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On One Point Singular Nonlinear Initial Boundary Value Problem for a Fractional Integro-Differential Equation via Fixed Point Theory

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Abstract: In this article, we focus on examining the existence, uniqueness, and continuous dependence of solutions on initial data for a specific initial boundary value problem which mainly arises from one-dimensional quasi-static contact problems in nonlinear thermo-elasticity. This problem concerns a fractional nonlinear singular integro-differential equation of order $\theta \in [0, 1]$. The primary methodology involves the application of a fixed point theorem coupled with certain a priori bounds. The feasibility of solving this problem is established under the context of data related to a weighted Sobolev space. Furthermore, an additional result related to the regularity of the solution for the formulated problem is also presented.

Keywords: integro-differential equation; fixed point theorem; fractional singular nonlinear problem; a priori bound; well-posedness; regularity of solution

MSC: 35L10; 35L20; 35L70

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1. Introduction

In recent years, both linear and nonlinear fractional as well as integer order partial differential equations have gained significant interest in theoretical and applied mathematics. This surge in attention, spanning over the last three decades, is primarily due to the instrumental role these equations play in modeling a great number of phenomena across various scientific and engineering disciplines. The inclusion of an integral component within these equations introduces a unique aspect, often interpreted as a memory or damping term, enriching the complexity and applicability of the models. The problem (1)–(3) models a general one-dimensional quasi-static contact problem in fractional thermo-elasticity (thermo-elasticity that uses the fractional heat equation with a Bessel operator). For ample information, the reader can refer to [1] and the references therein. We also mention that one-dimensional quasi-static contact problems for an integro-differential equation typically arise also in the context of solid mechanics, particularly when dealing with the contact between elastic or viscoelastic materials. The term quasi-static implies that the time-dependent effects are slow enough that inertial forces can be neglected, so the problem is treated as a series of static problems over time. We physically interpret the one-dimensional quasi-static contact problem by considering two bodies in contact along a single dimension (e.g., along a line or axis). The contact may involve compression, adhesion, friction, or other physical interactions between the two bodies. The integro-differential equation describes how the displacement (or other relevant physical quantities, such as stress or strain) varies with position along this dimension and over time, considering both local effects (differential terms) and nonlocal effects (integral terms). The study of integro-differential equations has thus become a focal point for researchers specializing in ordinary and partial differential equations, driven by the equations' extensive applications in diverse scientific domains.

These applications range from heat transfer to viscoelasticity, diffusion processes, and even epidemiology, as evidenced by numerous studies [2–14]. The profound impact of fractional integro-differential equations in physical and engineering sciences underscore their significance. The study of singular integro-differential equations has revealed a gap in various research outcomes, primarily due to the complexity and diverse nature of these equations. Singular integro-differential equations are characterized by their singularities, either in the coefficients, the integral part, or in the solution itself, making them significantly more challenging to analyze and solve compared to regular integro-differential equations. One of the key gaps is the inconsistency in theoretical results related to the existence and uniqueness of solutions. While some studies have successfully established these properties under certain conditions, others have found that slight variations in the equation or boundary conditions can lead to completely different outcomes. This variability is often attributed to the nature and type of singularities involved. Another gap is observed in the numerical methods used to solve these equations. Different approaches yield varying degrees of accuracy and efficiency, and in some cases, certain methods fail to converge or produce reliable results. Furthermore, the application of singular integro-differential equations in modeling real-world phenomena also presents discrepancies. Models developed for similar phenomena using these equations sometimes yield divergent results, reflecting the sensitivity of the equations to initial conditions and parameter values. In summary, the gaps in research on singular integro-differential equations are evident in the theoretical understanding of their properties, the efficacy and reliability of numerical methods, and the application of these equations in practical situations. For an in-depth understanding of the qualitative properties of solutions of fractional integro-differential equations, especially those with local and nonlocal boundary conditions encompassing aspects like existence, uniqueness, continuous dependence, stability, and controllability, a comprehensive review of the literature is available in references [15–28]. This body of work provides essential insights into the complex nature of these equations and their practical implications in various fields.

Recent studies have made significant progress in understanding the well-posedness of nonlocal initial boundary value problems for singular integro-differential equations. These studies contribute significantly to the understanding of nonlocal initial boundary value problems for singular integro-differential equations, providing insights into their well-posedness and solution methodologies [29–31].

In the rectangle $Q_T = (0,1) \times [0,T]$, where $0 < T < \infty$, we consider the fractional nonlinear singular second-order integro-differential equation

$$\partial_t^\sigma \theta - \frac{\partial^2 \theta}{\partial x^2} - \frac{1}{x}\frac{\partial \theta}{\partial x} + \theta = \max\left(\int_0^x \eta \theta(\eta, t) d\eta, 0 \right) + \beta(x,t), \tag{1}$$

where $\partial_t^\sigma \theta$ indicates the right Caputo fractional derivative of order σ, $0 < \sigma \leq 1$ [32] given by

$$\partial_t^\sigma \theta = \frac{1}{\Gamma(1-\sigma)} \int_0^t \frac{\theta_\tau(\tau)}{(t-\tau)^\sigma} d\tau, \quad \forall t \in [0,T].$$

The Equation (1) is supplemented by the initial condition

$$\theta(x,0) = Z(x), \ \ x \in (0,1), \tag{2}$$

and the one-point boundary condition

$$\theta_x(1,t) = 0, \qquad t \in [0,T], \tag{3}$$

where $Z(x) \in W^1_{\rho,2}((0,1))$, and $\beta \in L^2\big(0,T; L^2(0,1)\big)$

In Section 2, we will introduce several function spaces and tools that will be frequently utilized in the subsequent sections. Section 3 focuses on proving the uniqueness of the solution for the given problem within a specific fractional Sobolev space. In Section 4, we demonstrate the existence of a solution, with the proof primarily relying on the Schauder fixed point theorem. Finally, in the last section, we derive an a priori bound, which can assist in establishing certain regularity results for the solution to the problem described in Equations (1)–(3).

