

Then, the solution to problem (1) corresponds to the fixed point of $\mathcal{F}$, where

$$\begin{aligned}\mathcal{F}_{g}(u,w,y)(t)&=\frac{1}{\Gamma(g)}\int_0^t(t-\varrho)^{g-1}f_1(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\sigma_0t^{g-3}}{2(1-\sigma_0)\Gamma(g-2)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)^2f_1(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\varsigma_0t^{g-3}\left[t(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)\right]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)f_1(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\tau_0t^{g-2}\left[t(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)\right]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\int_0^{\mathcal{T}}f_1(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\sigma_0\tau_0t^{g-3}\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\int_0^{\mathcal{T}}f_1(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\hbar}(u,w,y)(t)&=\frac{1}{\Gamma(\hbar)}\int_0^t(t-\varrho)^{\hbar-1}f_2(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\sigma_1t^{\hbar-3}}{2(1-\sigma_1)\Gamma(\hbar-2)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)^2f_2(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\varsigma_1t^{\hbar-3}\left[t(1-\sigma_1)+\sigma_1\mathcal{T}(\hbar-2)\right]}{(1-\sigma_1)(1-\varsigma_1)\Gamma(\hbar-1)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)f_2(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\tau_1t^{\hbar-2}\left[t(1-\varsigma_1)+\varsigma_1\mathcal{T}(\hbar-1)\right]}{(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar)}\int_0^{\mathcal{T}}f_2(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\sigma_1\tau_1t^{\hbar-3}\mathcal{T}^2(1+\varsigma_1)}{2(1-\sigma_1)(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar-2)}\int_0^{\mathcal{T}}f_2(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{\ell}(u,w,y)(t)&=\frac{1}{\Gamma(\ell)}\int_0^t(t-\varrho)^{\ell-1}f_3(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\sigma_2t^{\ell-3}}{2(1-\sigma_2)\Gamma(\ell-2)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)^2f_3(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\varsigma_2t^{\ell-3}\left[t(1-\sigma_2)+\sigma_2\mathcal{T}(\ell-2)\right]}{(1-\sigma_2)(1-\varsigma_2)\Gamma(\ell-1)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)f_3(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\tau_2t^{\ell-2}\left[t(1-\varsigma_2)+\varsigma_2\mathcal{T}(\ell-1)\right]}{(1-\varsigma_2)(1-\tau_2)\Gamma(\ell)}\int_0^{\mathcal{T}}f_3(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\\&+\frac{\sigma_2\tau_2t^{\ell-3}\mathcal{T}^2(1+\varsigma_2)}{2(1-\sigma_2)(1-\varsigma_2)(1-\tau_2)\Gamma(\ell-2)}\int_0^{\mathcal{T}}f_3(\varrho,u(\varrho),w(\varrho),y(\varrho))\mathrm{d}\varrho\end{aligned}$$

Incorporating the Leray–Schauder alternative theorem (1), the following theorem proves the existence of at least one solution to the given system (1).

Theorem 2. *Let $\Lambda_{f_i}, \psi_{f_i}, \theta_{f_i}, \Upsilon_{f_i}; (i = 1,2,3) : \mathcal{G} \to \mathcal{R}^+$ be functions, where for all $u, w, y \in \mathcal{R}$, the following conditions hold:*

$$\left|f_i(t,u(t),w(t),y(t))\right| \le \Lambda_{f_i}(t) + \psi_{f_i}(t)\left|u(t)\right| + \theta_{f_i}(t)\left|w(t)\right| + \Upsilon_{f_i}(t)\left|y(t)\right|,$$

with $\sup_{t\in\mathcal{G}}\Lambda_{f_i}(t) = \Lambda^*_{f_i}$, $\sup_{t\in\mathcal{G}}\psi_{f_i}(t) = \psi^*_{f_i}$, $\sup_{t\in\mathcal{G}}\theta_{f_i}(t) = \theta^*_{f_i}$, $\sup_{t\in\mathcal{G}}\Upsilon_{f_i}(t) = \Upsilon^*_{f_i}$ and $\Lambda^*_{f_1}, \Lambda^*_{f_2}, \Lambda^*_{f_3} > 0$.
In addition, it is supposed that $\mathfrak{Q}_\psi, \mathfrak{Q}_\theta, \mathfrak{Q}_\Upsilon < 1$. The system (1) under these conditions admits at least one solution.

Proof. To begin with, we establish the complete continuity of $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. Given the continuity of f_i $(i = 1, 2, 3)$, $\mathcal{F}$ also exhibits continuity. Let set $\mathfrak{B} \subseteq \mathcal{S}$ be defined as bounded. Consequently, there exist constants $\mathcal{N}_{f_i} > 0$, implying that $|f_i(t, u(t), w(t), y(t))| \leq \mathcal{N}_{f_i}$ $(i = 1, 2, 3)$, $\forall (u, w, y) \in \mathfrak{B}$. Consequently, for any $(u, w, y) \in \mathfrak{B}$, it follows that

$$
\begin{aligned}
& t^{3-g}|\mathcal{F}_g(u, w, y)(t)| \\
\leq\ & t^{3-g}\Big[\frac{1}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} \mathrm{d}\varrho + \Big|\frac{\sigma_0}{2(1-\sigma_0)\Gamma(g-2)}\Big| \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 \mathrm{d}\varrho \\
& + \Big|\frac{\varsigma_0[t(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big| \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)\mathrm{d}\varrho + \Big|\frac{\tau_0 t[t(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big| \int_0^{\mathcal{T}} \mathrm{d}\varrho \\
& + \Big|\frac{\sigma_0\tau_0\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big| \int_0^{\mathcal{T}} \mathrm{d}\varrho\Big] |f_1(\varrho, u(\varrho), w(\varrho), y(\varrho))|, \\
\leq\ & t^{3-g}\mathcal{N}_{f_1}\Big[\frac{1}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} \mathrm{d}\varrho + \Big|\frac{\sigma_0}{2(1-\sigma_0)\Gamma(g-2)}\Big| \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 \mathrm{d}\varrho \\
& + \Big|\frac{\varsigma_0[t(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big| \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)\mathrm{d}\varrho + \Big|\frac{\tau_0 t[t(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big| \int_0^{\mathcal{T}} \mathrm{d}\varrho \\
& + \Big|\frac{\sigma_0\tau_0\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big| \int_0^{\mathcal{T}} \mathrm{d}\varrho\Big],
\end{aligned}
$$

which implies that

$$
\|\mathcal{F}_g(u, w, y)\| \leq \mathcal{N}_{f_1}\mathfrak{C}_g. \tag{11}
$$

Similarly, we obtain

$$
\|\mathcal{F}_{\hbar}(u, w, y)\| \leq \mathcal{N}_{f_2}\mathfrak{C}_{\hbar} \tag{12}
$$

and

$$
\|\mathcal{F}_{\ell}(u, w, y)\| \leq \mathcal{N}_{f_3}\mathfrak{C}_{\ell}. \tag{13}
$$

Hence, the inequalities (11), (12), and (13) collectively establish the uniform boundedness of $\mathcal{F}$.

