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# Exploring Solitons Solutions of a (3+1)-Dimensional Fractional mKdV-ZK Equation

**Amjad E. Hamza [1], Osman Osman [2,*], Muhammad Umair Sarwar [3], Khaled Aldwoah [4,*], Hicham Saber [1] and Manel Hleili [5]**

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 55473, Saudi Arabia
[2] Department of Mathematics, College of Science, Qassim University, Buraydah 52571, Saudi Arabia
[3] Institute of Mathematics, University of Kassel, 34125 Kassel, Germany
[4] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Medina 42351, Saudi Arabia
[5] Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia; mhleili@ut.edu.sa
* Correspondence: o.osman@qu.edu.sa (O.O.); aldwoah@iu.edu.sa (K.A.)

**Abstract:** This study presents the application of the $\phi^6$ model expansion technique to find exact solutions for the (3+1)-dimensional space-time fractional modified KdV-Zakharov-Kuznetsov equation under Jumarie's modified Riemann–Liouville derivative (JMRLD). The suggested method captures dark, periodic, traveling, and singular soliton solutions, providing deep insights into wave behavior. Clear graphics demonstrate that the solutions are greatly affected by changes in the fractional order, deepening our understanding and revealing the hidden dynamics of wave propagation. The considered equation has several applications in fluid dynamics, plasma physics, and nonlinear optics.

**Keywords:** $\phi^6$ model expansion approach; solitons; fractional derivative



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## 1. Introduction

Differential equations play important role in the modelling of physical phenomena [1,2]. Besides this, fractional differential equations (FDEs) have gained important attention in recent years due to their growing application in modeling complex nonlinear phenomena across several fields of science, containing physics, biology, mathematics, economics, engineering, and others [3,4]. These equations are used to define real-world systems, which are then explained into mathematical models. As a result, the quest of exact solutions for FDEs is essential in scientific research [5–7].

FDEs are an expansion of classical differential equations. Differentiation orders may be any real number, instead of just integers as in traditional differential equations. This makes FDEs much better at modeling complex system dynamics, especially in cases that show non-locality or memory features [8]. Applications for FDEs are numerous and varied starting from modelling population dynamics and chemical reactions, all the way to involving behavioral phenomena of materials and systems in physics and engineering [9–11].

In the investigation of FDEs, new analytical and numerical methods are developed for solving these equations, with prevailing among them the Adomian decomposition method [12], homotopy perturbation method [13], variational iteration method [14], and matrix approach method [15], between others. The method selected usually depends on the specific problem at hand and the required level of accuracy.

Fractional soliton solutions, arising from FDEs, have diverse applications across scientific and engineering fields. They model phenomena in nonlinear optical fibers, plasma physics, and fluid dynamics, where standard solitons fall short due to non-local effects. In quantum mechanics, they describe wave functions with unique dispersion properties, while in biological systems, they capture wave propagation with memory effects. Their applications extend to financial modeling, acoustics, epidemiology, control systems, and material science, offering insights into complex behaviors and interactions in these areas [16–19].

Recently, nonlinear integrable systems and their soliton solutions have attracted the attention of the researchers. Various soliton solutions of integrable systems have been investigated in the literature. For instance, Chen–Lee–Liu equation[20], Sasa-Satsuma equation [21], Drinfel'd–Sokolov–Wilson equation [22], pKP-BKP integrable equation [23], complex Ginzburg–Landau equation [24], Chaffee–Infante equation [25], nonlinear Zakharov system [26], and many more [27,28]. The (3+1)-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov ((3+1)-D MKDV-ZK) equation is a mathematical equation that shows the propagation of nonlinear waves in four-dimensional space time. It's an extension of the classic KdV equation, which expresses solitary wave phenomena, and the Zakharov-Kuznetsov equation, which explanations for wave propagation in plasmas. The revised equation contains additional terms or modifications for a better explanation of actual phenomena. It's commonly used to examine various physical systems such as fluid dynamics and nonlinear optics where it gives an understanding of complex wave behaviors [29,30]. The following expression has been proposed to offer a generalized version of the classical KdV equation to have fractional derivatives taken as per the modified Riemann-Liouville derivative.

(3+1)-D MKDV-ZK.

$$\mathcal{D}_t^{\varphi}\mathrm{U} + \gamma\mathrm{U}^2\mathcal{D}_x^{\varphi}\mathrm{U} + \mathcal{D}_x^{3\varphi}\mathrm{U} + \mathcal{D}_x^{\varphi}\mathcal{D}_y^{2\varphi}\mathrm{U} + \mathcal{D}_x^{\varphi}\mathcal{D}_z^{2\varphi}\mathrm{U} = 0, \qquad 0 < \varphi \leq 1. \tag{1}$$

Here $\gamma$ is an arbitrary constant.

Various techniques have been developed to solve the (3+1)-D MKDV-ZK equation, containing the method of undetermined coefficients [31], the ansatz method [32], the functional variable method [33], and the exp-function method [34]. These methods were applied for getting different types of solutions like solitons, dark solitons etc. In addition to this, they have also been working on studying a number of different cases including (3+1)-D MKDV-ZK equation in different fields of physics related to other aspects of nonlinear waves, solitons, etc. The equation is also related to fractional calculus and its applications in modeling complex systems.

The method that we have suggested provides exact solutions to complicated equations such as the (3+1)-D MKDV-ZK equation. Taking into account JMRLD and $\phi^6$ model expansion method of Jumarie makes it a fairly extensive observation of wide soliton behaviors. The power of the proposed approach lies in its ability to solve fractional derivative nonlinear wave equations properly and effectively in order to capture different wave phenomena. This method has wide applications such as in fluid dynamics and nonlinear optics where exact control of nonlinear wave dynamics is useful in forecasting actual systems.

The unique thing about this research is how it addresses the research gap by modeling complex wave dynamics exactly. Previous attempts often needed accuracy and completeness to achieve broad soliton behaviors depending on numerical approximations or simplification techniques. This study contributes by providing exact solutions to the (3+1)-D MKDV-ZK equations, thereby filling the research gap. Thus, this investigation significantly improves our capability for modeling and understanding nonlinear wave phenomena through a new approach for obtaining exact solutions which are different from those used before.

This paper proposes employing JMRLD [35] in combination with $\phi^6$ model expansion method to obtain exact solutions for nonlinear FPDEs. The primary objective of this study is to demonstrate the efficiency of this approach by utilizing it to derive exact solutions for FPDEs in both spatial and temporal domains related to JMRLD. The study will propose future possibilities, like extending the method to new equations, refining techniques for complexity, and exploring practical applications. It will highlight how these efforts can advance scientific understanding and solve real-world problems effectively.

The JMRLD of order $\varphi$ is defined by the following expression [36]:

$$\mathcal{D}_x^{\varphi}\mathcal{F}(x) = \begin{cases} \frac{1}{\Gamma(1-\varphi)}\frac{d}{dx}\int_0^x (x-\varepsilon)^{-\varphi}(\mathcal{F}(\varepsilon)-\mathcal{F}(0))d\varepsilon, 0<\varphi<1, \\ (\mathcal{F}^{(\mathfrak{n})}(x))^{(\varphi-\mathfrak{n})}, \mathfrak{n}\le\varphi<\mathfrak{n}+1, \mathfrak{n}\le 1. \end{cases} \tag{2}$$

Here $\mathcal{F}:\mathrm{R}\to\mathrm{R}, x\to\mathcal{F}(x)$ describes a function that is continuous.

**Property 1.** *Suppose that $\mathcal{F}(x)$ describes a function that is continuous from* R → R. *We will use the considered equality for the integral via $(dx)^{\varphi}$:*

$$\mathcal{D}_x^{\varphi}\mathcal{F}(x) = \frac{1}{\Gamma(\varphi)}\int_0^x (x-\varepsilon)^{\alpha-1}f(\varepsilon)d\varepsilon = \frac{1}{\Gamma(1+\varphi)}\frac{d}{dx}\int_0^x f(\varepsilon)(d\varepsilon)^{\varphi}, 0<\varphi\le 1. \tag{3}$$

**Property 2.**

$$\mathcal{D}_x^{\varphi}x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\varphi)}x^{r-\varphi}. \tag{4}$$

**Property 3.**

$$\mathcal{D}_x^{\varphi}(c\mathcal{F}(x)) = cD_x^{\varphi}\mathcal{F}(x), \ \ \textit{where } c \textit{ is a constant}. \tag{5}$$

**Property 4.**

$$\mathcal{D}_x^{\varphi}a\mathcal{F}(x)+bg(x) = a\mathcal{D}_x^{\varphi}\mathcal{F}(x)+bD_x^{\varphi}g(x), \tag{6}$$

$$\textit{here } a \textit{ and } b \textit{ are constants}.$$

**Property 5.**

$$\mathcal{D}_x^{\varphi}c = 0, \tag{7}$$

*where c is constant.*

## 2. Methodology

Here in the section, we provide briefly discuss the $\phi^6$ model expansion technique. Let us take a nonlinear partial differential equation(PDE).

$$\mathtt{U}(\mathtt{f},\mathtt{f}_x,\mathtt{f}_t,\mathtt{f}_{xx},\mathtt{f}_{tt},\dots)=0, \tag{8}$$

where, $\mathtt{U}$ is a polynomial with $\mathtt{f}(x,t)$ is a unknown partial derivatives.

