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Multiple Solutions to the Fractional p-Laplacian Equations of Schrödinger–Hardy-Type Involving Concave–Convex Nonlinearities

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Abstract: This paper is concerned with nonlocal fractional p-Laplacian Schrödinger–Hardy-type equations involving concave–convex nonlinearities. The first aim is to demonstrate the L^∞-bound for any possible weak solution to our problem. As far as we know, the global a priori bound for weak solutions to nonlinear elliptic problems involving a singular nonlinear term such as Hardy potentials has not been studied extensively. To overcome this, we utilize a truncated energy technique and the De Giorgi iteration method. As its application, we demonstrate that the problem above has at least two distinct nontrivial solutions by exploiting a variant of Ekeland's variational principle and the classical mountain pass theorem as the key tools. Furthermore, we prove the existence of a sequence of infinitely many weak solutions that converges to zero in the L^∞-norm. To derive this result, we employ the modified functional method and the dual fountain theorem.

Keywords: fractional p-Laplacian; a priori bounds; De Giorgi iteration; variational methods

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1. Introduction

Research on elliptic problems involving nonlocal fractional Laplacian or more general integro-differential operators has gained attention due to their relevance in terms of pure or applied mathematical theories that are used to illustrate some concrete phenomena, such as the image process, minimal surfaces and the Levy process, quasi-geostrophic flows, the thin obstacle problem, and multiple scattering. In addition, comprehensive studies on this topic can be found in works such as [1–6].

Meanwhile, in recent years, considerable attention has been paid to the investigation of stationary problems related to singular nonlinearities, because they can be used to describe a model for applied economical models and several physical phenomena; see [7–9] for more comprehensive details and examples. Furthermore, some recent papers [10–19] dealing with the existence and multiplicity of solutions to elliptic problems with singular coefficients have captured the attention of many mathematicians in the past few decades.

In this paper, we are concerned with the Schrödinger–Hardy-type nonlinear equation driven by the nonlocal fractional p-Laplacian as follows:

$$\mathcal{L}v(y) + \mathfrak{b}(y)|v|^{p-2}v = \mu\frac{|v|^{p-2}v}{|y|^{sp}} + \lambda a(y)|v|^{r-2}v + \theta g(y,v) \quad \text{in} \quad \mathbb{R}^N, \tag{1}$$

where $s \in (0,1)$, $p \in (1,+\infty)$, $sp < N$, and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition with superlinear nonlinearity and $a, \mathfrak{b}$ are potential functions that is specified later. Here, $\mathcal{L}$ is a nonlocal operator defined pointwise as

$$\mathcal{L}v(y) = 2\int_{\mathbb{R}^N} |v(y) - v(z)|^{p-2}(v(y) - v(z))\mathcal{K}(y,z)dz \quad \text{for all } y \in \mathbb{R}^N,$$

where $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ is a kernel function that fulfills the following conditions:
$(\mathcal{L}1)$ $m\mathcal{K} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $m(y,z) = \min\{|y-z|^p, 1\}$;

($\mathcal{L}2$) There exists a positive constant γ_0 such that $\mathcal{K}(y,z) \geq \gamma_0|y-z|^{-(N+sp)}$ for almost all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $y \neq z$;
($\mathcal{L}3$) $\mathcal{K}(y,z) = \mathcal{K}(z,y)$ for all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$.

When $\mathcal{K}(y,z) = |y-z|^{-(N+sp)}$, the operator $\mathcal{L}$ becomes the fractional p-Laplacian operator $(-\Delta)_p^s$ defined as

$$(-\Delta)_p^s v(y) = 2\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(y)} \frac{|v(y)-v(z)|^{p-2}(v(y)-v(z))}{|y-z|^{N+sp}}\,dz, \quad y \in \mathbb{R}^N,$$

where $B_\varepsilon(y) := \{y \in \mathbb{R}^N : |y-z| \leq \varepsilon\}$.

In this regard, the first aim of this paper is to provide the L^∞-bound for any possible weak solutions to Problem (1). As far as we know, the uniform boundedness of any possible weak solutions to the nonlocal fractional p-Laplacian problems of Schrödinger type with a singular coefficient such as Hardy potentials has not been studied extensively, and we are only aware of the study in [20]. In [20], Choudhuri leveraged the bootstrap argument known as the Moser iteration technique (for example, see [21,22]) as the main tool to obtain an a priori bound of weak solutions to the homogeneous Dirichlet boundary value problem of a fractional p-Kirchhoff type involving singular nonlinearity. In contrast to the approach in [20], the De Giorgi iteration method and a truncated energy technique are utilized as key tools; these were first suggested in [23]. This approach is based on the recent studies in [16,24]. However, this elliptic equation of the fractional p-Laplacian involving Hardy potential has more complex nonlinearities than the problem without such a potential and thus requires more challenging analyses to be carried out carefully. In particular, our approach is more useful than the Moser iteration technique as it is applicable to p-Laplacian or double-phase problems involving the Hardy potential; see [16,24]. This is one of novelties of this paper.

As its application, we demonstrate two multiplicity results of nontrivial weak solutions to the Schrödinger–Hardy-type nonlinear equation driven by the nonlocal fractional p-Laplacian. From a mathematical point of view, such elliptic problems with a singular coefficient have some technical difficulties because this operator is not homogeneous and the energy functional does not guarantee the compactness condition of the Palais–Smale type. In particular, it is not easy to show that the Palais–Smale-type sequence has the compactness property in the desired function space because of the appearance of the Hardy potential. Related to this fact, the authors in [11,14,15,19] discussed the multiplicity results of solutions by employing various critical point theorems in [25,26] without proving the Palais–Smale compactness condition. The authors in [11] studied the existence of at least one nontrivial weak solution to a nonlinear elliptic equation with a Dirichlet boundary condition:

$$\begin{cases} -\Delta_p v = \mu \frac{|v|^{p-2}v}{|y|^p} + \lambda g(y,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$ and $\mu \geq 0$ are two real parameters, $1 < p < N$, and $h : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. Inspired by this paper, Khodabakhshi et al. [15] determined the existence of at least three distinct generalized solutions when $\mu = -1$ in (2). In this case, we also cite the study in [14] for infinitely many solutions and the study in [19] for the existence of three solutions to elliptic equations driven by p-Laplacian-like operators. In this direction, concerning the elliptic problem involving the fractional p-Laplacian

$$\begin{cases} (-\Delta)_p^s v(y) = \mu \frac{|v|^{p-2}v}{|y|^p} + \lambda g(y,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

the authors of [10] proved the existence of at least three solutions to Problem (3) with $\mu = -1$. Furthermore, based on the study in [13], which is a result in a local setting, they proved the existence of two solutions to Problem (3) with $\mu = 1$ by demonstrating the Palais–Smale compactness property, which is essential in applying the critical point theorem in [27]. However, in this case, if we consider a standard argument, it is not difficult to show this property for a Palais–Smale-type sequence because can we easily show some topological properties for the energy functional corresponding to the principal part in (3) with $\mu \leq 0$. Very recently, in a different approach from [10,11,13–15,19], Kim and coworkers [17,18] presented several existence results for infinitely many solutions to Kirchhoff–Hardy-type nonlinear elliptic problems as some extension of Problems (2) and (3) when $\mu \leq 0$.

In this respect, as mentioned earlier, the present paper is dedicated to establishing two multiplicity results of solutions to (1) when μ belongs to the interval $(-\infty, \mu^*)$ for some positive constant μ^*. The first is to prove the existence of at least two distinct nontrivial solutions that belong to the L^∞ space by exploiting a variant of the Ekeland variational principle in [28] and the mountain pass theorem in [29] instead of the critical points theorems in [25–27]. To this end, by analyzing the boundedness of a Palais–Smale-type sequence and the Hardy inequality for the fractional Sobolev space, which is inspired by recent papers in [12,16,30], we overcome the lack of compactness of the Euler–Lagrange functional, which is the main difficulty. This is another novelty of this paper, which is different from previous studies [10,11,13–15,19]. In [12], Fiscella provided an existence result for at least one nontrivial solution to the Schrödinger–Kirchhoff-type fractional p-Laplacian involving Hardy potentials:

$$\left(a + b[v]_{s,p}^{p(\theta-1)}\right)(-\Delta)_p^s v(y) + \mathfrak{b}(y)|v|^{p-2}v = \mu \frac{|v|^{p-2}v}{|y|^{sp}} + \lambda g(y, v) \quad \text{in} \quad \mathbb{R}^N,$$

where $a > 0$, $b \geq 0$, μ is a real parameter, and g is a continuous function verifying the Ambrosetti–Rabinowitz condition in [29]. The main tool for obtaining this result is the classical mountain pass theorem. The existence of at least one nontrivial solution to a double-phase problem involving Hardy potential can be found in [30]. To obtain this, he proved the Palais–Smale compactness condition using the cut-off function method. Motivated by this work, the authors of [16] demonstrated several multiplicity results and a priori bounds of nontrivial weak solutions to Kirchhoff–Schrödinger–Hardy-type nonlinear problems with the p-Laplacian:

$$-K\left(\int_{\mathbb{R}^N} |\nabla v|^p \, dy\right) \operatorname{div}(|\nabla v|^{p-2}\nabla v) + \mathfrak{b}(y)|v|^{p-2}v = \mu \frac{|v|^{p-2}v}{|y|^p} + g(y, v) \text{ in } \mathbb{R}^N,$$

where $1 < p < p^*$, $K \in C(\mathbb{R}_0^+)$ is a real function, $\mathfrak{b} : \mathbb{R}^N \to (0, \infty)$ is a potential function satisfying some conditions, and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is the Carathéodory function that does not satisfy the Ambrosetti–Rabinowitz condition.

Finally, as an application of the L^∞-bound for weak solutions, which is our first main result, we derive the existence of a sequence of infinitely many small energy solutions converging to 0 in L^∞-norm. This is based on related studies [18,31–36] without the Hardy potential; for the Hardy potential, see [18]. To the best of our knowledge, for nonlinear elliptic problems with Hardy potentials, the L^∞-bound for weak solutions converging to zero has not been studied extensively, and we are only aware of the study in [18]. However, even considering the Kirchhoff–Hardy-type nonlinear equations in [18], the present paper obtains this multiplicity result for the case in which μ belongs to the interval $(-\infty, \mu^*)$ for a positive constant μ^*, which, in a sense, is an extension of the study in [18]. In this respect, we combine the modified functional method with the dual fountain theorem as in [18,32] to provide the final main result. For this reason, our approach is different from previously related works [31,35,36] that used the global variational formulation given in [37]. Moreover, our problem has a nonlocal operator and the Hardy potential, which requires us to perform more complex analyses than those of [18,32,33].

This paper is structured as follows: In Section 2, we review some necessary preliminary knowledge for the fractional Sobolev spaces that we use throughout the paper. Section 3 demonstrates the L^∞-bound for any possible weak solution to Problem (1). As its application, in Section 4, we offer the existence of at least two nontrivial solutions belonging to L^∞-space by showing some auxiliary results related to Problem (1). Finally, we offer the existence of a sequence of solutions converging to zero in the L^∞-norm.

2. Preliminaries

In this section, we briefly present some definitions and essential properties of the fractional Sobolev spaces to be used in the present paper. We let $0 < s < 1 < p < +\infty$ be real numbers and p_s^* be the fractional critical Sobolev exponent, that is,

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ \psi \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(y) - \psi(z)|^p}{|y - z|^{N+ps}} \, dydz < +\infty \right\},$$

endowed with the norm

$$\|\psi\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|\psi\|_{L^p(\mathbb{R}^N)}^p + |\psi|_{W^{s,p}(\mathbb{R}^N)}^p \right)^{\frac{1}{p}},$$

where

$$\|\psi\|_{L^p(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N} \mathfrak{b}(y)|\psi(y)|^p \, dy \quad \text{and} \quad |\psi|_{W^{s,p}(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(y) - \psi(z)|^p}{|y - z|^{N+ps}} \, dydz.$$

Then, $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space. Also, space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, that is, $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ (see, e.g., [38,39]).

Lemma 1 ([39,40])**.** *Let $0 < s < 1 < p < +\infty$ be such that $ps < N$. Then, there exists a positive constant $C > 0$ depending on s, p, and N such that*

$$\|\psi\|_{L^{p_s^*}(\mathbb{R}^N)} \leq C \, |\psi|_{W^{s,p}(\mathbb{R}^N)}$$

for all $\psi \in W^{s,p}(\mathbb{R}^N)$. Also, space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, p_s^*]$. Moreover, the embedding

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow L^t_{loc}(\mathbb{R}^N)$$

is compact for $t \in [p, p_s^*)$.

Now, let us consider the space $W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)$ defined as follows:

$$W_{\mathcal{K}}^{s,p}(\mathbb{R}^N) := \left\{ \psi \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi(y) - \psi(z)|^p \mathcal{K}(y,z) \, dydz < +\infty \right\},$$

where a kernel function $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0, +\infty)$ satisfies conditions ($\mathcal{L}1$)–($\mathcal{L}3$). By ($\mathcal{L}1$), the function

$$(y,z) \mapsto (\psi(y) - \psi(z))\mathcal{K}^{\frac{1}{p}}(y,z) \in L^p(\mathbb{R}^{2N})$$

for any $\psi \in C_0^\infty(\mathbb{R}^N)$. Let us denote by $W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\psi\|_{W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)} := \left(\|\psi\|^p_{L^p(\mathbb{R}^N)} + |\psi|^p_{W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

where

$$|\psi|^p_{W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y)-\psi(z)|^p \mathcal{K}(y,z)\,dydz.$$

Lemma 2 ([41])**.** *Let $0 < s < 1 < p < +\infty$ be such that $ps < N$, and let $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ satisfy assumptions ($\mathcal{L}$1)–($\mathcal{L}$3). If $\psi \in W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)$, then $\psi \in W^{s,p}(\mathbb{R}^N)$. Moreover,*

$$\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \le \max\{1, \gamma_0^{-\frac{1}{p}}\}\|\psi\|_{W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)},$$

where γ_0 is given in ($\mathcal{L}$2).

Next, we assume that the potential function $\mathfrak{b}$ fulfills the condition

(V) $\mathfrak{b} \in C(\mathbb{R}^N)$, $\inf_{y\in\mathbb{R}^N} \mathfrak{b}(y) > 0$, and $\mathrm{meas}\{y \in \mathbb{R}^N : \mathfrak{b}(y) \le \mathfrak{b}_0\} < +\infty$ for all $\mathfrak{b}_0 \in \mathbb{R}$.

On the linear subspace,

$$X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) := \left\{\psi \in L^p(\mathfrak{b},\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y)-\psi(z)|^p \mathcal{K}(y,z)\,dydz < +\infty\right\},$$

we equip the norm

$$\|\psi\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} := \left([\psi]^p_{p,\mathcal{K}} + \|\psi\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

where

$$[\psi]^p_{p,\mathcal{K}} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y)-\psi(z)|^p \mathcal{K}(y,z)\,dydz \quad \text{and} \quad \|\psi\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)} := \int_{\mathbb{R}^N} \mathfrak{b}(y)|\psi|^p\,dy.$$

Then, $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ is continuously embedded into $W^{s,p}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N), \|\cdot\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)})$ is also a separable reflexive Banach space.

