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Stability Analysis Study of Time-Fractional Nonlinear Modified Kawahara Equation Based on the Homotopy Perturbation Sadik Transform

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Abstract: In this manuscript, we survey a numerical algorithm based on the combination of the homotopy perturbation method and the Sadik transform for solving the time-fractional nonlinear modified shallow water waves (called Kawahara equation) within the frame of the Caputo–Prabhakar (CP) operator. The nonlinear terms are handled with the assistance of the homotopy polynomials. The stability analysis of the implemented method is studied by using S-stable mapping and the Banach contraction principle. Also, we use the fixed-point method to determine the existence and uniqueness of solutions in the given suggested model. Finally, some numerical simulations are illustrated to display the accuracy and efficiency of the present numerical method. Moreover, numerical behaviors are captured to validate the reliability and efficiency of the scheme.

Keywords: homotopy perturbation transform method; modified Kawahara equation; Sadik transform method; Caputo–Prabhakar fractional derivative

MSC: 26A33; 65L05; 34A08; 45J99; 65R20

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1. Introduction

The integral and derivatives of fractional order play a vital role and are basic in many branches of engineering and physical sciences [1–5]. The differential equations containing fractional operators have substantial applications in different scopes of applied sciences [6,7], engineering [8,9], and physics [10–15]. The Kawahara differential models of fractional order are important nonlinear models that play a vital role in mathematical and engineering sciences. These partial differential models describe physical phenomena such as water transfer and shallow water waves. Due to the important applications of these types of nonlinear equations in physical sciences [16], these equations are considered in this work. This paper studies the fractional expansion of the modified Kawahara model by applying the Caputo–Prabhakar fractional derivative in the sense of the three-parameter Mittag–Leffler function. The Kawahara model is one of the several influential models and is important in ocean engineering and physics. To study the water wave equations in the long-wave regime, it was introduced and developed by Kawahara as a type of modest value of Weber numbers (WNs) near $\frac{1}{3}$ and the surface tension [17]. Moreover, ref. [18] and Amick and Kirchgässner [19] derived the stimulating results associated with the model of water wave problems. The Kawahara equation and its modified forms have recently been discussed by many authors [20,21]. In this manuscript, we consider the following time-fractional modified equal-width wave equation [22–24]:

$$
\begin{aligned}
&{}^{C}\mathfrak{D}_t^{\mu} z(x,t) + z^2(x,t) z_x(x,t) + c\, z_{xxx}(x,t) + d\, z_{xxxxx} = 0, \\
&z(x,0) = z_0(x),
\end{aligned}
\tag{1}
$$

in which $\Re(\mu) > 0$, and c and d are non-zero real constants. Here, the symbol ${}^{C}\mathfrak{D}_t^{\mu}$ is Caputo–Prabhakar fractional operator of order μ and is defined by the following formula:

$$
{}^{C}\mathfrak{D}_t^{\mu} z(x,t) = \mathcal{I}^{1-\mu,-\eta} \frac{d}{dt} z(x,t), \tag{2}
$$

in which $\mathcal{I}^{1-\mu,-\eta}$ is the Prabhakar fractional operator and given as [25]

$$
(\mathcal{I}^{\mu,\eta} z)(x,t) = \int_0^t (t-s)^{\mu-1} \mathbb{E}_{\lambda,\mu}^{\eta}(\sigma(t-s)^{\rho}) z(x,s)\, ds, \; \sigma, \rho \in \mathbb{C}, \tag{3}
$$

in which $\mathbb{E}_{\lambda,\mu}^{\eta}$ is the Mittag–Leffler (ML) function and for $z \in \mathbb{C}$, defined by [25,26]

$$
\mathbb{E}_{\lambda,\mu}^{\eta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\eta+n)}{\Gamma(\eta)\, n!\, \Gamma(\lambda n + \mu)} z^n, \; \Re(\lambda), \Re(\mu) > 0, \; \eta > 0. \tag{4}
$$

In general, for all the fractional differential equations (FDEs), the process of finding the analytical solution is not easy to compute. Hence, researchers are drawn towards obtaining an approximate or numerical solution of these types of equations, which they find to be valuable. To gain the approximate solutions of Equation (1), we express a substantial and reliable method based on a combination of the Sadik transform (ST) with the homotopy perturbation (HP) technique. Prabhakar [25] showed a novel derivative with the ML kernel as an extension of Riemann–Liouville and Caputo derivatives and displayed the essence of generalizing the models associated with probability and mathematical physics. The main reason for choosing the ML function in this manuscript is related to its main applications in various models, such as heterogeneous [27], Havriliak–Negami [28,29], viscoelastic [30], and stochastic [31] models studied in physical sciences [32–34].

One of the numerical methods for solving the FDEs is the Sadik transform. Recently, Shaikh [35,36] discussed a modern integral representation called the Sadik transform (ST). In [35], Shaikh found some properties of this transform, like the existing theorem and duality theorem of the Sadik transform. Moreover, the author illustrated that Sumudu, Laplace, and Laplace–Carson transforms are particular cases of ST. Soon after, Shaikh in [37] presented a shifting theorem for ST and its properties for the derivative of functions. Also, for the dynamical system, the transfer function is obtained using control theory via ST. The integral transform technique is an effective tool and useful for solving fractional differential equations. However, it is difficult to find the exact solution for all kinds of FDEs using the integral transform method (see [38,39]). In this connection, the authors of [40] studied some basic notions and properties of ST, and then employed them to investigate nonlinear models. The integral transformation method with different structures has been performed to solve numerous types of differential equations in various scientific fields, for example, in the Adomian decomposition method [41–43], homotopy-based schemes [44–46], the iterative perturbation method [47], numerical methods based on the Petrov–Galerkin [48], the Petrov–Galerkin finite method [49], the variational iteration method [50], and the methods studied in the references [22,51–55].

With this aim, the rest of this paper is presented as follows. Section 2 contains a summary of the definitions and mathematical preliminaries of fractional calculus. We describe the suggested method for solving Equation (1) in Section 3. The convergence, stability, existence, and uniqueness of the suggested method are studied in Section 4. To verify and demonstrate the proposed approach, we solve the proposed model in Section 5. Finally, some conclusions are expressed in Section 6.

2. Important Preliminaries

This part studies the theorems and lemmas which will be applied in the following sections.