2. Notations and Preliminaries

Lemma 1 ([33])**.** *Let a nonnegative absolutely continuous function $\mathcal{P}(t)$ satisfy the inequality*

$$ {}^{C}\partial_t^{\alpha}\mathcal{P}(t) \leq C\mathcal{P}(t) + k(t), \quad 0 < \alpha < 1, $$

for almost all $t \in [0,T]$, where C is positive and $k(t)$ is an integrable nonnegative function on $[0,T]$. Then,

$$ \mathcal{P}(t) \leq \mathcal{P}(0)E_{\alpha}(Ct^{\alpha}) + \Gamma(\alpha)E_{\alpha,\alpha}(Ct^{\alpha})D_t^{-\alpha}k(t), $$

where

$$ E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}, \text{ and } E_{\delta,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\delta n+\alpha)}, $$

are the Mittag–Leffler functions.

Lemma 2 ([33])**.** *For any absolutely continuous function $v(t)$ on $[0,T]$, the following inequality holds:*

$$ v(t)^{C}\partial_t^{\alpha}v(t) \geq \frac{1}{2}{}^{C}\partial_t^{\alpha}v^2(t), \quad 0 < \alpha < 1. $$

Lemma 3 ([33])**.** *For $F \in L^2\left(0,T;L^2{}_{\rho}(0,1)\right)$, the following inequality holds:*

$$ D_t^{-\alpha}\| F \|^2_{L^2{}_{\rho}((0,1))} \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \| F \|^2_{L^2{}_{\rho}((0,1))} d\tau. $$

To study the problem (1)–(3), we use some important function spaces: Let $L^2\left(0,T;L^2_{\gamma}(0.1)\right)$, $L^2\left(0,T;L^2_{\rho}(0.1)\right)$ be the weighted Hilbert spaces of square integrable functions on Q_T with $\gamma = x^2$, and $\rho = x$. The inner products in $L^2\left(0,T;L^2_{\gamma}(0.1)\right)$, $L^2\left(0,T;L^2_{\rho}(0.1)\right)$ are, respectively, denoted by $(.,.)_{L^2(0,T;L^2_{\gamma}(0.1))}$, $(.,.)_{L^2(0,T;L^2_{\rho}(0.1))}$ such that

$$ (u,v)_{L^2(0,T;L^2_{k(x)}(0.1))} = \int_0^T (u,v)_{L^2_{k(x)}(0.1)}dx, \ k(x) = \gamma, \rho. $$

We also introduce the fractional derivative spaces: the space $L^2\left(0,T;H^{\sigma}_{\rho}(0,1)\right)$ to be the space of functions $U \in L^2\left(0,T;L^2_{\rho}(0.1)\right)$ having σ-order Caputo derivative $\partial_t^{\sigma}U \in L^2\left(0,T;L^2_{\rho}(0.1)\right)$, having the norm

$$ \| U \|^2_{L^2(0,T;H^{\sigma}_{\rho}(0,1))} = \| U \|^2_{L^2(0,T;L^2_{\rho}(0.1))} + \| \partial_t^{\sigma}U \|^2_{L^2(0,T;L^2_{\rho}(0.1))}, $$

and the space $L^2\left(0,T;H_\gamma^{2,\sigma}(0,1)\right)$ to be the space of functions $V \in L^2\left(0,T;L_\gamma^2(0.1)\right)$ such that $V_x, V_{xx} \in L^2\left(0,T;L_\gamma^2(0.1)\right)$ and having σ-order Caputo derivative $\partial_t^\sigma V \in L^2\left(0,T;L_\gamma^2(0.1)\right)$, associated with the norm

$$\begin{aligned} \| V \|^2_{L^2(0,T;H_\gamma^{\sigma,2}(0,1))} &= \| \partial_t^\sigma V \|^2_{L^2(0,T;L_\gamma^2(0.1))} + \| V \|^2_{L^2(0,T;L_\gamma^2(0.1))} \\ &+ \| V_x \|^2_{L^2(0,T;L_\gamma^2(0.1))} + \| V_{xx} \|^2_{L^2(0,T;L_\gamma^2(0.1))}. \end{aligned}$$

The function spaces $L^2\left(0,T;H_\rho^\sigma(0,1)\right)$, and $L^2\left(0,T;H_\gamma^{2,\sigma}(0,1)\right)$ can be defined as the closure of $C^\infty\left(0,T;L^2(0.1)\right)$ with respect to the norms (2) and (4), respectively. We denote by $C\left(0,T;L^2(0,1)\right)$ the Banach space of the set of functions $U:[0,T] \to L^2(0.1)$ equipped with the norm

$$\| U \|_{C([0,T;L^2(0.1))} = \max_{0\le t\le T} \| U(.,t) \|_{L^2(0,1)}.$$

Let $W^1_{\gamma,2}(0,1)$ be the set of functions ξ such that $\xi,\ \xi_x \in L^2_\gamma(0,1)$ with the norm

$$\| u \|^2_{W^1_{\gamma,2}(0,1)} = \| u \|^2_{L^2_\gamma(0.1)} + \| u_x \|^2_{L^2_\gamma(0.1)}.$$

The following inequalities are needed:

(1) Cauchy $\varepsilon-$inequality which holds for all $\varepsilon > 0$ and for arbitrary λ and μ

$$\lambda\mu \le \frac{\varepsilon}{2}\lambda^2 + \frac{1}{2\varepsilon}\mu^2, \tag{4}$$

(2) A Poincaré type inequality (see [34]).

$$\begin{cases} \| \Lambda_x(\xi u) \|^2_{L^2(0,1)} \le \frac{1}{2} \| u \|^2_{L^2_\rho(Q_T)}, \\ \| \Lambda_x^2(\xi u) \|^2_{L^2(0,1)} \le \frac{1}{2} \| \Lambda_x(\xi u) \|^2_{L^2(Q_T)}, \end{cases} \tag{5}$$

where

$$\Lambda_x(\xi u) = \int_0^x \xi u(\xi,t)d\xi.$$

(3) Gronwall's Belman inequality (see [28] Lemma 4.1).