From Lemmas 1 and 2, we can offer the following consequence directly.

Lemma 3 ([41])**.** *Let $0 < s < 1 < p < +\infty$ be such that $ps < N$, and let $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ satisfy assumptions ($\mathcal{L}$1)–($\mathcal{L}$3). Then, there exists a positive constant $C_0 = C_0(s,p,N)$ such that for any $\psi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ and $1 \le q \le p^*_s$,*

$$\begin{aligned}\|\psi\|^p_{L^q(\mathbb{R}^N)} &\le C_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(y)-\psi(z)|^p}{|y-z|^{N+ps}}dydz \\ &\le \frac{C_0}{\gamma_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y)-\psi(z)|^p \mathcal{K}(y,z)\,dydz,\end{aligned}$$

where γ_0 is given in ($\mathcal{L}$2). In addition, the space $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p^*_s]$ and the embedding

$$X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

is compact for $q \in [p, p^*_s)$.

The following assertion is the fractional Hardy inequality, which is given in [42].

Lemma 4. *Let $N \geq 1$, $0 < s < 1 \leq p$ and let $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ fulfil conditions ($\mathcal{L}1$)–($\mathcal{L}3$). Then, for any $\psi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, when $sp < N$, and for any $\psi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)\setminus\{0\}$, when $sp > N$,*

$$\|\psi\|^p_{H_p} := \int_{\mathbb{R}^N} \frac{|\psi(y)|^p}{|y|^{sp}} dy \leq c_H \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(y) - \psi(z)|^p}{|y - z|^{N+sp}} dydz$$
$$\leq \frac{c_H}{\gamma_0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi(y) - \psi(z)|^p \mathcal{K}(y,z) dydz,$$

where $c_H := c_H(N,s,p)$ is a positive constant.

Throughout this paper, the kernel function $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ ensures assumptions ($\mathcal{L}1$)–($\mathcal{L}3$). Moreover, $\langle \cdot, \cdot \rangle$ denotes the pairing of $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ and its dual $(X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N))^*$.

3. Variational Setting and a Priori Bound of Solution

In this section, we present the variational framework related to the given problem and then provide the L^∞-bound of any possible solutions to (1) when μ belongs to the interval $(-\infty, \mu^*)$ for some positive constant μ^*.

Definition 1. *Let $0 < s < 1 < p < +\infty$ be such that $ps < N$. We say that $v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ is a weak solution of Problem (1) if*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^{p-2}(v(y) - v(z))(\varphi(y) - \varphi(z))K(y,z)\, dydz + \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^{p-2}v\varphi\, dy$$
$$= \mu \int_{\mathbb{R}^N} \frac{|v|^{p-2}v}{|y|^{sp}} \varphi\, dy + \lambda \int_{\mathbb{R}^N} a(y)|v|^{r-2}v\varphi\, dy + \theta \int_{\mathbb{R}^N} g(y,v)\varphi\, dy$$

for all $\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$.

Let us define a functional $\Phi_{s,p} : X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi_{s,p}(v) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^p \mathcal{K}(y,z)\, dydz + \frac{1}{p} \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\, dy.$$

Then, it is obvious that the functional $\Phi_{s,p}$ is well defined on $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, $\Phi_{s,p} \in C^1(X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N), \mathbb{R})$ and its Fréchet derivative is given by, for any $\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$,

$$\langle \Phi'_{s,p}(v), \varphi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^{p-2}(v(y) - v(z))(\varphi(y) - \varphi(z))\mathcal{K}(y,z)\, dydz$$
$$+ \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^{p-2}v\varphi\, dy.$$

Denoting $G(y,t) = \int_0^t g(y,s)\, ds$, we suppose that

(A1) $1 < r < p < q < p_s^*$ and $0 \leq a \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p}{p-r}}(\mathbb{R}^N)$.

(G1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exists a non-negative function $b \in L^\infty(\mathbb{R}^N)$ such that

$$|g(y,\xi)| \leq b(y)|\xi|^{q-1}$$

for all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}$.

Under assumptions (A1) and (G1), we define the functional $\Psi_{\lambda,\mu} : X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi_{\lambda,\mu}(v) = \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}}\,dy + \frac{\lambda}{r}\int_{\mathbb{R}^N} a(y)|v|^r\,dy + \theta\int_{\mathbb{R}^N} G(y,v)\,dy.$$

Then, it follows that $\Psi_{\lambda,\mu} \in C^1(X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\left\langle \Psi'_{\lambda,\mu}(v),\varphi\right\rangle &= \mu\int_{\mathbb{R}^N} \frac{|v|^{p-2}v}{|y|^{sp}}\varphi\,dy + \lambda\int_{\mathbb{R}^N} a(y)|v|^{r-2}v\varphi\,dy \\ &\quad + \theta\int_{\mathbb{R}^N} g(y,v)\varphi\,dy\end{aligned}$$

for any $v,\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$. Next, we define a functional $\mathcal{I}_{\mu,\lambda} : X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{I}_{\mu,\lambda}(v) = \Phi_{s,p}(v) - \Psi_{\lambda,\mu}(v).$$

Then, we know that the functional $\mathcal{I}_{\mu,\lambda} \in C^1(X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle \mathcal{I}'_{\mu,\lambda}(v),\varphi\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^{p-2}(v(y)-v(z))(\varphi(y)-\varphi(z))\mathcal{K}(y,z)\,dydz \\ &\quad + \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^{p-2}v\varphi\,dy - \mu\int_{\mathbb{R}^N} \frac{|v|^{p-2}v}{|y|^{sp}}\varphi\,dy \\ &\quad - \lambda\int_{\mathbb{R}^N} a(y)|v|^{r-2}v\varphi\,dy - \theta\int_{\mathbb{R}^N} g(y,v)\varphi\,dy\end{aligned}$$

for any $v,\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$.

To obtain Theorem 1, which is our first main result, we need the following notable Lemma introduced in (Lemma 2.2 [23]).

Lemma 5. *Let $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ be a sequence of positive numbers satisfying the recursion inequality*

$$\mathcal{Z}_{n+1} \le c\tau^n Z_n^{1+\delta}, \qquad n = 0,1,2,\cdots$$

for some $\tau > 1, c > 0$ and $\delta > 0$. If $\mathcal{Z}_0 \le \min\{1, c^{(-1)/\delta}\tau^{(-1)/\delta^2}\}$, then $\mathcal{Z}_n \le 1$ for some $n \in \mathbb{N}\cup\{0\}$. Moreover,

$$\mathcal{Z}_n \le \min\left\{1, c^{(-1)/\delta}\tau^{(-1)/\delta^2}\tau^{(-n)/\delta}\right\}$$

for any $n \ge n_0$, where n_0 is the smallest $n \in \mathbb{N}\cup\{0\}$ satisfying $\mathcal{Z}_n \le 1$. In particular, $\mathcal{Z}_n \to 0$ as $n \to \infty$.

Now, we show the regularity-type result via the De Giorgi iteration argument and the localization method. The fundamental idea of the proof of this consequence follows from the study in [16]; see also [33].

Theorem 1. *We assume that* (V), (A1), *and* (G1) *hold. If v is a weak solution of Problem* (1)*, then there is a positive constant $\mu^* > 0$ such that $v \in L^\infty(\mathbb{R}^N)$ and there are positive constants $\eta, \mathcal{C}$ independent of v such that*

$$\|v\|_{L^\infty(\mathbb{R}^N)} \le \mathcal{C}\|v\|^{\eta}_{L^q(\mathbb{R}^N)}$$

for any $\mu \in (-\infty,\mu^*)$, where μ appears in Problem (1)*.*

Proof. Let $\mathcal{A}_m = \{y \in \mathbb{R}^N : v(y) > m\}$, $\widetilde{\mathcal{A}}_m = \{y \in \mathbb{R}^N : -v(y) > m\}$ for $m > 0$. We note that $|\mathcal{A}_m|$ and $|\widetilde{\mathcal{A}}_m|$ are finite for any $m \in \mathbb{N}$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$. Taking a test function $u = (v-m)_+ \in X(\mathbb{R}^N)$ in (1) and integrating over $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^{p-2}(v(y)-v(z))(u(y)-u(z))\mathcal{K}(y,z)\,dydz+\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^{p-2}vu\,dy$$
$$=\mu\int_{\mathbb{R}^N}\frac{|v|^{p-2}v}{|y|^{sp}}u\,dy+\lambda\int_{\mathbb{R}^N}a(y)|v|^{r-2}vu\,dy+\theta\int_{\mathbb{R}^N}g(y,v)u\,dy.$$

Using inequality $|\alpha-\beta|^{\gamma-2}(\alpha-\beta)(\alpha_+-\beta_+)\geq|\alpha_+-\beta_+|^{\gamma}$ for all $\alpha,\beta,\gamma\in\mathbb{R}$ with $\gamma>1$ and (F1), we deduce from the last equality that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)-u(z)|^p\mathcal{K}(y,z)\,dydz+\int_{\mathcal{A}_m}\mathfrak{b}(y)|v(y)|^{p-2}v(v-m)\,dy\\&\quad-\mu\int_{\mathcal{A}_m}\frac{|v|^{p-2}v}{|y|^{sp}}(v-m)\,dy\\&\leq\lambda\int_{\mathcal{A}_m}a(y)|v|^{r-2}v(v-m)\,dy+\theta\int_{\mathcal{A}_m}g(y,v)(v-m)\,dy.\end{aligned}$$

Let us set $\mu^*=\gamma_0{c_H}^{-1}$, where c_H is given in Lemma 4 and γ_0 is given in ($\mathcal{L}$2).

Case 1. $\mu\in(0,\mu^*)$.

Since $v\geq v-m>0$ on $\mathcal{A}_m$, by Lemma 4, we infer that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)-u(z)|^p\mathcal{K}(y,z)\,dydz\\&\quad+\int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy-\mu\int_{\mathcal{A}_m}\frac{|v|^{p-2}v}{|y|^{sp}}(v-m)\,dy\\&\geq\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)-u(z)|^p\mathcal{K}(y,z)\,dydz+\int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy-\mu\int_{\mathcal{A}_m}\frac{|v|^p}{|y|^{sp}}\,dy\\&\geq\int_{\mathcal{A}_m}\int_{\mathcal{A}_m}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dydz+\int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy-\mu\int_{\mathcal{A}_m}\frac{|v|^p}{|y|^{sp}}\,dy\\&\geq\left(1-\frac{\mu c_H}{\gamma_0}\right)\left(\int_{\mathcal{A}_m}\int_{\mathcal{A}_m}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dydz+\int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy\right).\end{aligned}\tag{4}$$

and thus, in accordance with (A1), (G1), and the Hölder inequality, we have

$$\begin{aligned}&\left(1-\frac{\mu c_H}{\gamma_0}\right)\left(\int_{\mathcal{A}_m}\int_{\mathcal{A}_m}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dydz+\int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy\right)\\&\qquad\leq\lambda\int_{\mathcal{A}_m}a(y)|v|^{r-2}v(v-m)\,dy+\theta\int_{\mathcal{A}_m}g(y,v)(v-m)\,dy\\&\qquad\leq\lambda\|a\|_{L^\infty(\mathbb{R}^N)}\int_{\mathcal{A}_m}|v|^r\,dy+\theta\|b\|_{L^\infty(\mathbb{R}^N)}\int_{\mathcal{A}_m}|v|^q\,dy\\&\qquad\leq\left(1+m^{r-q}\right)\left(\lambda\|a\|_{L^\infty(\mathbb{R}^N)}+\theta\|b\|_{L^\infty(\mathbb{R}^N)}\right)\int_{\mathcal{A}_m}|v|^q dy.\end{aligned}\tag{5}$$

We put $m_n:=m_*(2-1/2^n), n=0,1,2,\cdots$, with $m_*>0$ specified later and

$$\mathcal{Z}_n:=\int_{\mathcal{A}_{m_n}}(v-m_n)^q\,dy.$$

Since $m_*\leq m_n\leq m_{n+1}\leq 2m_*$ for all $n\in\mathbb{N}$, we have

$$\int_{\mathcal{A}_{m_n}}(v-m_n)^q\,dy\geq\int_{\mathcal{A}_{m_{n+1}}}|v|^q\left(1-\frac{m_n}{m_{n+1}}\right)^q dy\geq\int_{\mathcal{A}_{m_{n+1}}}\frac{|v|^q}{2^{q(n+2)}}\,dy$$

and therefore

$$\mathcal{Z}_n\geq\int_{\mathcal{A}_{m_{n+1}}}\frac{|v|^q}{2^{q(n+2)}}\,dy.$$

Thus,

$$\int_{\mathcal{A}_{m_{n+1}}} |v|^q \, dy \le \ell_1^{n+2} \mathcal{Z}_n, \tag{6}$$

where $\ell_1 := 2^q > 1$. For the Lebesgue measure of $\mathcal{A}_{m_{n+1}}$, we deduce that

$$|\mathcal{A}_{m_{n+1}}| \le \int_{\mathcal{A}_{m_{n+1}}} \left(\frac{v - m_n}{m_{n+1} - m_n} \right)^q dy \le \int_{\mathcal{A}_{m_n}} \left(\frac{2^{n+1}}{m_*} \right)^q (v - m_n)^q \, dy.$$

Thus, we have

$$|\mathcal{A}_{m_{n+1}}| \le \frac{\ell_1^{n+1}}{m_*^q} \mathcal{Z}_n. \tag{7}$$