Definition 1 ([56]). *Let $\psi(t)$ be a piecewise continuous function under the interval $[0, B]$ such that $|\psi(t)| \le C_1 \exp(at)$, where $B > 0$, $a > 0$, and $C_1 > 0$ are the real constants. Then, the Sadik transform for $\psi(t)$ is given by*

$$\mathcal{T}(\psi(t)) = \frac{1}{v^{\beta}} \int_0^{+\infty} e^{-tv^{\alpha}} \psi_1(t) dt.$$

Lemma 1 ([4]). *Under the assumptions of Equation (4), we have*

$$(\mathcal{I}^{\mu,\eta} t^{\nu-1})(x) = \Gamma(\nu) t^{\mu+\nu-1} \mathbb{E}^{\eta}_{\lambda,\mu+\nu}(\sigma t^{\lambda}). \tag{5}$$

Lemma 2 ([56]). *Suppose that $\Phi^{v}_{\alpha,\beta}$ is an ST of $\psi_1(t)$ and $\Psi^{v}_{\alpha,\beta}$ is an ST of $\psi_2(t)$. Then, ST of $(\psi_1 * \psi_2)(t)$ is displayed by*

$$\mathcal{T}[(\psi_1 * \psi_2)(t)] = v^{\beta} \Phi^{v}_{\alpha,\beta}.\Psi^{v}_{\alpha,\beta}, \tag{6}$$

in which the symbol $*$ shows the convolution operator. Here, we consider $\Phi^{v}_{\alpha,\beta} = \mathcal{T}(\psi_1(t)) = \frac{1}{v^{\beta}} \int_0^{+\infty} e^{-tv^{\alpha}} \psi_1(t) dt$ and $\Psi^{v}_{\alpha,\beta} = \mathcal{T}(\psi_2(t)) = \frac{1}{v^{\beta}} \int_0^{+\infty} e^{-tv^{\alpha}} \psi_2(t) dt$.

Lemma 3. *Under the assumptions of Lemma 1, we have*

$$\mathcal{T}\left(t^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma t^{\lambda}); v\right) = v^{-\alpha\mu-\beta}(1 - \sigma v^{-\lambda\alpha})^{-\eta}, \tag{7}$$

in which $\mathcal{T}$ is the Sadik transform of function.

Proof. With the help of the definition of ST and using Equation (4), we obtain

$$\begin{aligned} \mathcal{T}(t^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma t^{\lambda}); v) &= \frac{1}{v^{\beta}} \int_0^{+\infty} e^{-tv^{\alpha}} t^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma t^{\lambda}) dt \\ &= \frac{1}{v^{\beta}} Q(\lambda, \mu, \eta) \int_0^{+\infty} e^{-tv^{\alpha}} t^{\lambda n + \mu - 1} dt. \end{aligned} \tag{8}$$

in which $Q(\lambda, \mu, \eta) = \sum_{n=0}^{\infty} \frac{\Gamma(\eta+n)\sigma^n}{\Gamma(\lambda n + \mu)\Gamma(\eta) n!}$. By the use of the gamma function, the above equations can be expressed as follows:

$$\begin{aligned} \mathcal{T}(t^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma t^{\lambda}); v) &= \frac{1}{v^{\alpha\mu+\beta}} \sum_{n=0}^{\infty} \frac{(\eta)_n (\sigma v^{-\lambda\alpha})^n}{n!} \\ &= v^{-\alpha\mu-\beta}(1 - \sigma v^{-\lambda\alpha})^{-\eta}. \end{aligned} \tag{9}$$

In this case, Equation (7) is obtained. □

Theorem 1 ([56]). *Suppose $\Phi^{v}_{\alpha,\beta} = \mathcal{T}(\psi^{(i)}(t))$ and $\psi^{(i)}(t), i \in \mathbb{N} \bigcup \{0\}$ are continuous. Then,*

$$\mathcal{T}[\psi^{i}(t)] = v^{i\alpha} \Phi^{v}_{\alpha,\beta} - \sum_{k=0}^{i-1} v^{k\alpha-\beta} \psi^{i-k-1}(0). \tag{10}$$

Also, the integration of $\varphi^{(i)}(t)$ is achieved by using the ST displayed as follows:

$$\mathcal{T}[\int_0^t \psi(t)^{(i)}] = \frac{1}{v^{\alpha}}\Phi^{v}_{\alpha,\beta}. \tag{11}$$

Theorem 2. *The ST of fractional derivative defined in (2) for $i = 1$ is calculated by*

$$\mathcal{T}\Big({}^{C}\mathfrak{D}^{\mu}_{t} z(x,t)\Big) = v^{\alpha\,\mu}\,(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}\, Z(v,\,\alpha,\,\beta) - v^{\alpha\mu-(\alpha+\beta)}(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}\, z(\,x,\,0^{+}), \tag{12}$$

in which $Z(v,\alpha,\beta) = \mathcal{T}(z(x,t))$.

Proof. By using Equations (2), (6), (7) and (10), we have

$$\begin{aligned}
\mathcal{T}\Big({}^{C}\mathfrak{D}^{\mu}_{t} z(x,t)\Big) &= \frac{1}{v^{\beta}}\int_0^{\infty} e^{-tv^{\alpha}}\Big[{}^{C}\mathfrak{D}^{\mu}_{t} z(x,t)\Big]dt\\
&= \frac{1}{v^{\beta}}\int_0^{\infty} e^{-tv^{\alpha}}\Big\{\int_0^t (t-s)^{-\mu}\mathbb{E}^{-\eta}_{\lambda,1-\mu}(\sigma(t-s)^{\lambda})\frac{d}{ds}z(x,s)ds\Big\}dt\\
&= \frac{1}{v^{\beta}}\int_0^{\infty}\int_s^{\infty} e^{-tv^{\alpha}}(t-s)^{-\mu}\mathbb{E}^{-\eta}_{\lambda,1-\mu}(\sigma(t-s)^{\lambda})\frac{d}{ds}z(x,s)dtds\\
&= \frac{1}{v^{\beta}}\int_0^{\infty} e^{-sv^{\alpha}}\frac{d}{ds}z(x,s)\Big[\int_0^{\infty} e^{-yv^{\alpha}}y^{-\mu}\mathbb{E}^{-\eta}_{\lambda,1-\mu}(\sigma y^{\lambda})dy\Big]ds\\
&= v^{\beta}\mathcal{T}\Big(\frac{d}{dt}z(x,t)\Big)\times\mathcal{T}\Big(t^{-\mu}\mathbb{E}^{-\eta}_{\lambda,1-\mu}(\sigma t^{\lambda})\Big)\\
&= \Big[v^{\alpha}Z(v,\alpha,\beta) - v^{-\beta}z(x,0^{+})\Big]v^{\alpha(\mu-1)}(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}\\
&= v^{\alpha\,\mu}(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}Z(v,\,\alpha,\,\beta) - v^{\alpha\mu-(\alpha+\beta)}(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}z(x,\,0^{+}).
\end{aligned} \tag{13}$$

