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*Article*

# Existence Results of Nonlocal Fractional Integro-Neutral Differential Inclusions with Infinite Delay

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**Abstract:** This article addresses a new class of delayed fractional multivalued problems complemented with nonlocal boundary conditions. In view of infinite delay theory, we convert the inclusion problem into a fixed-point multivalued problem, defined in an appropriate phase space. Then, sufficient criteria for the existence of solutions are established for the convex case of the given problem using the nonlinear Leray–Schauder alternative type, while Covitz and Nadler's theorem is applied for nonconvex multivalued functions. Finally, the results are illustrated through examples.

**Keywords:** fractional derivative; inclusions; delay; fixed point; nonlocal boundary condition

**MSC:** 26A33; 34K05; 34A08

## 1. Introduction

Fractional calculus has emerged as a significant branch of mathematics over the past two decades. Unlike integer-order derivatives, fractional derivatives are expressed in terms of fractional integrals, leading to various types of fractional derivatives, such as Riemann–Liouville, Caputo, Erdélyi–Kober-type, Hadamard, Katugambola, Hillfer, and so on. The importance of these operators lies in their nonlocal nature and wide applications in various scientific areas, such as physics [1], econophysics [2], biological systems [3,4], viscoelasticity [5], and epidemiology [6]. For the theoretical analyses of fractional derivatives, refer to [7,8]. The investigation of fractional boundary value problems (FBVPs) with several types of boundary conditions (BCs) is a crucial subject in mathematical studies. The most realistic BCs are nonlocal conditions, as they accurately describe many scientific applications, including physical, biological, and chemical processes, where measurements are taken from the interior positions of the study domain. Currently, numerous applications in the literature are based on FBVPs with nonlocal BCs. For example, Ahmad et al. [9] derived the existence criteria for a class of multi-point sequential FBVPs involving the Hadamard fractional derivative. Ntouyas et al. [10] investigated a class of $(k, \psi)$-Hilfer nonlocal multi-point FBVPs involving differential inclusions and equations. In [11], the authors established the existence and stability of solutions for a class of multi-term implicit FBVPs with nonlocal BCs. Hussain et al. [12] discussed the controllability of impulsive Hilfer FBVPs with nonlocal BCs and infinite delay. More recently, Sarwar et al. [13] studied the controllability of non-instantaneous impulsive delayed differential equations involving the ABC fractional derivative.

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Differential inclusions (DIs) have garnered significant interest as a generalization of differential equations, serving as valuable tools for modeling problems where velocities may not be uniquely determined by the system's state, despite depending on it. In fact, the theory of DIs has rapidly grown since the 1960s as the best technique to deal with the dynamic system models of differential equations with discontinuous right-hand sides, such as the physical problems with dry fraction, see [14,15]. As a result, DIs have numerous applications, including control systems [16,17], stochastic processes [18], and synchronization of fractional order systems [19]. For more details and applications, refer to the books in [18,20] and the articles in [21–26].

In recent years, delay DIs have gained significant attention from researchers. Benchohra et al. [27] examined two classes of initial value problems involving delayed DIs with infinite delay and established existence results using fixed-point theorems of multivalued functions. Aissani et al. [28] utilized Bohnenblust–Karlin's theorem to establish existence criteria for a class of fractional DIs with impulse effects and infinite delay. Some existence results were investigated in [29] by utilizing the properties of the measure of noncompactness for impulsive DIs of fractional order with infinite delay. Zhou et al. [30] established controllability results for a class of delayed fractional neutral functional DIs in Banach spaces. In [31], a class of stochastic fractional DIs with non-instantaneous impulses and infinite delay was analyzed. Kavitha et al. [32] studied the controllability of Hilfer DIs with infinite delay. Williams et al. [33] examined the existence and controllability of a class of semi-linear fractional DIs with infinite delay. Recently, Alsheekhhussain et al. [34] derived the topological properties of the solutions for non-instantaneous impulsive DIs with infinite delay. Also, for some applications on delayed classical DIs, refer to the book in [35].

Motivated by the previous studies, we extend our results in [36] to the multivalued case. In precise terms, we establish the existence results for the following inclusion problem:

$$^{C}D^{\eta}_{0^+}[y(t) - \int_0^t \mathcal{K}(r, y_r)dr] \in \Psi(t, y_t), \;\; t \in \mathcal{I} := [0, q], \tag{1}$$

complemented with the following conditions:

$$y(t) = \beta(t), \;\; t \in (-\infty, 0], \;\; y(q) = \sum_{i=1}^{p} \vartheta_i \, {}^{C}D^{\mu}_{0^+} y(\kappa_i) + \sigma, \;\; \kappa_i \in (0, q), \tag{2}$$

where ${}^{C}D^{\eta}_{0^+}, {}^{C}D^{\mu}_{0^+}$ are the Caputo derivatives of order $1 < \eta \leq 2, 0 < \mu < 1$, respectively; $\Psi : \mathcal{I} \times \mathfrak{L} \to \mathcal{G}(\mathbb{R})$ is a multivalued function ($\mathcal{G}(\mathbb{R})$ is the collection of all nonempty subsets of $\mathbb{R}$); and $\beta \in \mathfrak{L}$, $\beta(0) = 0$, where $\mathfrak{L}$ is a phase space presented in detail in Section 2. The state $y_t : (-\infty, 0] \to \mathbb{R}$, is defined as $y_t(r) = y(t + r)$, $r \leq 0$, which belongs to a phase space $\mathfrak{L}$.

Compared to fractional differential equations, the literature on fractional DIs, particularly those with infinite delay, remains limited and requires further development and applications. Thus, this study aims to contribute to the research on fractional DIs by integrating the theory of multivalued mappings with unbounded delay effects. As almost all of the previous studies are addressed as initial value problems, see [27–32], we introduce in this research a new class of delayed FBVPs involving integro-DIs of order $\eta \in (1, 2)$ and complemented with nonlocal multi-term BCs. This will increase the variety of the applications on DIs and lead to more understanding of how to deal with DIs with infinite delay. Briefly, we transform problem (1)–(2) into a fixed-point problem, and, for the solution to be defined, we extend the state space $\mathfrak{L}$ to a new seminormed space $\mathfrak{L}_q$. As we cannot apply the fixed-point theorems directly on the seminormed space $\mathfrak{L}_q$, we split the solution

$y$ into two functions and generate the space $\tilde{\mathfrak{L}}_q$, which is indeed a Banach space. Finally, in view of the fixed-point theorems of convex and nonconvex multivalued functions, we establish our main results.

The rest of this article is presented as follows: Section 2 provides the concepts and lemmas related to multivalued functions. Section 3 establishes two existence results for solutions to inclusion problem (1)–(2): the first for Carathéodory functions and the second for Lipschitz functions, using the nonlinear Leray–Schauder alternative type for multivalued functions and Covitz and Nadler's theorem, respectively. Additionally, Section 3 includes two illustrative examples of the obtained results. Finally, Section 4 summarizes the findings of this study.

