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*Article*

# Consensus of T-S Fuzzy Fractional-Order, Singular Perturbation, Multi-Agent Systems

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**Abstract:** Due to system complexity, research on fuzzy fractional-order, singular perturbation, multi-agent systems (FOSPMASs) remains limited in control theory. This article focuses on the leader-following consensus of fuzzy FOSPMASs with orders in the range of $(0, 2)$. By employing the T-S fuzzy modeling approach, a fuzzy FOSPMAS is constructed. In order to achieve the consensus of a FOSPMAS with multiple time-scale characteristics, a fuzzy observer-based controller is designed, and the error system corresponding to each agent is derived. Through a series of equivalent transformations, the error system is decomposed into fuzzy singular fractional-order systems (SFOSs). The consensus conditions of the fuzzy FOSPMASs are obtained based on linear matrix inequalities (LMIs) without an equality constraint. The theorems provide a way to tackle the uncertainty and nonlinearity in FOSPMASs with orders in the range of $(0, 2)$. Finally, the effectiveness of the theorems is verified through an RLC circuit model and a numerical example.

**Keywords:** fuzzy systems; singular perturbation systems; multi-agent systems; consensus



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## 1. Introduction

In recent decades, the control of multi-agent systems (MASs) has become a leading research subject, stemming from the superior efficiency of multiple agents collaborating to execute tasks compared to an individual agent [1]. The applications of MASs are significant, spanning various domains, including service robotics [2,3], hazardous environment detection [4], and unmanned aerial vehicle formation flying [5]. Consensus control of MASs is a fundamental and core issue based on tracking control [6,7]. A significant amount of research has emerged on the consensus of MASs [8–12]. Ren [8] constructed MASs with second-order integrator dynamics by analyzing the swarming model and designed a consensus protocol. Tian and Liu [9] obtained two decentralized consensus conditions of MASs with diverse input and communication delays. Wen et al. [10] introduced an innovative protocol designed by using synchronous intermittent local feedback for second-order consensus of MASs. Zhang et al. [11] proposed event-trigger output feedback control approaches, enabling all connected communication graphs to reach a consensus. Tan et al. [12] derived the consensus criteria for cyber-physical systems under sampled data control, employing a suitable Lyapunov function. The above studies predominantly concentrate on achieving consensus of MASs with integer order, which encounters challenges in describing actual systems in nature and industry.

Fractional-order systems (FOSs) are capable of more accurately modeling and computing genetic and memory effects in various complex processes than integer-order systems [13]. Singular fractional-order systems (SFOSs), also called descriptor systems, have a broader range of applications. In physics, SFOSs are employed to accurately simulate a range of complex physical phenomena, including anomalous diffusion [14] and wave propagation [15]. In engineering, SFOSs are used to enhance the efficiency and accuracy of both signal processing and image recognition [16]. In addition, the consensus of fractional-order MASs (FOMASs) has attracted widespread interest [17–22]. Su and Ye proposed a control strategy with input delays to achieve the consensus of general linear and nonlinear FOMASs under event-triggered conditions in [17,18], respectively. Yang et al. [19] considered the consensus of nonlinear distributed and input-delayed FOMASs and further explored the performance of FOMASs in terms of leader-following and leaderless global consensus in [20]. Hu et al. [21] developed an adaptive controller that employs an event-triggered scheme without Zeno behavior, aiming to realize the consensus of FOMASs. Bahrampour et al. [22] proposed new Lyapunov-based LMI conditions to determine the state feedback controller gains on the distributed consensus control of heterogeneous FOMASs with interval uncertainties. However, many practical MASs exhibit multiple time-scale characteristics that refer to the coupled coexistence of fast dynamics and slow dynamics. The design of controllers for these systems frequently encounters difficulties due to the presence of high dimensionality and pathological values [23,24].

Singular perturbation systems (SPSs) have multiple time scales and inherently pathological dynamical properties [25–27]. SPSs with a certain parasitic parameter ($\varepsilon$) are modeled to describe real systems. In power system modeling, $\varepsilon$ is used to represent transient phenomena in machine reactors or voltage regulators [28]. In industrial control systems, it signifies small time constants between control and response [29]. Numerous scholars have intensively studied SPSs [30–35]. On one hand, two commonly employed strategies for solving control problems of SPSs are the quasi-steady-state method [30] and the block diagonalization method [31], which decompose the system into slow and fast subsystems. But these methods rely on the assumption that the fast subsystem matrix is non-singular, and they are not applicable to non-standard SPSs that cannot be easily decomposed. On the other hand, Yang et al. [32], Gao et al. [33], and Liu et al. [34] proposed the integral sliding mode control method for full-order SPSs with mismatched disturbances, uncertainty, and nonlinear input, respectively. Their methods are based on a full-order model, which eliminates the need to decompose the system. Furthermore, techniques such as the use of Lyapunov functions and LMIs are also applied to system analysis. Fridman [35] derived the LMI criteria for the stability of SPSs for delays proportional to $\varepsilon$ and delays independent of $\varepsilon$. Additionally, for singular perturbation MASs (SPMASs), both Ben Rejeb et al. [36] and Tognetti et al. [37] designed decentralized controllers, enabling systems to synchronize and ensuring global performance. Xu et al. [38] presented a sliding-mode controller with memory output for to address the consensus of SPMASs in finite time. Zhang et al. [39] achieved global Mittag–Leffler consensus tracking for fractional SPMASs modeled by a discontinuous function with a non-decreasing property. However, in practical applications, the exact value of the $\varepsilon$ parameter is often difficult to obtain directly. By analyzing the background information of specific problems in depth, the reasonable change range of $\varepsilon$ can be effectively estimated. Given $\varepsilon$ in a known interval, the design of controllers to achieve the consensus of nonlinear FOSPMASs remains an open problem in the field of control theory.

T-S fuzzy models possess the ability to approximate nonlinear dynamics; therefore, the well-established control methods for linear systems can be extended to the analysis and design of nonlinear systems. Wang et al. [40] proposed a parallel distributed compensation (PDC) method based on a T-S fuzzy model for the stability of nonlinear systems, where the fuzzy controller adopts the same fuzzy set as the fuzzy system. Using the PDC method, Wang et al. [41] further analyzed the effectiveness of using a T-S fuzzy model to approximate nonlinear functions. Numerous scholars have undertaken extensive research endeavors focusing on T-S fuzzy SPSs [42–45]. Yang and Zhang [42] proposed a design method for a state feedback controller depending on $\varepsilon$ for T-S fuzzy SPSs. Chen et al. [43] focused on nonlinear SPSs and presented two novel methods to design a static output feedback

($H_\infty$) controller based on LMIs. Visavakitcharoen et al. [44] designed an event-triggered controller based on integral feedback for nonlinear SPSs with a fuzzy model. Zhang and Han [45] proposed two diverse feedback controllers aiming to obtain the stabilization criteria of fuzzy FOSPSs with $\alpha \in (0,1)$. Nevertheless, research on the consensus control of fuzzy FOSPMASs is still relatively limited.

Inspired by previous discussions, this paper focuses on filling this research gap. The following is an overview of the main contributions of this research:

1 To provide a more accurate portrayal of complex systems in practice, a T-S fuzzy FOSPMAS with $\alpha \in (0,2)$ is formulated to reduce the difficulty of directly studying nonlinear systems. Compared to integer-order systems, the constructed model exhibits enhanced accuracy and complexity. A fuzzy FOSPS with error as a variable is derived by designing a fuzzy observer-based controller.

2 The fuzzy FOSPS is analyzed by transforming it into a fuzzy SFOS using the system augmentation method. In comparison to the existing methods [46], the proposed approach not only relaxes the assumption that the fast subsystem matrix must be non-singular but also avoids the ill-conditioned issue arising from the $\varepsilon$ parameter.

3 The consensus conditions for fuzzy FOSPMASs with $\alpha \in (0,1)$ and $[1,2)$ are formulated in this study for any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$, where $\underline{\varepsilon}$ and $\bar{\varepsilon}$ are the lower and upper boundaries, respectively. The results are presented based on LMIs without equality constraints, reducing solution difficulties. It is demonstrated through an RLC circuit model that the proposed methods are effective in practice.

The remaining parts are structured in the following manner. Section 2 provides foundational definitions in graph theory and correlative lemmas. The establishment of the system model and the primary findings on the consensus of FOSPMASs are detailed in Section 3. Section 4 presents two practical examples. Lastly, Section 5 summarizes the study.