The proof is completed. □

Lemma 4 ([12]). *Let $\mathbb{S} : X \to X$ be a mapping and $(X, \|\ .\ \|)$ be a Banach space. Then, for $\forall w, v \in X$*

$$\| \mathbb{S}_w - \mathbb{S}_v \| \le \sigma_1 \| w - \mathbb{S}_w \| + \eta_1 \| w - v \|,$$

in which $\sigma_1 \ge 0$ and $0 \le \eta_1 \le 1$. Moreover, there is a fixed point of $\mathbb{S}$ and it is Picard's $\mathbb{S}$-stable.

3. The Projected Scheme

This part demonstrates the HP method and the ST for obtaining the following equation solutions:

$$\begin{aligned}
&{}^{C}\mathfrak{D}^{\mu}_{t} z(x,t) + \mathbb{M}z(x,t) + \mathbb{V}z(x,t) = w(x,t),\\
&z(x,0) = z^{0}(x),
\end{aligned} \tag{14}$$

in which $\mathbb{M}z(x,t)$ is a linear part of Equation (14), $\mathbb{V}z(x,t)$ is the nonlinear part of Equation (14), and $w(x,t)$ is a known function. By applying the ST and Equation (12), one can obtain

$$\begin{aligned}
v^{\alpha\,\mu}\,(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}\, Z(v,\,\alpha,\beta) - v^{\alpha\mu-(\alpha+\beta)}(1-\sigma\, v^{-\lambda\,\alpha})^{\eta}z^{0}(x) + \mathcal{T}\Big(\mathbb{M}z(x,\,t) + \mathbb{V}z(x,\,t)\Big)&\\
= \mathcal{T}(w(x,\,t)).&
\end{aligned} \tag{15}$$

By using the Sadik inverse transform, we obtain

$$\begin{aligned}
z(x,t) = \mathcal{T}^{-1}\Big[v^{-\alpha\,\mu}(1-\sigma\, v^{-\lambda\,\alpha})^{-\eta}\mathcal{T}(w(x,\,t)) + \frac{1}{v^{\alpha+\beta}}z^{0}(x)\Big] - \mathcal{T}^{-1}\Big[\big(\frac{1}{v^{\alpha\,\mu}}(1-\sigma\, v^{-\lambda\,\alpha})^{-\eta}\big)&\\
\times\Big[\mathcal{T}\Big(\mathbb{M}z(x,\,t) + \mathbb{V}z(x,\,t)\Big)\Big]\Big].&
\end{aligned} \tag{16}$$

To solve Equation (16), we consider the set of solutions by

$$z = \sum_{n=0}^{\infty} \beta^n z_n, \tag{17}$$

where β^n is the unknown coefficient and is calculated from Equation (16). Now, we consider the nonlinear part $\mathbb{V}z$ by

$$\mathbb{V}z = \sum_{n=0}^{\infty} \beta^n \mathbf{F}_n(z), \tag{18}$$

in which $\mathbf{F}_n(z(x,t)) = \left\{ \frac{1}{n!} \frac{\partial^n}{\partial \beta^n} \left[N\left(\sum_{i=0}^{\infty} \varrho^i z_i \right) \right]_{\beta=0} \right\}$ is the Adomian function that is defined in [57]. By inputting Equations (17) and (18) into Equation (16), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \beta^n z_n &= W - \mathcal{T}^{-1} \left[\left(\frac{1}{\upsilon^{\alpha\mu}} (1 - \sigma \upsilon^{-\lambda\alpha})^{-\eta} \right) \times \left[\mathcal{T}\left(\mathbb{M} \sum_{n=0}^{\infty} \beta^n z_n + \mathbb{V} \sum_{n=0}^{\infty} \beta^n z_n \right) \right] \right] \\ &= W - \mathcal{T}^{-1} \left[\left(\frac{1}{\upsilon^{\alpha\mu}} (1 - \sigma \upsilon^{-\lambda\alpha})^{-\eta} \right) \times \left[\mathcal{T}\left(\mathbb{M} \sum_{n=0}^{\infty} \beta^n z_n + \sum_{n=0}^{\infty} \beta^n \mathbf{F}_n(z(x,t)) \right) \right] \right] \end{aligned} \tag{19}$$

in which $W = \mathcal{T}^{-1} \left[\frac{1}{\upsilon^{\alpha\mu}} (1 - \sigma \upsilon^{-\lambda\alpha})^{-\eta} \mathcal{T}(w) + \frac{1}{\upsilon^{\alpha+\beta}} z^0(x) \right]$. For $n = 1, 2, \ldots$, we have

$$\begin{aligned} \beta^0 &: z_0 = W(x,t), \\ \beta^1 &: z_1 = -\mathcal{T}^{-1} \left[\left(\frac{1}{\upsilon^{\alpha\mu}} (1 - \sigma \upsilon^{-\lambda\alpha})^{-\eta} \right) \times \left[\mathcal{T}\left(\mathbb{M} z^0 + \mathbf{F}_0(z) \right) \right]; t \right], \\ \beta^2 &: z_2 = -\mathcal{T}^{-1} \left[\left(\frac{1}{\upsilon^{\alpha\mu}} (1 - \sigma \upsilon^{-\lambda\alpha})^{-\eta} \right) \times \left[\mathcal{T}\left(\mathbb{M} z_1 + \mathbf{F}_1(z) \right) \right]; t \right], \\ &\vdots \\ \beta^{n+1} &: z_{n+1} = -\mathcal{T}^{-1} \left[\left(\frac{1}{\upsilon^{\alpha\mu}} (1 - \sigma \upsilon^{-\lambda\alpha})^{-\eta} \right) \times \left[\mathcal{T}\left(\mathbb{M} z_n + \mathbf{F}_n(z) \right) \right]; t \right]. \end{aligned} \tag{20}$$