## 2. Preliminaries

In this work, the Banach space that consists of all continuous functions mapping $\mathcal{I}$ into $\mathbb{R}$, equipped with the supremum norm defined by $\|y\| = \sup\{|y(t)| : t \in \mathcal{I}\}$, is denoted by $C(\mathcal{I}, \mathbb{R})$. The space of all functions $y : \mathcal{I} \to \mathbb{R}$ that are Lebesgue-measurable with the norm $\|y\|_{L^1} = \int_0^q |y(t)|dt$ is denoted by $L^1(\mathcal{I}, \mathbb{R})$. Also, for a normed space $(Y, \|\cdot\|)$, the set of all nonempty bounded subsets of $Y$ is denoted by $\mathcal{G}_b(Y)$, the set of all nonempty bounded and closed subsets of $Y$ is denoted by $\mathcal{G}_{cl,b}(Y)$, and the set of all nonempty convex and compact subsets of $Y$ is denoted by $\mathcal{G}_{cp,c}(Y)$. Moreover, we recall the following definitions and relations; see [37]:

A multivalued function $\Psi : Y \to \mathcal{G}(Y)$ is defined as the following:

- *Closed (convex)* -valued if, for all $z \in Y$, $\Psi(z)$ is closed (convex).
- *Bounded on bounded sets* if, for all $\Omega \in \mathcal{G}_b(Y)$, $\Psi(\Omega) = \cup_{z \in \Omega} \Psi(z)$ is bounded in $Y$.
- *Upper semi-continuous* on $Y$ if, for each $z_0 \in Y$, $\mathcal{H}(z_0)$ is a closed nonempty subset of $Y$, and if for every open subset $R$ of $Y$ such that $\mathcal{H}(z_0) \subset R$, there is an open neighborhood $\mathcal{R}_0$ of $z_0$, such that $\Psi(\mathcal{R}_0) \subseteq R$.
- *Completely continuous* if, for every $\Omega \in \mathcal{G}_b(Y)$, $\Psi(\Omega)$ is relatively compact.
  If the multivalued function $\Psi$ is a compact and completely continuous map, then $\Psi$ is upper semicontinuous if and only if $\Psi$ has a closed graph, i.e., $z_n \to z_*$, $z_n \to z_*$, $z_n \in \Psi(z_n)$ imply that $z_* \in \Psi(z_*)$; see [37] [Proposition 1.2].
- *Measurable* if, for each $z \in Y$,

$$t \longmapsto d(z, \Psi(t)) = \inf\{|z - v| : v \in \Psi(t)\}$$

  is a measurable function.
- Said to have a fixed point if there is $z \in Y$ such that $z \in \Psi(z)$. $Fix\Psi$ denotes the fixed-point set of $\Psi$.

Now, to study the given delayed inclusion, a linear seminormed space including the functions that assign $(-\infty, 0]$ into $\mathbb{R}$ is denoted by $(\mathfrak{L}, \|.\|_{\mathfrak{L}})$, which satisfies the following axioms (see [38]):

$(\mathcal{S}_1)$ For every $t \in [0, q]$, if $y : (-\infty, q] \to \mathbb{R}$, and $y_0 \in \mathfrak{L}$, then the following are satisfied:

(1) $y_t$ belongs to $\mathfrak{L}$.

(2) $\|y_t\|_{\mathfrak{L}} \leq \lambda(t) \sup\{|y(r)| : 0 \leq r \leq t\} + \delta(t)\|y_0\|_{\mathfrak{L}}$,
where $\lambda, \delta : [0, \infty) \to [0, \infty)$, are defined such that $\lambda$ is a continuous function, $\delta$ is a locally bounded function, and $\lambda$, $\delta$ are independent of $y(.)$ with

$$\lambda_q = \sup\{|\lambda(t)| : t \in \mathcal{I}\}, \quad \delta_q = \sup\{|\delta(t)| : t \in \mathcal{I}\}. \tag{3}$$

(3) A constant $l \geq 0$ exists with $|y(t)| \leq l\|y_t\|_{\mathfrak{L}}$.

$(\mathcal{S}_2)$ For $y(.)$ satisfying $(\mathcal{S}_1)$, $y_t$ is a continuous $\mathfrak{L}-$valued function on $[0, q]$.

$(\mathcal{S}_3)$ $\mathfrak{L}$ is a complete space.

Then, we consider the space $\mathfrak{L}_q = \{y : (-\infty, q] \to \mathbb{R} : y|_{(-\infty,0]} \in \mathfrak{L}$ and $y|_{[0,q]} \in C(\mathcal{I}, \mathbb{R})\}$, and let $\|.\|_{\mathfrak{L}_q}$ be defined as a seminorm on $\mathfrak{L}_q$ such that $\|y\|_{\mathfrak{L}_q} = \|\varphi\|_{\mathfrak{L}} + \sup_{\tau\in\mathcal{I}} |y(\tau)|$, $y \in \mathfrak{L}_q$.

**Definition 1.** *The integral of fractional order $\chi > 0$ for a function $\psi : [0, \infty) \to \mathbb{R}$ is defined by*

$$I^{\chi}_{0^+}\psi(t) = \int_0^t \frac{(t-s)^{\chi-1}}{\Gamma(\chi)}\psi(s)ds, \quad t > 0.$$

**Definition 2.** *For $\psi : [0, \infty) \to \mathbb{R}$ with $\psi(t) \in AC^n[0, \infty)$, the Caputo fractional derivative of order $\chi$ is defined by*

$${}^{C}D^{\chi}_{0^+}\psi(t) = \frac{1}{\Gamma(n-\chi)}\int_0^t \frac{\psi^{(n)}(s)}{(t-s)^{\eta-n+1}}ds = I^{n-\chi}_{0^+}\psi^{(n)}(t), \;\; t > 0, \, \chi \in (n-1, n], n \in \mathbb{N}.$$

**Example 1.** *Let $\eta > 0$, $\varsigma > 0$; then,*

$$I^{\eta}_{a+}(t-a)^{\varsigma-1} = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\eta)}(t-a)^{\varsigma+\eta-1}. \tag{4}$$

*For $\eta > 0$, $n = [\eta] + 1$, $\varsigma > 0$; then,*

$${}^{C}D^{\eta}_{a+}(t-a)^{\varsigma-1} = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\eta)}(t-a)^{\varsigma-\eta-1}, \;\; \varsigma > n. \tag{5}$$

*In particular, for $i = 0, 1, \ldots, n-1$, we have*

$${}^{C}D^{\eta}_{a+}(t-a)^{i} = 0.$$

## 3. Main Results

This section is devoted to establishing the existence results for the solutions to inclusion problem (1)–(2). In view of [36] [Lemma 2.2], we introduce the following definition:

**Definition 3.** *For problem (1)–(2), a function $y \in \mathfrak{L}_q$ is a solution, if there is a function $\zeta \in L^1(\mathcal{I}, \mathbb{R})$ with $\zeta(t) \in \Psi(t, y_t)$ a.e. on $[0, q]$, such that $y(t) = \beta(t)$ for $t \in (-\infty, 0]$, $y(q) = \sum_{i=1}^{p} \vartheta_i {}^{C}D^{\mu}_{0^+}y(\kappa_i) + \sigma$ for $\kappa_i \in (0, q)$, and*

$$y(t) = \begin{cases} \beta(t), \;\; t \in (-\infty, 0], \\ \displaystyle\int_0^t \mathcal{K}(r, y_r)dr + \int_0^t \frac{(t-r)^{\eta-1}}{\Gamma(\eta)}\zeta(r)dr + \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr \\ \displaystyle +\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, y_r)dr - \int_0^q \mathcal{K}(r, y_r)dr - \int_0^q \frac{(q-r)^{\eta-1}}{\Gamma(\eta)}\zeta(r)dr + \sigma\Big), \;\; t \in \mathcal{I}. \end{cases}$$

*where*

$$\Lambda_1 = q - \sum_{i=1}^{p}\vartheta_i\frac{\kappa_i^{1-\mu}}{\Gamma(2-\mu)} \neq 0. \tag{6}$$

For the linear variant of problem (1)–(2), the solution exists and can be easily obtained in view of Definition 3 as in the following example.