3. Uniqueness of Solution

Theorem 1. *Let $Z \in W^1_{\rho,2}((0,1))$, and $\beta \in L^2\left(0,T;L^2_\rho(0,1)\right)$. Then, the posed problem (1)–(3) has at most one solution in $L^2(0,T;\ H^\sigma_\rho((0,1))$, if it exists.*

Proof. Let V_1 and V_2 be two solutions of problem (1)–(3), and let $\eta(x,t) = V_1(x,t) - V_2(x,t)$. Then, η satisfies the problem

$$\mathcal{L}\eta = \partial_t^\sigma \eta - \frac{\partial^2\eta}{\partial x^2} - \frac{1}{x}\frac{\partial\eta}{\partial x} + \eta = \gamma_1(x,t) - \gamma_2(x,t), \tag{6}$$

$$\eta_x(1,t) = 0,\ \ t \in (0,T), \tag{7}$$

$$\eta(x,0) = 0, \quad x \in (0,1), \tag{8}$$

where

$$\gamma_j(x,t) = \max\left(\int_0^x \eta V_j(\eta,t)d\eta, 0\right),\ \ j = 1,2. \tag{9}$$

By direct calculation, we have

$$
\begin{aligned}
&(\partial_t^\sigma \eta, \mathcal{L}\eta)_{L^2_\rho(0,1)} = \left(\partial_t^\sigma \eta, \partial_t^\sigma \eta - \tfrac{1}{x}\tfrac{\partial}{\partial x}\left(x\tfrac{\partial \eta}{\partial x}\right) + \eta\right)_{L^2_\rho(0,1)} \\
= \;& \| \partial_t^\sigma \eta \|^2_{L^2_\rho(0,1)} - \left(\partial_t^\sigma \eta, \tfrac{\partial}{\partial x}\left(x\tfrac{\partial \eta}{\partial x}\right)\right)_{L^2(0,1)} + (\partial_t^\sigma \eta, \eta)_{L^2_\rho(0,1)} \\
= \;& \| \partial_t^\sigma \eta \|^2_{L^2_\rho(0,1)} + (\partial_t^\sigma \eta, \eta)_{L^2_\rho(0,1)} + (\partial_t^\sigma \eta_x, \eta_x)_{L^2_\rho(0,1)} \\
= \;& (\partial_t^\sigma \eta, \gamma_1 - \gamma_2)_{L^2_\rho(0,1)}.
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
(\eta, \mathcal{L}\eta)_{L^2_\rho(0,1)} &= (\partial_t^\sigma \eta, \eta)_{L^2_\rho(0,1)} - \left(\eta, \tfrac{\partial}{\partial x}\left(x\tfrac{\partial \eta}{\partial x}\right)\right)_{L^2(0,1)} + \| \eta \|^2_{L^2_\rho(0,1)} \\
&= (\partial_t^\sigma \eta, \eta)_{L^2_\rho(0,1)} + \| \eta \|^2_{L^2_\rho(0,1)} + \| \eta_x \|^2_{L^2_\rho(0,1)} \\
&= (\eta, \gamma_1 - \gamma_2)_{L^2_\rho(0,1)}.
\end{aligned} \tag{11}
$$

Summing (10), and (11) yields

$$
\begin{aligned}
&\| \partial_t^\sigma \eta \|^2_{L^2_\rho(0,1)} + \| \eta \|^2_{L^2_\rho(0,1)} + \| \eta_x \|^2_{L^2_\rho(0,1)} \\
&+2(\partial_t^\sigma \eta, \eta)_{L^2_\rho(0,1)} + (\partial_t^\sigma \eta_x, \eta_x)_{L^2_\rho(0,1)} \\
= \;& (\partial_t^\sigma \eta, \gamma_1 - \gamma_2)_{L^2_\rho(0,1)} + (\eta, \gamma_1 - \gamma_2)_{L^2_\rho(0,1)}.
\end{aligned} \tag{12}
$$

Aplication of Lemma 2 to the last two terms on the left-hand side of Equation (12) yields

$$
2\int_0^1 x\eta(x,t)\ \partial_t^\sigma \eta(x,t)dx \ \geq \ \int_0^1 x\partial_t^\sigma \eta^2(x,t)dx = \partial_t^\sigma \int_0^1 x\eta^2(x,t)dx.
$$

That is,

$$
2(\partial_t^\sigma \eta, \eta)_{L^2_\rho(0,1)} \geq \partial_t^\sigma \| \eta \|^2_{L^2_\rho(0,1)}.
$$

Similarly,

$$
(\partial_t^\sigma \eta_x, \eta_x)_{L^2_\rho(0,1)} \geq \partial_t^\sigma \| \eta_x \|^2_{L^2_\rho(0,1)}.
$$

We now apply Cauchy $\varepsilon-$ inequality to the right-hand side of Equation (12), and using the two precedent inequalities, we obtain

$$
\begin{aligned}
&\| \partial_t^\sigma \eta \|^2_{L^2_\rho(0,1)} + \| \eta \|^2_{L^2_\rho(0,1)} + \tfrac{1}{2}\| \eta_x \|^2_{L^2_\rho(0,1)} \\
&+\partial_t^\sigma \| \eta \|^2_{L^2_\rho(0,1)} + \partial_t^\sigma \| \eta_x \|^2_{L^2_\rho(0,1)} \\
\leq \;& \tfrac{\delta_1}{2}\| \partial_t^\sigma \eta \|^2_{L^2_\rho(0,1)} + \tfrac{1}{2\delta_1}\| \gamma_1 - \gamma_2 \|^2_{L^2_\rho(0,1)} \\
& \tfrac{\delta_2}{2}\| \eta \|^2_{L^2_\rho(0,1)} + \tfrac{1}{2\delta_2}\| \gamma_1 - \gamma_2 \|^2_{L^2_\rho(0,1)}.
\end{aligned} \tag{13}
$$