Following, the equicontinuity of $\mathcal{F}$ is demonstrated. Consider $0 \leq t_2 \leq t_1 \leq \mathcal{T}$. Then we obtain

$$
\begin{aligned}
& |t_1^{3-g}\mathcal{F}_g(u, w, y)(t_1) - t_2^{3-g}\mathcal{F}_g(u, w, y)(t_2)| \\
=\ & \Big|\frac{1}{\Gamma(g)} \int_0^{t_1} [t_1^{3-g}(t_1-\varrho)^{g-1} - t_2^{3-g}(t_2-\varrho)^{g-1}] f_1(\varrho, u(\varrho), w(\varrho), y(\varrho))\mathrm{d}\varrho \\
& - \frac{1}{\Gamma(g)} \int_{t_1}^{t_2} t_2^{3-g}(t_2-\varrho)^{g-1} f_1(\varrho, u(\varrho), w(\varrho), y(\varrho))\mathrm{d}\varrho \\
& + \frac{\varsigma_0[(t_1-t_2)(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) f_1(\varrho, u(\varrho), w(\varrho), y(\varrho))\mathrm{d}\varrho \\
& + \frac{\tau_0(t_1-t_2)[(t_1-t_2)(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} \int_0^{\mathcal{T}} f_1(\varrho, u(\varrho), w(\varrho), y(\varrho))\mathrm{d}\varrho\Big|.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&\left|t_1^{3-g}\mathcal{F}_g(u,w,y)(t_1)-t_2^{3-g}\mathcal{F}_g(u,w,y)(t_2)\right|\\
&\leq \mathcal{N}_{f_1}\Big(\Big|\frac{1}{\Gamma(g)}\int_0^{t_1}\big[t_1^{3-g}(t_1-\varrho)^{g-1}-t_2^{3-g}(t_2-\varrho)^{g-1}\big]\mathrm{d}\varrho-\frac{1}{\Gamma(g)}\int_{t_1}^{t_2}t_2^{3-g}(t_2-\varrho)^{g-1}\mathrm{d}\varrho\Big|\\
&\quad+\Big|\frac{\varsigma_0\big[(t_1-t_2)(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)\big]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)\mathrm{d}\varrho\Big|\\
&\quad+\Big|\frac{\tau_0(t_1-t_2)\big[(t_1-t_2)(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)\big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\int_0^{\mathcal{T}}\mathrm{d}\varrho\Big|\Big)\longrightarrow 0\quad\text{as}\quad t_1\longrightarrow t_2.
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
&\left|t_1^{3-\hbar}\mathcal{F}_\hbar(u,w,y)(t_1)-t_2^{3-\hbar}\mathcal{F}_\hbar(u,w,y)(t_2)\right|\\
&\leq \mathcal{N}_{f_2}\Big(\Big|\frac{1}{\Gamma(\hbar)}\int_0^{t_1}\big[t_1^{3-\hbar}(t_1-\varrho)^{\hbar-1}-t_2^{3-\hbar}(t_2-\varrho)^{\hbar-1}\big]\mathrm{d}\varrho-\frac{1}{\Gamma(\hbar)}\int_{t_1}^{t_2}t_2^{3-\hbar}(t_2-\varrho)^{\hbar-1}\mathrm{d}\varrho\Big|\\
&\quad+\Big|\frac{\varsigma_1\big[(t_1-t_2)(1-\sigma_1)+\sigma_1\mathcal{T}(\hbar-2)\big]}{(1-\sigma_1)(1-\varsigma_1)\Gamma(\hbar-1)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)\mathrm{d}\varrho\Big|\\
&\quad+\Big|\frac{\tau_1(t_1-t_2)\big[(t_1-t_2)(1-\varsigma_1)+\varsigma_1\mathcal{T}(\hbar-1)\big]}{(1-\varsigma_1)(1-\tau_1)\Gamma(\hbar)}\int_0^{\mathcal{T}}\mathrm{d}\varrho\Big|\Big)\longrightarrow 0\quad\text{as}\quad t_1\longrightarrow t_2
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|t_1^{3-\ell}\mathcal{F}_\ell(u,w,y)(t_1)-t_2^{3-\ell}\mathcal{F}_\ell(u,w,y)(t_2)\right|\\
&\leq \mathcal{N}_{f_3}\Big(\Big|\frac{1}{\Gamma(\ell)}\int_0^{t_1}\big[t_1^{3-\ell}(t_1-\varrho)^{\ell-1}-t_2^{3-\ell}(t_2-\varrho)^{\ell-1}\big]\mathrm{d}\varrho-\frac{1}{\Gamma(\ell)}\int_{t_1}^{t_2}t_2^{3-\ell}(t_2-\varrho)^{\ell-1}\mathrm{d}\varrho\Big|\\
&\quad+\Big|\frac{\varsigma_2\big[(t_1-t_2)(1-\sigma_2)+\sigma_2\mathcal{T}(\ell-2)\big]}{(1-\sigma_2)(1-\varsigma_2)\Gamma(\ell-1)}\int_0^{\mathcal{T}}(\mathcal{T}-\varrho)\mathrm{d}\varrho\Big|\\
&\quad+\Big|\frac{\tau_2(t_1-t_2)\big[(t_1-t_2)(1-\varsigma_2)+\varsigma_2\mathcal{T}(\ell-1)\big]}{(1-\varsigma_2)(1-\tau_2)\Gamma(\ell)}\int_0^{\mathcal{T}}\mathrm{d}\varrho\Big|\Big)\longrightarrow 0\quad\text{as}\quad t_1\longrightarrow t_2.
\end{aligned}
$$

Hence, $\mathcal{F}(u,w,y)$ demonstrates equicontinuity, establishing it as completely continuous.