The scheme contains following steps:

**Step 1:** Taking the traveling wave transformation defined as:

$$\mathtt{f}(x,t)=\mathtt{U}(\varepsilon), \quad \varepsilon=\frac{ax^{\varphi}}{\Gamma(\varphi+1)}+\frac{by^{\varphi}}{\Gamma(\varphi+1)}+\frac{cz^{\varphi}}{\Gamma(\varphi+1)}-\frac{st^{\varphi}}{\Gamma(\varphi+1)}. \tag{9}$$

The transformation given above will turn Equation (8) into ordinary differential equation (ODE). where $a,\ b,\ c$ and $s$ are arbitrary constants.

$$\mathtt{N}(\mathtt{U},\mathtt{U}',\mathtt{U}'',\mathtt{U}''',\dots), \tag{10}$$

where $N$ is a polynomial of transformed ordinary derivatives $\mathtt{U}=\mathtt{U}(\varepsilon)$, $\mathtt{U}'=\frac{d\mathtt{U}}{d\varepsilon}$, $\mathtt{U}''=\frac{d^2\mathtt{U}}{d\varepsilon^2}$, $\mathtt{U}'''=\frac{d^3\mathtt{U}}{d\varepsilon^3},\dots$

**Step 2:** Let the formal solution of Equation (10) is:

$$\mathtt{U}(\varepsilon) = \sum_{\nu=0}^{2s} \alpha_\nu \theta^\nu(\varepsilon). \tag{11}$$

$\alpha_\nu$ are constants for $(\nu = 0, 1, 2, \ldots, 2k)$ and $\theta(\varepsilon)$ follows the below auxiliary non-linear ODE.

$$\begin{aligned} \theta'(\varepsilon)^2 &= \varkappa_0 + \varkappa_2\theta(\varepsilon)^2 + \varkappa_4\theta(\varepsilon)^4 + \varkappa_6\theta(\varepsilon)^6, \\ \theta''(\varepsilon) &= \varkappa_2\theta(\varepsilon) + 2\varkappa_4\theta(\varepsilon)^3 + 3\varkappa_6\theta(\varepsilon)^5, \end{aligned} \tag{12}$$

here, $\varkappa_\nu$ is a real constant for $\nu = 0, 2, 4, 6$.
**Step 3:** By Balancing Equation (10) to obtain the value of $s$ in Equation (11).
**Step 4:** Obtaining solution for Equation (8):

$$\theta(\varepsilon) = \frac{\mathbb{A}(\varepsilon)}{\sqrt{\mathsf{m}\mathbb{A}(\varepsilon)^2 + \mathsf{n}}}, \tag{13}$$

here $\mathbb{A}(\varepsilon)$ and $\mathsf{m}\mathbb{A}(\varepsilon)^2 + \mathsf{n} > 0$ satisfy the Jacobian Elliptic equation (JEE):

$$\mathbb{A}'^2 = \mathsf{t}_0 + \mathsf{t}_2\mathbb{A}^2(\varepsilon) + \mathsf{t}_4\mathbb{A}^4(\varepsilon), \tag{14}$$

where $\mathsf{t}_j$ are constants for $j = 0, 2, 4$.
The values of $m$ and $n$ are defined as:

$$\mathsf{m} = \frac{\varkappa_4(\mathsf{t}_2 - \varkappa_2)}{(\mathsf{t}_2 - \varkappa_2)^2 - 2\mathsf{t}_2(\mathsf{t}_2 - \varkappa_2) + 3\mathsf{t}_0\mathsf{t}_4}, \tag{15}$$

$$\mathsf{n} = \frac{3\varkappa_4\mathsf{t}_0}{(\mathsf{t}_2 - \varkappa_2)^2 - 2\mathsf{t}_2(\mathsf{t}_2 - \varkappa_2) + 3\mathsf{t}_0\mathsf{t}_4}, \tag{16}$$

under the constraint condition:

$$\varkappa_4^2(\mathsf{t}_2 - \varkappa_2)[9\mathsf{t}_0\mathsf{t}_4 - (\mathsf{t}_2 - \varkappa_2)(\varkappa_2 + 2\mathsf{t}_2)] + 3\varkappa_6\Big[3\mathsf{t}_0\mathsf{t}_4 - \Big(\mathsf{t}_2^2 - \varkappa_2^2\Big)\Big]^2 = 0. \tag{17}$$

**Step 5:** The JEE for Equation (14), are given in the table below.

| Sr. No. | $\mathsf{t}_0$ | $\mathsf{t}_2$ | $\mathsf{t}_4$ | $\mathtt{U}(\varepsilon)$ |
|---|---|---|---|---|
| 1 | $1$ | $-(1+\mu^2)$ | $\mu^2$ | $sn(\varepsilon)$ or $cd(\varepsilon)$ |
| 2 | $1-\mu^2$ | $2\mu^2-1$ | $-\mu^2$ | $cn(\varepsilon)$ |
| 3 | $\mu^2-1$ | $2-\mu^2$ | $-1$ | $dn(\varepsilon)$ |
| 4 | $\mu^2$ | $-(1+a^2)$ | $1$ | $ns(\varepsilon)$ or $dc(\varepsilon)$ |
| 5 | $-\mu^2$ | $2\mu^2-1$ | $1-\mu^2$ | $nc(\varepsilon)$ |
| 6 | $-1$ | $2-\mu^3$ | $-(1-a^2)$ | $nd(\varepsilon)$ |
| 7 | $1$ | $2-\mu^2$ | $1-\mu^2$ | $sc(\varepsilon)$ |
| 8 | $1$ | $2\mu^2-1$ | $-\mu^2(1-\mu^2)$ | $sd(\varepsilon)$ |
| 9 | $1-\mu^2$ | $2-\mu^2$ | $1$ | $cs(\varepsilon)$ |
| 10 | $-\mu^2(1-\mu^2)$ | $2\mu^2-1$ | $1$ | $ds(\varepsilon)$ |
| 11 | $\frac{1-\mu^2}{4}$ | $\frac{1+\mu^2}{2}$ | $\frac{1-\mu^2}{4}$ | $nc(\varepsilon) \pm sc(\varepsilon)$ or $\frac{cn(\varepsilon)}{1\pm sn(\varepsilon)}$ |
| 12 | $\frac{-(1-\mu^2)^2}{4}$ | $\frac{1+\mu^2}{2}$ | $\frac{-1}{4}$ | $ncn(\varepsilon) \pm dn(\varepsilon)$ |
| 13 | $\frac{1}{4}$ | $\frac{1-2\mu^2}{2}$ | $\frac{1}{4}$ | $\frac{sn(\varepsilon)}{1\pm cn(\varepsilon)}$ |
| 14 | $\frac{1}{4}$ | $\frac{1+\mu^2}{2}$ | $\frac{(1-\mu^2)^2}{4}$ | $\frac{sn(\varepsilon)}{cn(\varepsilon)\pm dn(\varepsilon)}$ |

To obtain analytic solutions to the equation, jacobian elliptic functions (JEF) limitations are given in the table.

| function | $\mu \to 1$ | $\mu \to 0$ | function | $\mu \to 1$ | $\mu \to 0$ |
|---|---|---|---|---|---|
| $sn(\varepsilon, \mu)$ | $tanh(\varepsilon)$ | $sin(\varepsilon)$ | $ns(\varepsilon, \mu)$ | $coth(\varepsilon)$ | $csc(\varepsilon)$ |
| $cd(\varepsilon)$ | 1 | $cos(\varepsilon)$ | $dc(\varepsilon)$ | 1 | $sec(\varepsilon)$ |
| $cn(\varepsilon)$ | $sech(\varepsilon)$ | $cos(\varepsilon)$ | $nc(\varepsilon)$ | $cosh(\varepsilon)$ | $sec(\varepsilon)$ |
| $dn(\varepsilon)$ | $sech(\varepsilon)$ | 1 | $nd(\varepsilon)$ | $cosh(\varepsilon)$ | 1 |
| $sc(\varepsilon)$ | $sinh(\varepsilon)$ | $tan(\varepsilon)$ | $cs(\varepsilon)$ | $csch(\varepsilon)$ | $cot(\varepsilon)$ |
| $sd(\varepsilon)$ | $sinh(\varepsilon)$ | $sin(\varepsilon)$ | $ds(\varepsilon)$ | $csch(\varepsilon)$ | $csc(\varepsilon)$ |

**Step 6:** By substituting Equations (12) and (13) into Equation (11) one will achieve the JEF solutions of Equation (8).

## 3. Implementation of the Expansion Method

Through balancing the term $[\mathtt{U}'', \mathtt{U}^3]$, we get $s = 2$. After substituting the value obtained for $s$, we get formal solution expansion as:

$$\mathtt{U}(\varepsilon) = \alpha_0 + \alpha_1 \theta(\varepsilon) + \alpha_2 \theta^2(\varepsilon), \tag{18}$$

where $\alpha_0$, $\alpha_1$, and $\alpha_2$ are constants, and $\alpha_2 \neq 0$. For getting algebraic equations we make use of Equation (18), Equation (13), and Equation (10), by setting them equal to zero, we obtain.

$$\begin{aligned}
&\theta^0 : \frac{1}{3}\alpha_0^3 \delta\kappa + 2\alpha_2 \lambda^2 \kappa \varkappa_0 + \lambda + 2\alpha_2 \gamma^2 \kappa \varkappa_0 + 2\alpha_2 \kappa^3 \varkappa_0 - \alpha_0 \sigma = 0, \\
&\theta^1 : \alpha_0^2 \alpha_1 \delta\kappa + \alpha_1 c^2 \kappa \varkappa_2 + \alpha_1 \gamma^2 \kappa \varkappa_2 + \alpha_1 \kappa^3 \varkappa_2 - \alpha_1 \sigma = 0, \\
&\theta^2 : \alpha_0 \alpha_1^2 \delta\kappa + \alpha_0^2 \alpha_2 \delta\kappa + 4\alpha_2 c^2 \kappa \varkappa_2 + 4\alpha_2 \gamma^2 \kappa \varkappa_2 + 4\alpha_2 \kappa^3 \varkappa_2 - \alpha_2 \sigma = 0, \\
&\theta^3 : \frac{1}{3}\alpha_1^3 \delta\kappa + 2\alpha_0 \alpha_1 \alpha_2 \delta\kappa + 2\alpha_1 c^2 \kappa \varkappa_4 + 2\alpha_1 \gamma^2 \kappa \varkappa_4 + 2\alpha_1 \kappa^3 \varkappa_4 = 0, \\
&\theta^4 : \alpha_0 \alpha_2^2 \delta\kappa + \alpha_1^2 \alpha_2 \delta\kappa + 6\alpha_2 c^2 \kappa \varkappa_4 + 6\alpha_2 \gamma^2 \kappa \varkappa_4 + 6\alpha_2 \kappa^3 \varkappa_4 = 0, \\
&\theta^5 : \alpha_1 \alpha_2^2 \delta\kappa + 3\alpha_1 c^2 \kappa \varkappa_6 + 3\alpha_1 \gamma^2 \kappa \varkappa_6 + 3\alpha_1 \kappa^3 \varkappa_6 = 0.
\end{aligned} \tag{19}$$

We use Mathematica 13.0 for computing these constants from the above-mentioned algebraic equations.