Therefore, the solution of Equation (14) is obtained as

$$z = \sum_{n=0}^{\infty} z_n. \tag{21}$$

4. Existence and Uniqueness of Solutions with Caputo–Prabhakar Derivative

In this section, we used the fixed-point theorem to show the existence and uniqueness of solutions for the time-fractional modified Kawahara equation. To this aim, by applying the Prabhakar fractional integral given in Equation (3) on both sides of Equation (1), we obtain

$$\begin{aligned} z(x,t) - z(x,0) = -\Big[\int_0^t (t-s)^{\mu-1} \mathbb{E}_{\lambda,\mu}^{\eta}(\omega(t-s)^{\lambda}) \Big(& z^2(x,s) z_x(x,s) \\ & + c\, z_{xxx}(x,s) + d\, z_{xxxxx}(x,s) \Big) \Big] ds. \end{aligned} \tag{22}$$

For simplicity, we assume that

$$\zeta(z,x,t) = z^2(x,t) z_x(x,t) + c\, z_{xxx}(x,t) + d\, z_{xxxxx}(x,t).$$

Then, we have

$$z(x,t) - z^0 = -\Big[\int_0^t (t-s)^{\mu-1} \mathbb{E}_{\lambda,\mu}^{\eta}(\sigma(t-s)^{\lambda}) \zeta(z,x,s) \Big] ds. \tag{23}$$

Now, we show that the function $\zeta(Z,x,t)$ admits the Lipschitz condition when $\zeta(Z,x,t)$ is bounded. For this aim, we assume that the continuous functions U and V are upper bounded; thus, we have

$$\begin{aligned}
&\| \zeta(U,x,t) - \zeta(V,x,t) \| \\
&= \| (U^2U_x - V^2V_x) + c(U_{xxx} - V_{xxx}) + d(U_{xxxx} - V_{xxxx}) \| \\
&\leq \| U^2U_x - V^2V_x \| + c \| U_{xxx} - V_{xxx} \| + d \| U_{xxxxx} - V_{xxxxx} \| \\
&= \frac{1}{3} \| \frac{\partial}{\partial x}(U^3 - V^3) \| + a \| \frac{\partial}{\partial x^3}(U - V) \| + d \| \frac{\partial}{\partial x^5}(U - V) \| \\
&= \frac{1}{3} \| \frac{\partial}{\partial x}(U - V)(U^2 + UV + V^2) \| + c \| \frac{\partial}{\partial x^3}(U - V) \| + d \| \frac{\partial}{\partial x^5}(U - V) \| \\
&\leq \frac{cL_1}{3}(l_1^2 + l_1l_2 + l_2^2) \| U - V \| + cL_2 \| U - V \| + dL_3 \| U - V \| \\
&= \Big(\frac{cL_1}{3}(l_1^2 + l_1l_2 + l_2^2) + + cL_2 + dL_3 \Big) \| U - V \| \\
&= \varrho \| U - V \|,
\end{aligned} \tag{24}$$

in which $\| U \| \leq l_1$, $\| V \| \leq l_2$ and $\alpha = \frac{cL_1}{3}(l_1^2 + l_1l_2 + l_2^2) + + cL_2 + dL_3$. Then, the Lipschitz condition for $\zeta(Z,x,t)$ is satisfied and if $0 \leq \alpha < 1$, it is a contraction. From Equation (23), we use the recursive relation

$$z_n = -\Big[\int_0^t (t-s)^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) \zeta(z_{n-1},x,s) \Big] ds, \tag{25}$$

with initial condition $z_0 = z(x,0)$. Furthermore, we consider the following differences:

$$\begin{aligned}
\Psi^{n+1}(x,t) &= z_{n+1} - z_n \\
&- \Big[\int_0^t (t-s)^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) (\zeta(z_n,x,\tau) - \zeta(z_{n-1},x,s)) \Big] ds,
\end{aligned} \tag{26}$$

by taking the norm on both sides of Equation (26), one can obtain

$$\begin{aligned}
\| \Psi^{n+1}(x,t) \| &= \| z_{n+1} - z_n \| \\
&\| \Big[\int_0^t (t-s)^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) (\zeta(z_n,x,s) - \zeta(z_{n-1},x,s)) \Big] d\tau \| \\
&\leq \int_0^t (t-s)^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) \| \zeta(z_n,x,s) - \zeta(z_{n-1},x,s) \| ds.
\end{aligned} \tag{27}$$

Applying Equation (27), we show the existence of a solution for the projected equation.

Theorem 3. *The time-fractional modified Kawahara equation with Caputo–Prabhakar derivative (1) has a solution, provided that the following holds true:*

$$M\varrho < 1. \tag{28}$$

Proof. Let the function $\Phi_n(x,t) = Z_{n+1} - Z + Z(x,0)$. Then, by applying Equations (24) and (27) and Lemma 1, we obtain

$$\begin{aligned}
\| \Phi_n(x,t) \| &= \| z_{n+1} - z + z(x,0) \| \\
&\| \Big[\int_0^t (t-s)^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) (\zeta(z_n,x,s) - \zeta(z,x,s)) \Big] ds \| \\
&\leq \int_0^t (t-s)^{\mu-1} \mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) \| \zeta(z_n,x,s) - \zeta(z,x,s) \| ds \\
&= M\varrho \| z_n - z \| \\
&\leq (M\varrho)^n \| z - z_1 \|,
\end{aligned} \tag{29}$$

in which Lemma 2.1 in [6] ensures the existence of $\mathbb{E}^{\eta}_{\lambda,\mu+1}(\sigma t^{\lambda})$. Then, we can put $M = \mathbb{E}^{\eta}_{\lambda,\mu+1}(\sigma t^{\lambda})$. Thus, Equation (29) implies that the functions $\Phi_n(x,t) \to 0$ when $n \to \infty$ for $M\varrho < 1$, which further gives that $\lim_{n\to\infty} z_{n+1} = z$. Consequently, solutions of the time-fractional modified Kawahara Equation (1) exist. □