**Example 2.** *Consider the following problem:*

$$\begin{cases} {}^{C}D^{3/2}_{0^+}[y(t) - \int_0^t r dr] \in \left[t^2, t^3\right], \ t \in [0,1], \\ y(t) = e^t - e^{2t}, \ t \in (-\infty, 0], \ \ y(1) = \frac{1}{4}{}^{C}D^{1/3}_{0^+}y(2/5) + \frac{3}{4}{}^{C}D^{1/3}_{0^+}y(3/4) + 1, \end{cases} \tag{7}$$

*where $\mathcal{K}(t, y_t) = t$ and $\Psi(t, y_t) = \left[t^2, t^3\right]$. Then, the solution of (7) is given by*

$$y(t) = \begin{cases} e^t - e^{2t}, \ \ t \in (-\infty, 0], \\ 0.5\, t^2 + 0.17194\, t^{7/2} + 4.624\, t, \ \ t \in [0,1]. \end{cases}$$

*Clearly, one can check that* ${}^{C}D^{3/2}_{0^+}[y(t) - \int_0^t r dr] = t^2 = \zeta(t) \in \Psi(t, y_t)$, *and* $y(1) = 5.29621388 = \frac{1}{4}{}^{C}D^{1/3}_{0^+}y(2/5) + \frac{3}{4}{}^{C}D^{1/3}_{0^+}y(3/4) + 1.$

Now, for the nonlinear problem, by utilizing [36] [Lemma 2.2], we transform problem (1)–(2) into its equivalent fixed-point problem by defining a multivalued operator, $\mathcal{E} : \mathfrak{L}_q \to \mathcal{G}(\mathfrak{L}_q)$, as the following:

$$\mathcal{E}(y) = \left\{ k \in \mathfrak{L}_q : k(t) = \begin{cases} \beta(t), \ \ t \in (-\infty, 0] \\ \int_0^t \mathcal{K}(r, y_r)dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}\zeta(r)dr \\ +\frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr \\ +\sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\mu_i - r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, y_r)dr \\ -\frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta(r)dr - \int_0^q \mathcal{K}(r, y_r)dr + \sigma\Big), \\ t \in \mathcal{I}, \ \zeta \in S_{\Psi,y} \end{cases} \right\}, \tag{8}$$

where the set $S_{\Psi,y} = \{\zeta \in L^1([0,q], \mathbb{R}) : \zeta(t) \in \Psi(t, y_t)\}$ refers to the set of selections of $\Psi$.

Let $w(.) : (-\infty, q] \to \mathbb{R}$ be defined as

$$w(t) = \begin{cases} \beta(t), \ \ t \in (-\infty, 0], \\ 0, \ \ \ t \in (0, q], \end{cases} \tag{9}$$

then $w_0 = \beta$. For every $m \in C(\mathcal{I}, \mathbb{R})$ with $m(0) = 0$, we define

$$\bar{m}(t) = \begin{cases} 0, \ \ t \in (-\infty, 0], \\ m(t), \ \ \ t \in (0, q]. \end{cases} \tag{10}$$

Now, the solution of (8), $y(.)$ can be decomposed as $y(t) = w(t) + \bar{m}(t)$, which yields $y_t = w_t + \bar{m}_t$ for $t \in \mathcal{I}$, where, for $\zeta \in S_{\Psi, w+\bar{m}}$, $m(.)$ satisfies

$$\begin{aligned} m(t) \ \ &= \ \ \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}\zeta(r)dr \\ &+\frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_r)dr \\ &-\frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta(r)dr - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr + \sigma\Big). \end{aligned} \tag{11}$$

Next, set $\bar{\mathfrak{L}}_q = \{m \in \mathfrak{L}_q : m_0 = 0\}$ to be a space equipped with a seminorm $\|.\|_{\bar{\mathfrak{L}}_q}$ defined by

$$\|m\|_{\bar{\mathfrak{L}}_q} = \sup_{t\in[0,q]} |m(t)| + \|m_0\|_{\mathfrak{L}} = \sup_{t\in[0,q]} |m(t)|,\ m \in \bar{\mathfrak{L}}_q.$$

Thus, $\|m\|_{\bar{\mathfrak{L}}_q}$ defines a norm on $\bar{\mathfrak{L}}_q$, which implies that $(\bar{\mathfrak{L}}_q, \|.\|_{\bar{\mathfrak{L}}_q})$ is a Banach space.

To deal with the existence of fixed points of $\mathcal{E}$, we introduce the operator $\mathcal{F} : \bar{\mathfrak{L}}_q \to \mathcal{G}(\bar{\mathfrak{L}}_q)$ by

$$\mathcal{F}(m) = \Big\{k \in \bar{\mathfrak{L}}_q : k(t) = \mathcal{J}(m)(t)\Big\}, \tag{12}$$

where

$$\begin{aligned}
\mathcal{J}(m)(t) \;=\; & \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)ds + \frac{1}{\Gamma(\eta)} \int_0^t (t-r)^{\eta-1}\zeta(r)dr \\
& + \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_r)dr \\
& - \frac{1}{\Gamma(\eta)} \int_0^q (q-r)^{\eta-1}\zeta(r)dr - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)ds + \sigma\Big),
\end{aligned} \tag{13}$$

for $\zeta \in S_{\Psi,w+\bar{m}}$. Clearly, $\mathcal{E}$ possesses a fixed point only if $\mathcal{F}$ does. So, we deal with the operator $\mathcal{F}$ to prove our existence results.

For the forthcoming analysis, we put

$$\Lambda_2 = \frac{q^\eta}{\Gamma(\eta+1)} + \frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^{p} \frac{\vartheta_i \kappa_i^{\eta-\mu}}{\Gamma(\eta-\mu+1)} + \frac{q^\eta}{\Gamma(\eta+1)}\Big), \tag{14}$$

and

$$\Lambda_3 = q + \frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^{p} \frac{\vartheta_i \kappa_i^{1-\mu}}{\Gamma(2-\mu)} + q\Big). \tag{15}$$

### *3.1. The Carathéodory Case*

Our goal in this subsection is to investigate problem (1)–(2) for the convex-valued and Carathéodory function $\Psi$.

**Definition 4** (Ref. [20]). *A multivalued function $\Psi : \mathcal{I} \times \mathfrak{L} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory if the following properties hold:*

(1) *For each $y \in \mathfrak{L}$, $\Psi(., y)$ is measurable;*
(2) *For almost all $t \in \mathcal{I}$, $\Psi(t, .)$ is upper semicontinuous;*

*Moreover, a Carathéodory multivaled function $\Psi$ is said to be $L^1-$Carathéodory if the following holds:*

(3) *There is a function $\varpi_\eta \in L^1(\mathcal{I}, \mathbb{R}^+)$, for each $\eta > 0$, such that*

$$\|\Psi(t, y)\| = \sup\{|\zeta| : \zeta \in \Psi(t, y)\} \le \varpi_\eta(t),\ \forall\ \|y\| \le \eta \text{ and for a.e. } t \in \mathcal{I}.$$

The set $G(\mathcal{Q}) = \{(z, x) \in U \times V, x \in \mathcal{Q}(z)\}$ denotes the graph of $\mathcal{Q}$.