$$\begin{aligned}
&\alpha_0 = \alpha_0, \quad \alpha_2 = \alpha_2, \quad \alpha_1 = 0, \quad \varkappa_2 = \frac{\sigma - \alpha_0^2 \delta\kappa}{4\kappa(c^2 + \gamma^2 + \kappa^2)}, \quad \varkappa_4 = -\frac{\alpha_0 \alpha_2 \delta}{6(c^2 + \gamma^2 + \kappa^2)}, \\
&\varkappa_6 = -\frac{\alpha_2^2 \delta}{24(c^2 + \gamma^2 + \kappa^2)}, \quad \lambda = \frac{1}{3}\Big(-\alpha_0^3 \delta\kappa - 6\alpha_2 c^2 \kappa \varkappa_0 - 6\alpha_2 \gamma^2 \kappa \varkappa_0 - 6\alpha_2 \kappa^3 \varkappa_0 + 3\alpha_0 \sigma\Big).
\end{aligned} \tag{20}$$

The obtained analytic answers of Equation (1) are:
**Case 1:** If $\mathrm{t}_0 = 1; \mathrm{t}_2 = -\left(1 + \mu^2\right); \mathrm{t}_4 = \mu^2;$ *then* $\mathbb{A}(\varepsilon) = \mathrm{sn}(\varepsilon)$ or $\mathrm{cd}(\varepsilon)$, $0 < \mu < 1$ we obtain the JEF solutions as:

$$\mathtt{U}_1 = \alpha_0 + \frac{\alpha_2 \mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2 + \mathsf{n}}, \tag{21}$$

where $\mathsf{m}$ and $\mathsf{n}$ are given below as

$$\begin{aligned}
\mathsf{m} &= \frac{\varkappa_4(-\mu^2 - \varkappa_2 - 1)}{(-\mu^2 - \varkappa_2 - 1)^2 - 2(-\mu^2 - 1)(-\mu^2 - \varkappa_2 - 1) + 3\mu^2}, \\
\mathsf{n} &= \frac{3\varkappa_4}{(-\mu^2 - \varkappa_2 - 1)^2 - 2(-\mu^2 - 1)(-\mu^2 - \varkappa_2 - 1) + 3\mu^2},
\end{aligned}$$

subject to condition obtained as:

$$\varkappa_4^2\Big(-\mu^2-\varkappa_2-1\Big)\Big(9\mu^2-\Big(-\mu^2-\varkappa_2-1\Big)\Big(2\Big(-\mu^2-1\Big)+\varkappa_2\Big)\Big)+3\varkappa_6\Big(3\mu^2-\Big(-\mu^2-1\Big)^2+\varkappa_2^2\Big)^2=0,$$

when $\mu\to 1$

As $\mathbb{A}(\varepsilon)=\mathrm{sn}(\varepsilon)=\tanh(\varepsilon)$ then we obtained:

$$\mathtt{U}_{1,1}=\alpha_0+\alpha_2\left(-\frac{3\tanh^2(\varepsilon)\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\Big(12\kappa(c^2+\gamma^2+\kappa^2)-\tanh^2(\varepsilon)\big(\alpha_0^2(-\delta)\kappa+8\kappa(c^2+\gamma^2+\kappa^2)+\sigma\big)\Big)}\right), \tag{22}$$

or $\mathbb{A}(\varepsilon)=\mathrm{cd}(\varepsilon)=1$ then we get:

$$\mathtt{U}_{1,2}=\alpha_0+\alpha_2\left(-\frac{3\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\big(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\big)}\right). \tag{23}$$

Subject to condition computed as:

$$\varkappa_4^2(-\varkappa_2-2)(9-(-\varkappa_2-2)(\varkappa_2-4))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

When $\mu\to 0$

As $\mathbb{A}(\varepsilon)=\mathrm{sn}(\varepsilon)=\sin(\varepsilon)$ then we obtain:

$$\mathtt{U}_{1,3}=\alpha_0+\alpha_2\left[-\frac{3\sin^2(\varepsilon)\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\big(12\kappa(c^2+\gamma^2+\kappa^2)-\sin^2(\varepsilon)\big(\alpha_0^2(-\delta)\kappa+8\kappa(c^2+\gamma^2+\kappa^2)+\sigma\big)\big)}\right]. \tag{24}$$

or $\mathbb{A}(\varepsilon)=\mathrm{cd}(\varepsilon)=\cos(\varepsilon)$ then we get:

$$\mathtt{U}_{1,4}=\alpha_0+\alpha_2\left(-\frac{3\cos^2(\varepsilon)\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\big(12\kappa(c^2+\gamma^2+\kappa^2)-\cos^2(\varepsilon)\big(\alpha_0^2(-\delta)\kappa+8\kappa(c^2+\gamma^2+\kappa^2)+\sigma\big)\big)}\right). \tag{25}$$

Under the condition interpreted as:

$$\varkappa_4^2(-\varkappa_2-1)(-(-\varkappa_2-1)(\varkappa_2-2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

**Case 2:** If $\mathrm{t}_0=1-\mu^2$; $\mathrm{t}_2=2\mu^2-1$; $\mathrm{t}_4=-\mu^2$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)$ = $\mathrm{cn}(\varepsilon)$ the JEF solution is as:

$$\mathtt{U}_2=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2+\mathsf{n}}, \tag{26}$$

where $m$ and $n$ are of the form:

$$\mathsf{m}=\frac{\varkappa_4\big(2\mu^2-\varkappa_2-1\big)}{(2\mu^2-p_2-1)^2-2(2\mu^2-1)(2\mu^2-\varkappa_2-1)-3(1-\mu^2)\mu^2},$$
$$\mathsf{n}=\frac{3\big(1-\mu^2\big)\varkappa_4}{(2\mu^2-\varkappa_2-1)^2-2(2\mu^2-1)(2\mu^2-\varkappa_2-1)-3(1-\mu^2)\mu^2}.$$

Under the condition which interpret as:

$$\varkappa_4^2\Big(2\mu^2-\varkappa_2-1\Big)\Big(-\Big(2\mu^2-\varkappa_2-1\Big)\Big(2\Big(2\mu^2-1\Big)+\mu_2\Big),$$
$$-9\Big(1-\mu^2\Big)\mu^2\Big)+3\varkappa_6\Big(-3\Big(1-\mu^2\Big)\mu^2-\Big(2\mu^2-1\Big)^2+\varkappa_2^2\Big)^2=0.$$

When $\mu\to 1$
As $\mathbb{A}(\varepsilon)=\text{sn}(\varepsilon)=\text{sech}(\varepsilon)$ then we obtained

$$\mathtt{U}_{2,1}=\alpha_0+\alpha_2\left(\frac{3\big(\alpha_0^2(-\delta)\kappa+4\kappa\big(c^2+\gamma^2+\kappa^2\big)+\sigma\big)}{2\alpha_0\alpha_2\delta\kappa}\right), \tag{27}$$

subject to condition which computes as:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

Now when $\mu\to 0$
As $\mathbb{A}(\varepsilon)=\text{sn}(\varepsilon)=\cos(\varepsilon)$ then we obtained:

$$\mathtt{U}_{2,2}=\alpha_0+\alpha_2\left(\frac{3\big(\alpha_0^2(-\delta)\kappa+4\kappa\big(c^2+\gamma^2+\kappa^2\big)+\sigma\big)}{2\alpha_0\alpha_2\delta\kappa}\right), \tag{28}$$

subject to condition which is interpreted as:

$$\varkappa_4^2(-\varkappa_2-1)[-(-\varkappa_2-1)(\varkappa_2-2)]+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

**Case 3:** If $\mathtt{t}_0=\mu^2-1$, $\mathtt{t}_2=2-\mu^2$, $\mathtt{t}_4=-1$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)=\text{dn}(\varepsilon)$ we obtain the JEF solution as:

$$\mathtt{U}_3=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2+\mathsf{n}}, \tag{29}$$

where $\mathsf{m}$ and $\mathsf{n}$ are as:

$$\mathsf{m}=\frac{\varkappa_4\big(-\mu^2-\varkappa_2+2\big)}{(-\mu^2-\varkappa_2+2)^2-2(2-\mu^2)(-\mu^2-\varkappa_2+2)-3(\mu^2-1)},$$
$$\mathsf{n}=\frac{3\big(\mu^2-1\big)\varkappa_4}{(-\mu^2-\varkappa_2+2)^2-2(2-\mu^2)(-\mu^2-\varkappa_2+2)-3(\mu^2-1)}.$$

Under condition:

$$\varkappa_4^2\Big(-\mu^2-\varkappa_2+2\Big)\Big(-\Big(-\mu^2-\varkappa_2+2\Big)\Big(2\Big(2-\mu^2\Big)+\varkappa_2\Big)-9\Big(\mu^2-1\Big)\Big)+3\varkappa_6\Big(-\Big(2-\mu^2\Big)^2-3\Big(\mu^2-1\Big)+\varkappa_2^2\Big)^2=0.$$

If $\mu\to 1$
As $\mathbb{A}(\varepsilon)=\text{dn}(\varepsilon)=\text{sech}(\varepsilon)$ then we get:

$$\mathtt{U}_{3,1}=\alpha_0+\alpha_2\left(\frac{3\big(\alpha_0^2(-\delta)\kappa+4\kappa\big(c^2+\gamma^2+\kappa^2\big)+\sigma\big)}{2\alpha_0\alpha_2\delta\kappa}\right), \tag{30}$$

subject to interpreted constraint:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

Now when $\mu\to 0$

As $\mathbb{A}(\varepsilon) = \mathrm{sn}(\varepsilon) = 1$ then we obtained:

$$\mathtt{U}_{3,2} = \alpha_0 + \alpha_2\left(\frac{3\left(\alpha_0^2(-\delta)\kappa + 4\kappa\left(c^2+\gamma^2+\kappa^2\right)+\sigma\right)}{2\alpha_0\alpha_2\delta\kappa}\right), \tag{31}$$

subject to a condition which is interpreted as:

$$\varkappa_4^2(2-\varkappa_2)(9-(2-\varkappa_2)(\varkappa_2+4)) + 3\varkappa_6\left(\varkappa_2^2-1\right)^2 = 0.$$

**Case 4:** If $\mathrm{t}_0 = \mu^2$, $\mathrm{t}_2 = -\left(\mu^2+1\right)$, $\mathrm{t}_4 = 1$, $0 < \mu < 1$ then $\mathbb{A}(\varepsilon)$ = $\mathrm{ns}(\varepsilon)$ or $\mathrm{dc}(\varepsilon)$ the JEF solution can given as:

$$\mathtt{U}_4 = \alpha_0 + \frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2 + \mathsf{n}}, \tag{32}$$

here $\mathsf{m}$ and $\mathsf{n}$ are as:

$$\mathsf{m} = \frac{\varkappa_4\left(-\mu^2-\varkappa_2-1\right)}{(-\mu^2-\varkappa_2-1)^2 - 2(-\mu^2-1)(-\mu^2-\varkappa_2-1)+3\mu^2},$$
$$\mathsf{n} = \frac{3\mu^2\varkappa_4}{(-\mu^2-\varkappa_2-1)^2 - 2(-\mu^2-1)(-\mu^2-\varkappa_2-1)+3\mu^2}.$$