Theorem 4. *Equation (1) has a unique solution when true:*

$$1 - M\varrho \geq 0. \tag{30}$$

Proof. For the uniqueness solution of the time-fractional modified Kawahara equation with Caputo–Prabhakar derivative (1), we use the contrary path for the proof. Suppose that u_1 is the other solution of (1); then, we have

$$z - z_1 = -\Big[\int_0^t (t-s)^{\mu-1}\mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda})\zeta(z-z_1,x,s)\Big]ds. \tag{31}$$

Therefore, we use the norm, for which we thus have

$$\begin{aligned} \| z - z_1 \| &= \| \Big[\int_0^t (t-s)^{\mu-1}\mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda})\zeta(z-z_1,x,s)\Big]ds \| \\ &\leq \int_0^t (t-s)^{\mu-1}\mathbb{E}^{\eta}_{\lambda,\mu}(\sigma(t-s)^{\lambda}) \| \zeta(z-z_1,x,s) \| ds \\ &\leq M\varrho \| z - z_1 \|, \end{aligned} \tag{32}$$

which implies

$$\| z - z_1 \| \Big(1 - M\varrho\Big) \leq 0. \tag{33}$$

Thus, Equation (33) implies that the functions $\| z - z_1 \| \to 0$ when $1 - M\varrho \geq 0$. Consequently, $z = z_1$. Thus, the solution of the time-fractional modified Kawahara equation with Caputo–Prabhakar derivative (1) is unique. □

Theorem 5. *Suppose that z and z_n are the exact and numerical solutions of Equation (14), such that $\| z_{n+1} \| \leq \varpi \| z_n \|, \varpi \in (0,1), n \in \mathbb{N}$. Then, the series given by (21) converges.*

Proof. From $\| z_{n+1} \| \leq \varpi \| z_n \|$, we obtain

$$\begin{aligned} \| z_1 \| &\leq \varpi \| z_0 \| = \varpi \| z^0(x) \|, \\ \| z_2 \| &\leq \varpi \| z_1 \| \leq \varpi\varpi \| z^0(x) \|, \\ &\vdots \\ \| z_{n+1} \| &\leq \varpi \| z_n \| \leq \varpi^{n+1} \| z^0(x) \| . \end{aligned} \tag{34}$$

Then,

$$\sum_{i=n+1}^{\infty} \| z_i(x,t) \| \leq \sum_{i=n+1}^{\infty} \varpi^i \| z^0(x) \| = \| z^0(x) \| \sum_{i=n+1}^{\infty} \varpi^i. \tag{35}$$

Therefore,

$$\begin{aligned}\| z - z_{n+1} \| &= \| \sum_{i=n+1}^{\infty} z_i \| \leq \sum_{i=n+1}^{\infty} \| z_i \| \\ &\leq \sum_{i=n+1}^{\infty} \varpi^i \| z^0(x) \| = \| z^0(x) \| \sum_{i=n+1}^{\infty} \varpi^i \\ &\leq \frac{\varpi^{n+1}}{1-\varpi} \| z^0(x) \| . \end{aligned} \tag{36}$$

Because $\varpi \in (0,1)$, then $\| z - z_n \| \to 0$ as $n \to \infty$. □

Stability Analysis

In this part, we study the stability of the projected scheme. To achieve the stability of the method proposed in Equation (1), we define a self-mapping $\mathbb{T}$ as follows:

$$\begin{aligned}\mathbb{T}(z_i) = z_{i+1} &= \mathcal{T}^{-1}\Big[\frac{1}{v^{\alpha+\beta}} z^0(x)\Big] - \mathcal{T}^{-1}\Big[(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta}) \\ &\times \Big[\mathcal{T}\Big(cz_{xxx} + dz_{xxxxx} + z^2(x,t) z_x\Big)\Big]\Big] \\ &= z_i - \mathcal{T}^{-1}\Big[(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta}) \\ &\times \Big[\mathcal{T}\Big(cz_{xxx} + dz_{xxxxx} + z^2 z_x\Big)\Big]\Big], \end{aligned} \tag{37}$$

where $\mathcal{T}(1) = \frac{1}{v^{\alpha+\beta}}$. To obtain the required result, we consider the following for $(i,j) \in \mathbb{N} \times \mathbb{N}$:

$$\begin{aligned}\mathbb{T}(z_i) - \mathbb{T}(z_j) &= z_i - z_j - \Big[\mathcal{T}^{-1}\Big[(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta}) \\ &\times \Big[\mathcal{T}\Big(c\Big((z_i)_{xxx} - (z_j)_{xxx}\Big) + d\Big((z_i)_{xxxxx} - (z_j)_{xxxxx}\Big) + z_i^2 (z_i)_x - z_j^2 (z_j)_x\Big)\Big]\Big]\Big] \\ &= z_i(x,t) - z_j(x,t) - \Bigg[\mathcal{T}^{-1}\Big[(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta}) \\ &\times \Big[\mathcal{T}\Big(\frac{c}{3}\frac{\partial^3}{\partial x^3}\Big(z_i - u_j\Big) + d\frac{\partial^5}{\partial x^5}\Big(z_i - z_j\Big) + \frac{1}{3}\frac{\partial}{\partial x}\Big(z_i^3 - z_j^3\Big)\Big)\Big]\Big]\Bigg]. \end{aligned} \tag{38}$$

On employing the norm on Equation (38) and after simplifying, one can obtain

$$\begin{aligned}&\| \mathbb{T}(z_i) - \mathbb{T}(z_j) \| \\ &\leq \| z_i - z_j \| + \Bigg[\mathcal{T}^{-1}\Big[(v^{-\alpha\mu}(1-\sigma v^{-\lambda\alpha})^{-\eta}) \\ &\times \Big[\mathcal{T}\Big(\| \frac{c}{3}\frac{\partial^3}{\partial x^3}\Big(z_i - z_j\Big) \| + d \| \frac{\partial^5}{\partial x^5}\Big(z_i - z_j\Big) \| + \| \frac{1}{3}\frac{\partial}{\partial x}\Big(z_i^3 - z_j^3\Big) \| \Big)\Big]\Big]\Bigg]. \end{aligned} \tag{39}$$