We can easily show that

$$
\| \gamma_1 - \gamma_2 \|^2_{L^2_\rho(0,1)} \leq \frac{1}{2}\| \eta \|^2_{L^2_\rho(0,1)}. \tag{14}
$$

By evoking (14) and choosing $\delta_1 = 1$, and $\delta_2 = 2$, the inequality (13) can be reduced to

$$
\begin{aligned}
&\| \partial_t^\sigma \eta \|^2_{L^2_\rho(0,1)} + \| \eta_x \|^2_{L^2_\rho(0,1)} + \partial_t^\sigma \| \eta_x \|^2_{L^2_\rho(0,1)} + \partial_t^\sigma \| \eta \|^2_{L^2_\rho(0,1)} \\
\leq \;& \tfrac{3}{8}\| \eta \|^2_{L^2_\rho(0,1)}.
\end{aligned} \tag{15}
$$

We infer from the inequality (15) that

$$\begin{aligned} & \| \partial_t^\sigma \eta \|^2_{L^2(0,t;L^2_\rho(0,1))} + \| \eta_x \|^2_{L^2(0,t;L^2_\rho(0,1))} + D_t^{\sigma-1} \| \eta_x \|^2_{L^2_\rho(0,1)} \\ & D_t^{\sigma-1} \| \eta \|^2_{L^2_\rho(0,1)} \\ \leq \ & \tfrac{3}{8} \int_0^t \| \eta(x,\tau) \|^2_{L^2_\rho(0,1)} d\tau. \end{aligned} \tag{16}$$

By ignoring the first three terms, applying Lemma 1, and taking

$$h(t) = \int_0^t \| \eta(x,\tau) \|^2_{L^2_\rho(0,1)} d\tau, \ \partial_t^\sigma h = D_t^{\sigma-1} \| \eta \|^2_{L^2_\rho(0,1)}, \ h(0) = 0, \tag{17}$$

we have

$$h(t) \leq \frac{3}{8} \Gamma(\sigma) E_{\sigma,\sigma} \left(\frac{3}{8} T^\sigma \right) D_t^{-\sigma}(0). \tag{18}$$

We infer from (17) and (18) that $\eta(x,t) = V_1(x,t) - V_2(x,t) = 0$ for all $t \in [0,T]$. This implies that $V_1(x,t) = V_2(x,t)$. Hence, we conclude the uniqueness of the solution of problem (1)–(3) in the fractional function space $L^2(0,T; H^\sigma_\rho((0,1))$. □

4. Existence of the Solution

Theorem 2. *Let $Z \in W^1_{\rho,2}((0,1))$, and $\beta \in L^2\left(0,T; L^2_\rho(0,1)\right)$ be given and satisfy*

$$\| Z \|^2_{W^1_{\rho,2}((0,1))} + \| \beta \|^2_{L^2(0,T;L^2_\rho(0,1))} \leq C_2, \tag{19}$$

for $C_2 > 0$ small enough and that

$$Z_x(1) = 0. \tag{20}$$

Then, problem (1)–(3) admit a unique solution $\theta \in L^2(0,T; H^\sigma_\rho((0,1))$.

Proof. Consider the class of functions

$$\Sigma(B) = \begin{cases} \theta \in L^2\left(0,T; L^2_\rho(0,1)\right), \ \| \theta \|_{L^2(0,T;H^\sigma_\rho((0,1))} \leq B, \\ \qquad \| \partial_t^\sigma \theta \|_{L^2(0,T;L^2_\rho(0,1))} \leq 2B, \Big\}. \end{cases} \tag{21}$$

where B is a positive constant. Then, for any $V \in \Sigma(B)$, we can solve the problem

$$\begin{cases} \partial_t^\sigma \theta - \frac{\partial^2 \theta}{\partial x^2} - \frac{1}{x} \frac{\partial \theta}{\partial x} + \theta = \mathcal{I}V + \beta(x,t), \\ \qquad \theta(x,0) = Z(x), \ x \in (0,1), \\ \qquad \theta_x(1,t) = 0, \qquad t \in (0,T), \end{cases} \tag{22}$$

where

$$\mathcal{I}V = \max\left(\int_0^x \eta V(\eta,t) d\eta, 0 \right). \tag{23}$$

Problem (22) has a unique solution $\theta \in L^2(0,T; H^\sigma_\rho((0,1))$ for any $V \in \Sigma(B)$; thus, we can define a mapping $\mathcal{P}$ such that $\theta = \mathcal{P}V$. If we prove that $\mathcal{P}$ has a fixed point θ in the closed bounded convex subset $\Sigma(B)$, then θ will be the solution of our problem (1)–(3). We first prove that $\mathcal{P}$ maps $\Sigma(B)$ to $\Sigma(B)$. Let $\theta = Y + J$, such that Y solves

$$\begin{cases} \mathcal{L}Y = \partial_t^\sigma Y - \frac{\partial^2 Y}{\partial x^2} - \frac{1}{x} \frac{\partial Y}{\partial x} + Y = \mathcal{I}V, \quad (x,t) \in Q \\ \qquad Y(x,0) = 0, \ Y_x(1,t) = 0. \end{cases} \tag{24}$$

and J solves

$$\begin{cases} \mathcal{L}J = \partial_t^\sigma J - \frac{\partial^2 J}{\partial x^2} - \frac{1}{x}\frac{\partial J}{\partial x} + J = \beta(x,t), \quad (x,t) \in Q \\ J(x,0) = Z(x), \;\; J_x(1,t) = 0. \end{cases} \tag{25}$$