Subject to interpreted condition:

$$\varkappa_4^2\left(-\mu^2-\varkappa_2-1\right)\left(9\mu^2-\left(-\mu^2-\varkappa_2-1\right)\left(2\left(-\mu^2-1\right)+\varkappa_2\right)\right)+3\varkappa_6\left(3\mu^2-\left(-\mu^2-1\right)^2+\varkappa_2^2\right)^2=0.$$

if $\mu \to 1$

As $\mathbb{A}(\varepsilon) = \mathrm{ns}(\varepsilon) = \coth(\varepsilon)$ then we get:

$$\mathtt{U}_{4,1} = \alpha_0 + \alpha_2\left(\frac{3\coth^2(\varepsilon)\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(12\kappa(c^2+\gamma^2+\kappa^2)-\coth^2(\varepsilon)\left(\alpha_0^2(-\delta)\kappa+8\kappa(c^2+\gamma^2+\kappa^2)+\sigma\right)\right)}\right). \tag{33}$$

Or $\mathbb{A}(\varepsilon) = \mathrm{dc}(\varepsilon) = 1$ so we get:

$$\mathtt{U}_{4,2} = \alpha_0 + \alpha_2\left(\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}\right), \tag{34}$$

subject to a condition which is interpreted as:

$$\varkappa_4^2(-\varkappa_2-2)(9-(-\varkappa_2-2)(\varkappa_2-4)) + 3\varkappa_6\left(\varkappa_2^2-1\right)^2 = 0.$$

Now when $\mu \to 0$

As $\mathbb{A}(\varepsilon) = \mathrm{ns}(\varepsilon) = \csc(\varepsilon)$ so we obtain:

$$\mathtt{U}_{4,3} = \alpha_0 + \alpha_2\left(-\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2(-\delta)\kappa+4\kappa(c^2+\gamma^2+\kappa^2)+\sigma\right)}\right), \tag{35}$$

Or $\mathbb{A}(\varepsilon) = \mathrm{dc}(\varepsilon) = \sec(\varepsilon)$ so

$$\mathtt{U}_{4,4} = \alpha_0 + \alpha_2\left(-\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2(-\delta)\kappa+4\kappa(c^2+\gamma^2+\kappa^2)+\sigma\right)}\right), \tag{36}$$

under constraint condition:

$$\varkappa_4^2(-\varkappa_2-1)(-(-\varkappa_2-1)(\varkappa_2-2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0.$$

**Case 5:** If $t_0=-\mu^2$, $t_2=2\mu^2-1$, $t_4=1-\mu^2$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)$ = nc$(\varepsilon)$ obtained the JEF solutions as:

$$U_5=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{m\mathbb{A}(\varepsilon)^2+n}, \tag{37}$$

here m and n are:

$$m=\frac{\varkappa_4(2\mu^2-\varkappa_2-1)}{(2\mu^2-\varkappa_2-1)^2-2(2\mu^2-1)(2\mu^2-\varkappa_2-1)-3(1-\mu^2)\mu^2},$$
$$n=-\frac{3\mu^2\varkappa_4}{(2\mu^2-\varkappa_2-1)^2-2(2\mu^2-1)(2\mu^2-\varkappa_2-1)-3(1-\mu^2)\mu^2}.$$

Subject to interpreted condition:

$$\varkappa_4^2\left(2\mu^2-\varkappa_2-1\right)\left(-\left(2\mu^2-\varkappa_2-1\right)\left(2\left(2\mu^2-1\right)+\varkappa_2\right),\right.$$
$$\left.-9\left(1-\mu^2\right)\mu^2\right)+3\varkappa_6\left(-3\left(1-\mu^2\right)\mu^2-\left(2\mu^2-1\right)^2+\varkappa_2^2\right)^2=0.$$

if $\mu\to1$

As $\mathbb{A}(\varepsilon)=\mathrm{nc}(\varepsilon)=\cosh(\varepsilon)$ we get:

$$U_{5,1}=\alpha_0+\alpha_2\left(\frac{3\cosh^2(\varepsilon)\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\cosh^2(\varepsilon)\left(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)-12\kappa(c^2+\gamma^2+\kappa^2)\right)}\right). \tag{38}$$

With the following constraint:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0.$$

When $\mu\to0$

As $\mathbb{A}(\varepsilon)=\mathrm{nc}(\varepsilon)=\sec(\varepsilon)$ then we get:

$$U_{5,2}=\alpha_0+\alpha_2\left(\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa-4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}\right), \tag{39}$$

under computed constraint condition:

$$\varkappa_4^2(-\varkappa_2-1)(-(-\varkappa_2-1)(\varkappa_2-2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0.$$

**Case 6:** If $t_0=-1$, $t_2=2-\mu^2$, $t_4=-(1-\mu^2)$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)$ = nc$(\varepsilon)$ the JEF solution is as:

$$U_6=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{m\mathbb{A}(\varepsilon)^2+n}, \tag{40}$$

where m and n are:

$$\mathsf{m} = \frac{\varkappa_4\left(-\mu^2 - \varkappa_2 + 2\right)}{(-\mu^2 - \varkappa_2 + 2)^2 - 2(2-\mu^2)(-\mu^2 - \varkappa_2 + 2) - 3(\mu^2 - 1)},$$
$$\mathsf{n} = -\frac{3\varkappa_4}{(-\mu^2 - \varkappa_2 + 2)^2 - 2(2-\mu^2)(-\mu^2 - \varkappa_2 + 2) - 3(\mu^2 - 1)}.$$

Under condition:

$$\varkappa_4^2\left(-\mu^2 - \varkappa_2 + 2\right)\left(-\left(-\mu^2 - \varkappa_2 + 2\right)\left(2\left(2 - \mu^2\right) + \varkappa_2\right) - 9\left(\mu^2 - 1\right)\right) + 3\varkappa_6\left(-\left(2 - \mu^2\right)^2 - 3\left(\mu^2 - 1\right) + \varkappa_2^2\right)^2 = 0,$$

when $\mu \to 1$

As $\mathbb{A}(\varepsilon) = \mathrm{nc}(\varepsilon) = \cosh(\varepsilon)$ then we obtained:

$$\mathtt{U}_{6,1} = \alpha_0 + \alpha_2\left(-\frac{3\cosh^2(\varepsilon)\left(-16\kappa^2\left(c^2 + \gamma^2 + \kappa^2\right)^2 + \sigma^2 + \alpha_0^2\delta\kappa\left(\alpha_0^2\delta\kappa - 2\sigma\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\cosh^2(\varepsilon)\left(\alpha_0^2\delta\kappa + 4\kappa(c^2 + \gamma^2 + \kappa^2) - \sigma\right) - 12\kappa(c^2 + \gamma^2 + \kappa^2)\right)}\right). \tag{41}$$

Under condition interpreted as:

$$\varkappa_4^2(1 - \varkappa_2)(-(1 - \varkappa_2)(\varkappa_2 + 2)) + 3\varkappa_6\left(\varkappa_2^2 - 1\right)^2 = 0.$$

If $\mu \to 0$

As $\mathbb{A}(\varepsilon) = \mathrm{nc}(\varepsilon) = 1$ then we get:

$$\mathtt{U}_{6,2} = \alpha_0 + \alpha_2\left(-\frac{3\left(-16\kappa^2\left(c^2 + \gamma^2 + \kappa^2\right)^2 + \sigma^2 + \alpha_0^2\delta\kappa\left(\alpha_0^2\delta\kappa - 2\sigma\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa - 4\kappa(c^2 + \gamma^2 + \kappa^2) - \sigma\right)}\right), \tag{42}$$

under interpret constraint:

$$\varkappa_4^2(2 - \varkappa_2)(9 - (2 - \varkappa_2)(\varkappa_2 + 4)) + 3\varkappa_6\left(\varkappa_2^2 - 1\right)^2 = 0.$$

**Case 7:** If $\mathrm{t}_0 = 1$, $\mathrm{t}_2 = 2 - \mu^2$, $\mathrm{t}_4 = 1 - \mu^2$, $0 < \mu < 1$ then $\mathbb{A}(\varepsilon)$ = sc($\varepsilon$) JEF solution as:

$$\mathtt{U}_7 = \alpha_0 + \frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2 + \mathsf{n}}, \tag{43}$$

here $\mathsf{m}$ and $\mathsf{n}$ are:

$$\mathsf{m} = \frac{\varkappa_4\left(-\mu^2 - \varkappa_2 + 2\right)}{(-\mu^2 - p_2 + 2)^2 - 2(2-\mu^2)(-\mu^2 - \varkappa_2 + 2) + 3(1 - \mu^2)},$$
$$\mathsf{n} = \frac{3\varkappa_4}{(-\mu^2 - \varkappa_2 + 2)^2 - 2(2-\mu^2)(-\mu^2 - \varkappa_2 + 2) + 3(1 - \mu^2)}.$$

Under the interpreted condition:

$$\varkappa_4^2\left(-\mu^2 - \varkappa_2 + 2\right)\left(9\left(1 - \mu^2\right) - \left(-\mu^2 - \varkappa_2 + 2\right)\left(2\left(2 - \mu^2\right) + \varkappa_2\right)\right) + 3\varkappa_6\left(-\left(2 - \mu^2\right)^2 + 3\left(1 - \mu^2\right) + \varkappa_2^2\right)^2 = 0.$$

when $\mu \to 1$

As $\mathbb{A}(\varepsilon) = \mathrm{sc}(\varepsilon) = \sinh(\varepsilon)$ then we get:

$$\mathtt{U}_{7,1} = \alpha_0 + \alpha_2\left(-\frac{3\sinh^2(\varepsilon)\left(-16\kappa^2\left(c^2 + \gamma^2 + \kappa^2\right)^2 + \sigma^2 + \alpha_0^2\delta\kappa\left(\alpha_0^2\delta\kappa - 2\sigma\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(12\kappa(c^2 + \gamma^2 + \kappa^2) + \sinh^2(\varepsilon)\left(\alpha_0^2\delta\kappa + 4\kappa(c^2 + \gamma^2 + \kappa^2) - \sigma\right)\right)}\right), \tag{44}$$

subject to computed constraint:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