We let the continuous function z_i be upper bounded; then,

$$\begin{aligned}&\| \mathbb{T}(z_i(x,t)) - \mathbb{T}(z_j(x,t)) \| \\ &\leq \Big(\frac{c}{3}\beta H_1(\lambda,\mu,\eta) + d\beta_1 H_2(\lambda,\mu,\eta) + \frac{1}{3}\beta_2(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) H_3(\rho,\mu,\gamma)\Big) \| z_i(x,t) - z_j(x,t) \|, \end{aligned} \tag{40}$$

in which $\parallel z_i(x,t) \parallel \leq \gamma_1$, $\parallel z_j(x,t) \parallel \leq \gamma_2$, $\beta = \frac{\partial^3}{\partial x^3}$, $\beta_1 = \frac{\partial^5}{\partial x^5}$, $\beta_2 = \frac{\partial}{\partial x}$ and $H_k(\lambda,\mu,\eta)$, $k = 1,2,3$ are functions retrieved from $\mathcal{T}^{-1}\Big[(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta})\Big[\mathcal{T}\Big(.\Big)\Big]\Big]$. The given operator in Equation (40) is a contraction mapping if the following relation is satisfied:

$$\Big(\frac{c}{3}\beta H_1(\lambda,\mu,\eta) + d\beta_1 H_2(\lambda,\mu,\eta) + \frac{1}{3}\beta_2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)H_3(\rho,\mu,\gamma)\Big) < 1. \tag{41}$$

Thus, we can state that the self-mapping $\mathbb{S}$ has a unique fixed point. Also, $\mathbb{S}$ proves Lemma 4 with

$$\sigma_1 = 0,\ \eta_1 = \Big(\frac{c}{3}\beta H_1(\lambda,\mu,\eta) + d\beta_1 H_2(\lambda,\mu,\eta) + \frac{1}{3}\beta_2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)H_3(\lambda,\mu,\eta)\Big).$$

Therefore, $\mathbb{S}$ is in agreement with Picard's $\mathbb{S}$-stable. Hence the numerical method is stable.

5. Computational Results

To illustrate the numerical application of the hired scheme to investigate its efficiency and applicability, we consider the numerical results and solve them by applying the present method. In our implementation, the proposed method was carried out by applying Matlab. The absolute error function $\parallel z - z_n \parallel$ is obtained by the $L^2 - norm$, where z_n is the numerical solution and z is the exact solution computed in the present method. To show the computational results, we consider the time-fractional Equation (1) with $z_0 = \frac{3c}{\sqrt{-10d}} sech^2\Big(\lambda' x\Big)$, where $\lambda' = \sqrt{\frac{-c}{20d}}$. The exact solution of this equation for the special case $\eta = 0, \mu = 1$, $c > 0$ and $d < 0$ is $z(x,t) = \frac{3c}{\sqrt{-10d}} sech^2\Big(\lambda'(x - c't)\Big)$, where $c' = \frac{25d-4c^2}{25d}$. By decomposing the presented method on Equation (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\beta^n z_n(x,t) = \frac{3c}{\sqrt{-10d}} sech^2\Big(\lambda' x\Big) - \mathcal{T}^{-1}\Big[(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta})\times \\ \Big[\mathcal{T}\Big(c(\sum_{n=0}^{\infty}\beta^n z_n(x,t))_{xxx}(x,t) + d\,(\sum_{n=0}^{\infty}\beta^n z_n(x,t))_{xxxxx} + \sum_{n=0}^{\infty}\beta^n \mathbf{F}_n(z(x,t))\Big)\Big];t\Big],\end{aligned} \tag{42}$$

in which $\mathbf{F}_n(z)$ is the nonlinear part defined by

$$\sum_{n=0}^{\infty}\beta^n \mathbf{F}_n(z) = z^2 z_x. \tag{43}$$

The polynomials $\mathbf{F}_n(z)$ for some of the factors are calculated as

$$\begin{aligned}\mathbf{F}_0(z) &= z_0^2(z_0)_x, \\ \mathbf{F}_1(z) &= z_0^2(z_1)_x + 2z_0 z_1(z_0)_x,\end{aligned} \tag{44}$$

We can compute p in Equation (44) as

$$
\begin{aligned}
&\beta^0 : z_0(x,t) = \frac{3c}{\sqrt{-10d}} sech^2\left(\lambda' x\right),\\
&\beta^1 : z_1 = \mathcal{T}^{-1}\Big[\big(\frac{1}{v^{\alpha\,\mu}}(1-\sigma v^{-\lambda\alpha})^{-\eta}\big) \times \Big[\mathcal{T}\Big(c(z_0)_{xxx} + d\,(z_0(x,t))_{xxxxx} + \mathbf{F}_0(z)\Big)\Big];t\Big]\\
&\quad = \Bigg[\frac{-24\lambda'^3 c^2}{\sqrt{-10d}} sech^2\left(\lambda' x\right) \tanh^3\left(\lambda' x\right) + \frac{24\lambda'^3 c^2}{\sqrt{-10d}} sech^4\left(\lambda' x\right) \tanh\left(\lambda' x\right)\\
&\quad + \frac{12\lambda'^3 c^2}{\sqrt{-10d}} sech^2\left(\lambda' x\right) \tanh\left(\lambda' x\right)\\
&\quad - \frac{96\lambda'^5 cd}{\sqrt{-10d}} sech^2\left(\lambda' x\right) \tanh^5\left(\lambda' x\right) + \frac{192\lambda'^5 cd}{\sqrt{-10d}} sech^4\left(\lambda' x\right) \tanh^3\left(\lambda' x\right)\\
&\quad + \frac{168\lambda'^5 cd}{\sqrt{-10d}} sech^2\left(\lambda' x\right) \tanh^3\left(\lambda' x\right)\\
&\quad - \frac{168\lambda'^5 cd}{\sqrt{-10d}} sech^2\left(\lambda' x\right) \tanh\left(\lambda' x\right) + \frac{384\lambda'^5 cd}{\sqrt{-10d}} sech^5\left(\lambda' x\right) \tanh^3\left(\lambda' x\right)\\
&\quad - \frac{336\lambda'^5 cd}{\sqrt{-10d}} sech^6\left(\lambda' x\right) \tanh\left(\lambda' x\right)\\
&\quad + \frac{27c^3}{5d\sqrt{-10d}} sech^6\left(\lambda' x\right) \tanh\left(\lambda' x\right)\Bigg] t^{\mu-1}\mathbb{E}^{\eta}_{\lambda,\mu}(\sigma t^{\lambda}),
\end{aligned}
\tag{45}
$$