## 2. Preliminaries

### *2.1. Notations*

$X > 0$ and $X \geqslant 0$ signify that the matrix $X$ is positive definite and positive semi-definite, respectively. $X^T$ stands for the transpose of the matrix ($X$), and $\text{sym}\{X\} = X + X^T$. spec $(E, A)$ is the spectrum of $\det(s^\alpha E - A) = 0$. The $*$ symbol represents the symmetric element of a matrix. $\otimes$ denotes the Kronecker product. For $\alpha \in (0,2)$, $a = \sin\left(\alpha\frac{\pi}{2}\right)$, $b = \cos\left(\alpha\frac{\pi}{2}\right)$, and $\Theta$ denotes $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. diag$(\cdot)$ represents a diagonal matrix. $\lceil\alpha\rceil$ stands for the rounding of $\alpha$ up to the nearest integer.

### *2.2. Graph Theory*

Consider the case of an MAS comprising a single leader and $N$ followers. The information exchanged between $N$ agents is presented as an undirected graph ($\mathcal{G}$). The Laplace matrix of the graph ($\mathcal{G}$) is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = \begin{cases} -a_{ij}, i = j \\ \sum_{j=1}^{N} a_{ij}, i \neq j \end{cases}$ and $a_{ij}$ is the element of weighted adjacency matrix $\mathcal{A}$ of graph $\mathcal{G}$. $a_{ij} > 0$ means that follower $i$ communicates with follower $j$; otherwise, $a_{ij} = 0$. When $\mathcal{G}$ is undirected, $\mathcal{A}$ is symmetric. Similarly, $h_i$ represents the communication between the leader and follower $i$, and $h_i > 0$ means follower $i$ receives information from the leader; otherwise, $h_i = 0$.

### *2.3. Preliminary Lemmas*

Consider a continuous linear SFOS with $\alpha \in (0,2)$ described by

$$ED^\alpha x(t) = Ax(t), \tag{1}$$

where $A, E \in \mathbb{R}^{n\times n}$ are the system matrices and $\mathrm{rank}(E) = m < n$. $x(t) \in \mathbb{R}^n$ represents the state. $D^\alpha$ denotes the Caputo fractional-order derivative, which is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{f^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha+1-\lceil\alpha\rceil}} d\tau,$$

where $\Gamma(\cdot)$ is the Euler Gamma function and (1) is represented by the triple $(E, A, \alpha)$.

When $E = I$, system (1) is simplified to a normal FOS as follows:

$$D^\alpha x(t) = Ax(t). \tag{2}$$

**Lemma 1** ([47]). *System (2) is stable iff* $|\arg(\mathrm{spec}(A))| > \alpha\frac{\pi}{2}$.

**Lemma 2** ([48]). $D^\alpha x(t) = f(t, x)$ *is asymptotically stable at the equilibrium points if all the eigenvalues* ($\lambda_{Ji}$, $i = 1, \cdots, n$) *of the Jacobian matrix* ($J = \partial f/\partial x$) *satisfy*

$$|\arg(\lambda_{Ji})| > \alpha\frac{\pi}{2},\ i = 1, \cdots, n,$$

*where* $f = [f_1, \ldots, f_n]$.

**Lemma 3** ([49]). *Choose two non-singular matrices* ($U$ *and* $V$) *such that*

$$UEV = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix},\ UAV = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}; \tag{3}$$

*then, system (1) is regular, impulse-free and stable, and defined as admissible iff* $A_4$ *is non-singular and*

$$\left|\arg\left(\mathrm{spec}\left(A_1 - A_2A_4^{-1}A_3\right)\right)\right| > \alpha\frac{\pi}{2}.$$

**Lemma 4** ([50]). *System (1) is admissible with* $\alpha \in [1, 2)$ *iff there exists a matrix* ($P \in \mathbb{R}^{n\times n}$) *satisfying*

$$EP = P^TE^T \geqslant 0,\ \mathrm{sym}\{\Theta \otimes AP\} < 0.$$

**Lemma 5** ([49]). *System (1) is admissible with* $\alpha \in (0, 1)$ *iff there exist two matrices* ($X, Y \in \mathbb{R}^{n\times n}$) *satisfying*

$$\begin{bmatrix} EX & EY \\ -EY & EX \end{bmatrix} = \begin{bmatrix} X^TE^T & -Y^TE^T \\ Y^TE^T & X^TE^T \end{bmatrix} \geqslant 0,\ \mathrm{sym}\{A(aX - bY)\} < 0.$$

**Remark 1.** According to Lemma 1, Figure 1 shows the stability region of system (2) with a fractional order of $\alpha \in (0, 1)$ or $[1, 2)$. Lemmas 4 and 5 are regarded as the natural extension of Lyapunov stability from normal integer-order systems to SFOSs [49,50].

**Lemma 6** ([51]). *Given a symmetric constant matrix* ($Z$) *and constant matrices* ($U, V$), *the inequality*

$$Z + UFV + V^TF^TU^T < 0$$

*holds for all* $F$ *satisfying* $F^TF \leqslant S$ *iff there exist some* $\rho > 0$ *such that*

$$Z + \begin{bmatrix} \rho^{-1}V^T & \rho U \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \rho^{-1}V \\ \rho U^T \end{bmatrix} < 0.$$

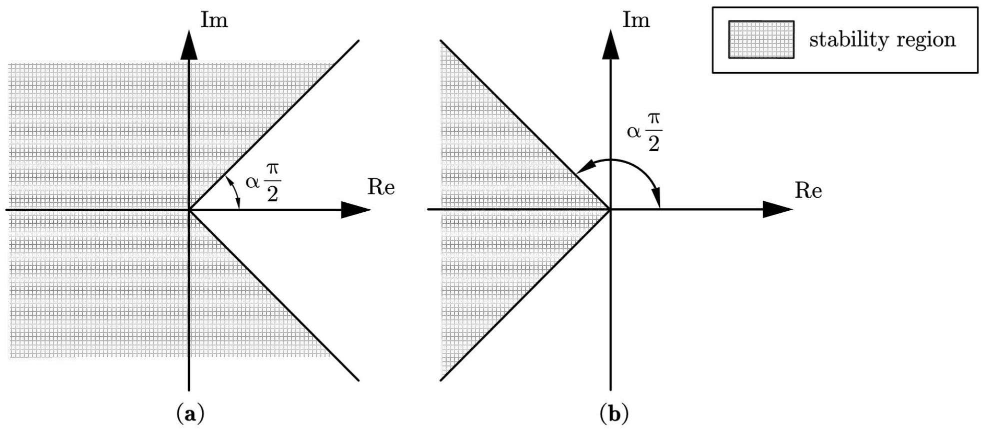


**Figure 1.** Stability region of system (2): (**a**) $\alpha \in (0,1)$; (**b**) $\alpha \in [1,2)$.

## 3. Main Results

### 3.1. System Model Description

Consider an MAS consisting of a leader and $N$ followers, with the dynamic of each agent modeled by a T-S fuzzy FOSPS. This nonlinear system is described by the fuzzy rules as follows:

Rule $k$: If $\xi_1(t)$ is $\Pi_{k1}$ and $\cdots$ and $\xi_p(t)$ is $\Pi_{kp}$,

Then, the dynamic description of each agent is written as

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_k x_i(t) + B_k u_i(t) \\ y_i(t) = C_k x_i(t) \end{cases} \quad (4)$$

$$\begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_k x_0(t) \\ y_0(t) = C_k x_0(t) \end{cases} \quad (5)$$

where

$$E(\varepsilon) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix},\ n_1 + n_2 = n,\ i = 1, 2, \cdots, N;$$

$x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^l$, and $y_i(t) \in \mathbb{R}^v$ represent the state, control input and output of follower $i$, respectively; $x_0(t) \in \mathbb{R}^n$ and $y_0(t) \in \mathbb{R}^v$ represent the state and output of the leader, respectively; and system matrices $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times l}$, and $C_k \in \mathbb{R}^{v\times n}$ are constant. Additionally, $\Pi_{kj}$ represents the fuzzy sets of the premise variables $\xi_j(t)$, where $j = 1, \cdots, p$ and $k = 1, \cdots, r$. Here, $r$ represents the number of rules.