We note that $1 + m_*^{r-q} \le 2(1 + m_*^{-q})$. Then, it follows from Relations (5)–(7) that we obtain

$$\begin{aligned}
&\left(1 - \frac{\mu c_H}{\gamma_0}\right) \left(\int_{\mathcal{A}_{m_{n+1}}} \int_{\mathcal{A}_{m_{n+1}}} |v(y) - v(z)|^p \mathcal{K}(y,z) \, dydz \right. \\
&\qquad \left. + \int_{\mathcal{A}_{m_{n+1}}} \mathfrak{b}(y) |v|^{p-2} v (v - m_{n+1}) \, dy \right) \\
&\le \left(1 + m_{n+1}^{r-q}\right) \left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \int_{\mathcal{A}_{m_{n+1}}} |v|^q dy \\
&\le \left(1 + m_*^{r-q}\right) \left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \ell_1^{n+2} \mathcal{Z}_n + |\mathcal{A}_{m_{n+1}}| \\
&\le \left(1 + m_*^{r-q}\right) \left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \ell_1^{n+2} \mathcal{Z}_n + \frac{\ell_1^{n+1}}{m_*^q} \mathcal{Z}_n \\
&\le \ell_1^n \mathcal{Z}_n \left[2\left(1 + m_*^{-q}\right) \left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \ell_1^2 + \ell_1 m_*^{-q} \right] \\
&\le \ell_1^n \mathcal{Z}_n \left[2\left(1 + m_*^{-q}\right) \left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \ell_1^2 + \ell_1 m_*^{-q} + 2\ell_1 + \ell_1 m_*^{-q} \right] \\
&\le 2\left(1 + m_*^{-q}\right) \left[\left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \ell_1^2 + \ell_1 \right] \ell_1^n \mathcal{Z}_n \\
&= \ell_2 \ell_1^n \mathcal{Z}_n,
\end{aligned} \tag{8}$$

where $\ell_2 := 2(1 + m_*^{-q}) C_1$ and $C_1 := \left(\lambda \|a\|_{L^\infty(\mathbb{R}^N)} + \theta \|b\|_{L^\infty(\mathbb{R}^N)} \right) \ell_1^2 + \ell_1$. We denote $q_0 := \frac{q+p^*}{2}$ and $v_n := (v - m_{n+1})_+$. Then, it follows from the Hölder inequality that

$$\begin{aligned}
\int_{\mathcal{A}_{m_{n+1}}} v_n^q \, dy &\le \left(\int_{\mathbb{R}^N} v_n^{q_0} \, dy \right)^{\frac{q}{q_0}} |\mathcal{A}_{m_{n+1}}|^{\frac{q_0 - q}{q_0}} \\
&= \|v_n\|^q_{L^{q_0}(\mathbb{R}^N)} |\mathcal{A}_{m_{n+1}}|^{1 - \frac{q}{q_0}} \\
&\le C_{imb}^q \|v_n\|_X^q |\mathcal{A}_{m_{n+1}}|^{1 - \frac{q}{q_0}},
\end{aligned} \tag{9}$$

where C_{imb} is an imbedding constant of $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N \hookrightarrow L^{q_0}(\mathbb{R}^N)$. By (8) and the fact that $v \ge v_n = v - m_{n+1} > 0$ on $\mathcal{A}_{m_{n+1}}$, we obtain

$$\begin{aligned}
\|v_n\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N} &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v_n(y) - v_n(z)|^p \mathcal{K}(y,z) \, dydz + \int_{\mathcal{A}_{m_{n+1}}} \mathfrak{b}(y) |v_n|^p \, dy \\
&= \int_{\mathcal{A}_{m_{n+1}}} \int_{\mathcal{A}_{m_{n+1}}} |v_n(y) - v_n(z)|^p \mathcal{K}(y,z) \, dydz + \int_{\mathcal{A}_{m_{n+1}}} \mathfrak{b}(y) |v_n|^p \, dy \\
&\le \int_{\mathcal{A}_{m_{n+1}}} \int_{\mathcal{A}_{m_{n+1}}} |v(y) - v(z)|^p \mathcal{K}(y,z) \, dydz \\
&\quad + \int_{\mathcal{A}_{m_{n+1}}} \mathfrak{b}(y) |v|^{p-2} v (v - m_{n+1}) \, dy
\end{aligned}$$

$$\leq C_2\ell_2\ell_1^n\mathcal{Z}_n, \tag{10}$$

where $C_2 := \left(1-\frac{\mu c_H}{\gamma_0}\right)^{-1}$. We deduce from (7), (9), and (10) that

$$\begin{aligned}
\mathcal{Z}_{n+1} &= \int_{\mathcal{A}_{m_{n+1}}} v_n^q\,dy \\
&\leq C_{imb}^q\|(v-m_{n+1})_+\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N}|\mathcal{A}_{m_{n+1}}|^{1-\frac{q}{q_0}} \\
&\leq C_{imb}^q(C_2\ell_2\ell_1^n\mathcal{Z}_n)^{\frac{q}{p}}|\mathcal{A}_{m_{n+1}}|^{1-\frac{q}{q_0}} \\
&\leq C_{imb}^q\left[2C_2(1+m_*^{-q})C_1\ell_1^n\mathcal{Z}_n\right]^{\frac{q}{p}}\left(\frac{\ell_1^{n+1}}{m_*^q}\mathcal{Z}_n\right)^{1-\frac{q}{q_0}} \\
&= C_{imb}^q(2C_2C_1)^{\frac{q}{p}}\left(1+m_*^{-q}\right)^{\frac{q}{p}}\ell_1^{n\frac{q}{p}+(n+1)\left(1-\frac{q}{q_0}\right)}m_*^{-q\left(1-\frac{q}{q_0}\right)}\mathcal{Z}_n^{1-\frac{q}{q_0}+\frac{q}{p}} \\
&\leq C_3C_{imb}^q(2C_2C_1)^{\frac{q}{p}}\ell_1^{1-\frac{q}{q_0}}\left(1+m_*^{-q\frac{q}{p}}\right)m_*^{-q\left(1-\frac{q}{q_0}\right)}\ell_1^{n\left(1-\frac{q}{q_0}+\frac{q}{p}\right)}\mathcal{Z}_n^{1-\frac{q}{q_0}+\frac{q}{p}}
\end{aligned}$$

for a positive constant C_3. We assert

$$\mathcal{Z}_{n+1} \leq \ell_{3+}\left(m_*^{-q\left(1-\frac{q}{q_0}\right)} + m_*^{-q\left(1-\frac{q}{q_0}+\frac{q}{p}\right)}\right)\ell_1^{n(1+\delta)}\mathcal{Z}_n^{1+\delta}, \quad n \in \mathbb{N}\cup\{0\},$$

where $\ell_{3+} := C_3C_{imb}^q(2C_2C_1)^{\frac{q}{p}}\ell_1^{1-\frac{q}{q_0}}$ and $\delta = \frac{q}{p} - \frac{q}{q_0}$. This implies

$$\mathcal{Z}_{n+1} \leq \ell_{3+}\left(m_*^{-\alpha_1}+m_*^{-\alpha_2}\right)b^n\mathcal{Z}_n^{1+\delta}, \tag{11}$$

where $0<\alpha_1 := q\left(1-\frac{q}{q_0}\right) < \alpha_2 := q\left(1-\frac{q}{q_0}+\frac{q}{p}\right)$ and $\tau := \ell_1^{1+\delta}$ for any $\mu\in(0,\mu^*)$.
Case 2. $\mu \in (-\infty, 0]$.

From similar arguments to those in (4) and (5), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)-u(z)|^p\mathcal{K}(y,z)\,dydz + \int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy \\
&\quad -\mu\int_{\mathcal{A}_m}\frac{|v|^{p-2}w}{|y|^{sp}}(v-m)\,dy \\
&\geq \int_{\mathcal{A}_m}\int_{\mathcal{A}_m}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dydz + \int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathcal{A}_m}\int_{\mathcal{A}_m}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dydz + \int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy \\
&\leq \left(1+m^{1-q}\right)\left(\lambda\|a\|_{L^\infty(\mathbb{R}^N)}+\|b\|_{L^\infty(\mathbb{R}^N)}\right)\int_{\mathcal{A}_m}|v|^q dy.
\end{aligned}$$

This, together with an analogous argument to that in (8), yields that

$$\int_{\mathcal{A}_m}\int_{\mathcal{A}_m}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dydz + \int_{\mathcal{A}_m}\mathfrak{b}(y)|v|^{p-2}v(v-m)\,dy \leq \ell_2\ell_1^n\mathcal{Z}_n,$$

where ℓ_1 and ℓ_2 are given in (8). This implies

$$\|v_n\|_X^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz + \int_{\mathcal{A}_{m_{n+1}}}\mathfrak{b}(y)|v_n|^p\,dy$$

$$\leq \int_{\mathcal{A}_{m_{n+1}}} \int_{\mathcal{A}_{m_{n+1}}} |v(y) - v(z)|^p \mathcal{K}(y,z)\, dydz$$
$$+ \int_{\mathcal{A}_{m_{n+1}}} \mathfrak{b}(y)|v|^{p-2} v(v - m_{n+1})\, dy$$
$$\leq \ell_2 \ell_1^n \mathcal{Z}_n.$$

Using an argument analogous to that used to derive (11), we attain

$$\mathcal{Z}_{n+1} \leq \ell_{3-} \left(m_*^{-\alpha_1} + m_*^{-\alpha_2} \right) \tau^n \mathcal{Z}_n^{1+\delta}, \tag{12}$$

where $\ell_{3-} := C_3 C_{imb}^q (2C_1)^{\frac{q}{p}} \ell_1^{1-\frac{q}{q_0}}$ for any $\mu \in (-\infty, 0]$.

Applying Lemma 5 with (11) and (12), we deduce that

$$\mathcal{Z}_n = \int_{\mathbb{R}^N} (v - m_n)_+^q\, dy \to 0 \quad \text{as} \quad n \to \infty, \tag{13}$$

provided that

$$\mathcal{Z}_0 \leq \min\left\{ 1, \ell_3^{-\frac{1}{\delta}} \left(m_*^{-\alpha_1} + m_*^{-\alpha_2} \right)^{-\frac{1}{\delta}} \tau^{-\frac{1}{\delta^2}} \right\},$$

where ℓ_3 is either ℓ_{3+} or ℓ_{3-}. We note that for a large enough m_*, $\mathcal{Z}_0 \leq 1$ since $|\mathcal{A}_{m_*}| \to 0$ as $m_* \to \infty$. Moreover, we observe that

$$\mathcal{Z}_0 = \int_{\mathcal{A}_{m_*}} (v - m_*)^q\, dy \leq \int_{\mathbb{R}^N} v_+^q\, dy. \tag{14}$$

Meanwhile,

$$\int_{\mathbb{R}^N} v_+^q\, dy \leq \ell_3^{-\frac{1}{\delta}} \left(m_*^{-\alpha_1} + m_*^{-\alpha_2} \right)^{-\frac{1}{\delta}} \tau^{-\frac{1}{\delta^2}}$$

is equivalent to

$$m_*^{-\alpha_1} + m_*^{-\alpha_2} \leq \ell_3^{-1} \tau^{-\frac{1}{\delta}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{-\delta}. \tag{15}$$

Moreover,

$$\begin{cases} 2m_*^{-\alpha_1} \leq \ell_3^{-1} \tau^{-\frac{1}{\delta}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{-\delta}, \\ 2m_*^{-\alpha_2} \leq \ell_3^{-1} \tau^{-\frac{1}{\delta}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{-\delta} \end{cases}$$

is equivalent to

$$\begin{cases} m_* \geq (2\ell_3)^{\frac{1}{\alpha_1}} \tau^{\frac{1}{\delta \alpha_1}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{\frac{\delta}{\alpha_1}}, \\ m_* \geq (2\ell_3)^{\frac{1}{\alpha_2}} \tau^{\frac{1}{\delta \alpha_2}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{\frac{\delta}{\alpha_2}}. \end{cases}$$

Hence, by choosing

$$m_* = \max\left\{ (2\ell_3)^{\frac{1}{\alpha_1}} \tau^{\frac{1}{\delta \alpha_1}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{\frac{\delta}{\alpha_1}}, (2\ell_3)^{\frac{1}{\alpha_2}} \tau^{\frac{1}{\delta \alpha_2}} \left(\int_{\mathbb{R}^N} v_+^q\, dy \right)^{\frac{\delta}{\alpha_2}} \right\},$$

we obtain Inequality (15). Combining this and (14), we derive Relation (13). Since $m_n \uparrow 2m_*$, Relation (13) and the Lebesgue dominated convergence theorem imply that

$$\int_{\mathbb{R}^N} (v - 2m_*)_+^q\, dy = 0.$$

Therefore, $(v-2m_*)_+ = 0$ almost everywhere in $\mathbb{R}^N$ and hence $\operatorname{ess\,sup}_{x\in\mathbb{R}^N} v(x) \le 2m_*$. By replacing v with $-v$ and $\mathcal{A}_m$ with $\widetilde{\mathcal{A}}_m$, we have analogously that v is bounded from below. Therefore,

$$\|v\|_{L^\infty(\mathbb{R}^N)} \le C_4 \max\left\{ \left(\int_{\mathbb{R}^N} |v|^q\,dy\right)^{\frac{\delta}{\alpha_1}}, \left(\int_{\mathbb{R}^N} |v|^q\,dy\right)^{\frac{\delta}{\alpha_2}} \right\},$$

where C_4 is a positive constant independent of v. The proof is complete. □

4. Applications

As an application of Theorem 1, we demonstrate two multiplicity results of nontrivial weak solutions to the Schrödinger–Hardy-type nonlinear equation driven by the nonlocal fractional p-Laplacian. First, we present useful auxiliaries that play a decisive role in proving the existence of at least two distinct nontrivial solutions to (1). The proof of the following assertion can be regarded as a modification of those of Lemma 5 in [33].

Lemma 6. *We assume that* (V), (A1), *and* (G1) *hold and the following is satisfied:*

(G2) *$G(y,\xi) \ge 0$ for all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}$ and $\lim_{|\xi|\to\infty} \frac{G(y,\xi)}{|\xi|^p} = \infty$ uniformly for almost all $y \in \mathbb{R}^N$.*

Then, for any $\theta > 0$, we have the following:

(i) There are constants $\lambda^* > 0$ and $\mu^* > 0$ such that, for any $\lambda \in (0,\lambda^*)$ and for any $\mu \in (-\infty,\mu^*)$, we can choose $\mathfrak{R} > 0$ and $0 < \tau < 1$ such that $\mathcal{I}_{\mu,\lambda}(v) \ge \mathfrak{R}$ for all $v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \tau$.

(ii) There exists an element ϕ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, $\phi > 0$ such that $\mathcal{I}_{\mu,\lambda}(\zeta\phi) \to -\infty$ as $\zeta \to +\infty$.

(iii) There is an element ψ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, $\psi > 0$ such that $\mathcal{I}_{\mu,\lambda}(\zeta\psi) < 0$ for all $\zeta \to 0^+$.

Proof. Let us show Condition (i). Through Lemma 1, there is a constant $d_1 > 0$ such that $\|v\|_{L^\gamma(\mathbb{R}^N)} \le d_1 \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}$ for $p \le \gamma < p_s^*$. We assume that $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} < 1$. We set $\mu^* = \gamma_0 {c_H}^{-1}$, where c_H and γ_0 are given in Lemma 4 and ($\mathcal{L}2$), respectively. First, we consider the case $\mu \in (0,\mu^*)$. Then, it follows from (A1), (G1), and Lemma 3 that

$$\begin{aligned}
\mathcal{I}_{\mu,\lambda}(v) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\,dy \\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}}\,dy - \frac{\lambda}{r}\int_{\mathbb{R}^N} a(y)|v|^r\,dy - \theta\int_{\mathbb{R}^N} G(y,v)\,dy \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\,dy \\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}}\,dy - \frac{\lambda d_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|v\|^q_{L^q(\mathbb{R}^N)} \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\,dy \\
&\quad - \frac{\mu c_H}{\gamma_0 p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dy\,dz \\
&\quad - \frac{\lambda d_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|v\|^q_{L^q(\mathbb{R}^N)} \\
&\ge \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\,dy \\
&\quad - \frac{\lambda d_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|v\|^q_{L^q(\mathbb{R}^N)}
\end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{1}{p}-\frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\frac{\lambda d_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad -\frac{\theta d_1}{q}\|b\|_{L^\infty(\mathbb{R}^N)}\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \left[\left(\frac{1}{p}-\frac{\mu c_H}{\gamma_0 p}\right)-\frac{\lambda d_2}{r}\|v\|^{r-p}_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\frac{\theta d_3}{q}\|v\|^{q-p}_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}\right]\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \qquad (16)
\end{aligned}$$

for positive constants d_2 and d_3.