Finally, the series numerical solution is calculated by

$$
\begin{aligned}
z &= \sum_{n=0}^{\infty} z_n\\
&= \frac{3c}{\sqrt{-10d}} sech^2\left(\lambda' x\right) + \Bigg[\frac{-24\lambda'^3 c^2}{\sqrt{-10d}} sech^2\left(\lambda' x\right) \tanh^3\left(\lambda' x\right) + \frac{24\lambda^3 c^2}{\sqrt{-10d}} sech^4\left(\lambda x\right) \tanh\left(\lambda' x\right)\\
&+ \frac{12\lambda'^3 c^2}{\sqrt{-10b}} sech^2\left(\lambda' x\right) \tanh\left(\lambda' x\right)\\
&- \frac{96\lambda'^5 cd}{\sqrt{-10b}} sech^2\left(\lambda' x\right) \tanh^5\left(\lambda' x\right) + \frac{192\lambda'^5 cd}{\sqrt{-10b}} sech^4\left(\lambda' x\right) \tanh^3\left(\lambda' x\right)\\
&+ \frac{168\lambda'^5 cd}{\sqrt{-10b}} sech^2\left(\lambda' x\right) \tanh^3\left(\lambda' x\right)\\
&- \frac{168\lambda'^5 cd}{\sqrt{-10b}} sech^2\left(\lambda' x\right) \tanh\left(\lambda' x\right) + \frac{384\lambda'^5 cd}{\sqrt{-10b}} sech^5\left(\lambda' x\right) \tanh^3\left(\lambda' x\right)\\
&- \frac{336\lambda'^5 cd}{\sqrt{-10b}} sech^6\left(\lambda' x\right) \tanh\left(\lambda' x\right)\\
&+ \frac{27c^3}{5d\sqrt{-10d}} sech^6\left(\lambda' x\right) \tanh\left(\lambda' x\right)\Bigg] t^{-\mu\alpha}\mathbb{E}^{\eta}_{-\lambda\alpha,1-\mu\alpha}(\sigma t^{-\lambda\alpha}) + \ldots.
\end{aligned}
\tag{46}
$$

Now, we study a comparative examination of the absolute error function results for classical Equation (1), for the special case $\gamma = 0, \mu = 1$. Table 1 presents the computational results given by the application of our proposed method and the other one obtained by the authors of [58,59] for the special case $\gamma = 0, \mu = 1$. The computational results for different values of μ are demonstrated in Figures 1 and 2 by selecting $\mu = 0.75, 0.85, 0.95$, and $\lambda = 0.5$, $\eta = 0.75, \sigma = 1$. In particular, the computational results for the solutions corresponding to $\lambda = 0.5, \eta = 0.75, \sigma = 1$ at disparate values of μ with $t = 0.5$ are presented in Figure 3. Figure 4 shows the graphs of the absolute error function for the case $\eta = 0, \mu = 1$. According to the established results, the achieved method is very accurate and effective in all cases.

Table 1. The exact solution values when $\mu = 1$ and comparison between the proposed method with the methods VIM [59], HPM [59], and HATM [58] corresponding to $\rho = 0.5, \gamma = 0.75, \omega = 1$ and $\mu = 0.99, \alpha = 0.5$ $a = 0.001$, $b = -1$.

x	t	**Exact Solution**	**VIM [59]**	**HPM [59]**	**HATM [58]**
-5	0.2	$9.474889415 \times 10^{-7}$	$9.474984315 \times 10^{-4}$	$9.474984315 \times 10^{-4}$	$9.474984315 \times 10^{-4}$
-2.5	0.4	$9.483773375 \times 10^{-6}$	$9.483868961 \times 10^{-4}$	$9.483868961 \times 10^{-4}$	$9.483868965 \times 10^{-4}$
0	0.6	$9.486831272 \times 10^{-7}$	$9.486832980 \times 10^{-4}$	$9.486832980 \times 10^{-4}$	$9.486832980 \times 10^{-4}$
2.5	0.8	$9.484055589 \times 10^{-6}$	$9.483868961 \times 10^{-4}$	$9.483868961 \times 10^{-4}$	$9.483868965 \times 10^{-4}$
5	1.0	$9.475453144 \times 10^{-6}$	$9.474984315 \times 10^{-4}$	$9.474984315 \times 10^{-4}$	$9.474984315 \times 10^{-4}$
x	t	Our Method			
-5	0.2	$9.745606079 \times 10^{-7}$			
-2.5	0.4	$1.526682172 \times 10^{-6}$			
0	0.6	$9.720373521 \times 10^{-7}$			
2.5	0.8	$1.076279876 \times 10^{-6}$			
5	1.0	$2.221689661 \times 10^{-6}$			