**Lemma 1** ([39]). *For a Banach space $Y$, assume the maltivalued map $\Psi : \mathcal{I} \times \mathbb{R} \to \mathcal{G}_{cp,c}(Y)$ is an $L^1$- Carathéodory and the function $\Pi$ is continuous and linear from $L^1(\mathcal{I}, Y)$ to $C(\mathcal{I}, Y)$. Then, the following operator*

$$\Pi \circ S_\Psi : C(\mathcal{I}, Y) \to \mathcal{G}_{cp,c}(C(\mathcal{I}, Y)),\ \ u \mapsto (\Pi \circ S_\Psi)(u) = \Pi(S_{\Psi,u})$$

*has a closed graph in $C(\mathcal{I}, Y) \times C(\mathcal{I}, Y)$.*

**Lemma 2** ((Nonlinear alternative Leray–Schauder for Kakutani maps [40]). *For a Banach space $Y$, assume $A$ is a convex closed subset of $Y$, and $\Omega$ is an open subset of $A$ with $0 \in \Omega$. Also, assume $\mathcal{J} : \overline{\Omega} \to \mathcal{G}_{cp,c}(A)$ is an upper semicontinuous and compact map. Then, only one of the following results is true:*

(1) *$\mathcal{J}$ possesses a fixed point in $\overline{\Omega}$;*
(2) *There is a $s \in \partial\Omega$ and $0 < \xi < 1$ with $s \in \xi\, \mathcal{J}(s)$.*

**Theorem 1.** *Assume the following assumptions are satisfied:*

$(C_1)$ *$\Psi : \mathcal{I} \times \mathfrak{L} \to \mathcal{G}_{cp,c}(\mathbb{R})$ is an $L^1$-Carathéodory multivalued function;*
$(C_2)$ *A nondecreasing function $\Phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a function $N \in L^1(\mathcal{I}, \mathbb{R}^+)$ exist such that*

$$\|\Psi(t,y)\| := \sup\{|x| : x \in \Psi(t,y)\} \leq N(t)\Phi(\|y\|_{\mathfrak{L}}) \ \text{for } t \in \mathcal{I} \text{ and each } y \in \mathfrak{L};$$

$(C_3)$ *$\mathcal{K} : \mathcal{I} \times \mathfrak{L} \to \mathbb{R}$ is continuous, and there are two constants, $0 \leq K_1 < 1$ and $K_2 > 0$, such that $|\mathcal{K}(t,y)| \leq K_1\|y\|_{\mathfrak{L}} + K_2$ for $t \in \mathcal{I}$ and each $y \in \mathfrak{L}$;*
$(C_4)$ *A constant $\widehat{\theta} > 0$ exists such that*

$$\frac{(1 - \lambda_q K_1 \Lambda_3)\widehat{\theta}}{(K_2 + K_1\delta_q\|\beta\|_{\mathfrak{L}})\Lambda_3 + \Phi(\lambda_q\widehat{\theta} + \delta_q\|\beta\|_{\mathfrak{L}})\, N^*\Lambda_2 + \dfrac{q}{|\Lambda_1|}|\sigma|} > 1, \tag{16}$$

*where $N^* = sup_{t\in\mathcal{I}} N(t)$, $\lambda_q, \delta_q$, $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ are, respectively, introduced by (3), (6), (14), and (15). Then, there exists at least one solution on $(-\infty, q]$ to problem (1)–(2).*

**Proof.** To utilize the conclusion of Lemma 2, we need to show that the multivalued operator $\mathcal{F} : \bar{\mathfrak{L}}_q \to \mathcal{G}(\bar{\mathfrak{L}}_q)$ defined by (12) satisfies its hypotheses in several steps.

*Step 1.* We show that, for each $m \in \bar{\mathfrak{L}}_q$, $\mathcal{F}(m)$ is convex.

Indeed, if $k_1, k_2 \in \mathcal{F}(m)$, then there exist $\zeta_1, \zeta_2 \in S_{\Psi, w_r + \bar{m}_r}$ such that

$$\begin{aligned} k_i(t) &= \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}\zeta_i(r)dr \\ &+ \frac{t}{\Lambda_1}\Big(\sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta_i(r)dr + \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_r)ds \\ &- \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta_i(r)dr - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr + \sigma\Big), \ \ i = 1,2, \ \forall\, t \in \mathcal{I}. \end{aligned}$$

For $0 \leq \varrho \leq 1$ and $t \in \mathcal{I}$, we find

$$\begin{aligned} \varrho k_1(t) + (1-\varrho)k_2(t) &= \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}(\varrho\zeta_1(r) + (1-\varrho)\zeta_2(r))dr \\ &+ \frac{t}{\Lambda_1}\Big(\sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}(\varrho\zeta_1(r) + (1-\varrho)\zeta_2(r))dr \\ &+ \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_r)dr - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr \\ &- \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}(\varrho\zeta_1(r) + (1-\varrho)\zeta_2(r))dr + \sigma\Big), \end{aligned}$$

since $\Psi$ is convex-valued and $S_{\Psi, w+\bar{m}}$ is convex, then $\varrho k_1(t) + (1-\varrho)k_2(t) \in \mathcal{F}(m)$.

*Step 2.* To prove the boundedness of $\mathcal{F}$, let us consider the bounded set $\mathfrak{B}_\psi = \{m \in \bar{\mathfrak{L}}_q : \|m\|_{\bar{\mathfrak{L}}_q} \leq \psi\}$, where $\psi > 0$, and let $k \in \mathcal{F}(m)$ for $m \in \mathfrak{B}_\psi$. Then, there exists $\zeta \in S_{\Psi, w+\bar{m}}$ such that

$$\begin{aligned} k(t) &= \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}\zeta(r)dr \\ &+ \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - s)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr + \sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_r)dr \\ &- \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr - \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta(r)dr + \sigma\Big), \ \forall\ t \in \mathcal{I}. \end{aligned}$$

Then, we find, for each $t \in \mathcal{I}$, that

$$\begin{aligned} |k(t)| &\leq \int_0^t |\mathcal{K}(r, w_r + \bar{m}_r)|dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}|\zeta(r)|dr \\ &+ \frac{t}{|\Lambda_1|}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}|\zeta(r)|dr + \sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}|\mathcal{K}(r, w_r + \bar{m}_r)|dr \\ &+ \int_0^q |\mathcal{K}(r, w_r + \bar{m}_r)|dr + \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}|\zeta(r)|dr + |\sigma|\Big) \\ &\leq \int_0^t [K_1\|w_r + \bar{m}_r\|_{\mathfrak{L}} + K_2]dr + \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}[N(r)\Phi(\|w_r + \bar{m}_r\|_{\mathfrak{L}})]dr \\ &+ \frac{t}{|\Lambda_1|}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}[N(r)\Phi(\|w_r + \bar{m}_r\|_{\mathfrak{L}})]dr \\ &+ \sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}[K_1\|w_r + \bar{m}_r\|_{\mathfrak{L}} + K_2]dr + \int_0^q [K_1\|w_r + \bar{m}_r\|_{\mathfrak{L}} + K_2]dr \\ &+ \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}[N(s)\Phi(\|w_r + \bar{m}_r\|_{\mathfrak{L}})]dr + |\sigma|\Big) \\ &\leq [K_1(\lambda_q\psi + \delta_q\|\beta\|_{\mathfrak{L}}) + K_2]\Big(q + \frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}dr + q\Big)\Big) \\ &+ N^*\Phi(\lambda_q\psi + \delta_q\|\beta\|_{\mathfrak{L}})\Big(\frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}dr + \frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}dr \\ &+ \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}dr\Big)\Big) + \frac{q}{|\Lambda_1|}|\sigma|. \end{aligned}$$