According to the procedures of defuzzification, the global model of T-S fuzzy FOSP-MASs (4) and (5) is derived as follows:

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = \sum_{k=1}^{r}\eta_k(\xi)[A_k x_i(t) + B_k u_i(t)] \\ y_i(t) = \sum_{k=1}^{r}\eta_k(\xi)C_k x_i(t) \end{cases} \quad (6)$$

$$\begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = \sum_{k=1}^{r}\eta_k(\xi)A_k x_0(t) \\ y_0(t) = \sum_{k=1}^{r}\eta_k(\xi)C_k x_0(t) \end{cases} \quad (7)$$

where $\eta_k(\xi) = \frac{\prod_{j=1}^{p}\omega_{kj}(\xi_j(t))}{\sum_{k=1}^{r}\prod_{j=1}^{p}\omega_{kj}(\xi_j(t))}$ is the weighting function and $\omega_{kj}(\xi_j(t))$ is the membership function, satisfying $\eta_k(\xi(t)) \geqslant 0$, $\sum_{k=1}^{r}\eta_k(\xi(t)) = 1$, $\prod_{j=1}^{p}\omega_{kj}(\xi_j(t)) \geqslant 0$, and $\sum_{k=1}^{r}\prod_{j=1}^{p}\omega_{kj}(\xi_j(t)) > 0$.

Utilizing the complete state information for controller design is often challenging, owing to economic constraints and measurement limitations. To address this issue and design a consensus protocol for fuzzy FOSPMASs (6) and (7), a fuzzy observer is formulated as follows:

$$\begin{cases} E(\varepsilon)D^{\alpha}\hat{x}_i(t) = \sum_{k=1}^{r}\eta_k(\xi)A_k\hat{x}_i(t) + \sum_{k=1}^{r}\eta_k(\xi)B_k u_i(t) + z_i(t) \\ z_i(t) = \sum_{s=1}^{r}\eta_s(\xi)W_s\left[\sum_{j=1}^{N}a_{ij}\left(\tilde{y}_i(t) - \tilde{y}_j(t)\right) + h_i\tilde{y}_i(t)\right] \end{cases} \tag{8}$$

where $\hat{x}_i(t)$ signifies the estimated state of follower $i$ and $\tilde{y}_i(t) = y_i(t) - \sum_{k=1}^{r}\eta_k(\xi)C_k\hat{x}_i(t)$ represents the error between the actual output ($y_i$) and the weighted sum of estimated outputs. Furthermore, $W_s \in \mathbb{R}^{n\times v}$ denotes the gain matrix.

To achieve the consensus of (6) and (7), the following distributed control protocol based on (8) is designed:

$$u_i(t) = \sum_{q=1}^{r}\eta_q(\xi)K_q\left[\sum_{j=1}^{N}a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right) + h_i(\hat{x}_i(t) - x_0(t))\right]. \tag{9}$$

Let $x_{ei}(t) = x_i(t) - x_0(t)$ and $\hat{x}_{ei}(t) = x_i(t) - \hat{x}_i(t)$. By substituting (9) into (6) and subtracting (6) from (8), the error system is written as

$$\begin{aligned} E(\varepsilon)D^{\alpha}x_{ei}(t) = &\sum_{k=1}^{r}\eta_k(\xi)A_k x_{ei}(t) + \sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q \\ &\cdot\left[\sum_{j=1}^{N}a_{ij}\left(x_{ei}(t) - x_{ej}(t) - \left(\hat{x}_{ei}(t) - \hat{x}_{ej}(t)\right)\right) + h_i\left(x_{ei}(t) - \hat{x}_{ei}(t)\right)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} E(\varepsilon)D^{\alpha}\hat{x}_{ei}(t) = &\sum_{k=1}^{r}\eta_k(\xi)A_k\hat{x}_{ei}(t) - \sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)W_sC_k \\ &\cdot\left[\sum_{j=1}^{N}a_{ij}\left(\hat{x}_{ei}(t) - \hat{x}_{ej}(t)\right) + h_i\hat{x}_{ei}(t)\right]. \end{aligned} \tag{11}$$

Let $\bar{x}_e(t) = \left[x_e^T(t) \quad \hat{x}_e^T(t)\right]^T$, where

$$x_e(t) = \left[x_{e1}^T(t), \quad \cdots, \quad x_{eN}^T(t)\ \right]^T, \ \hat{x}_e(t) = \left[\hat{x}_{e1}^T(t), \quad \cdots, \quad \hat{x}_{eN}^T(t)\ \right]^T.$$

The compact form of systems (10) and (11) is

$$\begin{aligned} (I_N \otimes E(\varepsilon))D^{\alpha}x_e(t) = &\left(I_N \otimes \sum_{k=1}^{r}\eta_k(\xi)A_k\right)x_e(t) \\ &+ M \otimes \left(\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q\right)(x_e(t) - \hat{x}_e(t)), \end{aligned} \tag{12}$$

$$(I_N \otimes E(\varepsilon))D^{\alpha}\hat{x}_e(t) = \left(I_N \otimes \sum_{k=1}^{r}\eta_k(\xi)A_k\right)\hat{x}_e(t) - M \otimes \left(\sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)W_sC_k\right)\hat{x}_e(t). \tag{13}$$

where $M = \mathcal{L} + \mathrm{diag}(h_1, h_2, \cdots, h_N)$.

By combining (12) and (13), the error system is described as

$$\bar{E}(\varepsilon)D^{\alpha}\bar{x}_e(t) = \bar{A}\bar{x}_e(t), \tag{14}$$

where

$$\bar{E}(\varepsilon) = \begin{bmatrix} I_N \otimes E(\varepsilon) & 0 \\ 0 & I_N \otimes E(\varepsilon) \end{bmatrix}, \ \bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{bmatrix},$$
$$\bar{A}_1 = I_N \otimes \sum_{k=1}^{r} \eta_k(\xi) A_k + M \otimes \left( \sum_{k=1}^{r} \sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi) B_k K_q \right),$$
$$\bar{A}_2 = - M \otimes \left( \sum_{k=1}^{r} \sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi) B_k K_q \right),$$
$$\bar{A}_4 = I_N \otimes \sum_{k=1}^{r} \eta_k(\xi) A_k - M \otimes \left( \sum_{s=1}^{r} \sum_{k=1}^{r} \eta_s(\xi)\eta_k(\xi) W_s C_k \right).$$

*3.2. Equivalent Transformations*

In this section, equivalence conditions of the consensus of fuzzy FOSPMASs are derived by addressing the stability problem of system (14).

Based on graph theory, it is known that matrix $M$ is positive definite. Therefore, an orthogonal matrix ($V$) exists such that $V^T M V = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, and all eigenvalues ($\lambda_i$) of the matrix ($M$) possess positive real parts, where $i = 1, 2, \cdots, N$.

Let $\tilde{X}_e(t) = \begin{bmatrix} X_e(t)^T & \hat{X}_e(t)^T \end{bmatrix}^T$, where $X_e = (V^T \otimes I_N) x_e(t)$ and $\hat{X}_e = (V^T \otimes I_N) \hat{x}_e(t)$. According to the properties of the Kronecker product, system (14) is transformed into the following form:

$$\tilde{E}(\varepsilon) D^{\alpha} \tilde{X}_{ei}(t) = \tilde{A} \tilde{X}_{ei}(t), \tag{15}$$

where

$$\tilde{E}(\varepsilon) = \begin{bmatrix} E(\varepsilon) & 0 \\ 0 & E(\varepsilon) \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix},$$
$$\tilde{A}_1 = \sum_{k=1}^{r} \eta_k(\xi) A_k + \lambda_i \sum_{k=1}^{r} \sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi) B_k K_q,$$
$$\tilde{A}_2 = - \lambda_i \sum_{k=1}^{r} \sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi) B_k K_q,$$
$$\tilde{A}_4 = \sum_{k=1}^{r} \eta_k(\xi) A_k - \lambda_i \sum_{s=1}^{r} \sum_{k=1}^{r} \eta_s(\xi)\eta_k(\xi) W_s C_k.$$

In order to analyze the stability of system (15), two independent SPSs are constructed as follows:

$$E(\varepsilon) D^{\alpha} X_{ei}(t) = \tilde{A}_1 X_{ei}(t), \tag{16}$$
$$E(\varepsilon) D^{\alpha} \hat{X}_{ei}(t) = \tilde{A}_4 \hat{X}_{ei}(t). \tag{17}$$

**Lemma 7.** *System (15) is stable iff systems (16) and (17) are both stable.*

**Proof.** According to Lemma 1, system (15) is stable iff $\left|\arg\left(\mathrm{spec}\left(\tilde{E}^{-1}(\varepsilon)\tilde{A}\right)\right)\right| > \frac{\pi}{2}\alpha$.