Consider the scalar products in $L^2_\rho(0,1)$

$$(\mathcal{L}Y, M_1 Y)_{L^2_\rho(0,1)} = (\mathcal{I}V, M_1 Y)_{L^2_\rho(0,1)}, \tag{26}$$

$$(\mathcal{L}J, M_2 J)_{L^2_\rho(0,1)} = (\beta, M_2 J)_{L^2_\rho(0,1)}, \tag{27}$$

where

$$M_1 Y = x\partial_t^\sigma Y + xY + xY_x, \tag{28}$$

$$M_2 J = x\partial_t^\sigma J + xJ + xJ_x. \tag{29}$$

Equation (26) reads,

$$\begin{aligned} &\left(\partial_t^\sigma Y - \tfrac{1}{x}\left(x\tfrac{\partial Y}{\partial x}\right)_x + Y, \partial_t^\sigma Y\right)_{L^2_\rho(0,1)} + \left(\partial_t^\sigma Y - \tfrac{1}{x}\left(x\tfrac{\partial Y}{\partial x}\right)_x + Y, Y\right)_{L^2_\rho(0,1)} \\ &+\left(\partial_t^\sigma Y - \tfrac{1}{x}\left(x\tfrac{\partial Y}{\partial x}\right)_x + Y, Y_x\right)_{L^2_\rho(0,1)} \\ = \;& (\partial_t^\sigma Y + Y + Y_x, \mathcal{I}V)_{L^2_\rho(0,1)}. \end{aligned} \tag{30}$$

The initial and boundary conditions transform (30) into the equation

$$\begin{aligned} &\| \partial_t^\sigma Y \|^2_{L^2_\rho((0,1))} + (\partial_t^\sigma Y_x, Y_x)_{L^2_\rho((0,1))} + 2(\partial_t^\sigma Y, Y)_{L^2_\rho((0,1))} \\ &+\tfrac{1}{2}\| Y_x \|^2_{L^2_\rho((0,1))} + \| Y \|^2_{L^2_\rho((0,1))} + (\partial_t^\sigma Y, Y_x)_{L^2_\rho((0,1))} + (Y, Y_x)_{L^2_\rho((0,1))} \\ = \;& (\partial_t^\sigma Y + Y + Y_x, \mathcal{I}V)_{L^2_\rho(0,1)}. \end{aligned} \tag{31}$$

By evoking Lemma 2 and Young's inequality, we infer from equality (31) that

$$\begin{aligned} &\| \partial_t^\sigma Y \|^2_{L^2_\rho((0,1))} + \tfrac{1}{2}\partial_t^\sigma \| Y_x \|^2_{L^2_\rho((0,1))} + \partial_t^\sigma \| Y \|^2_{L^2_\rho((0,1))} \\ &+\tfrac{1}{2}\| Y_x \|^2_{L^2_\rho((0,1))} + \| Y \|^2_{L^2_\rho((0,1))} \\ \leq \;& \left(\tfrac{\gamma_1}{2} + \tfrac{\gamma_4}{2}\right)\| \partial_t^\sigma Y \|^2_{L^2_\rho((0,1))} + \left(\tfrac{\gamma_5}{2} + \tfrac{\gamma_2}{2}\right)\| Y \|^2_{L^2_\rho((0,1))} \\ &+\left(\tfrac{1}{2\gamma_5} + \tfrac{\gamma_3}{2} + \tfrac{1}{2\gamma_4}\right)\| Y_x \|^2_{L^2_\rho((0,1))} \\ &+\left(\tfrac{1}{2\gamma_1} + \tfrac{1}{2\gamma_3} + \tfrac{1}{2\gamma_2}\right)\| \mathcal{I}V \|^2_{L^2_\rho((0,1))}. \end{aligned} \tag{32}$$

After choosing $\gamma_1 = 1/2, \gamma_2 = 1/4, \gamma_3 = 1, \gamma_4 = 1, \gamma_5 = 1$, inequality (32) reduces to

$$\begin{aligned} &\| \partial_t^\sigma Y \|^2_{L^2_\rho((0,1))} + \| Y \|^2_{L^2_\rho((0,1))} + \partial_t^\sigma \| Y \|^2_{L^2_\rho((0,1))} + \partial_t^\sigma \| Y_x \|^2_{L^2_\rho((0,1))} \\ \leq \;& 14\left(\| Y_x \|^2_{L^2_\rho((0,1))} + \| \mathcal{I}V \|^2_{L^2_\rho((0,1))}\right). \end{aligned} \tag{33}$$

In the same fashion, we have

$$\begin{aligned} &\| \partial_t^\sigma J \|^2_{L^2_\rho((0,1))} + \| J \|^2_{L^2_\rho((0,1))} + \partial_t^\sigma \| J \|^2_{L^2_\rho((0,1))} + \partial_t^\sigma \| J_x \|^2_{L^2_\rho((0,1))} \\ \leq \;& 14\left(\| J_x \|^2_{L^2_\rho((0,1))} + \| \beta \|^2_{L^2_\rho((0,1))}\right). \end{aligned} \tag{34}$$

Integration of both sides of (33), and (34), respectively, yields the inequalities

$$\begin{aligned} &\int_0^t \left(\| \partial_\tau^\sigma Y(.,\tau) \|^2_{L^2_\rho((0,1))} + \| Y(.,\tau) \|^2_{L^2_\rho((0,1))} \right) d\tau \\ &+ D^{\sigma-1} \| Y \|^2_{L^2_\rho((0,1))} + D^{\sigma-1} \| Y_x \|^2_{L^2_\rho((0,1))} \\ \leq\ & 14 \left(\int_0^t \| Y_x(.,\tau) \|^2_{L^2_\rho((0,1))} d\tau + \int_0^t \| \mathcal{I}V \|^2_{L^2_\rho((0,1))} d\tau \right), \end{aligned} \tag{35}$$

$$\begin{aligned} &\int_0^t \left(\| \partial_\tau^\sigma J(.,\tau) \|^2_{L^2_\rho((0,1))} + \| J(.,\tau) \|^2_{L^2_\rho((0,1))} \right) d\tau \\ &+ D^{\sigma-1} \| J \|^2_{L^2_\rho((0,1))} + D^{\sigma-1} \| J_x \|^2_{L^2_\rho((0,1))} \\ \leq\ & 14 \left(\int_0^t \| J_x(.,\tau) \|^2_{L^2_\rho((0,1))} d\tau + \int_0^t \| \beta(.,\tau) \|^2_{L^2_\rho((0,1))} d\tau \right). \end{aligned} \tag{36}$$