Thus,

$$\begin{aligned} \|k\|_{\mathfrak{L}_q} &\leq [K_1\mathcal{L} + K_2]\Big(q + \frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^{p}\frac{\vartheta_i\kappa_i^{1-\mu}}{\Gamma(2-\mu)} + q\Big)\Big) \\ &+ N^*\Psi(\mathcal{L})\Big(\frac{q^\eta}{\Gamma(\eta+1)} + \frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^{p}\frac{\vartheta_i\kappa_i^{\eta-\mu}}{\Gamma(\eta-\mu+1)} + \frac{q^\eta}{\Gamma(\eta+1)}\Big)\Big) + \frac{q}{|\Lambda_1|}|\sigma| \\ &= [K_1\mathcal{L} + K_2]\Lambda_3 + N^*\Phi(\mathcal{L})\Lambda_2 + \frac{q}{|\Lambda_1|}|\sigma| := \varpi, \end{aligned}$$

where

$$\|w_r + \bar{m}_r\|_{\mathfrak{L}} \leq \|w_r\|_{\mathfrak{L}} + \|\bar{m}_r\|_{\mathfrak{L}} \leq \lambda_q\psi + \delta_q\|\beta\|_{\mathfrak{L}} := \mathcal{L}.$$

*Step 3.* Let us take the bounded set $\mathfrak{B}_\psi$ that is defined in Step 2, and assume $t_1, t_2 \in (0, q]$, such that $t_1 < t_2$. Then, for $m \in \mathfrak{B}_\psi$ and $k \in \mathcal{F}(m)$, there exists $\zeta \in S_{\Psi, w+\bar{m}}$ such that

$$\begin{aligned}
& |k(t_2) - k(t_1)| \\
\leq & \Big| \int_0^{t_2} \mathcal{K}(r, w_r + \bar{m}_r) dr - \int_0^{t_1} \mathcal{K}(r, w_r + \bar{m}_r) dr \\
& + \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big( (t_2 - r)^{\eta-1} - (t_1 - r)^{\eta-1} \Big) \zeta(r) dr + \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2 - r)^{\eta-1} \zeta(r) dr \\
& + \frac{t_2 - t_1}{\Lambda_1} \Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta - \mu)} \zeta(r) dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1 - \mu)} \mathcal{K}(r, w_r + \bar{m}_r) dr \\
& + \int_0^q \mathcal{K}(r, w_r + \bar{m}_r) dr + \frac{1}{\Gamma(\eta)} \int_0^q (q - r)^{\eta-1} \zeta(r) dr + \sigma \Big) \Big| \\
\leq & \int_{t_1}^{t_2} [K_1 \mathcal{L} + K_2] dr + \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big( (t_2 - r)^{\eta-1} - (t_1 - r)^{\eta-1} \Big) N^* \, \Phi(\mathcal{L}) dr \\
& + \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2 - r)^{\eta-1} N^* \, \Phi(\mathcal{L}) dr + \frac{t_2 - t_1}{|\Lambda_1|} \Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta - \mu)} N^* \, \Phi(\mathcal{L}) dr \\
& + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1 - \mu)} [K_1 \mathcal{L} + K_2] dr + \int_0^q [K_1 \mathcal{L} + K_2] dr \\
& + \frac{1}{\Gamma(\eta)} \int_0^q (q - r)^{\eta-1} N^* \, \Phi(\mathcal{L}) dr + |\sigma| \Big).
\end{aligned}$$

Clearly, independently of $m \in \mathfrak{B}_\psi$, the inequality above tends to zero on its right-hand side as $t_1 \longrightarrow t_2$. From the outcomes of Steps 1–3 and in view of the conclusion of the Arzelá–Ascoli theorem, we confirm that $\mathcal{F} : \bar{\mathfrak{L}}_q \to \mathcal{G}(\bar{\mathfrak{L}}_q)$ is completely continuous.

*Step 4.* In this step, we establish that $\mathcal{F}$ is upper semicontinuous. As $\mathcal{F}$ is completely continuous and in view of [37] [Proposition 1.2], it just remains to prove that it has a closed graph.

Assume that $m_n \to m_*, k_n \in \mathcal{F}(m_n)$ and $k_n \to k_*$. We shall show that $k_* \in \mathcal{F}(m_*)$. Now, since $k_n \in \mathcal{F}(m_n)$, we can find, for each $t \in \mathcal{I}$, $\zeta_n \in S_{\Psi, w+\bar{m}_n}$ such that

$$\begin{aligned}
k_n(t) = & \int_0^t \mathcal{K}(r, w_r + \bar{m}_{n_r}) ds + \int_0^t \frac{(t - r)^{\eta-1}}{\Gamma(\eta)} \zeta_n(r) dr \\
& + \frac{t}{\Lambda_1} \Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta - \mu)} \zeta_n(r) dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1 - \mu)} \mathcal{K}(s, w_r + \bar{m}_{n_r}) dr \\
& - \int_0^q \mathcal{K}(s, w_r + \bar{m}_{n_r}) ds - \int_0^q \frac{(q - r)^{\eta-1}}{\Gamma(\eta)} \zeta_n(r) dr + \sigma \Big).
\end{aligned}$$

Thus, it is enough to show that, for $t \in \mathcal{I}$, there is $\zeta_* \in S_{\Psi, w+\bar{m}_*}$ such that

$$\begin{aligned}
k_*(t) = & \int_0^t \mathcal{K}(s, w_r + \bar{m}_{*_r}) dr + \frac{1}{\Gamma(\eta)} \int_0^t (t - r)^{\eta-1} \zeta_*(r) dr \\
& + \frac{t}{\Lambda_1} \Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta - \mu)} \zeta_*(r) dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{-\mu}}{\Gamma(1 - \mu)} \mathcal{K}(r, w_r + \bar{m}_{*_r}) dr \\
& - \int_0^q \mathcal{K}(s, w_r + \bar{m}_{*_r}) dr - \frac{1}{\Gamma(\eta)} \int_0^q (q - r)^{\eta-1} \zeta_*(r) dr + \sigma \Big).
\end{aligned}$$

Let us define $\Pi : L^1([0,q] \to \mathbb{R}) \to C([0,q] \to \mathbb{R})$ by

$$\begin{aligned}\zeta \mapsto \Pi\zeta(t) &= \int_0^t \frac{(t-r)^{\eta-1}}{\Gamma(\eta)}\zeta(r)dr + \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr \\ &\quad -\frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta(r)dr+\sigma\Big),\end{aligned}$$

as a linear operator, and we find

$$\begin{aligned}& k_n(t) - \int_0^t \mathcal{K}(r, w_r + \bar{m}_{n_r})dr \\ & -\frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-s)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r+\bar{m}_{n_r})dr - \int_0^q \mathcal{K}(r, w_r+\bar{m}_{n_r})dr\Big) \\ = \;& \frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}\zeta_n(r)dr \\ & +\frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta_n(r)dr - \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta_n(r)dr+\sigma\Big).\end{aligned}$$