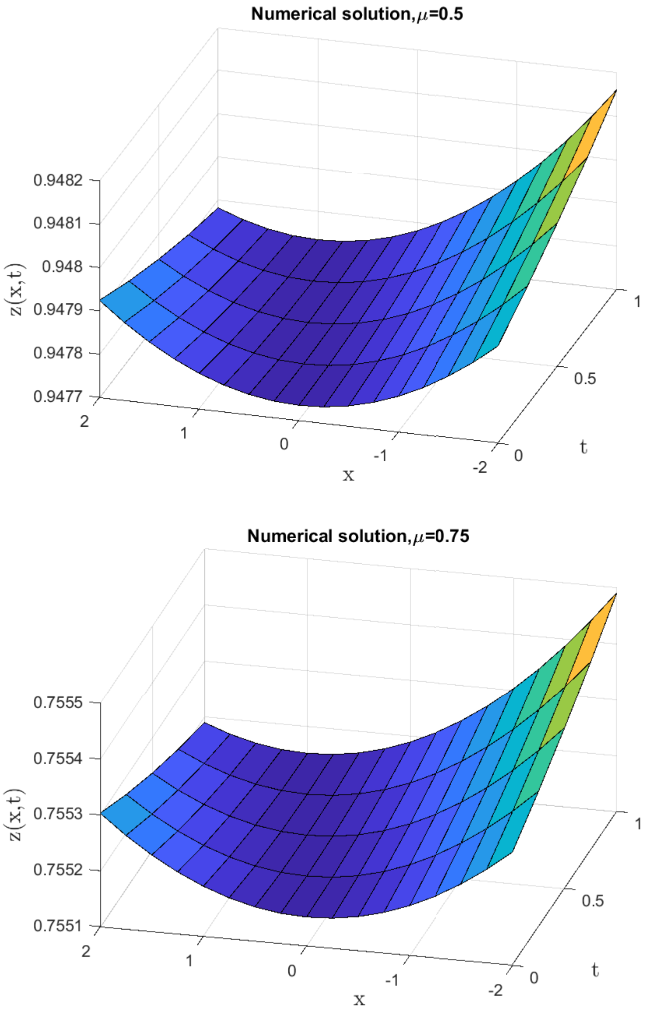

Figure 1. The approximate solution graph corresponding to $\lambda = 0.5, \eta = 0.75, \sigma = 1$ at different values of μ with $c = 1$, $d = -1$.

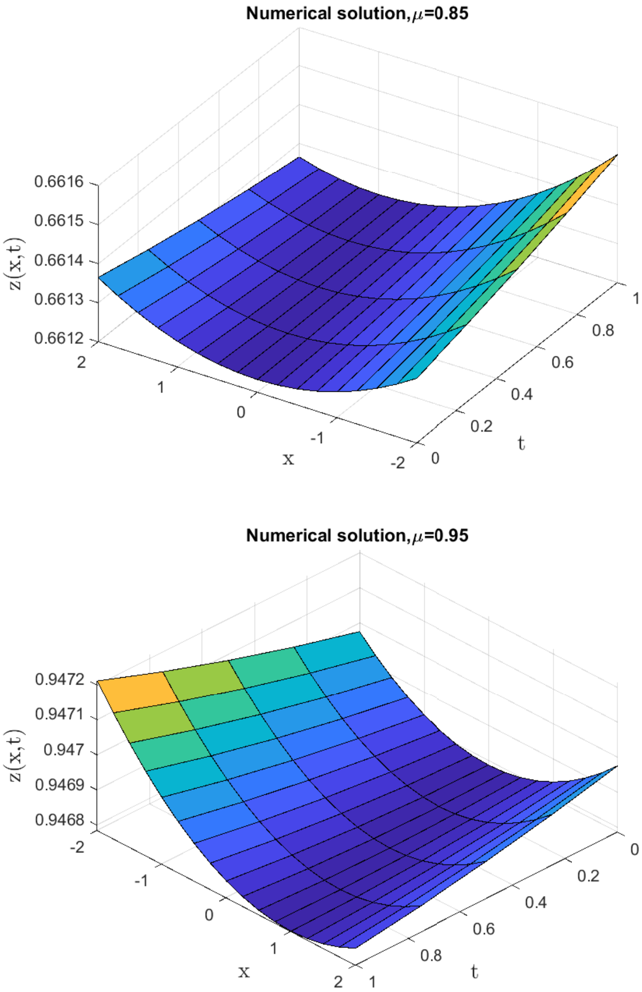

Figure 2. The approximate solution graph corresponding to $\lambda = 0.5, \eta = 0.75, \sigma = 1$ at different values of μ with $c = 1$, $d = -1$.

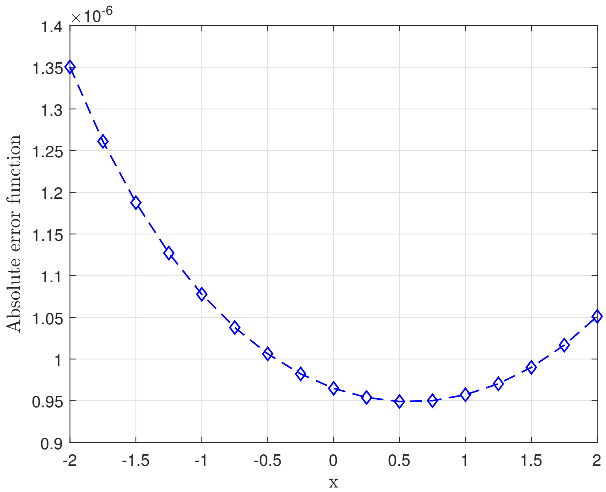

Figure 3. The absolute error function graph for $\eta = 0$ and $\mu = 1$ when $t = 0.5$ with $c = 1$, $d = -1$.

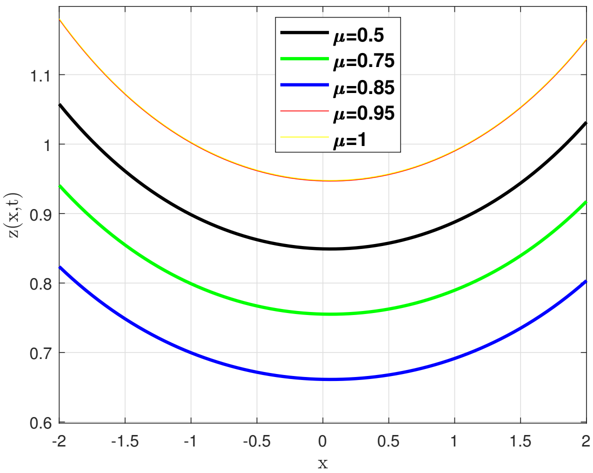

Figure 4. The approximate solution graph corresponding to $\lambda = 0.5, \eta = 0.75, \sigma = 1$ at different values of μ with $t = 0.5$, $c = 1$, $d = -1$.

6. Conclusions

In the present investigation, we studied the modified Kawahara equation associated with the Caputo–Prabhakar fractional operator. With the help of a unified computational technique with the homotopy perturbation algorithm and Sadik transform, the iterative solution was attained. To show and examine the stability, existence, and uniqueness of the solutions of the given suggested model, we used the fixed-point theory as well as the theory of $\mathbb{S}$-stable mapping. Some stimulating numerical results were obtained at different values of the parameters μ, ρ, γ, and ω. The attained results illustrate that the Caputo–Prabhakar fractional operator can be applied to demonstrate real-world problems.

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