On the other hand, we consider the case for $\mu \in (-\infty, 0]$. Then, we obtain

$$\begin{aligned}
\mathcal{I}_{\mu,\lambda}(v) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz+\frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\
&\quad -\frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|v|^p}{|y|^{sp}}\,dy-\frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|v|^r\,dy-\theta\int_{\mathbb{R}^N}G(y,v)\,dy \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz+\frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\
&\quad -\frac{\lambda d_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|v\|^q_{L^q(\mathbb{R}^N)} \\
&\geq \frac{1}{p}\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\frac{\lambda d_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\frac{\theta d_1}{q}\|b\|_{L^\infty(\mathbb{R}^N)}\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \left(\frac{1}{p}-\frac{\lambda d_2}{r}\|v\|^{r-p}_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\frac{\theta d_3}{q}\|v\|^{q-p}_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}. \qquad (17)
\end{aligned}$$

Let us define the function $f_\lambda : (0,\infty) \to \mathbb{R}$ by

$$f_\lambda(\xi) = \frac{\lambda d_2}{r}\xi^{r-p}+\frac{\theta d_3}{q}\xi^{q-p}.$$

Then, it is immediately clear that f_λ admits a local minimum at point $\xi_0 = \left(\frac{\lambda q d_2(p-r)}{r\theta_3(q-p)}\right)^{\frac{1}{q-r}}$ and so

$$\lim_{\lambda\to 0^+} f_\lambda(\xi_0) = 0.$$

Thus, it follows from (16) and (17) that there is a positive constant λ^*, such that for each $\lambda \in (0,\lambda^*)$ and for any $\mu \in (-\infty,\mu^*)$ we can choose $\mathfrak{R} > 0$ and small enough $\tau > 0$ such that $\mathcal{I}_{\mu,\lambda}(v) \geq \mathfrak{R} > 0$ for any $v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \tau$.

Next, we prove Statement (ii). By (G2), for any $\mathcal{C}_0 > 0$, there is a constant $\xi_0 > 0$ such that

$$G(y,\xi) \geq \mathcal{C}_0|\xi|^p \qquad (18)$$

for $|\xi| > \xi_0$ and for almost all $y \in \mathbb{R}^N$. We take $\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)\setminus\{0\}$. Then, Relation (18) yields

$$\begin{aligned}
\mathcal{I}_{\mu,\lambda}(\zeta\varphi) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\zeta^p|\varphi(y)-\varphi(z)|^p\mathcal{K}(y,z)\,dydz+\frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|\zeta\varphi|^p\,dy \\
&\quad -\frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|\zeta\varphi|^p}{|y|^{sp}}\,dy-\frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|\zeta\varphi|^r\,dy-\theta\int_{\mathbb{R}^N}G(y,\zeta\varphi)\,dy \\
&\leq \zeta^p\left(\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(y)-\varphi(z)|^p\mathcal{K}(y,z)\,dydz+\frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|\varphi|^p\,dy\right. \\
&\quad \left.-\frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|\varphi|^p}{|y|^{sp}}\,dy-\theta\mathcal{C}_0\int_{\mathbb{R}^N}|\varphi|^p\,dy\right)
\end{aligned}$$

for sufficiently large $\zeta > 1$. If $\mathcal{C}_0$ is large enough, then we arrive at $\mathcal{I}_{\mu,\lambda}(\zeta\varphi) \to -\infty$ as $\zeta \to \infty$. Hence, the functional $\mathcal{I}_{\mu,\lambda}$ is unbounded from below.

Finally, (iii) remains to be shown. We choose $\psi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ such that $\phi > 0$. For sufficiently small $\zeta > 0$, from (G2), we obtain

$$\begin{aligned}\mathcal{I}_{\mu,\lambda}(\zeta\phi) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\zeta^p|\phi(y)-\phi(z)|^p\mathcal{K}(y,z)\,dydz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|\zeta\phi|^p\,dy \\ &\quad - \frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|\zeta\phi|^p}{|y|^{sp}}\,dy - \frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|\zeta\phi|^r\,dy - \theta\int_{\mathbb{R}^N}G(y,\zeta\phi)\,dy \\ &\le \frac{\zeta^p}{p}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(y)-\phi(z)|^p\mathcal{K}(y,z)\,dydz + \int_{\mathbb{R}^N}\mathfrak{b}(y)|\phi|^p\,dy \\ &\quad - \mu\int_{\mathbb{R}^N}\frac{|\phi|^p}{|y|^{sp}}\,dy\bigg) - \frac{\lambda\zeta^r}{r}\int_{\mathbb{R}^N}a(y)|\phi|^r\,dy.\end{aligned}$$

Since $r < p$, it follows that $\mathcal{I}_{\mu,\lambda}(\zeta\phi) < 0$ as $\zeta \to 0^+$. This completes the proof. □

Now, we prove that the energy functional $\mathcal{I}_{\mu,\lambda}$ ensures the Cerami condition ((C)-condition for brevity), i.e., any sequence $\{v_n\}_{n\in\mathbb{N}} \subset X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ such that

$$\{\mathcal{I}_{\mu,\lambda}(v_n)\}_{n\in\mathbb{N}} \text{ is bounded} \quad \text{and} \quad \|\mathcal{I}'_{\mu,\lambda}(v_n)\|_{X^*_s(\mathbb{R}^N)}(1+\|v_n\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}) \to 0 \quad \text{as} \quad n \to \infty$$

has a convergent subsequence. The basic idea of the proofs of the following consequences follows analogous arguments to those in [12]; see also [16].

Lemma 7. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. We assume that* (A1) *and* (G1) *hold and that*

(G3) *there exist $\nu > p$ and $M > 0$ such that*

$$g(y,t)t - \nu G(y,t) \ge 0 \quad \text{for all} \quad y \in \mathbb{R}^N \quad \text{and} \quad |t| \ge M$$

is satisfied. Then, for any $\lambda, \theta > 0$ and for any $\mu \in (0,\mu^*)$, the functional $\mathcal{I}_{\mu,\lambda}$ satisfies the (C)-condition, where μ^* is given in Lemma 6.

Proof. Let $\{v_n\}_{n\in\mathbb{N}}$ be a (C)-sequence in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, i.e.,

$$\sup_{n\in\mathbb{N}}|\mathcal{I}_{\mu,\lambda}(v_n)| \le \mathfrak{C}_1 \text{ and } \left\langle \mathcal{I}'_{\mu,\lambda}(v_n), v_n\right\rangle = o(1) \to 0, \text{ as } n \to \infty, \tag{19}$$

where $\mathfrak{C}_1$ is a positive constant. From condition (V) and the same argument as in [43], we arrive at

$$\begin{aligned}&\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p dy - \theta C_5\int_{\{|v_n|\le M\}}(|v_n|^p + b(y)|v_n|^q)dy \\ &\qquad \ge \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p dy - \mathfrak{C}_0\end{aligned} \tag{20}$$

for any positive constant C_5 and for some positive constant $\mathfrak{C}_0$.

From (20), (A1), (G1), and (G3) and for any $\mu \in (0,\mu^*)$,

$$\begin{aligned}\mathfrak{C}_1 + 1 &\ge \mathcal{I}_{\mu,\lambda}(v_n) - \frac{1}{\nu}\left\langle\mathcal{I}'_{\mu,\lambda}(v_n), v_n\right\rangle \\ &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy \\ &\quad - \frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^{sp}}dy - \frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|v_n|^r\,dy - \theta\int_{\mathbb{R}^N}G(y,v_n)\,dy \\ &\quad - \frac{1}{\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz - \frac{1}{\nu}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu}{\nu}\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^{sp}}\,dy+\frac{\lambda}{\nu}\int_{\mathbb{R}^N}a(y)|v_n|^r\,dy+\frac{\theta}{\nu}\int_{\mathbb{R}^N}g(y,v_n)v_n\,dy\\
\geq\,&\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz\\
&+\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy+\frac{\theta}{\nu}\int_{\mathbb{R}^N}(g(y,v_n)v_n-\nu G(y,v_n))\,dy\\
&-\mu\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^{sp}}\,dy-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}a(y)|v_n|^r\,dy\\
\geq\,&\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz\\
&+\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy+\frac{\theta}{\nu}\int_{\{|v_n|\leq M\}}(g(y,v_n)v_n-\nu G(y,v_n))\,dy\\
&+\frac{\theta}{\nu}\int_{\{|v_n|\geq M\}}(g(y,v_n)v_n-\nu G(y,v_n))\,dy\\
&-\mu\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^{sp}}\,dy-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}a(y)|v_n|^r\,dy\\
\geq\,&\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz\\
&+\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy-\theta C_5\int_{\{|v_n|\leq M\}}(|v_n|^p+b(y)|v_n|^q)dy\\
&-\frac{c_H\mu}{\gamma_0}\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz\\
&-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}a(y)|v_n|^r\,dy\\
\geq\,&\left(\frac{1}{p}-\frac{1}{\nu}\right)\left(1-\frac{c_H\mu}{\gamma_0}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz\\
&+\frac{1}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}a(y)|v_n|^r\,dy-\mathfrak{C}_0\\
\geq\,&\min\left\{1-\frac{c_H\mu}{\gamma_0},\frac{1}{2}\right\}\left(\frac{1}{p}-\frac{1}{\mu}\right)\|v_n\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}\\
&-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)d_1\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\mathfrak{C}_0, \qquad (21)
\end{aligned}
$$

where d_1 is given in Lemma 6. Hence, we know that

$$
\begin{aligned}
&\mathfrak{C}_1+1+\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)d_1\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}+\mathfrak{C}_0\\
&\qquad\geq\min\left\{1-\frac{c_H\mu}{\gamma_0},\frac{1}{2}\right\}\left(\frac{1}{p}-\frac{1}{\mu}\right)\|v_n\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}. \qquad (22)
\end{aligned}
$$

Next, we consider the case for $\mu\in(-\infty,0]$. From an analogous argument to that in (21), it follows that

$$
\begin{aligned}
K_1+1\geq\,&\mathcal{I}_{\mu,\lambda}(v_n)-\frac{1}{\mu}\left\langle\mathcal{I}'_{\mu,\lambda}(v_n),v_n\right\rangle\\
\geq\,&\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz\\
&+\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^p\,dy-\mu\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^{sp}}\,dy\\
&+\frac{\theta}{\nu}\int_{\mathbb{R}^N}(g(y,v_n)v_n-\nu G(y,v_n))\,dy
\end{aligned}
$$

$$\geq \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\mu}\right)\|v_n\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)d_1\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \widetilde{\mathfrak{K}}_0.$$

Hence, we know that

$$K_1+1+\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)d_1\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} + \widetilde{\mathfrak{K}}_0 \geq \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\mu}\right)\|v_n\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}. \tag{23}$$

Therefore, from (22) and (23), we can state that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$.

From Lemmas 3 and 4 and the reflexivity of $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, there exists a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and $v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ such that

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N), \quad v_n \rightharpoonup v \text{ in } L^p(\mathbb{R}^N, |y|^{-sp}), \\ &v_n \to v \text{ a.e. in } \mathbb{R}^N, \quad v_n \to v \text{ in } L^{\nu}(\mathbb{R}^N), \quad \|v_n - v\|_{H_p} \to \ell \end{aligned} \tag{24}$$

for any $\nu \in [p, p_s^*)$ as $n \to \infty$. Then, the sequence

$$\left\{|v_n(y)-v_n(z)|^{p-2}(v_n(y)-v_n(z))\mathcal{K}(y,z)^{\frac{1}{p'}}\right\}_{n\in\mathbb{N}}$$

is bounded in $L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$, as well as almost everywhere in $\mathbb{R}^N\times\mathbb{R}^N$

$$\begin{aligned} &\mathcal{B}_n(y,z) := |v_n(y)-v_n(z)|^{p-2}(v_n(y)-v_n(z))\mathcal{K}(y,z)^{\frac{1}{p'}} \\ &\to \mathcal{B}(y,z) := |v(y)-v(z)|^{p-2}(v(y)-v(z))\mathcal{K}(y,z)^{\frac{1}{p'}} \quad \text{as } n\to\infty. \end{aligned}$$

Thus, proceeding, if necessary, to a further subsequence, we infer that $\mathcal{B}_n \rightharpoonup \mathcal{B}$ in $L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$ as $n\to\infty$. Furthermore, $|v_n|^{p-2}v_n \rightharpoonup |v|^{p-2}v$ in $L^{p'}(\mathfrak{b},\mathbb{R}^N)$. Hence, since $(y,z)\mapsto |\varphi(y)-\varphi(z)|\cdot|y-z|^{-(n+ps)/p} \in L^p(\mathbb{R}^N\times\mathbb{R}^N)$ and $\varphi\in L^p(\mathfrak{b},\mathbb{R}^N)$, we assert that for any $\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^{p-2}(v_n(y)-v_n(z))(\varphi(y)-\varphi(z))\mathcal{K}(y,z)\,dydz \\ &\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^{p-2}(v(y)-v(z))(\varphi(y)-\varphi(z))\mathcal{K}(y,z)\,dydz \end{aligned} \tag{25}$$

and

$$\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n(y)|^{p-2}v_n(y)\varphi(y)\,dy \to \int_{\mathbb{R}^N}\mathfrak{b}(y)|v(y)|^{p-2}v(y)\varphi(y)\,dy \tag{26}$$

as $n\to\infty$.

On the other hand, sequence

$$\left\{\frac{|v_n(y)|^{p-2}v_n(y)}{|y|^{\frac{sp}{p'}}}\right\}_{n\in\mathbb{N}}$$

is bounded in $L^{p'}(\mathbb{R}^N)$, as well as almost everywhere in $\mathbb{R}^N$

$$\frac{|v_n(y)|^{p-2}v_n(y)}{|y|^{\frac{sp}{p'}}} \to \frac{|v(y)|^{p-2}v(y)}{|y|^{\frac{sp}{p'}}} \quad \text{as } n\to\infty.$$

By (24), we have

$$|v_n|^{p-2}v_n \rightharpoonup |v|^{p-2}v \text{ in } L^{p'}(\mathbb{R}^N, |y|^{-sp}),$$

so that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|v_n|^{p-2}v_n}{|y|^{sp}}\varphi\,dy = \int_{\mathbb{R}^N}\frac{|v|^{p-2}v}{|y|^{sp}}\varphi\,dy \tag{27}$$

for any $\varphi \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$.