Obviously, since $k_n \to k_*$ and the function $\mathcal{K}$ is continuous, we obtain, as $n \to \infty$,

$$\begin{aligned}\|k_n(t) & - \int_0^t \mathcal{K}(r, w_r+\bar{m}_{n_r})dr - \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r+\bar{m}_{n_r})dr \\ & - \int_0^q \mathcal{K}(r, w_r+\bar{m}_{n_r})dr\Big) - \Big\{k_*(t) - \int_0^t \mathcal{K}(r, w_r+\bar{m}_{*_r})dr \\ & - \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r+\bar{m}_{*_r})dr - \int_0^q \mathcal{K}(r, w_r+\bar{m}_{*_r})dr\Big)\Big\}\| \to 0,\end{aligned}$$

which yields as $n \to \infty$,

$$\begin{aligned}&\|\frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}(\zeta_n(r)-\zeta_*(r))dr + \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}(\zeta_n(r)-\zeta_*(r))dr \\ &-\frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}(\zeta_n(r)-\zeta_*(r))dr\Big)\| \to 0,\end{aligned}$$

which means $\Pi \circ S_\Psi$ is continuous, and it implies from Lemma 1 that $\Pi \circ S_\Psi$ has a closed graph. In addition, we find $k_n(t) - A_n(t) \in \Pi(S_{\Psi, w+\bar{m}_n})$, where

$$A_n(t) = \int_0^t \mathcal{K}(r, w_r+\bar{m}_{n_r})dr - \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-s)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r+\bar{m}_{n_r})dr - \int_0^q h(s, w_s+\bar{m}_{n_s})ds\Big).$$

Since $m_n \to m_*$, therefore, there exists $\zeta_* \in S_{\Psi, w+\bar{m}_*}$, such that

$$\begin{aligned}& k_*(t) - \int_0^t \mathcal{K}(r, w_r+\bar{m}_{*_r})dr - \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-s)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r+\bar{m}_{*_r})dr \\ & - \int_0^q \mathcal{K}(r, w_s+\bar{m}_{*_s})ds\Big) \\ = \;& \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\zeta_*(s)ds + \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i\int_0^{\kappa_i}\frac{(\kappa_i-s)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta_*(r)dr - \int_0^q\frac{(q-s)^{\eta-1}}{\Gamma(\eta)}\zeta_*(r)dr+\sigma\Big),\end{aligned}$$

which proves that $k_* \in \mathcal{F}(m_*)$, and hence the operator $\mathcal{F}$ has a closed graph.

*Step 5.* In the last step, we prove that there exists an open set $\Omega \subseteq \bar{\mathfrak{L}}_q$ with $m \notin \gamma\mathcal{F}(m)$ for any $\gamma \in (0,1)$ and all $m \in \partial\Omega$. So, assume that $m \in \gamma\mathcal{F}(m)$ for $\gamma \in (0,1)$ and $m \in \bar{\mathfrak{L}}_q$. Then, there exists $\zeta \in L^1(\mathcal{I}, \mathbb{R})$ with $\zeta \in S_{\Psi, w+\bar{m}}$ such that we have

$$\begin{aligned} m(t) &= \gamma\Big\{ \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1}\zeta(r)dr \\ &\quad + \frac{t}{\Lambda_1}\Big( \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta(r)dr + \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_r)dr \\ &\quad - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr - \frac{1}{\Gamma(\eta)} \int_0^q (q-r)^{\eta-1}\zeta(s)ds + \sigma \Big) \Big\}, \text{ for } t \in \mathcal{I}. \end{aligned}$$

As we have completed in Step 2, for every $t \in \mathcal{I}$, we find

$$\begin{aligned} |m(t)| &\leq \int_0^t |\mathcal{K}(r, w_r + \bar{m}_r)|ds + \frac{1}{\Gamma(\eta)} \int_0^t (t-r)^{\eta-1}|\zeta(r)|dr \\ &\quad + \frac{t}{|\Lambda_1|}\Big( \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}|\zeta(r)|dr + \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{-\mu}}{\Gamma(1-\mu)}|\mathcal{K}(r, w_r + \bar{m}_r)|dr \\ &\quad + \int_0^q |\mathcal{K}(r, w_r + \bar{m}_r)|ds + \frac{1}{\Gamma(\eta)} \int_0^q (q-r)^{\eta-1}|\zeta(r)|dr + |\sigma| \Big) \\ &\leq \int_0^t [K_1\|w_r + \bar{m}_r\|_{\mathfrak{L}} + K_2]ds + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1}[N(s)\Phi(\|w_r + \bar{m}_r\|_{\mathfrak{L}})]dr \\ &\quad + \frac{t}{|\Lambda_1|}\Big( \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}[N(s)\Phi(\|w_r + \bar{m}_r\|_{\mathfrak{L}})]ds \\ &\quad + \sum_{i=1}^p \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}[K_1\|w_r + \bar{m}_r\|_{\mathfrak{L}} + K_2]dr + \int_0^q [K_1\|w_r + \bar{m}_r\|_{\mathfrak{L}} + K_2]dr \\ &\quad + \frac{1}{\Gamma(\eta)} \int_0^q (q-r)^{\eta-1}[N(r)\Phi(\|w_r + \bar{m}_r\|_{\mathfrak{L}})]dr + |\sigma| \Big) \\ &\leq [K_1(\lambda_q\|m\|_{\bar{\mathfrak{L}}_q} + \delta_q\|\beta\|_{\mathfrak{L}}) + K_2]\Big( q + \frac{q}{|\Lambda_1|}\Big( \sum_{i=1}^p \frac{\vartheta_i \kappa_i^{1-\mu}}{\Gamma(2-\mu)} + q \Big) \Big) \\ &\quad + \Phi(\lambda_q\|m\|_{\bar{\mathfrak{L}}_q} + \delta_q\|\beta\|_{\mathfrak{L}})\, N^* \\ &\quad \times \Big( \frac{q^\eta}{\Gamma(\eta+1)} + \frac{q}{|\Lambda_1|}\Big( \sum_{i=1}^p \frac{\vartheta_i \kappa_i^{\eta-\mu}}{\Gamma(\eta-\mu+1)} + \frac{q^\eta}{\Gamma(\eta+1)} \Big) \Big) + \frac{q}{|\Lambda_1|}|\sigma| \\ &\leq [K_1(\lambda_q\|m\|_{\bar{\mathfrak{L}}_q} + \delta_q\|\beta\|_{\mathfrak{L}}) + K_2]\Lambda_3 + \Phi(\lambda_q\|m\|_{\bar{\mathfrak{L}}_q} + \delta_q\|\beta\|_{\mathfrak{L}})\, N^*\, \Lambda_2 + \frac{q}{|\Lambda_1|}|\sigma|, \end{aligned}$$

which leads to

$$\frac{(1 - \lambda_q K_1 \Lambda_3)\|m\|_{\bar{\mathfrak{L}}_q}}{(K_2 + K_1\delta_q\|\beta\|_{\mathfrak{L}})\Lambda_3 + \Phi(\lambda_q\|m\|_{\bar{\mathfrak{L}}_q} + \delta_q\|\beta\|_{\mathfrak{L}})\, N^*\, \Lambda_2 + \frac{q}{|\Lambda_1|}|\sigma|} \leq 1.$$

By $(C_4)$, a constant $\widehat{\theta}$ can be found such that $\|m\|_{\bar{\mathfrak{L}}_q} \neq \widehat{\theta}$. Let us define the set $\Omega = \{m \in \bar{\mathfrak{L}}_q : \|m\|_{\bar{\mathfrak{L}}_q} < \widehat{\theta}\}$, and observe that $\mathcal{F} : \overline{\Omega} \to \mathcal{G}(\bar{\mathfrak{L}}_q)$ is upper semicontinuous and compact with closed and convex values. According to this choice of $\Omega$, we cannot find any $m \in \partial\Omega$ such that $m \in \gamma\mathcal{F}(m)$ for some $\gamma \in (0,1)$. As a result, by Lemma 2, we conclude that there exists a fixed point $m$ of $\mathcal{F}$ in $\overline{\Omega}$, which is a solution to problem (1)–(2). □

*3.2. The Lipschitz Case*

Let us define a metric $T_d : \mathcal{G}(Y) \times \mathcal{G}(Y) \to \mathbb{R} \cup \{\infty\}$ by

$$T_d(B,H) = \max\left\{\sup_{b\in B} d(b,H), \sup_{h\in H} d(B,h)\right\},$$

where $d(b,H) = \inf_{h\in H} d(b,h)$ and $d(B,h) = \inf_{b\in B} d(b,h)$, and $(Y,d)$ is a metric space generated from $(Y,\|\cdot\|)$. So, $(\mathcal{G}_{cl,b}(Y), T_d)$ forms a metric space (see [18]).