Now if $\mu \to 0$
As $\mathbb{A}(\varepsilon) = \mathrm{sc}(\varepsilon) = \tan(\varepsilon)$ then,

$$\mathtt{U}_{7,2}=\alpha_0+\alpha_2\left(-\frac{3\tan^2(\varepsilon)\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\big(12\kappa(c^2+\gamma^2+\kappa^2)+\tan^2(\varepsilon)\big(\alpha_0^2\delta\kappa+8\kappa(c^2+\gamma^2+\kappa^2)-\sigma\big)\big)}\right), \tag{45}$$

under condition:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

**Case 8:** If $\mathtt{t}_0 = 1$, $\mathtt{t}_2 = 2\mu^2-1$, $\mathtt{t}_4 = -\mu^2(1-\mu^2)$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)$ = sd($\varepsilon$) obtained JEF solution is as:

$$\mathtt{U}_8=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathtt{m}\mathbb{A}(\varepsilon)^2+\mathtt{n}}, \tag{46}$$

where $\mathtt{m}$ and $\mathtt{n}$ are:

$$\mathtt{m}=\frac{\varkappa_4(2\mu^2-\varkappa_2-1)}{(2\mu^2-\varkappa_2-1)^2-2(2\mu^2-1)(2\mu^2-\varkappa_2-1)-3(1-\mu^2)\mu^2}$$
$$\mathtt{n}=\frac{3\varkappa_4}{(2\mu^2-\varkappa_2-1)^2-2(2\mu^2-1)(2\mu^2-\varkappa_2-1)-3(1-\mu^2)\mu^2},$$

under condition interpreted as:

$$\varkappa_4^2\Big(2\mu^2-\varkappa_2-1\Big)\Big(-\Big(2\mu^2-\varkappa_2-1\Big)\Big(2\Big(2\mu^2-1\Big)+\varkappa_2\Big)-9\Big(1-\mu^2\Big)\mu^2\Big)+3\varkappa_6$$
$$\Big(-3\Big(1-\mu^2\Big)\mu^2-\Big(2\mu^2-1\Big)^2+\varkappa_2^2\Big)^2=0.$$

now if $\mu \to 1$
As $\mathbb{A}(\varepsilon) = \mathrm{sd}(\varepsilon) = \sinh(\varepsilon)$ then:

$$\mathtt{U}_{8,1}=\alpha_0+\alpha_2\left(-\frac{3\sinh^2(\varepsilon)\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\Big(12\kappa(c^2+\gamma^2+\kappa^2)+\sinh^2(\varepsilon)\big(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\big)\Big)}\right), \tag{47}$$

subject to the computed condition:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

If $\mu \to 0$
As $\mathbb{A}(\varepsilon) = \mathrm{sd}(\varepsilon) = \sin(\varepsilon)$ then

$$\mathtt{U}_{8,2}=\alpha_0+\alpha_2\left(-\frac{3\sin^2(\varepsilon)\Big(-16\kappa^2\big(c^2+\gamma^2+\kappa^2\big)^2+\sigma^2+\alpha_0^2\delta\kappa\big(\alpha_0^2\delta\kappa-2\sigma\big)\Big)}{2\alpha_0\alpha_2\delta\kappa\big(12\kappa(c^2+\gamma^2+\kappa^2)+\sin^2(\varepsilon)\big(\alpha_0^2\delta\kappa-4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\big)\big)}\right), \tag{48}$$

under interpreted constraint condition:

$$\varkappa_4^2(-\varkappa_2-1)(-(-\varkappa_2-1)(\varkappa_2-2))+3\varkappa_6\Big(\varkappa_2^2-1\Big)^2=0.$$

**Case 9:** If $t = 1 - \mu^2$, $t_2 = 2 - \mu^2$, $t_4 = 1$, $0 < \mu < 1$ then $\mathbb{A}(\varepsilon)$ = cs($\varepsilon$) the JEF solution is as:

$$U_9 = \alpha_0 + \frac{\alpha_2 \mathbb{A}(\varepsilon)^2}{m\mathbb{A}(\varepsilon)^2 + n}, \tag{49}$$

here m and n are as:

$$m = \frac{\varkappa_4(-\mu^2 - \varkappa_2 + 2)}{(-\mu^2 - \varkappa_2 + 2)^2 - 2(2-\mu^2)(-\mu^2 - \varkappa_2 + 2) + 3(1-\mu^2)},$$
$$n = \frac{3(1-\mu^2)\varkappa_4}{(-\mu^2 - \varkappa_2 + 2)^2 - 2(2-\mu^2)(-\mu^2 - \varkappa_2 + 2) + 3(1-\mu^2)}.$$

Subject to the condition:

$$\varkappa_4^2\left(-\mu^2 - \varkappa_2 + 2\right)\left(9\left(1-\mu^2\right) - \left(-\mu^2 - \varkappa_2 + 2\right)\left(2\left(2-\mu^2\right) + \varkappa_2\right)\right) + 3\varkappa_6\left(-\left(2-\mu^2\right)^2 + 3\left(1-\mu^2\right) + \varkappa_2^2\right)^2 = 0.$$

Now If $\mu \to 1$
As $\mathbb{A}(\varepsilon) = \text{cs}(\varepsilon) = \text{csch}(\varepsilon)$ then we get:

$$U_{9,1} = \alpha_0 + \alpha_2\left(\frac{3\left(16\kappa^2(c^2+\gamma^2+\kappa^2)^2 - \sigma^2 + \alpha_0^2\delta\kappa(2\sigma - \alpha_0^2\delta\kappa)\right)}{2\alpha_0\alpha_2\delta\kappa(\alpha_0^2\delta\kappa + 4\kappa(c^2+\gamma^2+\kappa^2) - \sigma)}\right), \tag{50}$$

under the constraint condition:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2)) + 3\varkappa_6\left(\varkappa_2^2 - 1\right)^2 = 0.$$

If $\mu \to 0$
As $\mathbb{A}(\varepsilon) = \text{cs}(\varepsilon) = \cot(\varepsilon)$ then we got:

$$U_{9,2} = \alpha_0 + \alpha_2\left(\frac{3\cot^2(\varepsilon)\left(16\kappa^2(c^2+\gamma^2+\kappa^2)^2 - \sigma^2 + \alpha_0^2\delta\kappa(2\sigma - \alpha_0^2\delta\kappa)\right)}{2\alpha_0\alpha_2\delta\kappa(12\kappa(c^2+\gamma^2+\kappa^2) + \cot^2(\varepsilon)(\alpha_0^2\delta\kappa + 8\kappa(c^2+\gamma^2+\kappa^2) - \sigma))}\right), \tag{51}$$

subject to interpreted condition:

$$\varkappa_4^2(2-\varkappa_2)(9 - (2-\varkappa_2)(\varkappa_2+4)) + 3\varkappa_6\left(\varkappa_2^2 - 1\right)^2 = 0.$$

**Case 10:** If $t_0 = -\mu^2(1-\mu^2)$, $t_2 = 2\mu^2 - 1$, $t_4 = 1$, $0 < \mu < 1$ then $\mathbb{A}(\varepsilon)$ = ds($\varepsilon$) we obtained the JEF solution as:

$$U_{10} = \alpha_0 + \frac{\alpha_2 \mathbb{A}(\varepsilon)^2}{m\mathbb{A}(\varepsilon)^2 + n}, \tag{52}$$

where m and n are,

$$m = \frac{\varkappa_4(2\mu^2 - \varkappa_2 - 1)}{(2\mu^2 - \varkappa_2 - 1)^2 - 2(2\mu^2 - 1)(2\mu^2 - \varkappa_2 - 1) - 3(1-\mu^2)\mu^2},$$
$$n = -\frac{3\mu^2(1-\mu^2)\varkappa_4}{(2\mu^2 - \varkappa_2 - 1)^2 - 2(2\mu^2 - 1)(2\mu^2 - \varkappa_2 - 1) - 3(1-\mu^2)\mu^2}.$$

Under condition:

$$\varkappa_4^2\left(2\mu^2-\varkappa_2-1\right)\left(-\left(2\mu^2-\varkappa_2-1\right)\left(2\left(2\mu^2-1\right)+\varkappa_2\right)-9\left(1-\mu^2\right)\mu^2\right)+3\varkappa_6\left(-3\left(1-\mu^2\right)\mu^2-\left(2\mu^2-1\right)^2+\varkappa_2^2\right)^2=0.$$

If $\mu\to1$
As $\Omega(\varepsilon)=\mathrm{ds}(\varepsilon)=\mathrm{csch}(\varepsilon)$ then we get:

$$\mathtt{U}_{10,1}=\alpha_0+\alpha_2\left(\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}\right), \tag{53}$$

under the interpreted constraint condition:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0.$$

Now if $\mu\to0$
As $\mathbb{A}(\varepsilon)=\mathrm{ds}(\varepsilon)=\csc(\varepsilon)$ we get:

$$\mathtt{U}_{10,2}=\alpha_0+\alpha_2\left(\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa-4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}\right), \tag{54}$$

under computed constraint condition:

$$\varkappa_4^2(-\varkappa_2-1)(-(-\varkappa_2-1)(\varkappa_2-2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0.$$

**Case 11:** If $\mathtt{t}_0=\frac{1}{4}(1-\mu^2)$, $\mathtt{t}_2=\frac{1}{2}(\mu^2+1)$, $\mathtt{t}_4=\frac{1}{4}(1-\mu^2)$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)=\mathrm{nc}(\varepsilon)\pm\mathrm{sc}(\varepsilon)$or$\frac{\mathrm{cn}(\varepsilon)}{1\pm\mathrm{sn}(\varepsilon)}$ then JEF solution obtained as:

$$\mathtt{U}_{11}=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathtt{m}\mathbb{A}(\varepsilon)^2+\mathtt{n}}, \tag{55}$$

here $\mathtt{m}$ and $\mathtt{n}$ are as:

$$\mathtt{m}=\frac{\varkappa_4\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)}{\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)^2-(\mu^2+1)\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)+\frac{3}{16}(1-\mu^2)^2},$$
$$\mathtt{n}=\frac{3(1-\mu^2)\varkappa_4}{4\left(\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)^2-(\mu^2+1)\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)+\frac{3}{16}(1-\mu^2)^2\right)}.$$

Subject to interpreted condition:

$$\varkappa_4^2\left(\frac{1}{2}\left(\mu^2+1\right)-\varkappa_2\right)\left(\frac{9}{16}\left(1-\mu^2\right)^2-\left(\frac{1}{2}\left(\mu^2+1\right)-\varkappa_2\right)\left(\mu^2+\varkappa_2+1\right)\right)+3\varkappa_6\left(\frac{3}{16}\left(1-\mu^2\right)^2-\frac{1}{4}\left(\mu^2+1\right)^2+\varkappa_2^2\right)^2=0.$$