If we discard the first three terms on the LHS of (35) and apply Lemma 1 with

$$\xi(t) = \int_0^t \| Y_x(.,\tau) \|^2_{L^2_\rho((0,1))} d\tau,\ \partial_t^\sigma \xi = D^{\sigma-1} \| Y_x \|^2_{L^2_\rho((0,1))},\ \xi(0) = 0, \tag{37}$$

we obtain the inequality

$$\begin{aligned} &\int_0^t \| Y_x(.,\tau) \|^2_{L^2_\rho((0,1))} d\tau \\ \leq\ & 14\Gamma(\sigma) E_{\sigma,\sigma}(14 t^\sigma) D_t^{-\sigma-1} \left(\| \mathcal{I}V \|^2_{L^2_\rho((0,1))} \right). \end{aligned} \tag{38}$$

By virtue of Lemma 3, inequality (38) and (35) reads

$$\begin{aligned} &\int_0^t \left(\| \partial_\tau^\sigma Y(.,\tau) \|^2_{L^2_\rho((0,1))} + \| Y(.,\tau) \|^2_{L^2_\rho((0,1))} \right) d\tau \\ &+ D^{\sigma-1} \| Y \|^2_{L^2_\rho((0,1))} + D^{\sigma-1} \| Y_x \|^2_{L^2_\rho((0,1))} \\ \leq\ & C^* \int_0^t \| \mathcal{I}V \|^2_{L^2_\rho((0,1))} d\tau, \end{aligned} \tag{39}$$

where

$$C^* = 14 \left\{ 1 + \frac{14 T^\sigma \Gamma(\sigma) E_{\sigma,\sigma}(14 t^\sigma)}{\Gamma(\sigma+1)} \right\}. \tag{40}$$

By symmetry, we also have

$$\begin{aligned} &\int_0^t \left(\| \partial_\tau^\sigma J(.,\tau) \|^2_{L^2_\rho((0,1))} + \| J(.,\tau) \|^2_{L^2_\rho((0,1))} \right) d\tau \\ &+ D^{\sigma-1} \| J \|^2_{L^2_\rho((0,1))} + D^{\sigma-1} \| J_x \|^2_{L^2_\rho((0,1))} \\ \leq\ & C^{**} \left(\int_0^t \| \beta \|^2_{L^2_\rho((0,1))} d\tau + \| Z \|^2_{W^1_{2,\rho}((0,1))} \right), \end{aligned} \tag{41}$$

where

$$C^{**} = Max\{C^*, 1\}. \tag{42}$$

We conclude from (39) and (41) that

$$\| Y \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} \leq C^* \int_0^T \| \mathcal{I}V \|^2_{L^2_\rho((0,1))} d\tau, \tag{43}$$

and

$$\| J \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} \le C^{**}\left(\int_0^T \| \beta \|^2_{L^2_\rho((0,1))} d\tau + \| Z \|^2_{W^1_{2,\gamma}((0,1))}\right). \tag{44}$$

Since $\theta = Y + J$ implies that

$$\begin{aligned} & \| \theta \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} \\ \le\ & \| Y \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} + \| J \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} \\ \le\ & C^* \int_0^T \| \mathcal{I}V \|^2_{L^2_\rho((0,1))} d\tau + C^{**}\left(\int_0^T \| \beta \|^2_{L^2_\rho((0,1))} d\tau + \| Z \|^2_{W^1_{2,\gamma}((0,1))}\right), \end{aligned} \tag{45}$$

and since

$$\begin{aligned} \mathcal{I}V &\in L^2\Big(0,T;L^2_\rho((0,1)),\ \beta \in L^2(0,T;L^2_\rho((0,1)) \\ \text{and}\ \ Z &\in W^1_{\rho,2}((0,1)), \end{aligned}$$

then, according to (45), we have

$$\| \theta \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} \le C^*C_1 + C^{**}C_2. \tag{46}$$

It also follows from (39) and (41) that

$$\| \partial_t^\sigma Y \|^2_{L^2(0,T;L^2_\rho((0,1))} \le C^* \int_0^T \| \mathcal{I}V \|^2_{L^2_\rho((0,1))} d\tau \le C^*C_1, \tag{47}$$

and

$$\| \partial_t^\sigma J \|^2_{L^2(0,T;L^2_\rho((0,1))} \le C^{**}\left(\int_0^T \| \beta \|^2_{L^2_\rho((0,1))} d\tau + \| Z \|^2_{W^1_{2,\gamma}((0,1))}\right) \le C^{**}C_2 \tag{48}$$

If we choose $B \ge \sqrt{C^*C_1 + C^{**}C_2}$, then we conclude from (46)–(48) that

$$\| \theta \|^2_{L^2(0,T;H^\sigma_\rho((0,1))} \le B,\ \text{and}\ \| \partial_t^\sigma \theta \|_{L^2(0,T;L^2(0,1))} \le 2B. \tag{49}$$

Hence, $\theta \in \Sigma(B)$, and consequently, the mapping $\mathcal{P}$ maps $\Sigma(B)$ into itself. □

We will now show that the mapping $\mathcal{P} : \Sigma(B) \to \Sigma(B)$ is continuous. Let $\theta_1,\theta_2 \in \Sigma(B)$, and let $\omega_1 = \mathcal{P}(\theta_1)$ and $\omega_2 = \mathcal{P}(\theta_2)$.