**Definition 5** ([41]). *A mapping $\mathcal{S}: Y \to \mathcal{G}_{cl}(Y)$ is said to be a contraction if and only if there is a constant $0 < \chi < 1$ such that*

$$T_d(\mathcal{S}(u), \mathcal{S}(v)) \le \chi d(u,v) \text{ for each } u, v \in Y.$$

The following Lemma is due to Covitz and Nadler [41].

**Lemma 3** ([41]). *If $\mathcal{S}: Y \to \mathcal{G}_{cl}(Y)$ is a contraction mapping in a complete metric space $(Y,d)$, then $Fix\mathcal{S} \neq \emptyset$.*

**Theorem 2.** *Let the following assumptions be satisfied:*

$(A_1)$ $\Psi : \mathcal{I} \times \mathfrak{L} \to \mathcal{G}_{cp}(\mathbb{R})$ *such that, for each* $y \in \mathbb{R}$, $\Psi(\cdot, y) : \mathcal{I} \to \mathcal{G}_{cp}(\mathbb{R})$ *is measurable.*

$(A_2)$ $T_d(\Psi(t,y), \Psi(t,y^*)) \le \varrho(t)\|y - y^*\|_{\mathfrak{L}}$ *for almost all* $t \in \mathcal{I}$ *and* $y, y^* \in \mathfrak{L}$ *with* $\varrho \in C(\mathcal{I}, \mathbb{R}^+)$, *and* $d(0, \Psi(t,0)) \le \varrho(t)$ *for almost all* $t \in \mathcal{I}$.

$(A_3)$ *A constant* $\mathcal{M} > 0$ *exists such that*

$$|\mathcal{K}(t,y) - \mathcal{K}(t,y^*)| \le \mathcal{M}\|y - y^*\|_{\mathfrak{L}}, \quad \forall\, t \in \mathcal{I} \text{ and every } y,\, y^* \in \mathfrak{L}.$$

*Then, there exists at least one solution on $(-\infty, q]$ to problem (1)–(2) if*

$$\lambda_q(\|\varrho\|\Lambda_2 + \mathcal{M}\Lambda_3) < 1, \tag{17}$$

*where $\lambda_q$, $\Lambda_2$, and $\Lambda_3$ are, respectively, given by* (3), (14), *and* (15).

**Proof.** Obviously, by hypothesis $(A_1)$, the set $S_{\Psi,w+\bar{m}}$ is nonempty for each $m \in \bar{\mathfrak{L}}_q$. So, in view of Theorem III.6 [42], $\Psi$ has a measurable selection. Now, we need to verify that the assumptions of Lemma 3 hold. Firstly, we show that for each $m \in \bar{\mathfrak{L}}_q$, $\mathcal{F}(m) \in \mathcal{G}_{cl}(\bar{\mathfrak{L}}_q)$. Let $\{z_n\}_{n\ge 0} \in \mathcal{F}(m)$ be a convergent sequence such that $z_n \to z$ in $\bar{\mathfrak{L}}_q$ as $n \to \infty$. So, $z \in \bar{\mathfrak{L}}_q$ and there is $\zeta_n \in S_{\Psi,w+\bar{m}_n}$ such that

$$\begin{aligned} z_n(t) \;=\; & \int_0^t \mathcal{K}(r, w_r + \bar{m}_{n_r})dr + \int_0^t \frac{(t-r)^{\eta-1}}{\Gamma(\eta)}\zeta_n(r)dr \\ & + \frac{t}{\Lambda_1}\Big(\sum_{i=1}^{p}\vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}\zeta_n(r)dr + \sum_{i=1}^{p}\vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)}\mathcal{K}(r, w_r + \bar{m}_{n_r})dr \\ & - \int_0^q \mathcal{K}(r, w_r + \bar{m}_{n_r})dr - \frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}\zeta_n(r)dr + \sigma\Big), \end{aligned}$$

for each $t \in \mathcal{I}$. By the compactness of the values of $\Psi$, take a subsequence to find that $\zeta_n \to \zeta$ in $L^1(\mathcal{I}, \mathbb{R})$. Hence, $\zeta \in S_{\Psi,w+\bar{m}}$ and we have

$$
\begin{aligned}
z_n(t) \to z(t) \quad = \quad & \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)} \int_0^t (t-r)^{\eta-1}\zeta(r)dr \\
& + \frac{t}{\Lambda_1}\Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)} \zeta(s)dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{-\mu}}{\Gamma(1-\mu)} \mathcal{K}(r, w_r + \bar{m}_r)dr \\
& - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr - \frac{1}{\Gamma(\eta)} \int_0^q (q-r)^{\eta-1}\zeta(r)dr + \sigma \Big),
\end{aligned}
$$

for each $t \in \mathcal{I}$. Consequently, $z \in \mathcal{F}(m)$.

In the next step, we investigate the existence of $0 < \chi < 1$ $(\chi = \lambda_q(\|\varrho\|\Lambda_2 + \mathcal{M}\Lambda_3))$ such that

$$
T_d(\mathcal{F}(m), \mathcal{F}(m^*)) \le \chi \|m - m^*\|_{\bar{\mathfrak{L}}_q} \text{ for each } m, m^* \in \bar{\mathfrak{L}}_q.
$$

Let $m, m^* \in \bar{\mathfrak{L}}_q$, and $k_1 \in \mathcal{F}(m)$. Then, there is $\zeta_1(t) \in \Psi(t, w_t + \bar{m}_t)$ such that

$$
\begin{aligned}
k_1(t) \quad = \quad & \int_0^t \mathcal{K}(r, w_r + \bar{m}_r)dr + \frac{1}{\Gamma(\eta)} \int_0^t (t-r)^{\eta-1}\zeta_1(r)dr \\
& + \frac{t}{\Lambda_1}\Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{\eta-\mu-1}}{\Gamma(\eta-\mu)} \zeta_1(r)dr + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{-\mu}}{\Gamma(1-\mu)} \mathcal{K}(r, w_r + \bar{m}_r)dr \\
& - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r)dr - \frac{1}{\Gamma(\eta)} \int_0^q (q-r)^{\eta-1}\zeta_1(r)dr + \sigma \Big),
\end{aligned}
$$

for each $t \in \mathcal{I}$. In view of $(A_2)$, we find

$$
T_d(\Psi(t, w_t + \bar{m}_t), \Psi(t, w_t + \bar{m}_t^*)) \le \varrho(t)\|m_t - m_t^*\|_{\mathfrak{L}}.
$$

Thus, we can select $u \in \Psi(t, w_t + \bar{m}_t^*)$ such that

$$
|\zeta_1(t) - u| \le \varrho(t)\|m_t - m_t^*\|_{\mathfrak{L}}, \quad t \in \mathcal{I}.
$$

Now, let us define $\Theta : \mathcal{I} \to \mathcal{P}(\mathbb{R})$ by

$$
\Theta(t) = \{u \in \mathbb{R} : |\zeta_1(t) - u| \le \varrho(t)\|m_t - m_t^*\|_{\mathfrak{L}}\}.
$$

According to Proposition III.4 [42] and since $\Theta(t) \cap \Psi(t, w + \bar{m}_t^*)$ is measurable, there exists, for each $t \in \mathcal{I}$, a measurable selection $\zeta_2(t)$ for $\Theta$ such that $\zeta_2(t) \in \Psi(t, w_t + m_t^*)$ and $|\zeta_1(t) - \zeta_2(t)| \le \varrho(t)\|m_t - m_t^*\|_{\mathfrak{L}}$.