From (25), (26), and (27), we derive that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^{p-2}(v_n(y)-v_n(z))(v(y)-v(z))\mathcal{K}(y,z)\,dydz=[v]^p_{p,\mathcal{K}} \tag{28}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^{p-2}v_nv\,dy=\|v\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)},\quad \lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|v_n|^{p-2}v_n}{|y|^{sp}}v\,dy=\int_{\mathbb{R}^N}\frac{|v|^p}{|y|^{sp}}dy. \tag{29}$$

By also considering Lemma 3, (24), Assumptions (A1) and (G1), and the Hölder inequality, we obtain

$$\left|\int_{\mathbb{R}^N}a(y)|v_n|^{r-2}v_n(v_n-v)dy\right|\le\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n-v\|^r_{L^p(\mathbb{R}^N)}\to 0 \tag{30}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^N}g(y,v_n)(v_n-v)dy\right|&\le\int_{\mathbb{R}^N}b(y)|v_n|^{q-1}|v_n-v|dy\\&\le\|b\|_{L^\infty(\mathbb{R}^N)}\|v_n\|^{q-1}_{L^q(\mathbb{R}^N)}\|v_n-v\|_{L^q(\mathbb{R}^N)}\to 0\end{aligned} \tag{31}$$

as $n\to\infty$. Furthermore, using (24) and (25) and the Brézis and Lieb lemma in Theorem 1 [44], we obtain

$$\begin{aligned}[v_n]^p_{p,\mathcal{K}}-[v_n-v]^p_{p,\mathcal{K}}&=[v_n]^p_{p,\mathcal{K}}+o(1),\\ \|v_n\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)}-\|v_n-v\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)}&=\|v\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)}+o(1)\end{aligned} \tag{32}$$

and

$$\|v_n\|^p_{H_p}-\|v_n-v\|^p_{H_p}=\|v\|^p_{H_p}+o(1) \tag{33}$$

as $n\to\infty$. Thus, by (19), (28)–(33), we obtain

$$\begin{aligned}o(1)&=\left\langle\mathcal{I}'_{\mu,\lambda}(v_n),v_n-v\right\rangle\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz+\int_{\mathbb{R}^N}\mathfrak{b}(y)|v_n|^{p-2}v_n(v_n-v)\,dy\\&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^{p-2}(v_n(y)-v_n(z))(v(y)-v(z))\mathcal{K}(y,z)\,dydz\\&\quad-\mu\int_{\mathbb{R}^N}\frac{|v_n|^{p-2}v_n}{|y|^{sp}}(v_n-v)dy-\lambda\int_{\mathbb{R}^N}a(y)|v_n|^{r-2}v_n(v_n-v)\,dy\\&\quad-\theta\int_{\mathbb{R}^N}g(y,v_n)(v_n-v)\,dy+o(1)\\&\ge[v_n]^p_{p,\mathcal{K}}+\|v_n\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)}-[v]^p_{p,\mathcal{K}}-\|v\|^p_{L^p(\mathfrak{b},\mathbb{R}^N)}\\&\quad-\mu\left(\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^p}dy-\int_{\mathbb{R}^N}\frac{|v|^p}{|y|^p}dy\right)+o(1)\\&\ge\|v_n\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\mu(\|v_n\|^p_{H_p}-\|v\|^p_{H_p})+o(1)\\&\ge\|v_n-v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}-\mu(\|v_n-v\|^p_{H_p})+o(1)\end{aligned} \tag{34}$$

as $n\to\infty$. Hence, it follows from (24) that

$$\begin{aligned}\|v_n-v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}&\le\mu\|v_n-v\|^p_{H_p}+o(1)\\&=\mu\ell+o(1)\end{aligned} \tag{35}$$

as $n \to \infty$. Now assume, for contradiction, that $\ell > 0$. Then, from Lemma 4, (35), and the fact that $\mu < \frac{\gamma_0}{c_H}$, we have

$$\begin{aligned}\lim_{n\to\infty}[v_n - v]^p_{p,\mathcal{K}} &\le \lim_{n\to\infty}\|v_n - v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\ &\le \mu \lim_{n\to\infty}\|v_n - v\|^p_{H_p} \\ &< \frac{\gamma_0}{c_H}\lim_{n\to\infty}\|v_n - v\|^p_{H_p} \\ &\le \lim_{n\to\infty}[v_n - v]^p_{p,\mathcal{K}},\end{aligned}$$

which is impossible. Therefore, $\ell = 0$; so, by (35), we have $v_n \to v$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$. This completes the proof. □

The following lemma, which is a variant of the Ekeland variational principle, plays a decisive role in obtaining our first main consequence.

Lemma 8 ([28]). *Let $\mathcal{E}$ be a Banach space and x_0 be a fixed point of $\mathcal{E}$. We suppose that $F : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in \mathcal{E}$ such that*

$$F(y) < \inf_{\mathcal{E}} F + \varepsilon,$$

and every $\lambda > 0$, there is a point $z \in \mathcal{E}$ such that

$$F(z) \le F(y), \quad \|z - x_0\|_{\mathcal{E}} \le (1 + \|y\|_{\mathcal{E}})(e^{\lambda} - 1),$$

and

$$F(x) \ge F(z) - \frac{\varepsilon}{\lambda(1 + \|z\|_{\mathcal{E}})}\|x - z\|_{\mathcal{E}} \quad \text{for all} \quad x \in \mathcal{E}.$$

With the help of Lemmas 6–8, we are in a position to derive our first major result. The proof is completely the same as that of Theorem 1 in [33].

Theorem 2. *We assume that* (V), (A1), *and* (G1)–(G3) *hold. Then, there is a constant $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ and for any $\mu \in (-\infty, \mu^*)$, Problem* (1) *has at least two different nontrivial solutions in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ that belong to L^∞-space, where μ^* is given in Lemma 6.*

Proof. By means of Lemmas 6 and 7, we choose positive numbers λ^* and μ^* such that $\mathcal{I}_{\lambda,\mu}$ has a mountain pass geometry and the (C)-condition for any $\lambda \in (0, \lambda^*)$ and for any $\mu \in (-\infty, \mu^*)$. The mountain pass theorem derives that $\mathcal{I}_{\mu,\lambda}$ has a critical point $v_0 \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ with $\mathcal{I}_{\mu,\lambda}(v_0) = \bar{\delta} > 0 = \mathcal{I}_{\mu,\lambda}(0)$. Thus, Problem (1) admits a nontrivial weak solution v_0. By virtue of Lemma 6, for a fixed $\lambda \in (0, \lambda^*)$ and $\mu \in (-\infty, \mu^*)$, we can choose $\mathfrak{R} > 0$ and $0 < \tau < 1$ such that $\mathcal{I}_{\mu,\lambda}(v) \ge \mathfrak{R}$ if $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \tau$. Let us denote $\delta := \inf_{u \in \overline{B}_\tau} \mathcal{I}_{\mu,\lambda}(u)$, where $B_\tau := \{v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) : \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} < \tau\}$ with a boundary ∂B_τ. Then, it follows from (16) and Lemma 6 (3) that $-\infty < \delta < 0$. Putting $0 < \epsilon < \inf_{u \in \partial B_\tau} \mathcal{I}_{\mu,\lambda}(u) - \delta$, invoking to Lemma 8, there is an element $v_\epsilon \in \overline{B}_\tau$ such that

$$\begin{cases} \mathcal{I}_{\mu,\lambda}(v_\epsilon) \le \delta + \epsilon, \\ \mathcal{I}_{\mu,\lambda}(v_\epsilon) < \mathcal{I}_{\mu,\lambda}(u) + \frac{\epsilon}{1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}}\|u - v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \end{cases} \tag{36}$$

for all $u \in \overline{B}_\tau$ with $u \neq v_\epsilon$. We set

$$\widehat{\mathcal{I}}_{\mu,\lambda}(u) = \mathcal{I}_{\mu,\lambda}(u) + \frac{\epsilon}{1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}}\|u - v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}.$$

Because $\mathcal{I}_{\mu,\lambda}(v_\epsilon) \le \delta + \epsilon < \inf_{u\in\partial B_\tau} \mathcal{I}_{\mu,\lambda}(u)$ we determine that $v_\epsilon \in B_\tau$. From these facts, we know that v_ϵ is a local minimum of $\widehat{\mathcal{I}}_{\mu,\lambda}$. Now, by taking $u = v_\epsilon + tv$ for $v \in B_1$ with small enough $t > 0$, from (36), we deduce

$$0 \le \frac{\widehat{\mathcal{I}}_{\mu,\lambda}(v_\epsilon + tv) - \widehat{\mathcal{I}}_{\mu,\lambda}(v_\epsilon)}{t} = \frac{\mathcal{I}_{\mu,\lambda}(v_\epsilon + tv) - \mathcal{I}_{\mu,\lambda}(v_\epsilon)}{t} + \frac{\epsilon}{1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}} \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}.$$

Therefore, letting $t \to 0+$, we obtain

$$\left\langle \mathcal{I}'_{\mu,\lambda}(v_\epsilon), v \right\rangle + \frac{\epsilon}{1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}} \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \ge 0.$$

Substituting $-v$ for v in the argument above, we derive

$$-\left\langle \mathcal{I}'_{\mu,\lambda}(v_\epsilon), v \right\rangle + \frac{\epsilon}{1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}} \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \ge 0.$$

Thus, we know

$$(1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}) \left| \left\langle \mathcal{I}'_{\mu,\lambda}(v_\epsilon), v \right\rangle \right| \le \epsilon \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}$$

for any $v \in \overline{B}_1$. Hence, we infer

$$(1 + \|v_\epsilon\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}) \|\mathcal{I}'_{\mu,\lambda}(v_\epsilon)\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)^*} \le \epsilon. \tag{37}$$

Combining (36) with (37), we can choose a sequence $\{v_n\}_{n\in\mathbb{N}} \subset B_\tau$ such that

$$\begin{cases} \mathcal{I}_{\mu,\lambda}(v_n) \to \delta \quad \text{as} \quad n \to \infty, \\ (1 + \|v_n\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}) \|\mathcal{I}'_{\mu,\lambda}(v_n)\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)^*} \to 0 \quad \text{as} \quad n \to \infty. \end{cases} \tag{38}$$

Thus, $\{v_n\}_{n\in\mathbb{N}}$ is a bounded Cerami sequence in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$. According to Lemma 7, $\{v_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ such that $v_{n_k} \to v_1$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ as $k \to \infty$. With the aid of this and (38), we determine that $\mathcal{I}_{\mu,\lambda}(v_1) = \delta$ and $\mathcal{I}'_{\mu,\lambda}(v_1) = 0$. Hence, v_1 is a nontrivial solution of Problem (1) with $\mathcal{I}_{\mu,\lambda}(v_1) < 0$, which is different from v_0. As a result, in accordance with Theorem 1, Problem (1) allows for at least two different nontrivial solutions in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, which belong to L^∞-space. The proof is completed. □

Finally, we demonstrate the existence of a sequence of infinitely many weak solutions to problem (1) which converges to 0 in the L^∞-norm. This requires the following additional conditions for g:

(G4) There is a constant $\xi_0 > 0$ such that $g(y,\xi)$ is odd in $\xi \in (-\xi_0, \xi_0)$ and $pG(y,\xi) - g(y,\xi)\xi > 0$ for almost all $y \in \mathbb{R}^N$ and for $0 < |\xi| < \xi_0$;

(G5) $\lim_{|\xi|\to 0} \frac{g(y,\xi)}{|\xi|^{p-2}\xi} = +\infty$ uniformly for all $y \in \mathbb{R}^N$.

Using the dual fountain theorem as the main tool, we consider the following decomposition lemma to obtain our final result. Let $\mathcal{E}$ be a separable and reflexive Banach space. Then, it is known (see [45,46]) that there are $\{e_n\}_{n\in\mathbb{N}} \subseteq \mathcal{E}$ and $\{h^*_n\}_{n\in\mathbb{N}} \subseteq \mathcal{E}^*$ such that

$$\mathcal{E} = \overline{\text{span}\{e_n : n = 1, 2, \cdots\}}, \quad \mathcal{E}^* = \overline{\text{span}\{h^*_n : n = 1, 2, \cdots\}}$$

and

$$\langle h^*_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

Let us denote $\mathcal{E}_n = \text{span}\{e_n\}_{n\in\mathbb{N}}$, $\mathfrak{Y}_n = \bigoplus_{k=1}^{n} \mathcal{E}_k$, and $\mathfrak{Z}_n = \overline{\bigoplus_{k=n}^{\infty} \mathcal{E}_k}$.

Lemma 9 (Dual Fountain Theorem [47])**.** *We assume that $(\mathcal{E}, \|\cdot\|)$ is a Banach space, and $\mathcal{H} \in C^1(\mathcal{E}, \mathbb{R})$ is an even functional. If there is $n_0 > 0$ so that, for each $n \geq n_0$, there exist $\beta_n > \alpha_n > 0$, the following hold:*

$(\mathcal{D}1)$ $\inf\{\mathcal{H}(\varpi) : \varpi \in \mathfrak{Z}_n, \|\varpi\| = \beta_n\} \geq 0$;

$(\mathcal{D}2)$ $\sigma_n := \max\{\mathcal{H}(\varpi) : \varpi \in \mathfrak{Y}_n, \|\varpi\| = \alpha_n\} < 0$;

$(\mathcal{D}3)$ $\psi_n := \inf\{\mathcal{H}(\varpi) : \varpi \in \mathfrak{Z}_n, \|\varpi\| \leq \beta_n\} \to 0$ *as* $n \to \infty$;

$(\mathcal{D}4)$ $\mathcal{H}$ *fulfills the* $(C)^*_c$*-condition for every* $c \in [\psi_{n_0}, 0)$,

then $\mathcal{H}$ *admits a sequence of negative critical values* $\psi_n < 0$ *satisfying* $\psi_n \to 0$ *as* $n \to \infty$.

Definition 2. We suppose that $(\mathcal{E}, \|\cdot\|)$ is a real reflexive and separable Banach space, $\mathcal{H} \in C^1(\mathcal{E}, \mathbb{R})$, $c \in \mathbb{R}$. We say that $\mathcal{H}$ fulfills the $(C)^*_c$-condition (with respect to $\mathfrak{Y}_n$) if any sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$ for which $v_n \in \mathfrak{Y}_n$, for any $n \in \mathbb{N}$,

$$\mathcal{H}(v_n) \to c \quad \text{and} \quad \|(\mathcal{H}|_{\mathfrak{Y}_n})'(v_n)\|_{\mathcal{E}^*}(1 + \|v_n\|_{\mathcal{E}}) \to 0 \text{ as } n \to \infty,$$

has a subsequence converging to a critical point of $\mathcal{H}$.

Let us introduce the following auxiliary results, which are useful in proving our final consequence.