Ultimately, we confirm the boundedness of the set $\mathcal{B}=\{(u,w,y)\in\mathcal{S}\,|\,(u,w,y)=\lambda\mathcal{F}(u,w,y),\ \lambda\in[0,1]\}$. Let us assume $(u,w,y)\in\mathcal{B}$, then $(u,w,y)=\lambda\mathcal{F}(u,w,y)$. For $t\in\mathcal{J}$, we have $u(t)=\lambda\mathcal{F}_g(u,w,y)(t)$, $w(t)=\lambda\mathcal{F}_\hbar(u,w,y)(t)$ and $y(t)=\lambda\mathcal{F}_\ell(u,w,y)(t)$. Then

$$
\begin{aligned}
&t^{3-g}|u(t)|\\
&\leq t^{3-g}\Big\{\frac{\mathcal{T}^3}{\Gamma(g+1)}+\Big|\frac{\sigma_0\mathcal{T}^3}{6(1-\sigma_0)\Gamma(g-2)}\Big|+\Big|\frac{\varsigma_0\mathcal{T}^3\big[1+|\sigma_0|(g-3)\big]}{2(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big|\\
&\quad+\Big|\frac{\tau_0\mathcal{T}^3\big[1+|\varsigma_0|(g-2)\big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big|+\Big|\frac{\sigma_0\tau_0\mathcal{T}^3(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big|\Big\}\\
&\quad\times\big(\Lambda_{f_1}(\varrho)+\psi_{f_1}(\varrho)|u(\varrho)|+\theta_{f_1}(\varrho)|w(\varrho)|+\Upsilon_{f_1}(\varrho)|y(\varrho)|\big).
\end{aligned}
\tag{14}
$$

Hence from (14), we have

$$
\|u\|\leq\mathfrak{C}_g\big(\Lambda_{f_1}^*+\psi_{f_1}^*\|u\|+\theta_{f_1}^*\|w\|+\Upsilon_{f_1}^*\|y\|\big).
$$

Similarly, we obtain

$$\|w\| \leq \mathcal{C}_{\hbar}\left(\Lambda^*_{f_2} + \psi^*_{f_2}\|u\| + \theta^*_{f_2}\|w\| + \Upsilon^*_{f_2}\|y\|\right),$$

and

$$\|y\| \leq \mathcal{C}_{\ell}\left(\Lambda^*_{f_3} + \psi^*_{f_3}\|u\| + \theta^*_{f_3}\|w\| + \Upsilon^*_{f_3}\|y\|\right),$$

which imply that

$$\|u\| + \|w\| + \|y\| = \left(\mathcal{C}_g\Lambda^*_{f_1} + \mathcal{C}_{\hbar}\Lambda^*_{f_2} + \mathcal{C}_{\theta}\Lambda^*_{f_3}\right) + \mathcal{Q}_{\psi}\|u\| + \mathcal{Q}_{\theta}\|w\| + \mathcal{Q}_{\Upsilon}\|y\|.$$

As a result, we have

$$\|(u,w,y)\| \leq \frac{\mathcal{C}_g\Lambda^*_{f_1} + \mathcal{C}_{\hbar}\Lambda^*_{f_2} + \mathcal{C}_{\theta}\Lambda^*_{f_3}}{\mathcal{C}_0}.$$

For each $t \in \mathcal{J}$, where $\mathcal{C}_0$ is provided in (9), signifying the boundedness of $\mathcal{B}$. Therefore, by virtue of the Leray–Schauder alternative, $\mathcal{F}$ possesses at least one fixed point, thereby ensuring the existence of at least one solution to problem (1). □

The second outcome hinges on the utilization of Banach's contraction principle (BCP).

Theorem 3. *Given the continuity assumption of functions f_i $(i = 1,2,3) : \mathcal{J} \times \mathcal{R}^3 \to \mathcal{R}$, and*

($\mathbf{H}_1$) *the existence of constants $\mathcal{K}_{f_i}, \mathcal{L}_{f_i}, \mathcal{M}_{f_i}$, such that for $u, w, y, u^*, w^*, y^* \in \mathcal{R}$, and $t \in \mathcal{J}$, it holds true that:*

$$\|f_i(t,u,w,y) - f_i(t,u^*,w^*,y^*)\| \\ \leq \mathcal{K}_{f_i}\|u - u^*\| + \mathcal{L}_{f_i}\|w - w^*\| + \mathcal{M}_{f_i}\|y - y^*\|.$$

In addition, suppose that

$$\mathcal{C}_g\aleph_{f_1} + \mathcal{C}_{\hbar}\aleph_{f_2} + \mathcal{C}_{\ell}\aleph_{f_3} < 1,$$

where

$$\begin{aligned}
\aleph_{f_1} &= \mathcal{K}_{f_1} + \mathcal{L}_{f_1} + \mathcal{M}_{f_1},\\
\aleph_{f_2} &= \mathcal{K}_{f_2} + \mathcal{L}_{f_2} + \mathcal{M}_{f_2},\\
\aleph_{f_3} &= \mathcal{K}_{f_3} + \mathcal{L}_{f_3} + \mathcal{M}_{f_3}.
\end{aligned}$$

Under these circumstances, the solution to problem (1) *will be unique.*

Proof. Let us define $\sup_{t\in\mathcal{J}} f_1(t,0,0,0) = \wp_g < \infty$, $\sup_{t\in\mathcal{J}} f_2(t,0,0,0) = \wp_{\hbar} < \infty$, and $\sup_{t\in\mathcal{J}} f_3(t,0,0,0) = \wp_{\ell} < \infty$, such that

$$r \geq \frac{\wp_g\mathcal{C}_g + \wp_{\hbar}\mathcal{C}_{\hbar} + \wp_{\ell}\mathcal{C}_{\ell}}{1 - [\mathcal{C}_g\aleph_{f_1} + \mathcal{C}_{\hbar}\aleph_{f_2} + \mathcal{C}_{\ell}\aleph_{f_3}]}.$$

We prove that $\mathcal{F}(\chi_r) \subset \chi_r$, where

$$\chi_r = \left\{(u,w,y) \in \mathcal{S} : \|(u,w,y)\| \leq r\right\}.$$

For $(u,w,y) \in \chi_r$, we have

$$
\begin{aligned}
& t^{3-g}\left|\mathcal{F}_g(u,w,y)(t)\right| \\
& \leq t^{3-g}\Big[\frac{1}{\Gamma(g)}\int_0^t (t-\varrho)^{g-1}\mathrm{d}\varrho + \Big|\frac{\sigma_0}{2(1-\sigma_0)\Gamma(g-2)}\Big|\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2\mathrm{d}\varrho \\
& \quad + \Big|\frac{\varsigma_0[t(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big|\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)\mathrm{d}\varrho + \Big|\frac{\tau_0 t[t(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big|\int_0^{\mathcal{T}} \mathrm{d}\varrho \\
& \quad + \Big|\frac{\sigma_0\tau_0\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big|\int_0^{\mathcal{T}} \mathrm{d}\varrho\Big] \\
& \quad \times \left(\left|f_1(\varrho,u(\varrho),w(\varrho),y(\varrho)) - f_1(\varrho,0,0,0)\right| + \left|f_1(\varrho,0,0,0)\right|\right), \\
& \leq \Big\{\frac{\mathcal{T}^3}{\Gamma(g+1)} + \Big|\frac{\sigma_0\mathcal{T}^3}{6(1-\sigma_0)\Gamma(g-2)}\Big| + \Big|\frac{\varsigma_0\mathcal{T}^3[1+|\sigma_0|(g-3)]}{2(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big| \\
& \quad + \Big|\frac{\tau_0\mathcal{T}^3[1+|\varsigma_0|(g-2)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big| + \Big|\frac{\sigma_0\tau_0\mathcal{T}^3(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big|\Big\} \\
& \quad \times \left(\mathcal{K}_{f_1}\|u\| + \mathcal{L}_{f_1}\|w\| + \mathcal{M}_{f_1}\|y\| + \wp_g\right), \\
& \leq \mathfrak{C}_g\left[\aleph_{f_1} r + \wp_g\right].
\end{aligned}
$$