The characteristic determinant of system (15) is factorized as

$$\det\left(s^{\alpha}\tilde{E}(\varepsilon) - \tilde{A}\right) = \det\left(s^{\alpha}E(\varepsilon) - \tilde{A}_1\right) \times \det\left(s^{\alpha}E(\varepsilon) - \tilde{A}_4\right),$$

which implies that the stability of system (15) is dependent on the stability of two subsidiary systems ((16) and (17)). Thus, system (15) is stable iff systems (16) and (17) are simultaneously stable. □

**Definition 1.** *The consensus of T-S fuzzy FOSPMASs (6) and (7) is achieved via protocol (9) if*

$$\lim_{t\to\infty}\|x_i(t)-x_0(t)\|=0,\ i=1,2,\cdots,N.$$

According to Definition 1 and Lemma 7, in order to achieve the consensus of fuzzy FOSPMASs (6) and (7), it is necessary for systems (16) and (17) to both be stable. Therefore, in order to derive the stability conditions of (16) and (17), the following equivalent transformation is presented.

Matrix $E(\varepsilon)$ is decomposed into

$$E(\varepsilon)=E_1+(\varepsilon-\beta)E_2,$$

where the scalar is $\beta>0$, $E_1=\begin{bmatrix} I_{n_1} & 0 \\ 0 & \beta I_{n_2}\end{bmatrix}$, and $E_2=\begin{bmatrix} 0 & 0 \\ 0 & I_{n_2}\end{bmatrix}$.

Let $f_1(t)=D^\alpha X_{ei}(t)$ and $x_z(t)=\left[X_{ei}^T(t),f_1^T(t)\right]^T$. Then, (16) is derived as

$$ED^\alpha x_z(t)=\left(\sum_{k=1}^r\eta_k(\xi)A_{k1}+(\varepsilon-\beta)A_2+\lambda_i\sum_{k=1}^r\sum_{q=1}^r\eta_k(\xi)\eta_q(\xi)\bar{B}_k\bar{K}_q\right)x_z(t),\tag{18}$$

where $E=\begin{bmatrix} I_n & 0 \\ 0 & 0\end{bmatrix}$, $A_{k1}=\begin{bmatrix} 0 & I_n \\ A_k & -E_1\end{bmatrix}$, $A_2=\begin{bmatrix} 0 & 0 \\ 0 & -E_2\end{bmatrix}$, $\bar{B}_k=\begin{bmatrix} 0 \\ B_k\end{bmatrix}$, and $\bar{K}_q=\begin{bmatrix} K_q & 0\end{bmatrix}$.

Similarly, $f_2(t)=D^\alpha\hat{X}_{ei}(t)$ and $\hat{x}_z(t)=\left[\hat{X}_{ei}^T(t),f_2^T(t)\right]^T$. Then, (17) is reformulated as

$$ED^\alpha\hat{x}_z(t)=\left(\sum_{k=1}^r\eta_k(\xi)\bar{A}_{k1}+(\varepsilon-\beta)A_2-\lambda_i\sum_{s=1}^r\sum_{k=1}^r\eta_s(\xi)\eta_k(\xi)\bar{W}_s\bar{C}_k\right)\hat{x}_z(t),\tag{19}$$

where $\bar{A}_{k1}=\begin{bmatrix} 0 & A_k \\ I_n & -E_1\end{bmatrix}$, $\bar{W}_q=\begin{bmatrix} W_q \\ 0\end{bmatrix}$, and $\bar{C}_k=\begin{bmatrix} 0 & C_k\end{bmatrix}$.

**Lemma 8.** *With $\varepsilon>0$, system (16) is stable iff system (18) is admissible.*

**Proof.** Based on the aforementioned analysis, system (18) is reformulated as

$$ED^\alpha x_z(t)=\begin{bmatrix} 0 & I_n \\ \sum_{k=1}^r\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^r\sum_{q=1}^r\eta_k(\xi)\eta_q(\xi)B_kK_q & -E(\varepsilon)\end{bmatrix}x_z(t).\tag{20}$$

According to Lemma 3, when $U=V=I$ in (3), then system (20) is admissible iff $E(\varepsilon)$ is non-singular and

$$\left|\arg\left(\operatorname{spec}\left(E^{-1}(\varepsilon)\left(\sum_{k=1}^r\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^r\sum_{q=1}^r\eta_k(\xi)\eta_q(\xi)B_kK_q\right)\right)\right)\right|>\frac{\pi}{2}\alpha.\tag{21}$$

Then, system (16) is rewritten as

$$D^\alpha X_{ei}(t)=E^{-1}(\varepsilon)\left(\sum_{k=1}^r\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^r\sum_{q=1}^r\eta_k(\xi)\eta_q(\xi)B_kK_q\right)X_{ei}(t).$$

According to Lemma 1, and (21), system (16) is stable iff system (18) is admissible. □

According to Lemma 8, the stability conditions of systems (16) and (17) are interpreted as the admissibility conditions for systems (18) and (19).

*3.3. Consensus Conditions of T-S Fuzzy FOSPMASs*

In this section, the LMI criteria for the consensus of fuzzy FOSPMASs (6) and (7) are proposed by studying the admissibility of systems (18) and (19).

**Assumption 1.** *$\xi_j(t)$ and $|\partial\eta_k(\xi)/\partial\xi_j(t)|$ are in the range of $[0,\rho)$, and $\rho$ is a sufficiently small scalar.*

**Theorem 1.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMASs (6) and (7) with $\alpha \in [1,2)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9) if there exist matrices ($P_1$, $P_2$, $H_q$, and $G_s$) and positive scalars ($\mu_1$ and $\mu_2$) such that*

$$EP_1 = P_1^T E^T \geqslant 0, \tag{22}$$

$$\Phi_{kk} < 0, k = 1, 2, \cdots, r, \tag{23}$$

$$\Phi_{kq} + \Phi_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{24}$$

$$E^T P_2 = P_2^T E \geqslant 0, \tag{25}$$

$$\Psi_{kk} < 0, k = 1, 2, \cdots, r, \tag{26}$$

$$\Psi_{ks} + \Psi_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{27}$$

*where*

$$\Phi_{kq} = \begin{bmatrix} \mathrm{sym}\{\Theta \otimes (A_{k1}P_1 + \lambda_i \bar{B}_k H_q)\} & I_2 \otimes P_1^T & \Theta \otimes \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_1 I_2 \end{bmatrix},$$

$$\Psi_{ks} = \begin{bmatrix} \mathrm{sym}\{\Theta \otimes (\bar{A}_{k1}^T P_2 - \lambda_i \bar{C}_k^T G_s^T)\} & I_2 \otimes P_2^T & \Theta \otimes \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_2 I_2 \end{bmatrix}.$$

*The gain matrices are derived as*

$$\bar{K}_q = H_q P_1^{-1},\ \bar{W}_s = P_2^{-T} G_s.$$

**Proof.** Then, (18) is reformulated with $\beta = \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}$ as

$$\begin{aligned} ED^\alpha x_z(t) &= \left(\sum_{k=1}^{r} \eta_k(\xi) A_{k1} + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) A_2 + \lambda_i \sum_{k=1}^{r}\sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi) \bar{B}_k \bar{K}_q\right) x_z(t) \\ &\triangleq \sum_{l=1}^{kq} \tau_l(x_z(t)) \mathbb{A}_l x_z(t). \end{aligned} \tag{28}$$

The Jacobian matrix of system (28) $J$ is

$$\partial \sum_{l=1}^{kq} \tau_l(x_z(t)) \mathbb{A}_l x_z(t) / \partial x_z(t) = \sum_{l=1}^{kq} \partial \tau_l(x_z(t)) / \partial x_z(t) \mathbb{A}_l x_z(t) + \tau_l(x_z(t)) \mathbb{A}_l.$$

Based on Assumption 1, the Jacobian matrix of system (28) is seen as a constant ($J = \sum_{l=1}^{kq} \tau_l \mathbb{A}_l$). Based on Lemma 2, system (28) is admissible at the equilibrium points if

$$|\arg(\mathrm{spec}(E, J))| > \alpha \frac{\pi}{2}. \tag{29}$$

According to $\eta_k(\xi) \geqslant 0$, (23) and (24) ensure that the following inequality holds:

$$\begin{bmatrix} \Delta_1 & I_2 \otimes P_1^T & \mu_1(\Theta \otimes A_2) \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_1 I_2 \end{bmatrix} < 0, \tag{30}$$

where

$$\Delta_1 = \text{sym}\left\{\Theta \otimes \left(\sum_{k=1}^{r} \eta_k(\xi) A_{k1} P_1 + \lambda_i \sum_{k=1}^{r}\sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi)\bar{B}_k H_q\right)\right\}.$$