Now if $\mu\to1$
As $\mathbb{A}(\varepsilon)=\mathrm{nc}(\varepsilon)\pm\mathrm{sc}(\varepsilon)=\cosh(\varepsilon)\pm\sinh(\varepsilon)$ then we get:

$$\mathtt{U}_{11,1}=\alpha_0+\alpha_2\left(\frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}\right), \tag{56}$$

also,

$$\mathtt{U}_{11,2} = \alpha_0 + \alpha_2 \left( \frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2 - \sigma^2 + \alpha_0^2\delta\kappa\left(2\sigma - \alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa + 4\kappa(c^2+\gamma^2+\kappa^2) - \sigma\right)} \right), \tag{57}$$

condition which computes as:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2)) + 3\varkappa_6\left(\varkappa_2^2 - 1\right)^2 = 0.$$

Now if $\mu \to 0$

As $\mathbb{A}(\varepsilon) = \mathrm{nc}(\varepsilon) \pm \mathrm{sc}(\varepsilon) = \sec(\varepsilon) \pm \tan(\varepsilon)$ then we get:

$$\mathtt{U}_{11,3} = \alpha_0 + \alpha_2 \left( \frac{3(\sec(\varepsilon) \pm \tan(\varepsilon))^2\left(\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2 - \sigma^2 + \alpha_0^2\delta\kappa\left(2\sigma - \alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(3\kappa(c^2+\gamma^2+\kappa^2) + (\sec(\varepsilon) \pm \tan(\varepsilon))^2\left(\alpha_0^2\delta\kappa + 2\kappa(c^2+\gamma^2+\kappa^2) - \sigma\right)\right)} \right). \tag{58}$$

or $\mathbb{A}(\varepsilon) = \frac{\mathrm{cn}(\varepsilon)}{1 \pm \mathrm{sn}(\varepsilon)} = \frac{\cos(\varepsilon)}{1 \pm \sin(\varepsilon)}$ then we get:

$$\mathtt{U}_{11,4} = \alpha_0 + \alpha_2 \left( \frac{3\cos^2(\varepsilon)\left(\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2 - \sigma^2 + \alpha_0^2\delta\kappa\left(2\sigma - \alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa(1 \pm \sin(\varepsilon))^2\left(3\kappa(c^2+\gamma^2+\kappa^2) + \frac{\cos^2(\varepsilon)\left(\alpha_0^2\delta\kappa + 2\kappa(c^2+\gamma^2+\kappa^2) - \sigma\right)}{(1 \pm \sin(\varepsilon))^2}\right)} \right), \tag{59}$$

under condition which interpreted as:

$$\varkappa_4^2\left(\frac{1}{2} - \varkappa_2\right)\left(\frac{9}{16} - \left(\frac{1}{2} - \varkappa_2\right)(\varkappa_2 + 1)\right) + 3\varkappa_6\left(\varkappa_2^2 - \frac{1}{16}\right)^2 = 0.$$

**Case 12:** if $\mathrm{t}_0 = -\frac{1}{4}\left(1-\mu^2\right)^2$, $\mathrm{t}_2 = \frac{1}{2}\left(\mu^2+1\right)$, $\mathrm{t}_4 = -\frac{1}{4}$, $0 < \mu < 1$ then $\Omega(\varepsilon) = \mu\mathrm{cn}(\varepsilon) \pm \mathrm{dn}(\varepsilon)$ then JEF solution is as:

$$\mathtt{U}_{12} = \alpha_0 + \frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2 + \mathsf{n}}, \tag{60}$$

here $\mathsf{m}$ and $\mathsf{n}$ are,

$$\mathsf{m} = \frac{\varkappa_4\left(\frac{1}{2}(\mu^2+1) - \varkappa_2\right)}{\left(\frac{1}{2}(\mu^2+1) - \varkappa_2\right)^2 - (\mu^2+1)\left(\frac{1}{2}(\mu^2+1) - \varkappa_2\right) + \frac{3}{16}(1-\mu^2)^2},$$

$$\mathsf{n} = -\frac{3\left(1-\mu^2\right)^2\varkappa_4}{4\left(\left(\frac{1}{2}(\mu^2+1) - \varkappa_2\right)^2 - (\mu^2+1)\left(\frac{1}{2}(\mu^2+1) - \varkappa_2\right) + \frac{3}{16}(1-\mu^2)^2\right)}.$$

Under condition:

$$\varkappa_4^2\left(\frac{1}{2}\left(\mu^2+1\right) - \varkappa_2\right)\left(\frac{9}{16}\left(1-\mu^2\right)^2 - \left(\frac{1}{2}\left(\mu^2+1\right) - \varkappa_2\right)\left(\mu^2 + p_2 + 1\right)\right) + 3\varkappa_6$$
$$\left(\frac{3}{16}\left(1-\mu^2\right)^2 - \frac{1}{4}\left(\mu^2+1\right)^2 + \varkappa_2^2\right)^2 = 0.$$

now if $\mu \to 1$

As $\mathbb{A}(\varepsilon) = \mu\mathrm{cn}(\varepsilon) \pm \mathrm{dn}(\varepsilon) = \mu\mathrm{sech}(\varepsilon) \pm \mathrm{sech}(\varepsilon)$ then we get:

$$\mathtt{U}_{12,1} = \alpha_0 + \alpha_2 \left( \frac{3\left(16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2 - \sigma^2 + \alpha_0^2\delta\kappa\left(2\sigma - \alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left(\alpha_0^2\delta\kappa + 4\kappa(c^2+\gamma^2+\kappa^2) - \sigma\right)} \right), \tag{61}$$

subject to interpreted condition:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0,$$

now if $\mu \to 0$

As $\mathbb{A}(\varepsilon)=\mu \mathrm{cn}(\varepsilon)\pm \mathrm{dn}(\varepsilon)=\mu\cos(\varepsilon)\pm 1$ then:

$$\mathtt{U}_{12,2}=\alpha_0+\alpha_2\left(\frac{3(0\pm 1)^2\left(\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2-\sigma^2+\alpha_0^2\delta\kappa\left(2\sigma-\alpha_0^2\delta\kappa\right)\right)}{2\alpha_0\alpha_2\delta\kappa\left((0\pm 1)^2\left(\alpha_0^2\delta\kappa+2\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)-3\kappa(c^2+\gamma^2+\kappa^2)\right)}\right), \tag{62}$$

under computed condition:

$$\varkappa_4^2\left(\frac{1}{2}-\varkappa_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-\varkappa_2\right)(\varkappa_2+1)\right)+3\varkappa_6\left(\varkappa_2^2-\frac{1}{16}\right)^2=0.$$

**Case 13:** If $\mathsf{t}_0=\frac{1}{4}$, $\mathsf{t}_2=\frac{1}{2}\left(1-2\mu^2\right)$, $\mathsf{t}_4=\frac{1}{4}$, $0<\mu<1$ then $\mathbb{A}(\varepsilon)=\frac{\mathrm{sn}(\varepsilon)}{1\pm\mathrm{cn}(\varepsilon)}$ then JEF solution is as:

$$\mathtt{U}_{13}=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2+\mathsf{n}}, \tag{63}$$

here $\mathsf{m}$ and $\mathsf{n}$ are,

$$\mathsf{m}=\frac{\varkappa_4\left(\frac{1}{2}(1-2\mu^2)-\varkappa_2\right)}{\left(\frac{1}{2}(1-2\mu^2)-\varkappa_2\right)^2-(1-2\mu^2)\left(\frac{1}{2}(1-2\mu^2)-\varkappa_2\right)+\frac{3}{16}},$$
$$\mathsf{n}=\frac{3\varkappa_4}{4\left(\left(\frac{1}{2}(1-2\mu^2)-\varkappa_2\right)^2-(1-2\mu^2)\left(\frac{1}{2}(1-2\mu^2)-\varkappa_2\right)+\frac{3}{16}\right)}.$$

Under interpreted condition:

$$\varkappa_4^2\left(\frac{1}{2}\left(1-2\mu^2\right)-\varkappa_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}\left(1-2\mu^2\right)-\varkappa_2\right)\left(-2\mu^2+\varkappa_2+1\right)\right)+3\varkappa_6\left(-\frac{1}{4}\left(1-2\mu^2\right)^2+\varkappa_2^2+\frac{3}{16}\right)^2=0.$$

now if $\mu \to 1$

As $\mathbb{A}(\varepsilon)=\frac{\mathrm{sn}(\varepsilon)}{1\pm\mathrm{cn}(\varepsilon)}=\frac{\tanh(\varepsilon)}{1\pm\mathrm{sech}(\varepsilon)}$ then

$$\mathtt{U}_{13,1}=\alpha_0+\alpha_2\left(-\frac{3\tanh^2(\varepsilon)\left(\alpha_0^4\delta^2\kappa^2-\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2+v^2-2\alpha_0^2\delta\kappa v\right)}{2\alpha_0\alpha_2\delta\kappa\left(3\kappa(c^2+\gamma^2+\kappa^2)(1\pm\mathrm{sech}(\varepsilon))^2-\tanh^2(\varepsilon)\left(\alpha_0^2(-\delta)\kappa+2\kappa(c^2+\gamma^2+\kappa^2)+v\right)\right)}\right), \tag{64}$$

under interpreted constraint condition:

$$\varkappa_4^2\left(-\varkappa_2-\frac{1}{2}\right)\left(\frac{9}{16}-\left(-\varkappa_2-\frac{1}{2}\right)(\varkappa_2-1)\right)+3\varkappa_6\left(\varkappa_2^2-\frac{1}{16}\right)^2=0.$$

Now if $\mu \to 0$

As $\mathbb{A}(\varepsilon)=\frac{\mathrm{sn}(\varepsilon)}{1\pm\mathrm{cn}(\varepsilon)}=\frac{\sin(\varepsilon)}{1\pm\cos(\varepsilon)}$ then

$$\mathtt{U}_{13,2}=\alpha_0+\alpha_2\left(-\frac{3\sin^2(\varepsilon)\left(-\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2+\sigma^2+\alpha_0^2\delta\kappa\left(\alpha_0^2\delta\kappa-2\sigma\right)\right)}{2\alpha_0\alpha_2\delta\kappa(1\pm\cos(\varepsilon))^2\left(3\kappa(c^2+\gamma^2+\kappa^2)+\frac{\sin^2(\varepsilon)\left(\alpha_0^2\delta\kappa+2\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}{(1\pm\cos(\varepsilon))^2}\right)}\right), \tag{65}$$

subject to interpreted condition:

$$\varkappa_4^2\left(\frac{1}{2}-\varkappa_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-\varkappa_2\right)(\varkappa_2+1)\right)+3\varkappa_6\left(\varkappa_2^2-\frac{1}{16}\right)^2=0.$$