Hence,

$$
\|\mathcal{F}_g(u,w,y)\| \leq \mathfrak{C}_g\left[\aleph_{f_1} r + \wp_g\right]. \tag{15}
$$

Similarly, we can obtain

$$
\|\mathcal{F}_h(u,w,y)\| \leq \mathfrak{C}_h\left[\aleph_{f_2} r + \wp_h\right] \tag{16}
$$

and

$$
\|\mathcal{F}_\ell(u,w,y)\| \leq \mathfrak{C}_\ell\left[\aleph_{f_3} r + \wp_\ell\right]. \tag{17}
$$

Combining the aforementioned inequalities (15), (16), and (17), we derive the following:

$$
\|\mathcal{F}(u,w,y)\| \leq r.
$$

Considering $(u,w,y),(u^*,w^*,y^*) \in \mathcal{S}$, and for any $t \in \mathcal{J}$, it follows that

$$
\begin{aligned}
& t^{3-g}\left|\mathcal{F}_g(u,w,y)(t) - \mathcal{F}_g(u^*,w^*,y^*)(t)\right| \\
& \leq t^{3-g}\Big[\frac{1}{\Gamma(g)}\int_0^t (t-\varrho)^{g-1}\mathrm{d}\varrho + \Big|\frac{\sigma_0}{2(1-\sigma_0)\Gamma(g-2)}\Big|\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2\mathrm{d}\varrho \\
& \quad + \Big|\frac{\varsigma_0[t(1-\sigma_0)+\sigma_0\mathcal{T}(g-2)]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big|\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)\mathrm{d}\varrho + \Big|\frac{\tau_0 t[t(1-\varsigma_0)+\varsigma_0\mathcal{T}(g-1)]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big|\int_0^{\mathcal{T}} \mathrm{d}\varrho \\
& \quad + \Big|\frac{\sigma_0\tau_0\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big|\int_0^{\mathcal{T}} \mathrm{d}\varrho\Big] \\
& \quad \times \left|f_1(\varrho,u(\varrho),w(\varrho),y(\varrho)) - f_1(\varrho,u^*(\varrho),w^*(\varrho),y^*(\varrho))\right|\mathrm{d}\varrho, \\
& \leq \mathfrak{C}_g\left(\mathcal{K}_{f_1}\|u-u^*\| + \mathcal{L}_{f_1}\|w-w^*\| + \mathcal{M}_{f_1}\|y-y^*\|\right)
\end{aligned}
$$

and hence, we obtain

$$
\|\mathcal{F}_g(u,w,y) - \mathcal{F}_g(u^*,w^*,y^*)\| \leq \mathfrak{C}_g\aleph_{f_1}\left(\|u-u^*\| + \|w-w^*\| + \|y-y^*\|\right). \tag{18}
$$

Similarly,

$$\|\mathcal{F}_{\hbar}(u,w,y)-\mathcal{F}_{\hbar}(u^{*},w^{*},y^{*})\|\leq \mathcal{C}_{\hbar}\aleph_{f_2}(\|u-u^{*}\|+\|w-w^{*}\|+\|y-y^{*}\|), \tag{19}$$

and

$$\|\mathcal{F}_{\ell}(u,w,y)-\mathcal{F}_{\ell}(u^{*},w^{*},y^{*})\|\leq \mathcal{C}_{\ell}\aleph_{f_3}(\|u-u^{*}\|+\|w-w^{*}\|+\|y-y^{*}\|). \tag{20}$$

The inequalities (18), (19), and (20) lead us to conclude that

$$\|\mathcal{F}(u,w,y)-\mathcal{F}(u^{*},w^{*},y^{*})\|\leq [\mathcal{C}_{g}\aleph_{f_1}+\mathcal{C}_{\hbar}\aleph_{f_2}+\mathcal{C}_{\ell}\aleph_{f_3}]\|(u,w,y)-(u^{*},w^{*},y^{*})\|.$$

As $\mathcal{C}_{g}\aleph_{f_1}+\mathcal{C}_{\hbar}\aleph_{f_2}+\mathcal{C}_{\ell}\aleph_{f_3}<1$, $\mathcal{F}$ qualifies as a contraction operator. By BCP, $\mathcal{F}$ possesses a unique fixed point, implying the uniqueness of the solution to system (1). □

4. Stability Results

Let us review definitions associated with Ulam stability:

Let Θ_{f_i} $(i=1,2,3):\mathcal{J}\to\mathcal{R}^{+}$ be non-decreasing functions, and let $\epsilon_{f_i}>0$. We consider the set of inequalities given below:

$$\begin{cases}\left|\mathcal{D}^{g}u(t)-f_1(t,u(t),w(t),y(t))\right|\leq \epsilon_{f_1}, & t\in\mathcal{J},\\ \left|\mathcal{D}^{\hbar}w(t)-f_2(t,u(t),w(t),y(t))\right|\leq \epsilon_{f_2}, & t\in\mathcal{J},\\ \left|\mathcal{D}^{\hbar}y(t)-f_3(t,u(t),w(t),y(t))\right|\leq \epsilon_{f_3}, & t\in\mathcal{J}.\end{cases} \tag{21}$$

Definition 3 ([60]). *The system (1) is identified as UH stable if certain positive constants $\mathbf{C}_{g,\hbar,\ell}=(C_g,C_{\hbar},C_{\ell})$ exist. These constants, along with $\epsilon=(\epsilon_g,\epsilon_{\hbar},\epsilon_{\ell})>0$, guarantee that for every solution $(u,w,y)\in\mathcal{S}$ of (21), there exists a unique solution $(v,\chi,z)\in\mathcal{S}$. This solution satisfies*

$$\left|(u,w,y)(t)-(v,\chi,z)(t)\right|\leq \mathbf{C}_{g,\hbar,\ell}\epsilon,\ \ t\in\mathcal{J}. \tag{22}$$

Definition 4 ([60]). *The system (1) is identified as generalized UH stable if a function $\Phi_{g,\hbar,\ell}\in C(\mathcal{R}^{+},\mathcal{R}^{+})$ with $\Phi_{g,\hbar,\ell}(0)=0$ exists. This function, given a solution $(u,w,y)\in\mathcal{S}$ of (21), ensures the existence of a unique solution $(v,\chi,z)\in\mathcal{S}$ for problem (1). Moreover, this solution satisfies*

$$\left|(u,w,y)(t)-(v,\chi,z)(t)\right|\leq \Phi_{g,\hbar,\ell}(\epsilon),\ \ t\in\mathcal{J}. \tag{23}$$