By pre- and post-multiplying (30) by $\text{diag}\left(I, \frac{I}{\sqrt{\mu_1}}, \frac{I}{\sqrt{\mu_1}}\right)$, it is transformed as follows:

$$\begin{bmatrix} \Delta_1 & \frac{I_2 \otimes P_1^T}{\sqrt{\mu_1}} & \sqrt{\mu_1}(\Theta \otimes A_2) \\ * & -\frac{4}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -I_2 \end{bmatrix} < 0. \tag{31}$$

According to the Schur complement, (31) is equivalent to

$$\Delta_1 + \begin{bmatrix} \frac{I_2 \otimes P_1^T}{\sqrt{\mu_1}} & \sqrt{\mu_1}(\Theta \otimes A_2) \end{bmatrix} \begin{bmatrix} \frac{(\bar{\varepsilon}-\underline{\varepsilon})^2}{4} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{I_2 \otimes P_1}{\sqrt{\mu_1}} \\ \sqrt{\mu_1}(\Theta \otimes A_2)^T \end{bmatrix} < 0. \tag{32}$$

Considering $\left|\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right| \leqslant \frac{\bar{\varepsilon}-\underline{\varepsilon}}{2}$ in (32) and Lemma 6 yields

$$\Delta_1 + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right)\text{sym}\{\Theta \otimes A_2 P_1\} < 0. \tag{33}$$

Substituting $\sum_{q=1}^{r} \eta_q(\xi) H_q = \sum_{q=1}^{r} \eta_q(\xi)\bar{K}_q P_1$ into (33) yields the following expression:

$$\text{sym}\left\{\Theta \otimes \left(\left(\sum_{k=1}^{r} \eta_k(\xi) A_{k1} + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) A_2 + \lambda_i \sum_{k=1}^{r}\sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi)\bar{B}_k\bar{K}_q\right) P_1\right)\right\} < 0. \tag{34}$$

According to Lemma 4, (29) is guaranteed, and system (28) with $\alpha \in [1,2)$ is admissible according to (22) and (34).

Similarly, substituting $\beta = \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}$ into system (19) yields

$$ED^{\alpha}\hat{x}_z(t) = \left(\sum_{k=1}^{r} \eta_k(\xi)\bar{A}_{k1} + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) A_2 - \lambda_i \sum_{s=1}^{r}\sum_{k=1}^{r} \eta_s(\xi)\eta_k(\xi)\bar{W}_s\bar{C}_k\right)\hat{x}_z(t). \tag{35}$$

Based on $\eta_k(\xi) \geqslant 0$, the following inequality is derived from (26) and (27):

$$\begin{bmatrix} \Delta_2 & I_2 \otimes P_2^T & \mu_2(\Theta \otimes A_2^T) \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_2 I_2 \end{bmatrix} < 0, \tag{36}$$

where

$$\Delta_2 = \text{sym}\left\{\Theta \otimes \left(\sum_{k=1}^{r} \eta_k(\xi)\bar{A}_{k1}^T P - \lambda_i \sum_{s=1}^{r}\sum_{k=1}^{r} \eta_s(\xi)\eta_k(\xi)\bar{C}_k^T G_s^T\right)\right\}.$$

By pre- and post-multiplying (36) by $\operatorname{diag}\left(I, \frac{I}{\sqrt{\mu_2}}, \frac{I}{\sqrt{\mu_2}}\right)$, it is transformed as

$$\begin{bmatrix} \Delta_2 & \frac{I_2 \otimes P_2^T}{\sqrt{\mu_2}} & \sqrt{\mu_2}(\Theta \otimes A_2^T) \\ * & -\frac{4}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -I_2 \end{bmatrix} < 0. \tag{37}$$

In the same way, (37) is equivalent to

$$\Delta_2 + \begin{bmatrix} \frac{I_2 \otimes P_2^T}{\sqrt{\mu_2}} & \sqrt{\mu_2}(\Theta \otimes A_2^T) \end{bmatrix} \begin{bmatrix} \frac{(\bar{\varepsilon}-\underline{\varepsilon})^2}{4} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{I_2 \otimes P_2}{\sqrt{\mu_2}} \\ \sqrt{\mu_2}(\Theta \otimes A_2^T)^T \end{bmatrix} < 0. \tag{38}$$

Lemma 6 and $\left|\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right| \leqslant \frac{\bar{\varepsilon}-\underline{\varepsilon}}{2}$ yield

$$\Delta_2 + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) \operatorname{sym}\left\{\Theta \otimes A_2^T P_2\right\} < 0. \tag{39}$$

Substituting $\sum_{s=1}^{r} \eta_s(\xi) G_s = P_2^T \sum_{q=1}^{r} \eta_q(\xi) \bar{W}_s$ into (39) yields

$$\operatorname{sym}\left\{\Theta \otimes \left(\left(\sum_{k=1}^{r} \eta_k(\xi) \bar{A}_{k1} + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) A_2 - \lambda_i \sum_{s=1}^{r} \sum_{k=1}^{r} \eta_s(\xi) \eta_k(\xi) \bar{C}_k \bar{W}_s\right)^T P_2\right)\right\} < 0. \tag{40}$$

Given the equivalence between the admissibility of $(E, A, \alpha)$ and that of $(E^T, A^T, \alpha)$, according to (25) and (40), it is deduced that system (35) with $\alpha \in [1,2)$ is admissible according to Lemma 4. In summary, (6) and (7) achieve leader-following consensus via observer-based protocol (9). □

**Remark 2.** The conditions in Theorem 1 involve LMIs with equality constraints, rendering them fragile and potentially prone to computational difficulty. Consequently, the subsequent theorem presents strict LMI conditions to address these issues and enhance computational accuracy.

**Theorem 2.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMASs (6) and (7) with $\alpha \in [1,2)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9) if there exist matrices ($X_1 > 0$, $X_2 > 0$, $Y_1$, $Y_2$, $H_q$, and $G_s$) and positive scalars ($\mu_1$ and $\mu_2$) such that*

$$\Gamma_{kk} < 0, k = 1, 2, \cdots, r, \tag{41}$$

$$\Gamma_{kq} + \Gamma_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{42}$$

$$\Upsilon_{kk} < 0, k = 1, 2, \cdots, r, \tag{43}$$

$$\Upsilon_{ks} + \Upsilon_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{44}$$

*where*

$$\Gamma_{kq} = \begin{bmatrix} \operatorname{sym}\{\Theta \otimes (A_{k1}(X_1 E^T + S_1 Y_1) + \lambda_i \bar{B}_k H_q)\} & I_2 \otimes (X_1 E^T + S_1 Y_1)^T & \Theta \otimes \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_1 I_2 \end{bmatrix},$$

$$\Upsilon_{ks} = \begin{bmatrix} \operatorname{sym}\{\Theta \otimes (\bar{A}_{k1}^T (X_2 E + S_2 Y_2) - \lambda_i \bar{C}_k^T G_s^T)\} & I_2 \otimes (X_2 E + S_2 Y_2)^T & \Theta \otimes \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_2 I_2 \end{bmatrix}.$$

*$S_1$, $S_2 \in \mathbb{R}^{n\times(n-m)}$ are arbitrary matrices with full column ranks satisfying $ES_1 = 0$ and $E^T S_2 = 0$. The gain matrices are derived as*

$$\bar{K}_q = H_q\left(X_1E^T + S_1Y_1\right)^{-1}, \ \bar{W}_s = (X_2E + S_2Y_2)^{-T}G_s.$$

**Proof.** Assume that there exist matrices ($X_1 > 0$, $X_2 > 0$, $Y_1$, $Y_2$, $H_q$, and $G_q$) and scalars ($\mu_1 > 0$ and $\mu_2 > 0$) such that (41) and (44) hold. Let

$$P_1 = X_1E^T + S_1Y_1 \ , \ P_2 = X_2E + S_2Y_2.$$

Then, using (41) and (44), it is easy to verify $P_1$, $P_2$, $H_q$, and $G_q$ and scalars $\mu_1 > 0$ and $\mu_2 > 0$ satisfying (22) and (27). Therefore, according to Theorem 1, (6) and (7) achieve consensus. □