**Case 14:** If $\mathsf{t}_0=\frac{1}{4}, \mathsf{t}_2=\frac{1}{2}(\mu^2+1), \mathsf{t}_4=\frac{1}{4}(1-\mu^2)^2, 0<\mu<1$ then $mathbbA(\varepsilon)=\frac{\mathrm{sn}(\varepsilon)}{\mathrm{cn}(\varepsilon)\pm\mathrm{dn}(\varepsilon)}$ then JEF solution is as:

$$\mathtt{U}_{14}=\alpha_0+\frac{\alpha_2\mathbb{A}(\varepsilon)^2}{\mathsf{m}\mathbb{A}(\varepsilon)^2+\mathsf{n}}, \tag{66}$$

here $\mathsf{m}$ and $\mathsf{n}$ are as:

$$\mathsf{m}=\frac{\varkappa_4\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)}{\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)^2-(\mu^2+1)\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)+\frac{3}{16}(1-\mu^2)^2},$$
$$\mathsf{n}=\frac{3\varkappa_4}{4\left(\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)^2-(\mu^2+1)\left(\frac{1}{2}(\mu^2+1)-\varkappa_2\right)+\frac{3}{16}(1-\mu^2)^2\right)}.$$

Subject to an interpreted condition:

$$\varkappa_4^2\left(\frac{1}{2}\left(\mu^2+1\right)-\varkappa_2\right)\left(\frac{9}{16}\left(1-\mu^2\right)^2-\left(\frac{1}{2}\left(\mu^2+1\right)-\varkappa_2\right)\left(\mu^2+\varkappa_2+1\right)\right)+3\varkappa_6$$
$$\left(\frac{3}{16}\left(1-\mu^2\right)^2-\frac{1}{4}\left(\mu^2+1\right)^2+\varkappa_2^2\right)^2=0.$$

Now if $\mu\to 1$
As $\mathbb{A}(\varepsilon)=\frac{\mathrm{sn}(\varepsilon)}{\mathrm{cn}(\varepsilon)\pm\mathrm{dn}(\varepsilon)}=\frac{\tanh(\varepsilon)}{\mathrm{sech}(\varepsilon)\pm\mathrm{sech}(\varepsilon)}$ then:

$$\mathtt{U}_{14,1}=\alpha_0+\alpha_2\left(-\frac{3\tanh^2(\varepsilon)\left(-16\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2+\sigma^2+\alpha_0^2\delta\kappa\left(\alpha_0^2\delta\kappa-2\sigma\right)\right)}{2\alpha_0\alpha_2\delta\kappa(\mathrm{sech}(\varepsilon)\pm\mathrm{sech}(\varepsilon))^2\left(3\kappa(c^2+\gamma^2+\kappa^2)+\frac{\tanh^2(\varepsilon)\left(\alpha_0^2\delta\kappa+4\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}{(\mathrm{sech}(\varepsilon)\pm\mathrm{sech}(\varepsilon))^2}\right)}\right), \tag{67}$$

under condition:

$$\varkappa_4^2(1-\varkappa_2)(-(1-\varkappa_2)(\varkappa_2+2))+3\varkappa_6\left(\varkappa_2^2-1\right)^2=0.$$

If $\mu\to 0$
As $\mathbb{A}(\varepsilon)=\frac{\mathrm{sn}(\varepsilon)}{\mathrm{cn}(\varepsilon)\pm\mathrm{dn}(\varepsilon)}=\frac{\sin(\varepsilon)}{\cos(\varepsilon)\pm 1}$ then

$$\mathtt{U}_{14,2}=\alpha_0+\alpha_2\left(-\frac{3\sin^2(\varepsilon)\left(-\kappa^2\left(c^2+\gamma^2+\kappa^2\right)^2+\sigma^2+\alpha_0^2\delta\kappa\left(\alpha_0^2\delta\kappa-2\sigma\right)\right)}{2\alpha_0\alpha_2\delta\kappa(\cos(\varepsilon)\pm 1)^2\left(3\kappa(c^2+\gamma^2+\kappa^2)+\frac{\sin^2(\varepsilon)\left(\alpha_0^2\delta\kappa+2\kappa(c^2+\gamma^2+\kappa^2)-\sigma\right)}{(\cos(\varepsilon)\pm 1)^2}\right)}\right), \tag{68}$$

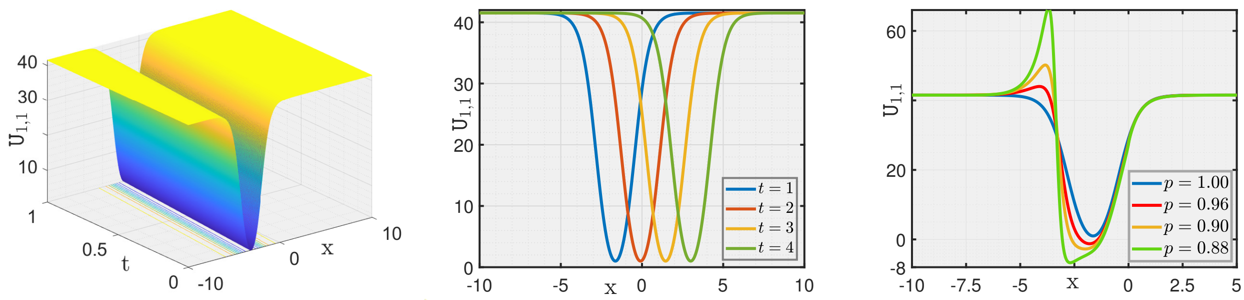
under defined constraint condition:

$$\varkappa_4^2\left(\frac{1}{2}-\varkappa_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-\varkappa_2\right)(\varkappa_2+1)\right)+3\varkappa_6\left(\varkappa_2^2-\frac{1}{16}\right)^2=0.$$

## 4. Graphical Illustration and Discussion

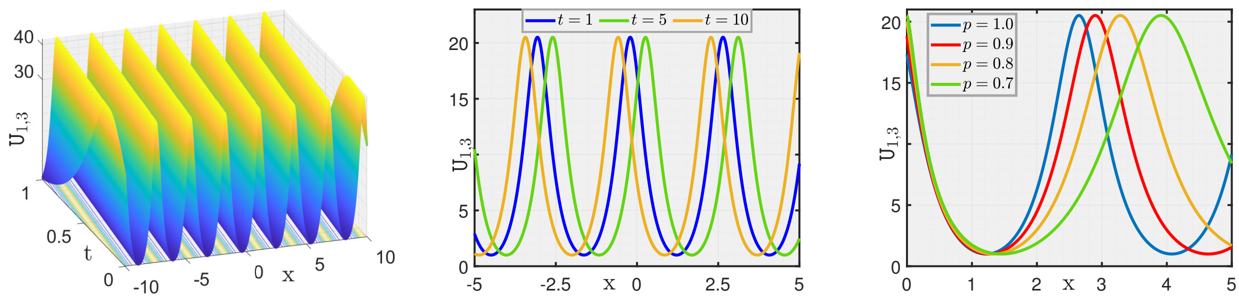
Here, we show a graphical illustration of some of the obtained new analytical soliton solutions for the (3+1)-dimensional fractional (mKDV-ZK) equation, including periodic soliton, bright soliton dark soliton, and singular periodic soliton. They are visualized in

Figures 1–8 via 3D and 2D for different values of $t$. We also show the effect of fractional parameters on the obtained solutions through 2D for different fractional order shown in each legend of the graph. In Figure 1, we plot the solution represented by $\mathtt{U}_{1,1}$, which shows the dark soliton behaviour. Moreover, we show periodic solitons through Figures 2–7 representing by $\mathtt{U}_{1,3}$, $\mathtt{U}_{1,4}$, $\mathtt{U}_{11,3}$, and $\mathtt{U}_{13,2}$ respectively. Additionally, we also plot singular solitons through Figures 4 and 6 representing by $\mathtt{U}_{7,2}$, $\mathtt{U}_{13,1}$. Lastly, we show Bright Soliton solution represented by $\mathtt{U}_{14,1}$ through Figure 8. The innovation of these solutions can be observed as that they provide new analytical expressions for soliton solutions of the fractional mKdV-ZK equation, which has not been widely studied in the literature. These solutions will be more helpful in the understanding of nonlinear wave dynamics narrated by the considered equation, which is applicable in many physical systems such as fluid dynamics, plasma physics, and nonlinear optics. The gained soliton solutions can be applied to model and examine the propagation of periodic wave patterns, localized wave packets, and singular wave profiles in systems reign over by the fractional mKdV-ZK equation, together with applications in plasma turbulence, shallow water waves, and optical communications.