Remark 1. *We designate $(u,w,y)\in\mathcal{S}$ as a solution to inequality (21) provided that there exist functions Ψ_{f_i} $(i=1,2,3)\in C(\mathcal{J},\mathcal{R})$, hinging on u,w,y, respectively, satisfying the conditions*

(A_1) $|\Psi_{f_i}(t)|\leq\epsilon_{f_i}$, $t\in\mathcal{J}$;

(A_2) *For $t\in\mathcal{J}$, the system of equations is described as follows:*

$$\begin{cases}\mathcal{D}^{g}u(t)=f_1(t,u(t),w(t),y(t))+\Psi_{f_1}(t),\\ \mathcal{D}^{\hbar}w(t)=f_2(t,u(t),w(t),y(t))+\Psi_{f_2}(t),\\ \mathcal{D}^{\ell}y(t)=f_3(t,u(t),w(t),y(t))+\Psi_{f_3}(t).\end{cases}$$

Lemma 5. *If $(u,w,y)\in\mathcal{S}$ constitutes a solution to inequality (21), then we obtain*

$$\begin{cases}\|u-n_1\|\leq\mathcal{C}_{g}\epsilon_{f_1}, & t\in\mathcal{J},\\ \|w-n_2\|\leq\mathcal{C}_{\hbar}\epsilon_{f_2}, & t\in\mathcal{J},\\ \|y-n_3\|\leq\mathcal{C}_{\ell}\epsilon_{f_3}, & t\in\mathcal{J}.\end{cases}$$

Proof. Using (A_2) from Remark 1 and considering $t \in \mathcal{G}$, it follows that

$$\begin{cases} \mathcal{D}^{g} u(t) = f_1(t, u(t), w(t), y(t)) + \Psi_{f_1}(t), \\ \mathcal{D}^{h} w(t) = f_2(t, u(t), w(t), y(t)) + \Psi_{f_2}(t), \\ \mathcal{D}^{\ell} y(t) = f_3(t, u(t), w(t), y(t)) + \Psi_{f_3}(t), \\ \mathcal{D}^{g-3} u(0) = \sigma_0 \mathcal{D}^{g-3} u(\mathcal{T}),\ \mathcal{D}^{g-2} u(0) = \varsigma_0 \mathcal{D}^{g-2} u(\mathcal{T}),\ \mathcal{D}^{g-1} u(0) = \tau_0 \mathcal{D}^{g-1} u(\mathcal{T}), \\ \mathcal{D}^{h-3} w(0) = \sigma_1 \mathcal{D}^{h-3} w(\mathcal{T}),\ \mathcal{D}^{h-2} w(0) = \varsigma_1 \mathcal{D}^{h-2} w(\mathcal{T}),\ \mathcal{D}^{h-1} w(0) = \tau_1 \mathcal{D}^{h-1} w(\mathcal{T}), \\ \mathcal{D}^{\ell-3} y(0) = \sigma_2 \mathcal{D}^{\ell-3} y(\mathcal{T}),\ \mathcal{D}^{\ell-2} y(0) = \varsigma_2 \mathcal{D}^{\ell-2} y(\mathcal{T}),\ \mathcal{D}^{\ell-1} y(0) = \tau_2 \mathcal{D}^{\ell-1} y(\mathcal{T}). \end{cases} \tag{24}$$

Therefore, considering Lemma 1, we can express the solution to the first equation in (24) as follows:

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} \big[f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) + \Psi_{f_1}(\varrho)\big] \mathrm{d}\varrho \\ & + \frac{\sigma_0 t^{g-3}}{2(1-\sigma_0)\Gamma(g-2)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 \big[f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) + \Psi_{f_1}(\varrho)\big] \mathrm{d}\varrho \\ & + \frac{\varsigma_0 t^{g-3}\big[t(1-\sigma_0) + \sigma_0 \mathcal{T}(g-2)\big]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) \big[f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) + \Psi_{f_1}(\varrho)\big] \mathrm{d}\varrho \\ & + \frac{\tau_0 t^{g-2}\big[t(1-\varsigma_0) + \varsigma_0 \mathcal{T}(g-1)\big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} \int_0^{\mathcal{T}} \big[f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) + \Psi_{f_1}(\varrho)\big] \mathrm{d}\varrho \\ & + \frac{\sigma_0 \tau_0 t^{g-3} \mathcal{T}^2 (1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)} \int_0^{\mathcal{T}} \big[f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) + \Psi_{f_1}(\varrho)\big] \mathrm{d}\varrho, \end{aligned} \tag{25}$$

From Equation (25), we have

$$\begin{aligned} t^{3-g}\big|u(t) - n_1(t)\big| \le {} & \frac{t^{3-g}}{\Gamma(g)} \int_0^t (t-\varrho)^{g-1} \big|\Psi_{f_1}(\varrho)\big| \mathrm{d}\varrho \\ & + \frac{\sigma_0}{2(1-\sigma_0)\Gamma(g-2)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 \big|\Psi_{f_1}(\varrho)\big| \mathrm{d}\varrho \\ & + \frac{\varsigma_0\big[t(1-\sigma_0) + \sigma_0 \mathcal{T}(g-2)\big]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)} \int_0^{\mathcal{T}} (\mathcal{T}-\varrho) \big|\Psi_{f_1}(\varrho)\big| \mathrm{d}\varrho \\ & + \frac{\tau_0 t\big[t(1-\varsigma_0) + \varsigma_0 \mathcal{T}(g-1)\big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)} \int_0^{\mathcal{T}} \big|\Psi_{f_1}(\varrho)\big| \mathrm{d}\varrho \\ & + \frac{\sigma_0 \tau_0 \mathcal{T}^2 (1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)} \int_0^{\mathcal{T}} \big|\Psi_{f_1}(\varrho)\big| \mathrm{d}\varrho. \end{aligned} \tag{26}$$

Here, $n_1(t)$ represents terms devoid of Ψ_{f_1}. Utilizing (6) alongside (A_1) from Remark 1, (26) transforms into:

$$\|u - n_1\| \le \mathfrak{C}_g \epsilon_{f_1}.$$

Applying a similar approach to the second equation of (25), we arrive at

$$\|w - n_2\| \le \mathfrak{C}_h \epsilon_{f_2},$$

and

$$\|y - n_3\| \le \mathfrak{C}_\ell \epsilon_{f_3}.$$

□

Theorem 4. *Under the hypothesis* $(\mathbf{H}_1)$ *and if*

$$\Delta = 1 - \big[\Omega_{f_2}\mathcal{M}_{f_2}\Omega_{f_3}\mathcal{L}_{f_3} + \Omega_{f_1}\mathcal{L}_{f_1}(\Omega_{f_2}\mathcal{K}_{f_2} + \Omega_{f_3}\mathcal{K}_{f_3}\Omega_{f_2}\mathcal{M}_{f_2}) + \Omega_{f_1}\mathcal{M}_{f_1}(\Omega_{f_3}\mathcal{K}_{f_3} + \Omega_{f_2}\mathcal{K}_{f_2}\Omega_{f_3}\mathcal{L}_{f_3})\big] > 0. \tag{27}$$

Then problem (1) *is UH stable, where* $\Omega_{f_1} = \frac{\mathfrak{C}_g}{(1-\mathfrak{C}_g\mathcal{K}_{f_1})}$, $\Omega_{f_2} = \frac{\mathfrak{C}_\hbar}{(1-\mathfrak{C}_\hbar\mathcal{L}_{f_2})}$ *and* $\Omega_{f_3} = \frac{\mathfrak{C}_\ell}{(1-\mathfrak{C}_\ell\mathcal{M}_{f_3})}$.