**Theorem 3.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMASs (6) and (7) with $\alpha \in (0,1)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9) if there exist matrices ($X_1, X_2, Y_1, Y_2$, $H_q$, and $G_s$) and positive scalars ($\mu_1$ and $\mu_2$) such that*

$$\begin{bmatrix} EX_1 & EY_1 \\ -EY_1 & EX_1 \end{bmatrix} = \begin{bmatrix} X_1^TE^T & -Y_1^TE^T \\ Y_1^TE^T & X_1^TE^T \end{bmatrix} \geqslant 0, \tag{45}$$

$$\Pi_{kk} < 0, k = 1, 2, \cdots, r, \tag{46}$$

$$\Pi_{kq} + \Pi_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{47}$$

$$\begin{bmatrix} E^TX_2 & E^TY_2 \\ -E^TY_2 & E^TX_2 \end{bmatrix} = \begin{bmatrix} X_2^TE & -Y_2^TE \\ Y_2^TE & X_2^TE \end{bmatrix} \geqslant 0, \tag{48}$$

$$\Omega_{kk} < 0, k = 1, 2, \cdots, r, \tag{49}$$

$$\Omega_{ks} + \Omega_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{50}$$

*where*

$$\Pi_{kq} = \begin{bmatrix} \mathrm{sym}\{A_{k1}(aX_1 - bY_1) + \lambda_i \bar{B}_k H_q\} & (aX_1 - bY_1)^T & \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I & 0 \\ * & * & -\mu_1 I \end{bmatrix},$$

$$\Omega_{ks} = \begin{bmatrix} \mathrm{sym}\{\bar{A}_{k1}^T(aX_2 - bY_2) - \lambda_i \bar{C}_k^T G_s^T\} & (aX_2 - bY_2)^T & \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I & 0 \\ * & * & -\mu_2 I \end{bmatrix}.$$

*The the gain matrices are chosen as*

$$\bar{K}_q = H_q(aX_1 - bY_1)^{-1}, \ \bar{W}_s = (aX_2 - bY_2)^{-T}G_s.$$

**Proof.** The proof parallels that of Theorem 1 based on Lemma 5, which is omitted here for brevity. □

**Theorem 4.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMASs (6) and (7) with $\alpha \in (0,1)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9) if there exist matrices ($X_1$, $X_2$, $Y_1$, $Y_2$, $Q_1$, $Q_2$, $H_q$, and $G_s$), and positive scalars ($\mu_1$ and $\mu_2$) such that*

$$\begin{bmatrix} X_1 & Y_1 \\ -Y_1 & X_1 \end{bmatrix} > 0, \tag{51}$$

$$\Xi_{kk} < 0, k = 1, 2, \cdots, r, \tag{52}$$

$$\Xi_{kq} + \Xi_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{53}$$

$$\begin{bmatrix} X_2 & Y_2 \\ -Y_2 & X_2 \end{bmatrix} > 0, \tag{54}$$

$$\Lambda_{kk} < 0, k = 1, 2, \cdots, r, \tag{55}$$

$$\Lambda_{ks} + \Lambda_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{56}$$

*where*

$$\Xi_{kq} = \begin{bmatrix} \mathrm{sym}\{A_{k1}((aX_1 - bY_1)E^T + S_1Q_1) + \lambda_i \bar{B}_k H_q\} & \left((aX_1 - bY_1)E^T + S_1Q_1\right)^T & \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\varepsilon)^2} I & 0 \\ * & * & -\mu_1 I \end{bmatrix},$$

$$\Lambda_{ks} = \begin{bmatrix} \mathrm{sym}\{\bar{A}_{k1}^T((aX_2 - bY_2)E + S_2Q_2) - \lambda_i \bar{C}_k^T G_s^T\} & ((aX_2 - bY_2)E + S_2Q_2)^T & \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\varepsilon)^2} I & 0 \\ * & * & -\mu_2 I \end{bmatrix}.$$

*$S_1$ and $S_2$ satisfy the conditions in Theorem 2. The gain matrices are chosen as*

$$\bar{K}_q = H_q\left((aX_1 - bY_1)E^T + S_1Q_1\right)^{-1}, \ \bar{W}_s = ((aX_2 - bY_2)E + S_2Q_2)^{-T} G_s.$$

**Proof.** Assume that there exist matrices ($X_1$, $X_2$, $Y_1$, $Y_2$, $Q_1$, $Q_2$, $H_q$, and $G_q$) and scalars ($\mu_1 > 0$ and $\mu_2 > 0$) such that (51)–(56) hold. Let

$$\bar{X}_1 = X_1E^T + a^{-1}S_1Q_1, \ \bar{Y}_1 = Y_1E^T, \ \bar{X}_2 = X_2E + a^{-1}S_2Q_2, \ \bar{Y}_2 = Y_2E.$$

Using (51) and (56), it is easy to verify $\bar{X}_1$, $\bar{X}_2$, $\bar{Y}_1$, $\bar{Y}_2$, $H_q$, and $G_q$ and scalars $\mu_1 > 0$ and $\mu_2 > 0$, satisfying (45) and (50). Therefore, according to Theorem 3, the consensus of (6) and (7) is achieved. □

**Remark 3.** Table 1 shows that this paper focuses on FOSPMASs with $\alpha \in (0,1)$ or $[1,2)$, which are more complex and have a wider range of order $\alpha$. Some early literature, including [30,36], decomposed SPSs into fast subsystems and slow subsystems. However, this approach needs to assume that the fast subsystem matrix is non-singular, which is not applicable to non-standard SPSs. In contrast, the method proposed in this paper is based on a full-order model, which overcomes the limitation of decomposition and avoids the above assumption.

**Table 1.** Comparison of existing methods.

| Ref. | Range of $\alpha$ | Observer | Fuzzy | MASs | SPSs | Non-Limitations to SPSs |
|---|---|---|---|---|---|---|
| [30] | 1 | × | ✓ | × | ✓ | × |
| [34] | 1 | ✓ | × | × | ✓ | ✓ |
| [36] | 1 | × | × | × | ✓ | × |
| [38] | 2 | ✓ | × | × | ✓ | ✓ |
| [39] | (0, 1) | ✓ | × | ✓ | ✓ | ✓ |
| [45] | (0, 1) | × | ✓ | × | ✓ | ✓ |
| [52] | (0, 2) | ✓ | ✓ | ✓ | × | - |
| [53] | (0, 1) | × | ✓ | × | × | - |
| [54] | (0, 1) | × | × | × | ✓ | ✓ |
| Ours | (0, 1), [1, 2) | ✓ | ✓ | ✓ | ✓ | ✓ |

## 4. Numerical Examples

This section presents two demonstrative instances that highlight the effectiveness of the control protocol in achieving the consensus of fuzzy FOSPMASs with an orders of $\alpha$ in $(0,1)$ and $[1,2)$.

**Example 1.** *The capacitor and inductor have fractional characteristics in the circuit; the inductance value is very small and prone to pathological problems. The volt-ampere characteristic of the diode is nonlinear. This nonlinear circuit model is taken as a node of an MAS. With parasitic parameters and fractional-order characteristics, the MAS is very complex, and the consensus problem is difficult to solve. Thus, by approximating the volt-ampere characteristic as a quadratic function and constructing a T-S fuzzy model, Theorem 4 is used to achieve the consensus of the nonlinear FOSPMASs.*

*Consider a T-S fuzzy FOSPMAS composed of one leader and four followers; the behavior of each agent is described by the fractional-order RLC circuit model as shown in Figure 2.*

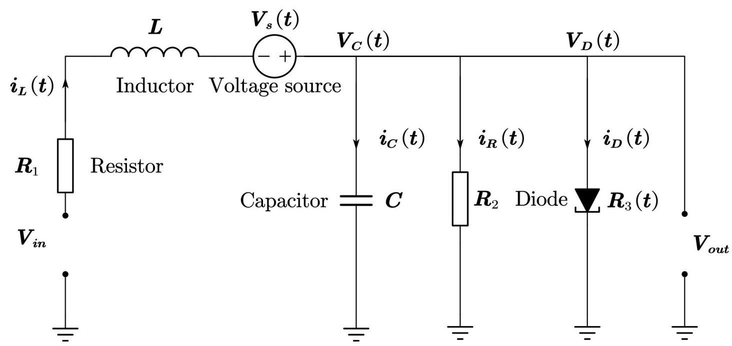


**Figure 2.** RLC circuit with a diode.

*The capacitor and inductor have fractional characteristics with an order of $\alpha$. L represents a very small parasitic inductance. $R_1$, and $R_2$ denote the resistances of the corresponding resistors. $R_3$ is a diode, and its characteristic function is approximated as $R_3 = 1/0.4 + 0.15V_D^2(t)$. It is known that the relationships between voltages are $V_D(t) = V_C(t)$ and $V_S(t) = -V_C(t)$. The dynamic of each agent is subsequently described as follows:*

$$\begin{cases} CD^{\alpha}V_C(t) = -\frac{V_C(t)}{R_2} - \frac{V_C(t)}{R_3} + i_L(t) \\ LD^{\alpha}i_L(t) = V_S(t) - V_C(t) + R_1 i_L(t) + u(t) \end{cases} \tag{57}$$