Lemma 10. *If* (G1) *holds and*

$$pG(y,t) - g(y,t)t > 0 \text{ for all } y \in \mathbb{R}^N \text{ and for } t \neq 0, \tag{39}$$

then we have

$$\mathcal{I}_{\mu,\lambda}(v) = \left\langle \mathcal{I}'_{\mu,\lambda}(v), v \right\rangle = 0 \text{ if and only if } v = 0. \tag{40}$$

Proof. Let $\mathcal{I}_{\mu,\lambda}(v) = \left\langle \mathcal{I}'_{\mu,\lambda}(v), v \right\rangle = 0$. Then, we see that

$$\begin{aligned} 0 &= -p\mathcal{I}_{\mu,\lambda}(v) \\ &= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\, dy\, dz - \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\, dy \\ &\quad + \mu \int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}}\, dy + \frac{p\lambda}{r}\int_{\mathbb{R}^N} a(y)|v|^r\, dy + \theta p \int_{\mathbb{R}^N} \widetilde{G}(y,v)\, dy \\ &\geq -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\, dy\, dz - \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\, dy \\ &\quad + \mu \int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}}\, dy + \lambda\int_{\mathbb{R}^N} a(y)|v|^r\, dy + \theta \int_{\mathbb{R}^N} pG(y,v)\, dy \end{aligned} \tag{41}$$

and

$$\begin{aligned} \left\langle \mathcal{I}'_{\mu,\lambda}(v), v \right\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\, dy\, dz + \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p\, dy \\ &\quad - \mu \int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}}\, dy - \lambda\int_{\mathbb{R}^N} a(y)|v|^r\, dy - \theta \int_{\mathbb{R}^N} G(y,v)\, dy = 0. \end{aligned} \tag{42}$$

It follows from Relations (41) and (42) that

$$\int_{\mathbb{R}^N} (pG(y,v) - g(y,v)v)\, dy \leq 0.$$

Consequently, Assumption (39) implies that $v = 0$. The converse is clear from the definition of $\mathcal{I}_{\mu,\lambda}$. □

Remark 1. *By* (G4) *and* (G5)*, for any* $\mathfrak{C}_2 > 0$*, there exists* $\xi_2 \in (0, min\{\xi_1, 1\})$ *such that*

$$G(y, \xi) \geq \mathfrak{C}_2 |\xi|^p \quad \textit{for a.e. } y \in \mathbb{R}^N \textit{ and all } |\xi| < \xi_2. \tag{43}$$

We fix $\xi_3 \in (0, \xi_2/2)$ *and let* $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ *be such that* φ *is even,* $\varphi(\xi) = 1$ *for* $|\xi| \leq \xi_3$*,* $\varphi(\xi) = 0$ *for* $|\xi| \geq 2\xi_3$*,* $|\varphi'(\xi)| \leq 2/\xi_3$*, and* $\varphi'(\xi)\xi \leq 0$*. We then define the modified function* $\widetilde{g} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *as*

$$\widetilde{g}(y, \xi) := \frac{\partial}{\partial \xi} \widetilde{G}(y, \xi),$$

where

$$\widetilde{G}(y, \xi) := \varphi(\xi) G(y, \xi) + (1 - \varphi(\xi))\eta |\xi|^p \tag{44}$$

for some fixed $\eta \in \left(0, \min\left\{\frac{1}{p}, \frac{1}{q C^p_{p,imb}}\right\}\right)$ *with* $C_{p,imb}$ *being the imbedding constant for the imbedding* $\mathfrak{E} \hookrightarrow L^p(\mathbb{R}^N)$*. Then, there exists a positive constant* ξ_2 *and* $\widetilde{g} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *such that* $\widetilde{g}(y, \xi)$ *is odd in* ξ*,*

$$p\widetilde{G}(y, \xi) - \widetilde{g}(y, \xi)\xi \geq 0 \textit{ for almost all } y \in \mathbb{R}^N \textit{ and all } \xi \in \mathbb{R} \tag{45}$$

and

$$p\widetilde{G}(y, \xi) - \widetilde{g}(y, \xi)\xi = 0 \quad \textit{iff} \quad \xi \equiv 0 \quad \textit{or} \quad |\xi| \geq 2\xi_2. \tag{46}$$

In view of Remark 1, let us define the modified energy functional $\widetilde{\mathcal{I}}_{\mu,\theta} : X \to \mathbb{R}$ by

$$\widetilde{\mathcal{I}}_{\mu,\theta}(v) := \Phi(v) - \widetilde{\Psi}_{\mu,\theta}(v),$$

where

$$\widetilde{\Psi}_{\mu,\theta}(v) = \frac{\mu}{p} \int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}} \, dy + \frac{\lambda}{r} \int_{\mathbb{R}^N} a(y)|v|^r \, dy + \theta \int_{\mathbb{R}^N} \widetilde{G}(y, v) \, dy.$$

Then, it is clear that $\widetilde{\mathcal{I}}_{\mu,\theta} \in C^1(X, \mathbb{R})$ is an even functional.

Lemma 11. *We assume that* (V), (A1), (G1), (G4), *and* (G5) *hold. Then, for any* $\lambda > 0$ *and for any* $\mu \in (0, \mu^*)$*, there exists an interval* Γ *such that* $\widetilde{\mathcal{I}}_{\mu,\theta}$ *is coercive for every* $\theta \in \Gamma$*, where* μ^* *is given in Lemma 6.*

Proof. Let $v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \geq 1$. We set $\Lambda_1 := \{y \in \mathbb{R}^N : |v(y)| \leq \xi_3\}$, $\Lambda_2 := \{y \in \mathbb{R}^N : \xi_3 \leq |v(y)| \leq 2\xi_3\}$, and $\Lambda_3 := \{y \in \mathbb{R}^N : 2\xi_3 \leq |v(y)|\}$, where ξ_3 is given in Remark 1. Let us consider $\mu \in (0, \mu^*)$. Since $\mu < \gamma_0 {c_H}^{-1}$, taking into account Lemma 4, (G1), (44), and the definition of φ, we have

$$\begin{aligned}
\widetilde{\mathcal{I}}_{\mu,\theta}(v) &= \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^p \mathcal{K}(y, z) \, dy \, dz + \frac{1}{p} \int_{\mathbb{R}^N} \mathfrak{b}(y)|v|^p \, dy \\
&\quad - \frac{\mu}{p} \int_{\mathbb{R}^N} \frac{|v|^p}{|y|^{sp}} \, dy - \frac{\lambda}{r} \int_{\mathbb{R}^N} a(y)|v|^r \, dy - \theta \int_{\mathbb{R}^N} \left|\widetilde{G}(y, v)\right| dy \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right) \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} \|v\|^r_{L^p(\mathbb{R}^N)} - \theta \int_{\Lambda_1} |G(y, v)| \, dy \\
&\quad - \theta \int_{\Lambda_2} \varphi(v)|G(y, v)| + (1 - \varphi(v))\eta |v|^p \, dy - \theta \int_{\Lambda_3} \eta |v|^p \, dy \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right) \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} C^r_{p,imb} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad - \theta \int_{\Lambda_1 \cup \Lambda_2} |G(y, v)| \, dy - \theta \int_{\Lambda_2 \cup \Lambda_3} \eta |v|^p \, dy \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right) \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} C^r_{p,imb} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}
\end{aligned}$$

$$
\begin{aligned}
& - \theta \int_{\Lambda_1 \cup \Lambda_2} \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} |v|^q \, dy - \theta \int_{\Lambda_2 \cup \Lambda_3} \eta |v|^p \, dy \\
\geq & \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p} \right) \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} C^r_{p,imb} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
& - \theta \left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta \right) \int_{\mathbb{R}^N} |v|^p \, dy \\
\geq & \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p} \right) \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} C^r_{p,imb} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
& - \theta \left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta \right) \|v\|^p_{L^p(\mathbb{R}^N)} \\
\geq & \left[\left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p} \right) - \theta \left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta \right) C_{p,imb} \right] \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
& - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} C^r_{p,imb} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)},
\end{aligned} \tag{47}
$$

where $C_{m,imb}$ is an imbedding constant of $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N)$ for any m with $p \leq m < p^*$. Also, if $\mu \in (-\infty, 0]$, then it follows in a similar way to in (47) that

$$
\begin{aligned}
\widetilde{\mathcal{I}}_{\mu,\theta}(v) \geq & \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^p \mathcal{K}(y,z) \, dy \, dz + \frac{1}{p} \int_{\mathbb{R}^N} \mathfrak{b}(y) |v|^p \, dy \\
& - \frac{\lambda}{r} \int_{\mathbb{R}^N} a(y) |v|^r \, dy - \theta \int_{\mathbb{R}^N} \left| \widetilde{G}(y,v) \right| dy \\
\geq & \left[\frac{1}{p} - \theta \left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta \right) C_{p,imb} \right] \|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
& - \frac{1}{r} \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} C^r_{p,imb} \|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}.
\end{aligned} \tag{48}
$$

We set

$$
\Gamma_1 = \left(0, \frac{q(\gamma_0 - \mu c_H)}{p \gamma_0 (\|b\|_{L^\infty(\mathbb{R}^N)} + q\eta) C_{p,imb}} \right)
$$

and

$$
\Gamma_2 = \left(0, \frac{q}{p(\|b\|_{L^\infty(\mathbb{R}^N)} + q\eta) C_{p,imb}} \right).
$$

Therefore, we arrive through (47) and (48) that the functional $\widetilde{\mathcal{I}}_{\mu,\lambda}$ is coercive in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, that is, $\widetilde{\mathcal{I}}_{\mu,\lambda}(v) \to \infty$ as $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \to \infty$ for any $\mu \in (-\infty, \mu^*)$ and for any $\theta \in \Gamma$, where Γ is either Γ_1 or Γ_2. □

Lemma 12. *We assume that* (V), (A1), (G1), (G4), *and* (G5) *hold. Then, for any $\lambda > 0$ and for any $\mu \in (0, \mu^*)$, the functional $\widetilde{\mathcal{I}}_{\mu,\theta}$ ensures the $(C)_c$-condition for every $\theta \in \Gamma$, where Γ and μ^* are given in Lemma 6 and Lemma 11, respectively.*

Proof. For any $c \in \mathbb{R}$, we let $\{v_n\}_{n\in\mathbb{N}}$ be a $(C)_c$-sequence in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ satisfying (19). From the coercivity of $\widetilde{\mathcal{I}}_{\mu,\theta}$, we infer the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ and thus $\{v_n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$. Without loss of generality, we suppose that

$$
v_n \rightharpoonup v_0 \text{ in } X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) \text{ as } n \to \infty.
$$

So, there is a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and a function v_0 in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ such that (24) is satisfied. By the definition of φ and (G1), we deduce that

$$|\widetilde{g}(y,\xi)| \le C_1\Big(b(y)|\xi|^{q-1} + \eta p|\xi|^{p-1}\Big) \tag{49}$$

for a positive constant C_1. Due to (24) and (49), we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} \widetilde{g}(y,v_n)(v_n - v_0)dy\right| \\ &\le C_1 \int_{\mathbb{R}^N}\Big(b(y)|v_n|^{q-1} + \eta p|v_n|^{p-1}\Big)|v_n - v_0|dy \\ &\le C_1\Big(\|b\|_{L^\infty(\mathbb{R}^N)}\|v_n\|^{q-1}_{L^q(\mathbb{R}^N)}\|v_n - v_0\|_{L^q(\mathbb{R}^N)} + \eta p\|v_n\|^{p-1}_{L^p(\mathbb{R}^N)}\|v_n - v_0\|_{L^p(\mathbb{R}^N)}\Big) \to 0\end{aligned} \tag{50}$$

as $n \to \infty$. From analogous arguments to those in Lemma 7, we state that $v_n \to v_0$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ as $n \to \infty$. □

Lemma 13. *Let us denote*

$$\chi_{\kappa,n} = \sup_{\|u\|=1, u\in\mathfrak{G}_n} \|u\|_{L^\kappa(\mathbb{R}^N)}$$

and

$$\chi_n = \max\{\chi_{q,n}, \chi_{p,n}\}. \tag{51}$$

Then, $\chi_n \to 0$ as $n \to \infty$ (see [47]).

With the help of Theorem 1 and Lemmas 10–12 and Remark 1, we are in a position to demonstrate our final main assertion.

Theorem 3. *We assume that* (V), (A1), (G1), (G4), *and* (G5) *hold. Then, for any $\lambda > 0$ and for any $\mu \in (0,\mu^*)$, Problem* (1) *has a sequence of nontrivial solutions $\{v_n\}_{n\in\mathbb{N}}$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ whose $\widetilde{\mathcal{I}}_{\mu,\theta}(v_n) \to 0$ and $\|v_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for every $\theta \in \Gamma$, where μ^* and Γ are given in Lemma 6 and Lemma 11, respectively.*

Proof. If all conditions ($\mathcal{D}$1)–($\mathcal{D}$4) of Lemma 9 are ensured, then for any $\mu \in (0,\mu^*)$ and for every $\theta \in \Gamma$, $\widetilde{\mathcal{I}}_{\mu,\theta}$ admits a sequence of negative critical values c_n for satisfying $c_n \to 0$ as $n \to \infty$. This, together with Lemma 12, yields that for any $\{v_n\}_{n\in\mathbb{N}} \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ with $\widetilde{\mathcal{I}}_{\mu,\theta}(v_n) = c_n$ and $\|\widetilde{\mathcal{I}}'_{\mu,\theta}(v_n)\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)^*} = 0$, we know that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a $(C)_0$-sequence of $\widetilde{\mathcal{I}}_{\mu,\theta}$ and $\{v_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence. Thus, up to a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, we have $v_n \to v$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ as $n \to \infty$. From Lemma 10 and Remark 1, we infer that zero is the only critical point with zero energy and $\{v_n\}_{n\in\mathbb{N}}$ has to converge to zero in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$; so, $\|v_n\|_{L^m(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for any m with $p \le m \le p^*$. In accordance with Theorem 1, any weak solution ω of (1) belongs to space $L^\infty(\mathbb{R}^N)$ and there exist positive constants $\eta, \mathcal{C}$ independent of ω such that

$$\|\omega\|_{L^\infty(\mathbb{R}^N)} \le \mathcal{C}\|\omega\|^{\eta}_{L^q(\mathbb{R}^N)}.$$

From this fact, we know $\|v_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, and thus, by Lemma 10 and Remark 1 again, we arrive at $\|v_n\|_{L^\infty(\mathbb{R}^N)} \le \xi_3$ for large n. Hence, $\{v_n\}_{n\in\mathbb{N}}$ with large enough n is a sequence of weak solutions to (1), as desired. From this point of view, we check that all conditions of Lemma 9 hold.