Proof. Let $(u, w, y) \in \mathcal{S}$ represent the solution to (21), while $(v, \chi, z) \in \mathcal{S}$ denote the unique solution to the provided system:

$$\begin{cases} \mathcal{D}^g v(t) = f_1(t, v(t), \chi(t), z(t)), \ t \in \mathcal{J}, \\ \mathcal{D}^\hbar \chi(t) = f_2(t, v(t), \chi(t), z(t)), \ t \in \mathcal{J}, \\ \mathcal{D}^\ell z(t) = f_3(t, v(t), \chi(t), z(t)), \ t \in \mathcal{J}, \\ \mathcal{D}^{g-3}v(0) = \sigma_0\mathcal{D}^{g-3}v(\mathcal{T}), \ \mathcal{D}^{g-2}v(0) = \varsigma_0\mathcal{D}^{g-2}v(\mathcal{T}), \ \mathcal{D}^{g-1}v(0) = \tau_0\mathcal{D}^{g-1}v(\mathcal{T}), \\ \mathcal{D}^{\hbar-3}\chi(0) = \sigma_1\mathcal{D}^{\hbar-3}\chi(\mathcal{T}), \ \mathcal{D}^{\hbar-2}\chi(0) = \varsigma_1\mathcal{D}^{\hbar-2}\chi(\mathcal{T}), \ \mathcal{D}^{\hbar-1}\chi(0) = \tau_1\mathcal{D}^{\hbar-1}\chi(\mathcal{T}), \\ \mathcal{D}^{\ell-3}z(0) = \sigma_2\mathcal{D}^{\ell-3}z(\mathcal{T}), \ \mathcal{D}^{\ell-2}z(0) = \varsigma_2\mathcal{D}^{\ell-2}z(\mathcal{T}), \ \mathcal{D}^{\ell-1}z(0) = \tau_2\mathcal{D}^{\ell-1}z(\mathcal{T}). \end{cases} \tag{28}$$

In light of Lemma 1, for $t \in \mathcal{J}$, the solution to the first equation of (28) takes the form:

$$\begin{aligned} v(t) = {}& \frac{1}{\Gamma(g)}\int_0^t (t-\varrho)^{g-1} f_1(\varrho, v(\varrho), \chi(\varrho), z(\varrho))\mathrm{d}\varrho \\ &+ \frac{\sigma_0 t^{g-3}}{2(1-\sigma_0)\Gamma(g-2)}\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2 f_1(\varrho, v(\varrho), \chi(\varrho), z(\varrho))\mathrm{d}\varrho \\ &+ \frac{\varsigma_0 t^{g-3}\big[t(1-\sigma_0) + \sigma_0\mathcal{T}(g-2)\big]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\int_0^{\mathcal{T}} (\mathcal{T}-\varrho) f_1(\varrho, v(\varrho), \chi(\varrho), z(\varrho))\mathrm{d}\varrho \\ &+ \frac{\tau_0 t^{g-2}\big[t(1-\varsigma_0) + \varsigma_0\mathcal{T}(g-1)\big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\int_0^{\mathcal{T}} f_1(\varrho, v(\varrho), \chi(\varrho), z(\varrho))\mathrm{d}\varrho \\ &+ \frac{\sigma_0\tau_0 t^{g-3}\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\int_0^{\mathcal{T}} f_1(\varrho, v(\varrho), \chi(\varrho), z(\varrho))\mathrm{d}\varrho. \end{aligned} \tag{29}$$

Consider

$$t^{3-g}\big|u(t) - v(t)\big| \le t^{3-g}\big|u(t) - n_1(t)\big| + t^{3-g}\big|n_1(t) - v(t)\big|. \tag{30}$$

Using Lemma 5 in the above inequality (30), we obtain

$$\begin{aligned} & t^{3-g}\big|u(t) - v(t)\big| \\ \le {}& \mathfrak{C}_g\epsilon_{f_1} + t^{3-g}\Big[\frac{1}{\Gamma(g)}\int_0^t (t-\varrho)^{g-1}\mathrm{d}\varrho + \Big|\frac{\sigma_0}{2(1-\sigma_0)\Gamma(g-2)}\Big|\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)^2\mathrm{d}\varrho \\ &+ \Big|\frac{\varsigma_0\big[t(1-\sigma_0) + \sigma_0\mathcal{T}(g-2)\big]}{(1-\sigma_0)(1-\varsigma_0)\Gamma(g-1)}\Big|\int_0^{\mathcal{T}} (\mathcal{T}-\varrho)\mathrm{d}\varrho + \Big|\frac{\tau_0 t\big[t(1-\varsigma_0) + \varsigma_0\mathcal{T}(g-1)\big]}{(1-\varsigma_0)(1-\tau_0)\Gamma(g)}\Big|\int_0^{\mathcal{T}} \mathrm{d}\varrho \\ &+ \Big|\frac{\sigma_0\tau_0\mathcal{T}^2(1+\varsigma_0)}{2(1-\sigma_0)(1-\varsigma_0)(1-\tau_0)\Gamma(g-2)}\Big|\int_0^{\mathcal{T}} \mathrm{d}\varrho\Big] \\ &\times \big|f_1(\varrho, u(\varrho), w(\varrho), y(\varrho)) - f_1(\varrho, v(\varrho), \chi(\varrho), z(\varrho))\big|. \end{aligned} \tag{31}$$

Using $(\mathbf{H}_1)$ and (6) in (31), we have

$$\|u - v\| \le \mathfrak{C}_g\epsilon_{f_1} + \mathfrak{C}_g\big(\mathcal{K}_{f_1}\|u - v\| + \mathcal{L}_{f_1}\|w - \chi\| + \mathcal{M}_{f_1}\|y - z\|\big).$$

So, we obtain

$$\|u-v\| \leq \Omega_{f_1}\left[\epsilon_{f_1} + \mathcal{L}_{f_1}\|w-\chi\| + \mathcal{M}_{f_1}\|y-z\|\right]. \tag{32}$$