*Let $x_{i1}(t) = V_C(t)$, $x_{i2}(t) = i_L(t)$, and $\varepsilon = L$; then, the circuit model (57) is reformulated as follows:*

$$\begin{bmatrix} D^{\alpha}x_{i1}(t) \\ \varepsilon D^{\alpha}x_{i2}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_2C} - \frac{1}{C}(0.4 + 0.15x_{i1}^2(t)) & \frac{1}{C} \\ -2 & R_1 \end{bmatrix} \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \tag{58}$$

*where $i = 1,2,3,4$.*

*The parameters are chosen as $C = 0.3$ F, $\varepsilon = 0.02$ H, $R_1 = 0.1\ \Omega$, $R_2 = 0.5\ \Omega$, and $\alpha = 0.5$. It is assumed that $x_{11}(t)$ belongs to $[-2,2]$. Subsequently, the fuzzy rules are set as follows:*

*Rule 1: If the value of $x_{11}(t)$ is approximately 0, then*

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_1x_i(t) + B_1u_i(t) \\ y_i(t) = C_1x_i(t) \end{cases} \quad \text{and} \quad \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_1x_0(t) \\ y_0(t) = C_1x_0(t) \end{cases}$$

*Rule 2: If the value of $x_{11}(t)$ is approximately $\pm 2$, then*

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_2x_i(t) + B_2u_i(t) \\ y_i(t) = C_2x_i(t) \end{cases} \quad \text{and} \quad \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_2x_0(t) \\ y_0(t) = C_2x_0(t), \end{cases}$$

*where*

$$E(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, A_1 = \begin{bmatrix} -8 & 3 \\ -2 & 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} -10 & 3 \\ -2 & 0.1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, C_1 = C_2 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}.$$

*The fuzzy weighting function is selected as* $\eta_1(\xi(t)) = 1 - 0.25x_{11}^2(t)$, $\eta_2(\xi(t)) = 1 - \eta_1(\xi(t))$. *The matrix (M) is expressed as follows:*

$$M = \mathcal{L} + \text{diag}(h_1, h_2, 0, 0) = \begin{bmatrix} 3 & -0.5 & 0 & -0.5 \\ -0.5 & 3 & -0.5 & 0 \\ 0 & -0.5 & 2 & -0.5 \\ -0.5 & 0 & -0.5 & 2 \end{bmatrix}.$$

*Based on the modeling and analysis of the power system, the connection relationships and interactions of the system are abstracted into a undirected topology graph, as depicted in Figure 3.*

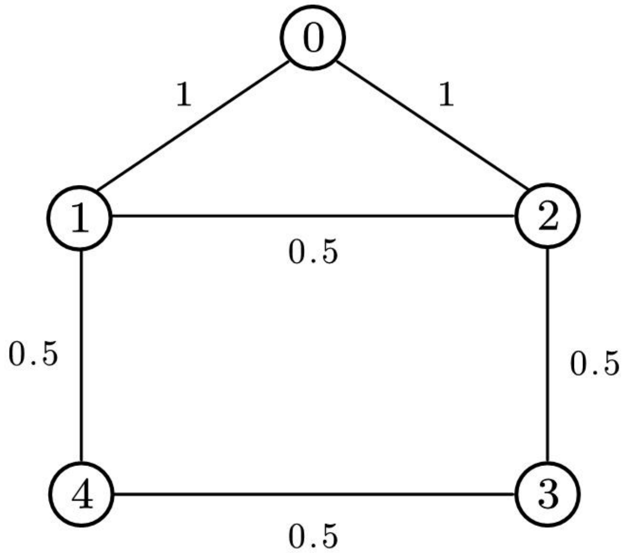


**Figure 3.** Weighted undirected graph.

*With* $\underline{\varepsilon} = 0.001$ *and* $\bar{\varepsilon} = 0.04$, *by solving LMIs (51) and (56) in Theorem 4, the feasible solutions are obtained as*

$$\bar{K}_1 = \begin{bmatrix} 0.4367 & -0.0916 & 0 & 0 \end{bmatrix},$$

$$\bar{K}_2 = \begin{bmatrix} 0.4371 & -0.0916 & 0 & 0 \end{bmatrix},$$

$$\bar{W}_1 = \begin{bmatrix} 1.4511 & 0.1933 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{W}_2 = \begin{bmatrix} 1.4562 & 0.1938 & 0 & 0 \end{bmatrix}^T.$$

*Let* $\varepsilon = 0.02$. *The tracking errors and estimation errors are depicted in Figures 4 and 5, respectively. A consensus error of zero means that each follower converges toward the leader, which demonstrates that the system achieves consensus by using the observer-based protocol (9), indicating the practical applicability and efficacy of the proposed method with* $\alpha \in [1, 2)$. *Model simulation is shown in the Appendix A.*

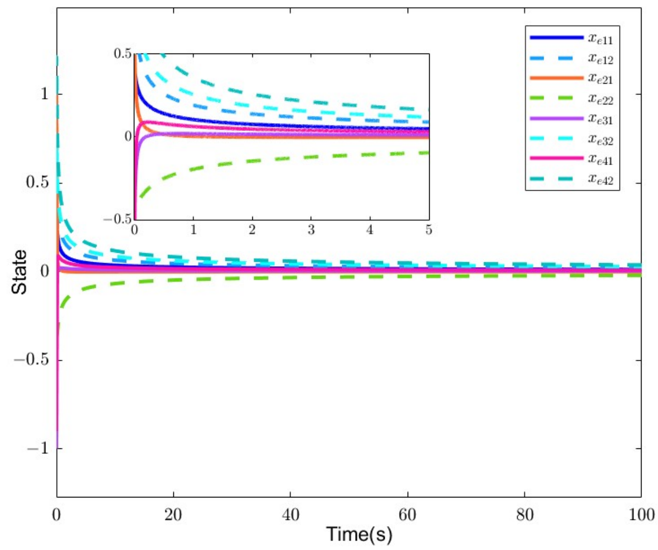


**Figure 4.** State errors between leader and followers in Example 1.

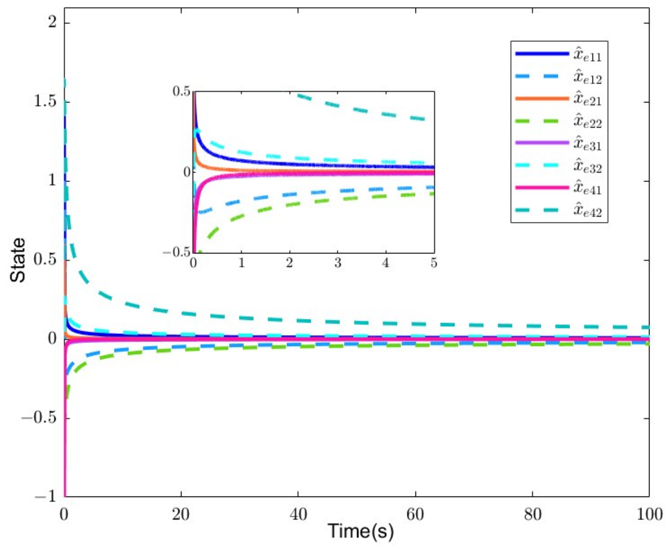


**Figure 5.** Estimation errors of followers in Example 1.

**Example 2.** *Considering T-S fuzzy FOSPMASs (6) and (7) within the topology presented in Figure 3, the fuzzy rules of the system are established in the following manner:*

*Rule 1: If $x_{e11}(t)$ is $\Pi_{k1}$, then*

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_1x_i(t) + B_1u_i(t) \\ y_i(t) = C_1x_i(t) \end{cases} \quad \text{and} \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_1x_0(t) \\ y_0(t) = C_1x_0(t) \end{cases}$$

*Rule 2: If $x_{e11}(t)$ is $\Pi_{k2}$, then*

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_2x_i(t) + B_2u_i(t) \\ y_i(t) = C_2x_i(t) \end{cases} \quad \text{and} \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_2x_0(t) \\ y_0(t) = C_2x_0(t). \end{cases}$$