($\mathcal{D}$1): By (G5), (44), and the definition of φ,

$$\left|\widetilde{G}(y,\xi)\right| \le |G(y,\xi)| + \eta|\xi|^p \tag{52}$$

for almost all $y \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}$. Let $\chi_n < 1$ for large enough n. First, let us consider $\mu \in (0, \mu^*)$. Then, it follows from (52) that for any $v \in X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \geq 1$,

$$\begin{aligned}
\widetilde{\mathcal{I}}_{\mu,\theta}(v) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|v|^p}{|y|^{sp}}\,dy - \frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|v|^r\,dy - \theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,v)\right|dy \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\int_{\mathbb{R}^N}a(y)|v|^r\,dy \\
&\quad - \theta\int_{\mathbb{R}^N}(|G(y,v)| + \eta|v|^p)\,dy \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{L^p(\mathbb{R}^N)} \\
&\quad - \theta\int_{\mathbb{R}^N}|G(y,v)|\,dy - \theta\eta\int_{\mathbb{R}^N}|v|^p\,dy \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{L^p(\mathbb{R}^N)} \\
&\quad - \theta\int_{\mathbb{R}^N}\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}|v|^q\,dy - \theta\eta\chi_n^p\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{L^p(\mathbb{R}^N)} \\
&\quad - \frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|v\|^q_{L^q(\mathbb{R}^N)} - \theta\eta\chi_n^p\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\chi_n^r\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad - \frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\chi_n^q\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \theta\eta\chi_n^p\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\chi_n^r\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad - \theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\chi_n^p\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}
\end{aligned} \tag{53}$$

for large enough n. Let us choose

$$\beta_{1,n} = \left[\theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\frac{2\gamma_0 p\chi_n^p}{\gamma_0 - \mu c_H}\right]^{\frac{1}{p-2q}} \tag{54}$$

and let $v \in \mathfrak{Z}_n$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \beta_{1,n} > 1$ for sufficiently large n. Since $\mu \in (0, \mu^*)$ and $\beta_{1,n} \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\widetilde{\mathcal{I}}_{\mu,\theta}(v) &\geq \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\chi_n^r\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad - \theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\chi_n^p\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \left(\frac{1}{2p} - \frac{\mu c_H}{2\gamma_0 p}\right)\beta_{1,n}^p \\
&\quad - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\left[\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\frac{2\theta\gamma_0 p}{\gamma_0 - \mu c_H}\right]^{\frac{r}{p-2q}}\chi_n^{\frac{2r(p-q)}{p-2q}}
\end{aligned}$$

$$\geq 0$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

On the other hand, if $\mu \in (-\infty, 0]$ and $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \geq 1$, then it follows from (G1), the definition of χ_n, and similar arguments to those in (53) that

$$\begin{aligned}
\widetilde{\mathcal{I}}_{\mu,\theta}(v) &\geq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\
&\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|v|^r\,dy - \theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,v)\right|dy \\
&\geq \frac{1}{p}\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\chi_n^r\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad - \theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\chi_n^p\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}
\end{aligned} \tag{55}$$

for sufficiently large n. We choose

$$\beta_{2,n} = \left(2p\theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\chi_n^p\right)^{\frac{1}{p-2q}}. \tag{56}$$

Then, we know $\lim_{n\to\infty}\beta_{2,n} = \infty$. Let $v \in \mathfrak{Z}_n$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \beta_{2,n} > 1$ for large enough n. Then, using (55), we choose an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\widetilde{\mathcal{I}}_{\mu,\theta}(v) &\geq \frac{1}{p}\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\chi_n^p\|v\|^q_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\quad - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\chi_n^r\|v\|^r_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\
&\geq \frac{1}{2p}\beta_{2,n}^p - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\left(2p\theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q} + \eta\right)\right)^{\frac{r}{p-2q}}\chi_n^{\frac{2r(p-q)}{p-2q}} \\
&\geq 0
\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

Let β_n be either $\beta_{1,n}$ or $\beta_{2,n}$, which is given in (54) and (56), respectively. Then, we conclude

$$\inf\{\mathcal{I}_{\mu,\theta}(v) : v \in \mathfrak{Z}_n, \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \beta_n\} \geq 0$$

for any $\mu \in (-\infty, \mu^*)$.

($\mathcal{D}2$): We note that $\|\cdot\|_{L^\infty(\mathbb{R}^N)}$, $\|\cdot\|_{L^p(\mathbb{R}^N)}$ and $\|\cdot\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}$ are equivalent on $\mathfrak{Y}_n$. Then, there are constants $\widetilde{\varrho}_{1,n} > 0$ and $\widetilde{\varrho}_{2,n} > 0$ such that

$$\widetilde{\varrho}_{1,n}\|v\|_{L^\infty(\mathbb{R}^N)} \leq \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \leq \widetilde{\varrho}_{2,n}\|v\|_{L^p(\mathbb{R}^N)} \tag{57}$$

for any $v \in \mathfrak{Y}_n$. From (G4) and (G5), for any $\mathfrak{C}_3 > 0$, there exists $\xi_3 \in (0, \xi_2/2)$ such that

$$G(y,\xi) \geq \frac{\mathfrak{C}_3\widetilde{\varrho}^p_{2,n}}{p}|\xi|^p \tag{58}$$

for almost all $y \in \mathbb{R}^N$ and all $|\xi| \leq \xi_3$. We choose $\alpha_n := \min\{\frac{1}{2}, \xi_3\widetilde{\varrho}_{1,n}\}$ for all $n \in \mathbb{N}$. Then, we determine that $\|v\|_{L^\infty(\mathbb{R}^N)} \leq \xi_3$ for $v \in \mathfrak{Y}_n$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \alpha_n$, and thus $\widetilde{G}(y,v) = G(y,v)$.

First, we consider $\mu \in (0, \mu^*)$. Then, we determine by (57) and (58) that

$$\begin{aligned}\widetilde{\mathcal{I}}_{\mu,\theta}(v) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\ &\quad - \frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|v|^p}{|y|^{sp}}\,dy - \frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|v|^r\,dy - \theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,v)\right|dy \\ &\le \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\ &\quad - \theta\int_{\mathbb{R}^N}\frac{\mathfrak{C}_3\widetilde{\varrho}_{2,n}^p}{p}|v|^p\,dy \\ &\le \frac{1}{p}\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{\theta\mathfrak{C}_3\widetilde{\varrho}_{2,n}^p}{p}\|v\|^p_{L^p(\mathbb{R}^N)} \\ &\le \frac{1}{p}\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} - \frac{\theta\mathfrak{C}_3}{p}\|v\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\ &\le \frac{p-\theta\mathfrak{C}_3}{p}\alpha_n^p \end{aligned} \tag{59}$$

for any $v \in \mathfrak{Y}_n$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \alpha_n$.

Next, if $\mu \in (-\infty, 0]$, we have

$$\begin{aligned}\widetilde{\mathcal{I}}_{\mu,\theta}(v) &\le \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)-v(z)|^p\mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|v|^p\,dy \\ &\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N}a(y)|v|^r\,dy - \theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,v)\right|dy \\ &\le \frac{p-\theta\mathfrak{C}_3}{p}\alpha_n^p. \end{aligned} \tag{60}$$

If we choose a large enough $\mathfrak{C}_3$ such that $1 < \theta\mathfrak{C}_3$, then, through (59) and (60),

$$\sigma_n = \max\{\widetilde{\mathcal{I}}_{\mu,\theta}(v) : v \in \mathfrak{Y}_n, \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = \alpha_n\} < 0$$

for any $\mu \in (-\infty, \mu^*)$. If necessary, we can replace n_0 with a larger value, so that $\beta_n > \alpha_n > 0$ for all $n \ge n_0$.

$(\mathcal{D}3)$: Let β_n be either $\beta_{1,n}$ or $\beta_{2,n}$, which is given in (54) and (56), respectively. Because $\mathfrak{Y}_n \cap \mathfrak{Z}_n \neq \emptyset$ and $0 < \alpha_n < \beta_n$, we have $\psi_n \le \sigma_n < 0$ for all $n \ge n_0$. Let $v \in \mathfrak{Z}_n$ with $\|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} = 1$ and $0 < t < \beta_n$. With a similar argument to that in (53), we have, for any $\mu \in (0, \mu^*)$,

$$\begin{aligned}\widetilde{\mathcal{I}}_{\mu,\theta}(tv) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|tv(y)-tv(z)|^p\mathcal{K}(y,z)\,dy\,dz + \frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|tv|^p\,dy \\ &\quad - \frac{\mu}{p}\int_{\mathbb{R}^N}\frac{|tv|^p}{|y|^{sp}}\,dy - \frac{1}{r}\int_{\mathbb{R}^N}a(y)|tv|^r\,dy - \theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,tv)\right|dy \\ &\ge \left(\frac{1}{p} - \frac{\mu c_H}{\gamma_0 p}\right)\|tv\|^p_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \\ &\quad - \frac{1}{r}\int_{\mathbb{R}^N}a(y)|tv|^r\,dy - \theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,tv)\right|dy \\ &\ge -\frac{1}{r}\beta_{1,n}^r\int_{\mathbb{R}^N}a(y)|v|^r\,dy - \theta\int_{\mathbb{R}^N}(|G(y,tv)| + \eta|tv|^p)\,dy \\ &\ge -\frac{1}{r}\beta_{1,n}^r\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{L^p(\mathbb{R}^N)} \\ &\quad - \theta\int_{\mathbb{R}^N}G(y,tv)\,dy - \theta\eta\int_{\mathbb{R}^N}|tv|^p\,dy \end{aligned}$$

$$\geq -\frac{1}{r}\beta_{1,n}^{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|_{L^p(\mathbb{R}^N)}^{r}$$
$$-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\int_{\mathbb{R}^N}|tv|^q\,dy-\theta\eta\int_{\mathbb{R}^N}|tv|^p dy$$
$$\geq -\frac{1}{r}\beta_{1,n}^{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|_{L^p(\mathbb{R}^N)}^{r}$$
$$-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\beta_{1,n}^{q}\int_{\mathbb{R}^N}|v|^q\,dy-\theta\eta\beta_{1,n}^{p}\int_{\mathbb{R}^N}|v|^p\,dy$$
$$\geq -\frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\beta_{1,n}^{r}\chi_n^{r}-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\beta_{1,n}^{q}\chi_n^{q}-\theta\eta\beta_{1,n}^{p}\chi_n^{p}, \tag{61}$$

where χ_n is given in (51). Hence, from this and the definition of $\beta_{1,n}$, we infer

$$0>\psi_n\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\beta_{1,n}^{r}\chi_n^{r}-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\beta_{1,n}^{q}\chi_n^{q}-\theta\eta\beta_{1,n}^{p}\chi_n^{p}$$
$$\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\left[\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}+\eta\right)\frac{2\theta\gamma_0 p}{\gamma_0-\mu c_H}\right]^{\frac{r}{p-2q}}\chi_n^{\frac{2r(p-q)}{p-2q}}$$
$$-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\left[\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}+\eta\right)\frac{2\theta\gamma_0 p}{\gamma_0-\mu c_H}\right]^{\frac{q}{p-2q}}\chi_n^{\frac{2q(p-q)}{p-2q}}$$
$$-\theta\eta\left[\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}+\eta\right)\frac{2\theta\gamma_0 p}{\gamma_0-\mu c_H}\right]^{\frac{p}{p-2q}}\chi_n^{\frac{2p(p-q)}{p-2q}}. \tag{62}$$

On the other hand, we let $\mu\in(-\infty,0]$. Then, it follows from a similar proceeding to that in (61) that

$$\widetilde{\mathcal{I}}_{\mu,\theta}(tv)\geq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|tv(y)-tv(z)|^p\mathcal{K}(y,z)\,dy\,dz+\frac{1}{p}\int_{\mathbb{R}^N}\mathfrak{b}(y)|tv|^p\,dy$$
$$-\frac{1}{r}\int_{\mathbb{R}^N}a(y)|tv|^r\,dy-\theta\int_{\mathbb{R}^N}\left|\widetilde{G}(y,tv)\right|dy$$
$$\geq -\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}t^r\chi_n^r\|v\|_{X_{s,\mathfrak{b}}^{\mathcal{K}}(\mathbb{R}^N)}^{r}-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}t^q\chi_n^q\|v\|_{X_{s,\mathfrak{b}}^{\mathcal{K}}(\mathbb{R}^N)}^{q}$$
$$\geq -\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\beta_{2,n}^{r}\chi_n^{r}-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\beta_{2,n}^{q}\chi_n^{q}$$

for large enough n. This, together with the definition of $\beta_{2,n}$, yields

$$0>\psi_n\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\beta_{2,n}^{r}\chi_n^{r}-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\beta_{2,n}^{q}\chi_n^{q}-\theta\eta\beta_{2,n}^{p}\chi_n^{p}$$
$$\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\left[2p\theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}+\eta\right)\right]^{\frac{r}{p-2q}}\chi_n^{\frac{2r(p-q)}{p-2q}}$$
$$-\frac{\theta\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\left[2p\theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}+\eta\right)\right]^{\frac{q}{p-2q}}\chi_n^{\frac{2q(p-q)}{p-2q}}$$
$$-\theta\eta\left[2p\theta\left(\frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}+\eta\right)\right]^{\frac{p}{p-2q}}\chi_n^{\frac{2p(p-q)}{p-2q}}. \tag{63}$$

Because $p < q$ and $\chi_n \to 0$ as $n \to \infty$, we conclude by (62) and (63) that

$$\psi_n = \{\widetilde{\mathcal{I}}_{\mu,\theta}(v) : v \in \mathfrak{Z}_n, \|v\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)} \leq \beta_n\} \to 0$$

as $n \to \infty$ for any $\mu \in (0, \mu^*)$.

($\mathcal{D}4$): Let $c \in \mathbb{R}$ and let the sequence $\{v_n\}_{n\in\mathbb{N}}$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ be such that $v_n \in \mathfrak{Y}_n$ for any $n \in \mathbb{N}$,

$$\widetilde{\mathcal{I}}_{\mu,\theta}(v_n) \to c \quad \text{and} \quad \|(\widetilde{\mathcal{I}}_{\mu,\theta}|_{\mathfrak{Y}_n})'(v_n)\|_{(X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N))^*}(1 + \|v_n\|_{X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)}) \to 0 \text{ as } n \to \infty.$$

Since $\widetilde{\mathcal{I}}_{\mu,\theta}$ is coercive for any $\mu \in (0, \mu^*)$ and for every $\theta \in \Gamma$, by Lemma 11 it follows that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ for every $\theta \in \Gamma$. So, there is a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and a function v in $\widetilde{\mathcal{I}}_{\mu,\theta}$ such that (24) is satisfied.