Similarly, we can obtain

$$\|w-\chi\| \leq \Omega_{f_2}\left[\epsilon_{f_2} + \mathcal{K}_{f_2}\|u-v\| + \mathcal{M}_{f_2}\|y-z\|\right], \tag{33}$$

and

$$\|y-z\| \leq \Omega_{f_3}\left[\epsilon_{f_3} + \mathcal{K}_{f_3}\|u-v\| + \mathcal{L}_{f_3}\|w-\chi\|\right]. \tag{34}$$

We write Equations (32)–(34) as:

$$\begin{bmatrix} 1 & -\Omega_{f_1}\mathcal{L}_{f_1} & -\Omega_{f_1}\mathcal{M}_{f_1} \\ -\Omega_{f_2}\mathcal{K}_{f_2} & 1 & -\Omega_{f_2}\mathcal{M}_{f_2} \\ -\Omega_{f_3}\mathcal{K}_{f_3} & -\Omega_{f_3}\mathcal{L}_{f_3} & 1 \end{bmatrix} \begin{bmatrix} \|u-v\| \\ \|w-\chi\| \\ \|y-z\| \end{bmatrix} \leq \begin{bmatrix} \Omega_{f_1}\epsilon_{f_1} \\ \Omega_{f_2}\epsilon_{f_2} \\ \Omega_{f_3}\epsilon_{f_3} \end{bmatrix}.$$

Given the preceding matrices, the result is

$$\begin{bmatrix} \|u-v\| \\ \|w-\chi\| \\ \|y-z\| \end{bmatrix} \leq \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \Omega_{f_1}\epsilon_{f_1} \\ \Omega_{f_2}\epsilon_{f_2} \\ \Omega_{f_3}\epsilon_{f_3} \end{bmatrix}, \tag{35}$$

where Δ is given in (27) and

$$\begin{aligned} a_{11} &= \frac{1-\Omega_{f_2}\mathcal{M}_{f_2}\Omega_{f_3}\mathcal{L}_{f_3}}{\Delta}, & a_{12} &= \frac{\Omega_{f_1}\mathcal{L}_{f_1} + \Omega_{f_1}\mathcal{M}_{f_1}\Omega_{f_3}\mathcal{L}_{f_3}}{\Delta}, & a_{13} &= \frac{\Omega_{f_1}\mathcal{M}_{f_1} + \Omega_{f_1}\mathcal{L}_{f_1}\Omega_{f_2}\mathcal{M}_{f_2}}{\Delta}, \\ a_{21} &= \frac{\Omega_{f_2}\mathcal{K}_{f_2} + \Omega_{f_2}\mathcal{M}_{f_2}\Omega_{f_3}\mathcal{K}_{f_3}}{\Delta}, & a_{22} &= \frac{1-\Omega_{f_1}\mathcal{M}_{f_1}\Omega_{f_3}\mathcal{K}_{f_3}}{\Delta}, & a_{23} &= \frac{\Omega_{f_2}\mathcal{M}_{f_2} + \Omega_{f_1}\mathcal{M}_{f_1}\Omega_{f_2}\mathcal{K}_{f_2}}{\Delta}, \\ a_{31} &= \frac{\Omega_{f_3}\mathcal{K}_{f_3} + \Omega_{f_2}\mathcal{K}_{f_2}\Omega_{f_3}\mathcal{L}_{f_3}}{\Delta}, & a_{32} &= \frac{\Omega_{f_3}\mathcal{L}_{f_3} + \Omega_{f_1}\mathcal{L}_{f_1}\Omega_{f_3}\mathcal{K}_{f_3}}{\Delta}, & a_{33} &= \frac{1-\Omega_{f_1}\mathcal{L}_{f_1}\Omega_{f_2}\mathcal{K}_{f_2}}{\Delta}. \end{aligned}$$

From (35), we obtain

$$\begin{aligned} \|u-v\| &\leq a_{11}\Omega_{f_1}\epsilon_{f_1} + a_{12}\Omega_{f_2}\epsilon_{f_2} + a_{13}\Omega_{f_3}\epsilon_{f_3}, \\ \|w-\chi\| &\leq a_{21}\Omega_{f_1}\epsilon_{f_1} + a_{22}\Omega_{f_2}\epsilon_{f_2} + a_{23}\Omega_{f_3}\epsilon_{f_3}, \\ \|y-z\| &\leq a_{31}\Omega_{f_1}\epsilon_{f_1} + a_{32}\Omega_{f_2}\epsilon_{f_2} + a_{33}\Omega_{f_3}\epsilon_{f_3}, \end{aligned}$$

Following this, we obtain

$$\begin{aligned} \|u-v\| + \|w-\chi\| + \|y-z\| \leq{}& \Omega_{f_1}\epsilon_{f_1}(a_{11}+a_{21}+a_{31}) + \Omega_{f_2}\epsilon_{f_2}(a_{12}+a_{22}+a_{32}) \\ &+ \Omega_{f_3}\epsilon_{f_3}(a_{13}+a_{23}+a_{33}). \end{aligned} \tag{36}$$

Let $\epsilon = \max\{\epsilon_{f_1}, \epsilon_{f_2}, \epsilon_{f_3}\}$. Consequently, by (36), we arrive at

$$\|(u,w,y)-(v,\chi,z)\| \leq \mathbf{C}_{g,\hbar,\ell}\,\epsilon, \tag{37}$$

where

$$\mathbf{C}_{g,\hbar,\ell} = \left[\Omega_{f_1}(a_{11}+a_{21}+a_{31}) + \Omega_{f_2}(a_{12}+a_{22}+a_{32}) + \Omega_{f_3}(a_{13}+a_{23}+a_{33})\right].$$

□

Remark 2. *If we set $\Phi_{g,\hbar,\ell}(\epsilon) = \mathbf{C}_{g,\hbar,\ell}\epsilon$ with $\Phi_{g,\hbar,\ell}(0) = 0$ in (37), system (1) exhibits generalized UH stability, in accordance with Definition 4.*