*The remaining parameters are proposed as follows:*

$$\alpha = 1.2,\ E(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix},\ A_1 = \begin{bmatrix} -5 & 9 & 5 \\ 12 & -5 & 13 \\ 5 & -10 & -5 \end{bmatrix},\ A_2 = \begin{bmatrix} -4 & 5 & 5 \\ 5 & -18 & 15 \\ 5 & -10 & -5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}^T,\ B_2 = \begin{bmatrix} 0.5 & 0.5 & 1 \end{bmatrix}^T,$$
$$C_1 = \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.5 & 1 & 0.5 \end{bmatrix}.$$

*Assuming that state $x_{e11}(t)$ belongs to $[-1,1]$, the fuzzy weighting function is selected as $\eta_1(\xi(t)) = \cos^2(x_{e11})$, $\eta_2(\xi(t)) = 1 - \eta_1(\xi(t))$.*

*Considering $\underline{\varepsilon} = 0.001$ and $\bar{\varepsilon} = 0.03$, the feasible solutions are presented based on LMIs (41) and (44) in Theorem 2 as follows:*

$$\bar{K}_1 = \begin{bmatrix} 0.1193 & -0.2976 & -0.1530 & 0 & 0 & 0 \end{bmatrix},$$
$$\bar{K}_2 = \begin{bmatrix} -0.0783 & 0.1285 & 0.0181 & 0 & 0 & 0 \end{bmatrix},$$
$$\bar{W}_1 = \begin{bmatrix} -0.0009 & -0.5099 & 0.0147 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\bar{W}_2 = \begin{bmatrix} 0.0150 & 0.4288 & 0.0108 & 0 & 0 & 0 \end{bmatrix}^T.$$

*Figures 6 and 7 present the simulation results of the error system (14) with $\varepsilon = 0.02$. As depicted in Figure 6, the state of follower agents exhibits a successful tracking of the state of the leader agent, indicating that the consensus issue of fuzzy FOSPMASs (6) and (7) with $\alpha \in [1,2)$ is solved by the criteria in Theorem 2.*

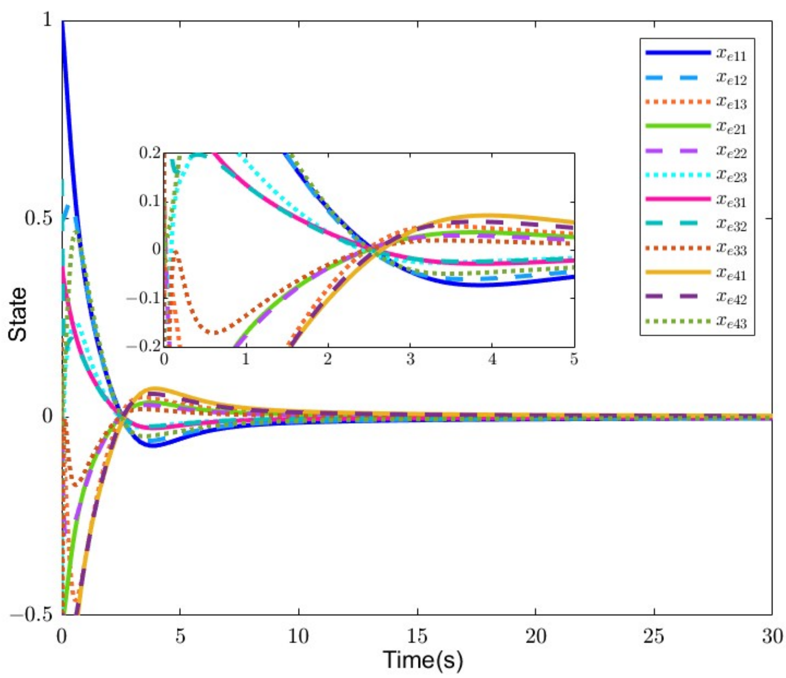


**Figure 6.** State errors between leader and followers in Example 2.

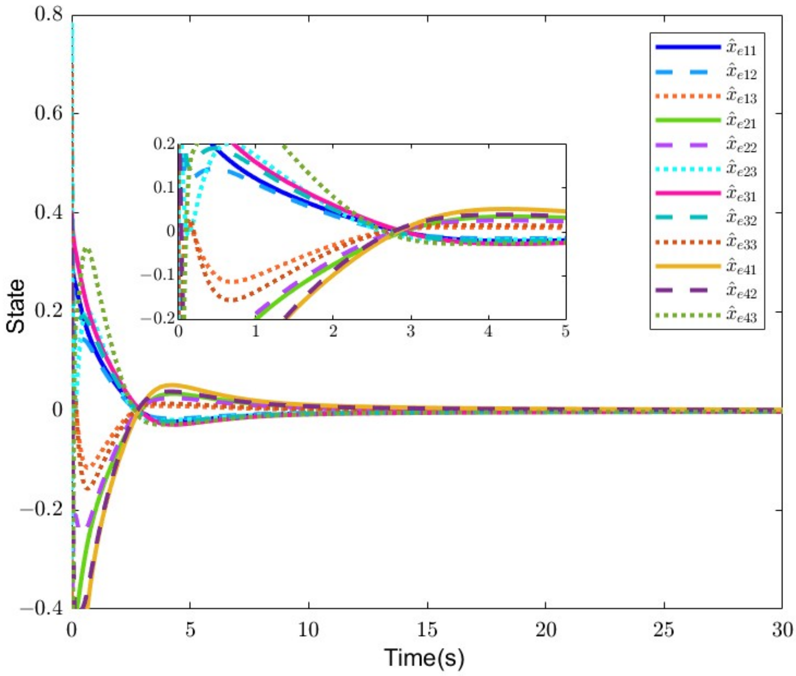


**Figure 7.** Estimation errors of followers in Example 2.

## 5. Conclusions

In this paper, a T-S fuzzy FOSPMAS with $\alpha \in (0,2)$ is modeled and studied to more accurately describe actual complex systems. The consensus problem of T-S fuzzy FOSPMASs is transformed into admissibility assessment of fuzzy SFOSs (18) and (19). In contrast to the methodologies in proposed in previous literature, the proposed method not only overcomes the pathological problem arising from multiple time scales but is also applicable to both standard and non-standard SPMASs. Theorems 1 and 3 provide sufficient conditions for achieving consensus with $\alpha \in (0,1)$ and $[1,2)$. Furthermore, strict LMI criteria are proposed in Theorems 2 and 4, which are solved easily with the LMI toolbox. In practical applications, the proposed method can be used to simulate the dynamic behavior of helicopter swarms in complex flight environments. There are still some difficulties in dealing with the consensus problem of fuzzy FOSPMASs with input saturation and actuator faults. In the future, the fault-tolerant control and fault detection problem of fuzzy FOSPMASs will be studied.

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**Conflicts of Interest:** The authors declare no conflicts of interest.

## Appendix A

The simulation of the fuzzy FOSPMAS model is depicted in Figure A1. This simulation model essentially encompasses fuzzy logic controllers, m functions, fractional-order operators, and integer-order integrators, the latter two of which are combined into fractional-order integrators with initial values. Notably, due to constraints in space, only a part of the simulation model pertaining to fuzzy FOSPMASs is shown. In practical cases, the precise number of state variables and fuzzy rules and the corresponding system description are tailored to suit actual scenarios or application requirements.

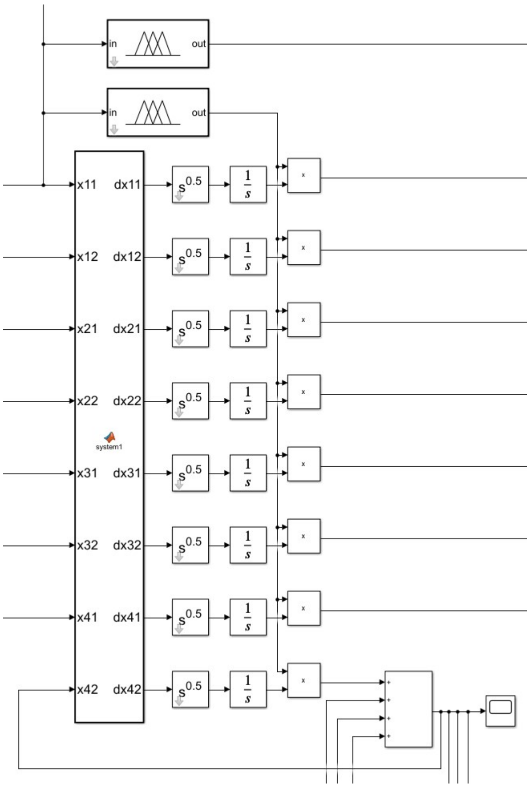


**Figure A1.** Fuzzy FOSPMAS model simulation.

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