To finish this proof, we prove that $v_n \to v$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ as $n \to \infty$ and also that v is a critical point of $\widetilde{\mathcal{I}}_{\mu,\theta}$. As $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N) = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{Y}_n}$, for $n \in \mathbb{N}$, we can choose $u_n \in \mathfrak{Y}_n$ such that $u_n \to v$ as $n \to \infty$. Hence, we know we have

$$\langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v_n), v_n - v\rangle = \langle (\widetilde{\mathcal{I}}_{\mu,\theta}|_{\mathfrak{Y}_n})'(v_n), v_n - u_n\rangle + \langle (\widetilde{\mathcal{I}}_{\mu,\theta}|_{\mathfrak{Y}_n})'(w_n), u_n - v\rangle.$$

Since $(\widetilde{\mathcal{I}}_{\mu,\theta}|_{\mathfrak{Y}_n})'(v_n) \to 0$, $u_n \to v$ and $v_n - u_n \to 0$ in $\mathfrak{Y}_n$ as $n \to \infty$, we have

$$\langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v_n), v_n - v\rangle \to 0 \text{ as } n \to \infty.$$

This, together with (19) and (28)–(33), yields Relation (34). From similar arguments to Lemma 7, we can state that $v_n \to v$ as $n \to \infty$. In addition, we have $\widetilde{\mathcal{I}}'_{\mu,\theta}(v_n) \to \widetilde{\mathcal{I}}'_{\mu,\theta}(v)$ as $n \to \infty$. Let us show that v is a critical point of $\widetilde{\mathcal{I}}_{\mu,\theta}$. In fact, we let $n_0 \in \mathbb{N}$ be fixed and take any $u \in \mathfrak{Y}_{n_0}$. For $n \geq n_0$, we have

$$\begin{aligned}
\langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v), u\rangle &= \langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v) - \widetilde{\mathcal{I}}'_{\mu,\theta}(v_n), u\rangle + \langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v_n), u\rangle \\
&= \langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v) - \widetilde{\mathcal{I}}'_{\mu,\theta}(v_n), u\rangle + \langle (\widetilde{\mathcal{I}}_{\mu,\theta}|_{\mathfrak{Y}_n})'(v_n), u\rangle,
\end{aligned}$$

so, passing the limit on the right side of the equation above, as $n \to \infty$, we arrive at

$$\langle \widetilde{\mathcal{I}}'_{\mu,\theta}(v), u\rangle = 0 \text{ for all } u \in \mathfrak{Y}_{n_0}.$$

As n_0 is taken arbitrarily and $\bigcup_{n\in\mathbb{N}} \mathfrak{Y}_n$ is dense in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$, we have $\widetilde{\mathcal{I}}'_{\mu,\theta}(v) = 0$, as claimed. Hence, we arrive at $v_n \to v$ in $X^{\mathcal{K}}_{s,\mathfrak{b}}(\mathbb{R}^N)$ as $n \to \infty$, and v is also a critical point of $\widetilde{\mathcal{I}}_{\mu,\theta}$. Accordingly, we know that the functional $\widetilde{\mathcal{I}}_{\mu,\theta}$ assures the $(C)^*_c$-condition for any $\mu \in (0, \mu^*)$ and for every $\theta \in \Gamma$. Condition ($\mathcal{D}4$) is proved. The proof is complete. □

5. Conclusions

The present paper is devoted to deriving the multiplicity and a priori bounds of solutions to the Schrödinger–Hardy-type nonlinear equation driven by the nonlocal fractional p-Laplacian. As far as we know, the uniform boundedness of any possible weak solutions to Schrödinger-type nonlocal fractional p-Laplacian problems with a singular coefficient, such as Hardy potentials, has not been studied extensively, and we are only aware of the study in [20]. However, our approach to obtain this regularity result is different from that in [20] because we employ the De Giorgi iteration method and a truncated energy technique. By applying these methods, we provide two multiplicity results of nontrivial weak solutions to our problem. To obtain these results, we consider a different approach to those in previous related studies [10,11,13–15,19,31,35,36]. More precisely, in contrast to the papers in [10,11,13–15,19], we show the existence of at least two distinct nontrivial solutions which belong to the L^∞-space by exploiting a variant of the Ekeland variational principle and the mountain pass theorem instead of the critical point theorems in [25–27]. By combining the modified functional method with the dual fountain theorem as in [18,32], we derive the existence of a sequence of infinitely many small-energy solutions that converge to zero in the L^∞-space. This approach is different from previously related works [31,35,36] that use the global variational formulation given in [37]. These are the novelties of this paper.

Furthermore, a new direction of research in strong relation is the investigation of the fractional $p(\cdot)$-Laplacian with the Hardy potential as follows:

$$M\Big([v]_{s,p(\cdot,\cdot)}\Big)\mathcal{L}v(y) + \mathfrak{b}(y)|v|^{p(y)-2}w = \mu\frac{|v|^{p(y)-2}v}{|y|^{sp(y)}} + \lambda g(y,v) \quad \text{in} \quad \mathbb{R}^N, \tag{64}$$

where

$$[v]_{s,p(\cdot,\cdot)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(y)-v(z)|^{p(y,z)}}{p(y,z)|y-z|^{N+sp(y,z)}}\,dy\,dz$$

and the operator $\mathcal{L}$ is defined by

$$\mathcal{L}v(y) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(y)} \frac{|v(y)-v(z)|^{p(y,z)-2}\,(v(y)-v(z))}{|y-z|^{N+sp(y,z)}}\,dy, \quad y\in\mathbb{R}^N,$$

where $s\in(0,1)$ and $B_\varepsilon(y) := \{z\in\mathbb{R}^N : |y-z|\le\varepsilon\}$. Furthermore, the Kirchhoff coefficient $M:[0,\infty)\to\mathbb{R}^+$ fulfills the following requirements:

(M1) $M\in C(\mathbb{R}^+)$ fulfills $\inf_{\zeta\in\mathbb{R}^+} M(\zeta)\ge\tau_0$ for a positive constant τ_0;

(M2) There exists a positive constant $\vartheta\ge 1$ such that

$$\vartheta\mathcal{M}(\zeta) = \vartheta\int_0^\zeta M(\eta)d\eta \ge M(\zeta)\zeta$$

for $\zeta\ge 0$.

To the best of our knowledge, there are no results on the existence of solutions to the fractional $p(\cdot)$-Laplacian with the Hardy potential due to the absence of the fractional Hardy inequality in variable Lebesgue space. However, the authors of [48] obtained the Hardy–Leray inequality and related various inequalities in variable Lebesgue spaces. An analysis of the results in [48] should yield some results regarding the existence of solutions to Problem (64).

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References

1. Bertoin, J. *Levy Processes Cambridge Tracts in Mathematics*; Cambridge University Press: Cambridge, UK, 1996.
2. Bjorland, C.; Caffarelli, L.; Figalli, A. A Non-local gradient dependent operators. *Adv. Math.* **2012**, *230*, 1859–1894. [CrossRef]
3. Caffarelli, L. Nonlocal equations, drifts and games. In *Nonlinear Partial Differential Equations Abel Symposia*; Springer: New York, NY, USA, 2012; Volume 7, pp. 37–52.
4. Gilboa, G.; Osher, S. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.* **2008**, *7*, 1005–1028. [CrossRef]
5. Laskin, N. Fractional quantum mechanics and Levy path integrals. *Phys. Lett. A* **2000**, *268*, 298–305. [CrossRef]
6. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2003**, *339*, 1–77. [CrossRef]
7. Diaz, J.I. Nonlinear Partial Differential Equations and Free Boundaries, V. I. Elliptic Equations. *Res. Notes Math.* **1985**, *1*, 106.

8. Diaz, J.I.; Morel, J.M.; Oswald, L. An elliptic equation with singular nonlinearity. *Comm. Partial. Differ. Equ.* **1987**, *12*, 1333–1344. [CrossRef]
9. Nachman, A.; Callegari, A. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math.* **1980**, *38*, 275–281. [CrossRef]
10. Achour, H.; Bensid, S. Existence results for singular elliptic problem involving a fractional p-Laplacian. *Fract. Calc. Appl. Anal.* **2023**, *26*, 2361–2391. [CrossRef]
11. Ferrara, M.; Bisci, G.M. Existence results for elliptic problems with Hardy potential. *Bull. Sci. Math.* **2014**, *138*, 846–859. [CrossRef]
12. Fiscella, A. Schrödinger-Kirchhoff-Hardy p-fractional equations without the Ambrosetti-Rabinowitz condition. *Discrete Contin. Dyn. Syst. Ser. S* **2020**, *13*, 1993–2007.
13. Khodabakhshi, M.; Afrouzi, G.A.; Hadjian, A. Existence of infinitely many weak solutions for some singular elliptic problems. *Complex Var. Elliptic Equ.* **2018**, *63*, 1570–1580. [CrossRef]
14. Khodabakhshi, M.; Aminpour, A.M.; Afrouzi, G.A.; Hadjian, A. Existence of two weak solutions for some singular elliptic problems. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* **2016**, *110*, 385–393. [CrossRef]
15. Khodabakhshi, M.; Hadjian, A. Existence of three weak solutions for some singular elliptic problems. *Complex Var. Elliptic Equ.* **2018**, *63*, 68–75. [CrossRef]
16. Kim, Y.-H.; Ahn, J.-H.; Lee, J.; Zeng, S. Multiplicity and a-priori bounds of solutions to Kirchhoff-Schrödinger-Hardy type equations involving the p-Laplacian. *Discret. Contin. Dyn. Syst.-S* **2024**, *submitted*.
17. Kim, Y.-H.; Na, H.Y. Multiplicity of solutions to non-local problems of Kirchhoff type involving Hardy potential. *AIMS Math.* **2023**, *8*, 26896–26921. [CrossRef]
18. Kim, Y.-H.; Park, C.Y.; Zeng, S. Infinitely many small energy solutions to the p-Laplacian problems of Kirchhoff type with Hardy potential. *Discrete Contin. Dyn. Syst. Ser. S* **2024**. . [CrossRef]
19. Liu, J.; Zhao, Z. Existence of triple solutions for elliptic equations driven by p-Laplacian-like operators with Hardy potential under Dirichlet-Neumann boundary conditions. *Bound. Value Probl.* **2023**, *2023*, 3. [CrossRef]
20. Choudhuri, D. Existence and Hölder regularity of infinitely many solutions to a p Kirchhoff type problem involving a singular nonlinearity without the Ambrosetti–Rabinowitz (AR) condition. *Z. Angew. Math. Phys.* **2021**, *72*, 36. [CrossRef]
21. Drábek, P. Nonlinear eigenvalue problem for p-Laplacian in $\mathbb{R}^N$. *Math. Nachr.* **1995**, *173*, 131–139. [CrossRef]
22. Drábek, P.; Kufner, A.; Nicolosi, F. *Quasilinear Elliptic Equations with Degenerations and Singularities*; de Gruyter: Berlin, Germany, 1997.
23. Vergara, V.; Zacher, R. A priori bounds for degenerate and singular evolutionary partial integro-differential equations. *Nonlinear Anal.* **2010**, *73*, 3572–3585. [CrossRef]
24. Ahn, J.-H.; Kim, I.H.; Kim, Y.-H.; Zeng, S. Existence results and L^∞-bound of solutions to Kirchhoff-Schrödinger-Hardy type equations involving double phase operators. *submitted*.
25. Ricceri, B. A general variational principle and some of its applications. *J. Appl. Math. Comput.* **2000**, *113*, 401–410. [CrossRef]
26. Ricceri, B. A further three critical points theorem. *Nonlinear Anal.* **2009**, *71*, 4151–4157. [CrossRef]
27. Bonanno, G. Relations between the mountain pass theorem and local minima. *Adv. Nonlinear Anal.* **2012**, *1*, 205–220. [CrossRef]
28. Bae, J.-H.; Kim, Y.-H. Critical points theorems via the generalized Ekeland variational principle and its application to equations of $p(x)$-Laplace type in $\mathbb{R}^N$. *Taiwanese J. Math.* **2019**, *23*, 193–229. [CrossRef]
29. Ambrosetti, A.; Rabinowitz, P. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **1973**, *14*, 349–381. [CrossRef]
30. Fiscella, A. A double phase problem involving Hardy potentials. *Appl. Math. Optim.* **2022**, *85*, 45. [CrossRef]
31. Ho, K.; Winkert, P. Infinitely many solutions to Kirchhoff double phase problems with variable exponents. *Appl. Math. Lett.* **2023**, *145*, 108783. [CrossRef]
32. Kim, Y.-H.; Jeong, T.-J. Multiplicity Results of Solutions to the Double Phase Problems of Schrödinger-Kirchhoff Type with Concave-Convex Nonlinearities. *Mathematics* **2024**, *12*, 60. [CrossRef]
33. Kim, Y.-H. Existence and Multiplicity of Solutions to a Class of Fractional p-Laplacian Equations of Schrödinger-Type with Concave-Convex Nonlinearities in $\mathbb{R}^N$. *Mathematics* **2020**, *8*, 1792.. [CrossRef]
34. Lee, J.I.; Kim, Y.-H. Multiplicity of Radially Symmetric Small Energy Solutions for Quasilinear Elliptic Equations Involving Nonhomogeneous Operators. *Mathematics* **2020**, *8*, 128. [CrossRef]
35. Tan, Z.; Fang, F. On superlinear $p(x)$-Laplacian problems without Ambrosetti and Rabinowitz condition. *Nonlinear Anal.* **2012**, *75*, 3902–3915.
36. Wang, Z.-Q. Nonlinear boundary value problems with concave nonlinearities near the origin. *NoDEA Nonlinear Differ. Equ. Appl.* **2001**, *8*, 15–33. [CrossRef]
37. Heinz, H.P. Free Ljusternik-Schnirelman theory and the bifurcation diagrams of certain singular nonlinear problems. *J. Differ. Equ.* **1987**, *66*, 263–300. [CrossRef]
38. Adams, R.A.; Fournier, J.J.F. *Sobolev Spaces*, 2nd ed.; Academic Press: New York, NY, USA; London, UK, 2003.
39. Nezza, E.D.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [CrossRef]
40. Perera, K.; Squassina, M.; Yang, Y. Bifurcation and multiplicity results for critical fractional p-Laplacian problems. *Math. Nachr.* **2016**, *289*, 332–342. [CrossRef]

41. Xiang, M.Q.; Zhang, B.L.; Ferrara, M., Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian. *J. Math. Anal. Appl.* **2015**, *424*, 1021–1041. [CrossRef]
42. Frank, R.L.; Seiringer, R. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.* **2008**, *255*, 3407–3430. [CrossRef]
43. Kim, I.H.; Kim, Y.-H. Infinitely many small energy solutions to nonlinear Kirchhoff-Schrödinger equations with the p-Laplacian. *Bull. Malays. Math. Sci. Soc.* **2024**, *47*, 99. [CrossRef]
44. Brezis, H.; Lieb, E. A Relation Between Pointwise Convergence of Functions and Convergence of Functionals. *Proc. Am. Math. Soc.* **1983**, *88*, 486–490. [CrossRef]
45. Fabian, M.; Habala, P.; Hajék, P.; Montesinos, V.; Zizler, V. *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*; Springer: New York, NY, USA, 2011.
46. Zhou, Y.; Wang, J.; Zhang, L. *Basic Theory of Fractional Differential Equations*, 2nd ed.; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2017.
47. Hurtado, E.J.; Miyagaki, O.H.; Rodrigues, R.S. Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti-Rabinowitz type conditions. *J. Dyn. Diff. Equ.* **2018**, *30*, 405–432. [CrossRef]
48. Cruz-Uribe, D.; Suragan, D. Hardy-Leray inequalities in variable Lebesgue spaces. *J. Math. Anal. Appl.* **2024**, *530*, 127747. [CrossRef]