5. Example

Example 1. *Let us examine the FDEs systems given below:*

$$\begin{cases} \mathcal{D}^{\frac{5}{2}}u(t) = e^{-3t} + \dfrac{1}{11}u(t)\cos(t) + \dfrac{e^{-t}}{20}w(t) + \dfrac{e^{-t}}{12}\sin y(t), \\ \mathcal{D}^{\frac{7}{3}}w(t) = t\sqrt{t^2+3} + \dfrac{1}{4\pi}u(t)\tan^{-1}(t) + \dfrac{1}{\sqrt{80+t^2}}\sin w(t) + \dfrac{1}{13}y(t)\sin(t), \\ \mathcal{D}^{\frac{9}{4}}y(t) = \dfrac{e^{-t}}{5} + \dfrac{e^{-t}}{12}\sin u(t) + \dfrac{1}{16+t}w(t) + \dfrac{e^{-t}}{15}y(t)\cos(t), \\ \mathcal{D}^{\frac{-1}{2}}u(0) = \dfrac{1}{2}\mathcal{D}^{\frac{-1}{2}}u(1),\ \mathcal{D}^{\frac{1}{2}}u(0) = \dfrac{1}{3}\mathcal{D}^{\frac{1}{2}}u(1),\ \mathcal{D}^{\frac{3}{2}}u(0) = -\mathcal{D}^{\frac{3}{2}}u(1), \\ \mathcal{D}^{\frac{-2}{3}}w(0) = \dfrac{1}{2}\mathcal{D}^{\frac{-2}{3}}w(1),\ \mathcal{D}^{\frac{1}{3}}w(0) = \dfrac{1}{3}\mathcal{D}^{\frac{1}{3}}w(1),\ \mathcal{D}^{\frac{4}{3}}w(0) = -\mathcal{D}^{\frac{4}{3}}w(1), \\ \mathcal{D}^{\frac{-3}{4}}y(0) = \dfrac{1}{2}\mathcal{D}^{\frac{-3}{4}}y(1),\ \mathcal{D}^{\frac{1}{4}}y(0) = \dfrac{1}{3}\mathcal{D}^{\frac{1}{4}}y(1),\ \mathcal{D}^{\frac{5}{4}}y(0) = -\mathcal{D}^{\frac{5}{4}}y(1). \end{cases} \tag{38}$$

where $t \in [0,1]$. From problem (38), we have $g = 5/2$, $\hbar = 7/3$, $\ell = 9/4$, $\mathcal{T} = 1$, $\sigma_0 = \sigma_1 = \sigma_2 = 1/2$, $\varsigma_0 = \varsigma_1 = \varsigma_2 = 1/3$ and $\tau_0 = \tau_1 = \tau_2 = -1$. Moreover, $\Lambda^*_{f_1} = 1/e^3$, $\psi^*_{f_1} = 1/11$, $\theta^*_{f_1} = 1/20e$, $\Upsilon^*_{f_1} = 1/12e$, $\Lambda^*_{f_2} = 2$, $\psi^*_{f_2} = 1/16$, $\theta^*_{f_2} = 1/9$, $\Upsilon^*_{f_2} = 1/13$, $\Lambda^*_{f_3} = 1/5e$, $\psi^*_{f_3} = 1/12e$, $\theta^*_{f_3} = 1/17$, and $\Upsilon^*_{f_3} = 1/15e$. Then the conditions of the Theorem 2:

$$\mathfrak{Q}_\psi = \mathfrak{C}_g\psi^*_{f_1} + \mathfrak{C}_\hbar\psi^*_{f_2} + \mathfrak{C}_\ell\psi^*_{f_3} \simeq 0.31519492 < 1.$$
$$\mathfrak{Q}_\theta = \mathfrak{C}_g\theta^*_{f_1} + \mathfrak{C}_\hbar\theta^*_{f_2} + \mathfrak{C}_\ell\theta^*_{f_3} \simeq 0.31558696 < 1.$$
$$\mathfrak{Q}_\Upsilon = \mathfrak{C}_g\Upsilon^*_{f_1} + \mathfrak{C}_\hbar\Upsilon^*_{f_2} + \mathfrak{C}_\ell\Upsilon^*_{f_3} \simeq 0.22346367 < 1.$$

are satisfied. For Theorem 3, we can see in problem (38), that $\mathcal{K}_{f_1} = 1/11$, $\mathcal{L}_{f_1} = 1/20e$, $\mathcal{M}_{f_1} = 1/12e$, $\mathcal{K}_{f_2} = 1/16$, $\mathcal{L}_{f_2} = 1/9$, $\mathcal{M}_{f_2} = 1/13$ and $\mathcal{K}_{f_3} = 1/12e$, $\mathcal{L}_{f_3} = 1/17$, $\mathcal{M}_{f_3} = 1/15e$. Therefore,

$$\mathfrak{C}_g\aleph_{f_1} + \mathfrak{C}_\hbar\aleph_{f_2} + \mathfrak{C}_\ell\aleph_{f_3} \simeq 0.85424557 < 1.$$

Thus, the system (38) possesses a unique solution.
Moreover, the condition of Theorem 4:

$$\Delta \simeq 0.95104343 > 0$$

is also fulfilled. Consequently, the problem (38) demonstrates UH stability and generalized UH stability.

6. Conclusions

This manuscript has effectively demonstrated the existence and uniqueness of solutions for problem (1) using the fixed point theory. Moreover, it has derived the essential criteria for UH and generalized UH stability. By illustrating an example, the practical implications of these findings have been highlighted, underscoring the significance of the research in broader applications.

The findings are new and intriguing. Specifically, by setting $\sigma_\ell = \varsigma_\ell = \tau_\ell = 0$ $(\ell = 0, 1, 2)$ and $\mathscr{g}, \hbar, \ell = 3$ in the proposed system (1), the following third-order ODEs system alongside initial conditions, is derived.

$$\begin{cases} u'''(t) = f_1(t, u(t), w(t), y(t)); \ t \in \mathcal{J}, \\ w'''(t) = f_2(t, u(t), w(t), y(t)); \ t \in \mathcal{J}, \\ y'''(t) = f_3(t, u(t), w(t), y(t)); \ t \in \mathcal{J}, \\ u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \\ w(0) = 0, \ w'(0) = 0, \ w''(0) = 0, \\ y(0) = 0, \ y'(0) = 0, \ y''(0) = 0. \end{cases}$$

Similarly, by setting $\sigma_\ell = \varsigma_\ell = \tau_\ell = -1$ $(\ell = 0, 1, 2)$ and $\mathscr{g}, \hbar, \ell = 3$ in system (1), a system of ODEs of third-order alongside anti-periodic BCs is obtained, given as

$$\begin{cases} u'''(t) = f_1(t, u(t), w(t), y(t)); \ t \in \mathcal{J}, \\ w'''(t) = f_2(t, u(t), w(t), y(t)); \ t \in \mathcal{J}, \\ y'''(t) = f_3(t, u(t), w(t), y(t)); \ t \in \mathcal{J}, \\ u(0) = -u(\mathcal{T}), \ u'(0) = -u'(\mathcal{T}), \ u''(0) = -u''(\mathcal{T}), \\ w(0) = -w(\mathcal{T}), \ w'(0) = -w'(\mathcal{T}), \ w''(0) = -w''(\mathcal{T}), \\ y(0) = -y(\mathcal{T}), \ y'(0) = -y'(\mathcal{T}), \ y''(0) = -y''(\mathcal{T}). \end{cases}$$

As far as we know, this is the first manuscript addressing a nonlocal generalized fractional order BVP involving a tripled system of nonlinear FDEs. Furthermore, this manuscript is the first to obtain solutions for a third-order ODEs system alongside initial and anti-periodic BCs involving three equations using the RL fractional derivative. Future research directions include exploring alternative fractional operators, integrating fractal-fractional derivatives for more comprehensive modeling, examining other types of stability, and extending the study to multi-point boundary conditions.

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