We observe that $\omega = \omega_1 - \omega_2$ satisfies

$$\partial_t^\sigma \omega - \frac{\partial^2 \omega}{\partial x^2} - \frac{1}{x}\frac{\partial \omega}{\partial x} + \omega = U(t,x), \tag{50}$$

where

$$\begin{cases} U(t,x) = \max\left(\int_0^x \eta\theta_1(\eta,t)d\eta, 0\right) - \max\left(\int_0^x \eta\theta_2(\eta,t)d\eta, 0\right) \\ \omega(x,0) = 0,\ \omega_x(1,t) = 0. \end{cases} \tag{51}$$

It is clear that

$$\| U(x,t) \|^2_{L^2(0,T;L^2_\rho((0,1))} \le \| \theta_1 - \theta_2 \|^2_{L^2(0,T;L^2_\rho((0,1))}, \tag{52}$$

and

$$\| \omega(x,t) \|^2_{L^2(0,T;L^2_\rho((0,1))} \le \| \theta_1 - \theta_2 \|^2_{L^2(0,T;L^2_\rho((0,1))}. \tag{53}$$

That is,

$$\| \mathcal{P}(\theta_1) - \mathcal{P}(\theta_2) \|_{L^2(0,T;L^2_\rho((0,1))} \le \| \theta_1 - \theta_2 \|_{L^2(0,T;L^2_\rho((0,1))}. \tag{54}$$

Consequently, the mapping $\mathcal{P} : \Sigma(B) \to \Sigma(B)$ is continuous. The set $\overline{\Sigma(B)}$ is compact, due to the following:

Theorem 3. *Let $F_0 \subset F \subset F_1$ with compact embedding (see [35]). Suppose that $\alpha, \lambda \in (0, \infty)$ and $T > 0$. Then,*

$$W = \left\{\theta : \theta \in L^{\alpha}(0, T; F),\ \partial_t^{\delta}\theta \in L^{\lambda}(0, T; F_1)\right\}$$

is compactly embedded in $L^{\alpha}(0, T; F)$, that is the bounded sets are relatively compact in $L^{\alpha}(0, T; F)$.

Remark that $\mathcal{P}(\Sigma(F)) \subset \Sigma(F) \subset L^2(0, T; L^2_{\rho}((0, 1))$. Then apply Schauder fixed point theorem to conclude that mapping $\mathcal{P}$ has a fixed point $\theta \in \Sigma(B)$.

5. A Priori Estimate for the Solution

We will establish a priori estimate for the solution of the posed problem (1)–(3) in the function space $L^2\left(0, T; H^{2,\sigma}_{\gamma}(0, 1)\right)$. That is, we may expect the solution of (1)–(3) to be in $L^2\left(0, T; H^{2,\sigma}_{\gamma}(0, 1)\right)$, with $\gamma = x^2$.

Theorem 4. *Let $L^2\left(0, T; H^{2,\sigma}_{\gamma}(0, 1)\right)$, solve (1)–(3). Then, the following a priori estimate is true*

$$\| \theta \|^2_{L^2(0,T;H^{2,\sigma}_{\gamma}(0,1))} \leq D\left(\| Z \|^2_{W^1_{2,\gamma}((0,1))} + \| \beta \|^2_{L^2(0,T;L^2_{\rho}(0,1))}\right), \tag{55}$$

where

$$D = \max\left\{2, \frac{T^{1-\sigma}}{(1-\sigma)\Gamma(1-\sigma)}\right\}. \tag{56}$$

Proof. Note that

$$\int_0^1 x^2\left(\partial_t^{\sigma}\theta - \frac{\partial^2\theta}{\partial x^2} - \frac{1}{x}\frac{\partial\theta}{\partial x} + \theta\right)^2 dx = \int_0^1 \left[x \max\left(\int_0^x \eta\theta(\eta, t)d\eta, 0\right) + x\beta\right]^2 dx. \tag{57}$$

Then,

$$\begin{aligned} &\int_0^1 x^2(\partial_t^{\sigma}\theta)^2 dx + \int_0^1 \left[\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right]^2 dx + \int_0^1 x^2\theta^2 dx + 2(\partial_t^{\sigma}\theta, \theta)_{L^2_{\gamma}(0,1)} \\ &-2\left(x\theta, \frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right)_{L^2(0,1)} - 2\left(x\partial_t^{\sigma}\theta, \frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right)_{L^2(0,1)} \\ = &\int_0^1 \left[x \max\left(\int_0^x \eta\theta(\eta, t)d\eta, 0\right) + x\beta\right]^2 dx. \end{aligned} \tag{58}$$

The last two terms on the LHS of (58) can be evaluated as

$$-2\left(x\theta, \frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right)_{L^2(0,1)} = 2\| \theta_x \|^2_{L^2_{\gamma}(0,1)} + 2(x\theta, \theta_x)_{L^2(0,1)}, \tag{59}$$

$$\begin{aligned} &-2\left(x\partial_t^{\sigma}\theta, \frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right)_{L^2(0,1)} \\ = \ &2(\partial_t^{\sigma}\theta_x, \theta_x)_{L^2_{\gamma}(0,1)} + 2(x\partial_t^{\sigma}\theta, \theta_x)_{L^2(0,1)}, \end{aligned} \tag{60}$$

and

$$\int_0^1 \left[\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right]^2 dx = \| \theta_{xx} \|^2_{L^2_{\gamma}(0,1)}. \tag{61}$$

We now evaluate the expression

$$-2(x\mathcal{L}\theta,\theta_x)_{L^2(0,1)} = -2\left(\max\left(\int_0^x \eta\theta(\eta,t)d\eta, 0\right) + \beta, x\theta_x\right)_{L^2(0,1)}. \tag{62}$$