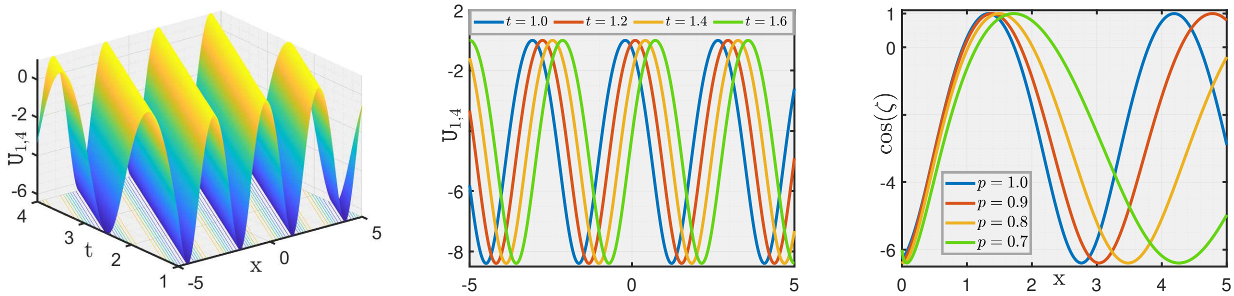

(**a**) (**b**) (**c**)

**Figure 1.** (**a**–**c**) Visualization of $\mathtt{U}_{1,1}$ for $\sigma = 15,\ \delta = 2.1\ , \kappa = 1.7, a_0 = 1,\ a_2 = 3,\ \gamma = 2.1,\ c = 1,\ p = 1,$ $a = 1.1,\ b = 1.5,\ s = 1.7,\ y = 1,\ z = 2.$



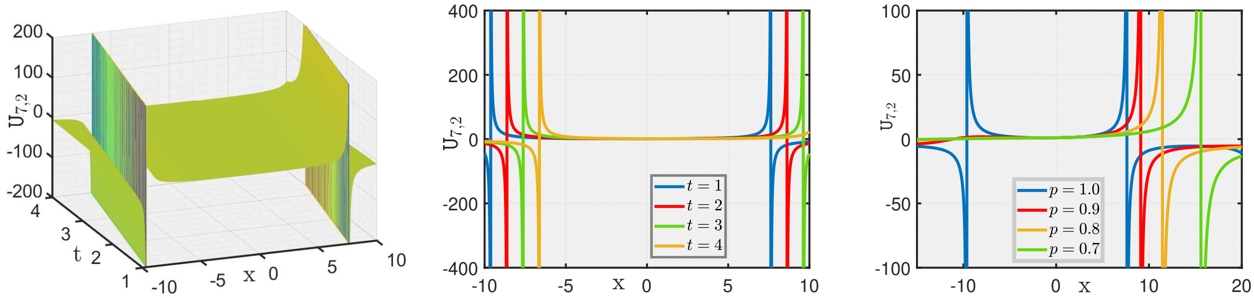

(**a**) (**b**) (**c**)

**Figure 2.** (**a**–**c**) Visualization of $\mathtt{U}_{1,3}$ for $\sigma = 15,\ \delta = 2.1\ , \kappa = 1.7, a_0 = 1,\ a_2 = 3,\ \gamma = 2.1,\ c = 1,\ p = 1,$ $a = 1.1,\ b = 1.5,\ s = 1.7,\ y = 1,\ z = 2.1.$



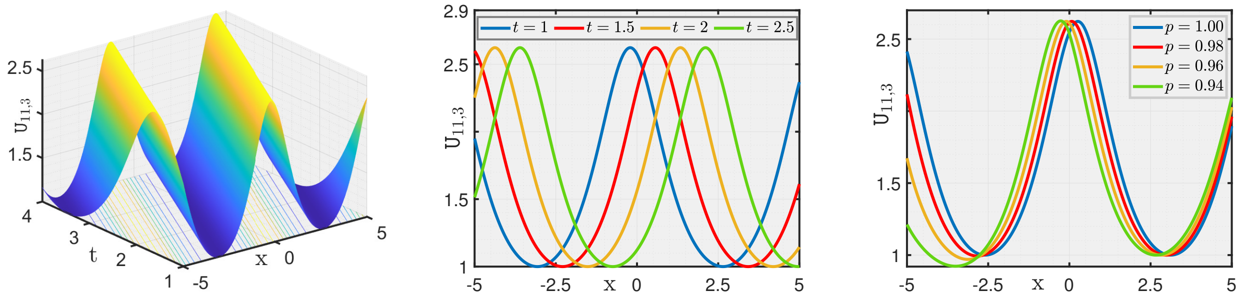

(**a**) (**b**) (**c**)

**Figure 3.** (**a**–**c**) Visualization of $U_{1,4}$ for $\sigma = 15,\ \delta = 2.1\ , \kappa = 1.7, a_0 = 1,\ a_2 = 3,\ \gamma = 2.1,\ c = 1,\ p = 1,$ $a = 1.1,\ b = 1.5,\ s = 1.7,\ y = 1,\ z = 2.1.$



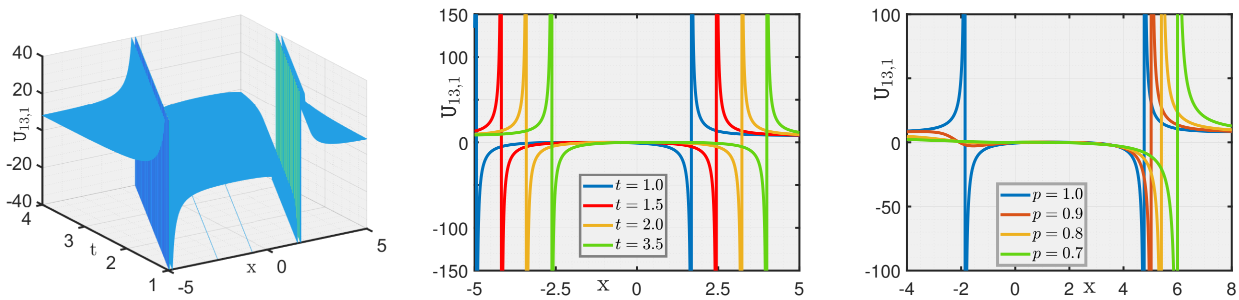

(**a**) (**b**) (**c**)

**Figure 4.** (**a**–**c**) Visualization of $U_{7,2}$ for $\sigma = 15,\ \delta = 2.1\ , \kappa = 1.7, a_0 = 1,\ a_2 = 3,\ \gamma = 2.1,\ c = 1,\ p = 1,$ $a = 1.1,\ b = 1.5,\ s = 1.7,\ y = 1,\ z = 2.1.$




(**a**) (**b**) (**c**)

**Figure 5.** (**a**–**c**) Visualization of $U_{11,3}$ for $\sigma = 15,\ \delta = 2.1\ , \kappa = 1.7, a_0 = 1,\ a_2 = 3,\ \gamma = 2.1,\ c = 1,$ $p = 1,\ a = 1.1,\ b = 1.5,\ s = 1.7,\ y = 1,\ z = 2.1.$




(**a**) (**b**) (**c**)

**Figure 6.** (**a**–**c**) Visualization of $U_{13,1}$ for $\sigma = 15,\ \delta = 2.1\ , \kappa = 1.7, a_0 = 1,\ a_2 = 3,\ \gamma = 2.1,\ c = 1,$ $p = 1,\ a = 1.1,\ b = 1.5,\ s = 1.7,\ y = 1,\ z = 2.1.$

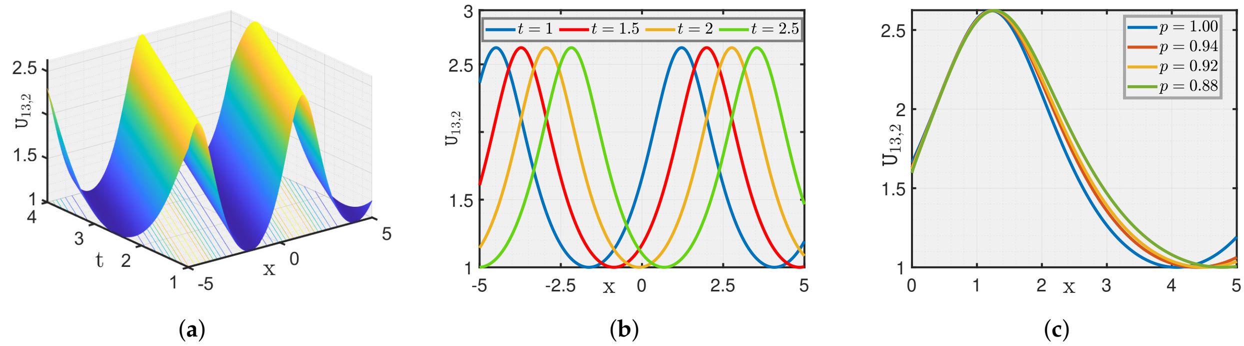



**Figure 7.** (**a**–**c**) Visualization of $\mho_{13,2}$ for $\sigma = 15$, $\delta = 2.1$ , $\kappa = 1.7$, $a_0 = 1$, $a_2 = 3$, $\gamma = 2.1$, $c = 1$, $p = 1$, $a = 1.1$, $b = 1.5$, $s = 1.7$, $y = 1$, $z = 2.1$.

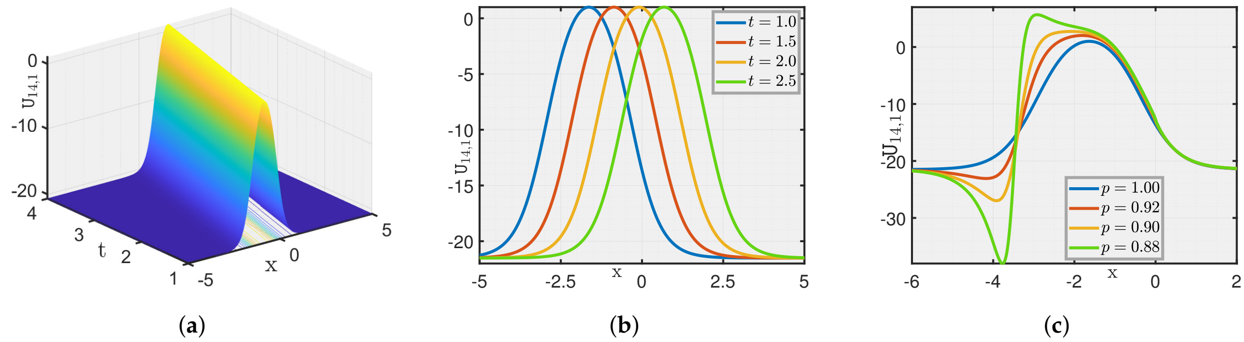



**Figure 8.** (**a**–**c**) Visualization of $\mho_{14,1}$ for $\sigma = 15$, $\delta = 2.1$ , $\kappa = 1.7$, $a_0 = 1$, $a_2 = 3$, $\gamma = 2.1$, $c = 1$, $p = 1$, $a = 1.1$, $b = 1.5$, $s = 1.7$, $y = 1$, $z = 2.1$.

## 5. Conclusions

This study has well focused on the challenge of solving the (3+1)-dimensional space-time fractional modified KdV-ZK equation by using the $\phi^6$ model expansion method with Jumarie's modified RL derivative. The method has been confirmed effective in arising exact solutions that widely capture some soliton behaviors, containing dark, periodic, traveling, and singular solitons. The effect of the fractional order on the propagation of waves has been clearly presented via graphs. The dynamics of the soliton solutions were different when we varied the fractional order. These results significantly contribute to proceeding with our understanding of nonlinear wave dynamics and have wide implications across scientific disciplines, including fluid dynamics, plasma physics, and nonlinear optics. The exact solutions obtained surpass previous efforts and present a strong framework for exactly modeling complex wave phenomena. Moving forward, further research could explore extending the method to other equations, refining techniques for handling complexity, and exploring practical applications in specific fields.

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