Computation of the terms on the LHS of (62) leads to

$$\begin{aligned} & 2\| \theta_x \|^2_{L^2(0,1)} - 2(x\partial_t^\sigma\theta,\theta_x)_{L^2(0,1)} - 2(x\theta,\theta_x)_{L^2(0,1)} \\ = & -2\left(\max\left(\int_0^x \eta\theta(\eta,t)d\eta, 0\right) + \beta, x\theta_x\right)_{L^2(0,1)}. \end{aligned} \tag{63}$$

Observe that the combination of (58)–(63) and the use of Lemma 2 yield

$$\begin{aligned} & \| \partial_t^\sigma\theta \|^2_{L^2_\gamma(0,1)} + \| \theta \|^2_{L^2_\gamma(0,1)} + 2\| \theta_x \|^2_{L^2_\gamma(0,1)} + 2\| \theta_x \|^2_{L^2(0,1)} \\ & + \| \theta_{xx} \|^2_{L^2_\gamma(0,1)} + \partial_t^\sigma \| \theta_x \|^2_{L^2_\gamma(0,1)} + \partial_t^\sigma \| \theta \|^2_{L^2_\gamma(0,1)} \\ \leq & \int_0^1 \left[x\max\left(\int_0^x \eta\theta(\eta,t)d\eta, 0\right) + x\beta\right]^2 dx \\ & -2\left(\max\left(\int_0^x \eta\theta(\eta,t)d\eta, 0\right), x\theta_x\right)_{L^2(0,1)} - 2(x\theta_x,\beta)_{L^2(0,1)}. \end{aligned} \tag{64}$$

We now estimate the RHS of (64) in the following way:

$$\int_0^1 \left[x\max\left(\int_0^x \eta\theta(\eta,t)d\eta, 0\right) + x\beta\right]^2 dx \leq \frac{2}{3}\| \theta_x \|^2_{L^2(0,1)} + 2\| \beta \|^2_{L^2_\gamma(0,1)}, \tag{65}$$

$$-2\left(\max\left(\int_0^x \eta\theta(\eta,t)d\eta, 0\right), x\theta_x\right)_{L^2(0,1)} \leq \delta_1\frac{2}{3}\| \theta \|^2_{L^2_\gamma(0,1)} + \frac{1}{\delta_1}\| \theta_x \|^2_{L^2(0,1)}, \tag{66}$$

$$-2(x\theta_x,\beta)_{L^2(0,1)} \leq \delta_2\| \theta_x \|^2_{L^2(0,1)} + \frac{1}{\delta_{21}}\| \beta \|^2_{L^2_\gamma(0,1)}. \tag{67}$$

The insertion of (65)–(67) into (64) gives

$$\begin{aligned} & \| \partial_t^\sigma\theta \|^2_{L^2_\gamma(0,1)} + \| \theta \|^2_{L^2_\gamma(0,1)} + 2\| \theta_x \|^2_{L^2_\gamma(0,1)} + 2\| \theta_x \|^2_{L^2(0,1)} \\ & + \| \theta_{xx} \|^2_{L^2_\gamma(0,1)} + \partial_t^\sigma \| \theta_x \|^2_{L^2_\gamma(0,1)} + \partial_t^\sigma \| \theta \|^2_{L^2_\gamma(0,1)} \\ \leq & \Big(\tfrac{2}{3}\| \theta_x \|^2_{L^2_\gamma(0,1)} + 2\| \beta \|^2_{L^2_\gamma(0,1)} + \delta_1\tfrac{2}{3}\| \theta \|^2_{L^2_\gamma(0,1)} + \tfrac{1}{\delta_1}\| \theta_x \|^2_{L^2(0,1)} \\ & \delta_2\| \theta_x \|^2_{L^2(0,1)} + \tfrac{1}{\delta_2}\| \beta \|^2_{L^2_\gamma(0,1)}\Big). \end{aligned} \tag{68}$$

Upon the choices $\delta_1 = 1$,$\delta_2 = 1$, the inequality (68) reduces to

$$\| \theta \|^2_{H^{2,\sigma}_\gamma(0,1)} + \partial_t^\sigma \| \theta_x \|^2_{L^2_\gamma(0,1)} + \partial_t^\sigma \| \theta \|^2_{L^2_\gamma(0,1)} \leq 9\| \beta \|^2_{L^2_\gamma(0,1)}. \tag{69}$$

The integration of (69) yields

$$\begin{aligned} & \| \theta \|^2_{L^2(0,T;H^{2,\sigma}_\gamma(0,1))} + D^{\sigma-1}\| \theta_x \|^2_{L^2_\gamma(0,1)} + D^{\sigma-1}\| \theta \|^2_{L^2_\gamma(0,1)} \\ \leq & 9\| \beta \|^2_{L^2(0,t;L^2_\gamma(0,1))} + \tfrac{T^{1-\sigma}}{(1-\sigma)\Gamma(1-\sigma)}\Big(\| Z_x \|^2_{L^2_\gamma(0,1)} + \| Z \|^2_{L^2_\gamma(0,1)}\Big) \\ \leq & D\Big(\| Z \|^2_{W^1_{2,\gamma}((0,1))} + \| \beta \|^2_{L^2(0,T;L^2(0,1))}\Big). \end{aligned} \tag{70}$$

where $D > 0$ is given by (56). Once we drop the last two terms on the LHS, we obtain the a priori estimate (55) from which we deduce the uniqueness of the solution of problem (1)–(3) in the fractional Sobolev space $L^2\left(0, T; H_\gamma^{2,\sigma}(0,1)\right)$. $\square$

6. Conclusions

The well-posedness of a one-point IBVP for a one-dimensional fractional nonlinear integro-differential equation of order between zero and one is investigated. The Schauder fixed point theorem is applied to establish the existence of the solution. The feasibility of solving this problem is established under the context of data related to a weighted Sobolev space. Furthermore, an additional result related to the regularity of the solution for the formulated problem is also presented.

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