So, let us introduce

$$
\begin{aligned}
k_2(t) \quad = \quad & \int_0^t \mathcal{K}(s, w_r + \bar{m}_r^*)dr + \int_0^t \frac{(t-r)^{\eta-1}}{\Gamma(\eta)} \zeta_2(r)dr \\
& + \frac{t}{\Lambda_1}\Big( \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - r)^{\eta-\mu-1}}{\Gamma(\eta-\mu)} \zeta_2(r)ds + \sum_{i=1}^{p} \vartheta_i \int_0^{\kappa_i} \frac{(\kappa_i - s)^{-\mu}}{\Gamma(1-\mu)} \mathcal{K}(r, w_r + \bar{m}_r^*)dr \\
& - \int_0^q \mathcal{K}(r, w_r + \bar{m}_r^*)dr - \int_0^q \frac{(q-r)^{\eta-1}}{\Gamma(\eta)} \zeta_2(r)dr + \sigma \Big).
\end{aligned}
$$

Then,

$$\begin{aligned}
|k_1(t)-k_2(t)| \quad &\leq \quad \int_0^t |\mathcal{K}(r,w_r+\bar{m}_r)dr-\mathcal{K}(r,w_r+\bar{m}_r^*)|dr \\
&+\frac{1}{\Gamma(\eta)}\int_0^t (t-r)^{\eta-1}|\zeta_1(r)-\zeta_2(r)|dr \\
&+\frac{t}{|\Lambda_1|}\Big(\sum_{i=1}^p \vartheta_i \int_0^{\kappa_i}\frac{(\kappa_i-s)^{\eta-\mu-1}}{\Gamma(\eta-\mu)}|\zeta_1(s)-\zeta_2(r)|dr \\
&+\sum_{i=1}^p \vartheta_i \int_0^{\kappa_i}\frac{(\kappa_i-r)^{-\mu}}{\Gamma(1-\mu)}|\mathcal{K}(r,w_r+\bar{m}_r)dr-\mathcal{K}(r,w_r+\bar{m}_r^*)|dr \\
&+\int_0^q |\mathcal{K}(r,w_r+\bar{m}_r)dr-\mathcal{K}(r,w_r+\bar{m}_r^*)|dr \\
&+\frac{1}{\Gamma(\eta)}\int_0^q (q-r)^{\eta-1}|\zeta_1(r)-\zeta_2(r)|dr\Big) \\
&\leq \quad \Big[\|\varrho\|\Big(\frac{q^\eta}{\Gamma(\eta+1)}+\frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^p \frac{\vartheta_i\kappa_i^{\eta-\mu}}{\Gamma(\eta-\mu+1)}+\frac{q^\eta}{\Gamma(\eta+1)}\Big)\Big) \\
&+\mathcal{M}\Big(q+\frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^p \frac{\vartheta_i\kappa_i^{1-\mu}}{\Gamma(2-\mu)}+q\Big)\Big)\Big]\|m_t-m_t^*\|_{\mathfrak{L}} \\
&\leq \quad \Big[\|\varrho\|\Big(\frac{q^\eta}{\Gamma(\eta+1)}+\frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^p \frac{\vartheta_i\kappa_i^{\eta-\mu}}{\Gamma(\eta-\mu+1)}+\frac{q^\eta}{\Gamma(\eta+1)}\Big)\Big) \\
&+\mathcal{M}\Big(q+\frac{q}{|\Lambda_1|}\Big(\sum_{i=1}^p \frac{\vartheta_i\kappa_i^{1-\mu}}{\Gamma(2-\mu)}+q\Big)\Big)\Big]\lambda_q\|m-m^*\|_{\mathfrak{L}_q}.
\end{aligned}$$

Hence,

$$\|k_1-k_2\| \quad \leq \quad \lambda_q[\|\varrho\|\Lambda_2+L\Lambda_3]\|m-m^*\|_{\bar{\mathfrak{L}}_q},$$

which implies, by switching the roles of $m$ and $m^*$, that

$$T_d(\mathcal{F}(m),\mathcal{F}(m^*)) \quad \leq \quad \lambda_q[\|\varrho\|\Lambda_2+L\Lambda_3]\,\|m-m^*\|_{\bar{\mathfrak{L}}_q}.$$

Thus, $\mathcal{F}$ is a contraction. Therefore, in view of Lemma 3, we conclude that $\mathcal{F}$ has a fixed point on $(-\infty,q]$, which is indeed a solution to (1)–(2). The conclusion of the theorem is proved. □

**Remark 1.** *Letting $\mathcal{K}(t,y_t)$ be identically zero ($\mathcal{K}(t,y_t)=0$ for all $t\in[0,q]$) leads to some new results related to the problem:*

$$\begin{cases} {}^{C}D_{0+}^{\eta}y(t)\in\Psi(t,y_t),\ t:=[0,q], \\ y(t)=\beta(t),\ \ y(q)=\sum_{i=1}^p \vartheta_i {}^{C}D_{0+}^{\mu}y(\kappa_i)+\sigma,\ \ \kappa_i\in(0,q), \end{cases} \tag{18}$$

*where the conditions $(C_3)$ and $(A_3)$ in Theorems 1 and 2, respectively, will be omitted and the conditions (16) and (17) will be modified by putting $\Lambda_3=0$.*

### 3.3. Examples

This subsection is devoted to demonstrating our main results.

Consider the following inclusion problem:

$$\begin{cases} {}^C D_{0^+}^{5/4}[y(t) - \int_0^t \mathcal{K}(r, y_r)dr] \in \Psi(t, y_t),\ t \in [0,2], \\ y(t) = \beta(t)\ t \in (-\infty, 0],\ \ y(2) = \frac{1}{4} {}^C D_{0^+}^{1/3} y(9/5) + \frac{3}{4} {}^C D_{0^+}^{1/3} y(7/4) + 3, \end{cases} \tag{19}$$

where $\eta = 5/4$, $\mu = 1/3$, $p = 2$, $\kappa_1 = 9/5$, $\kappa_2 = 7/4$, $\vartheta_1 = 1/4$, $\vartheta_2 = 3/4$, $\sigma = 3$, and $\Psi(t, y_t)$, $\mathcal{K}(t, y_t)$, $\beta(t)$ will be fixed later. We obtain, in view of the above data, that $\Lambda_1 = 0.383731105$, $\Lambda_2 = 20.84793204$, $\Lambda_3 = 22.09613910$, where $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ are, respectively, given by (6), (14),and (15).

Let us define, for a continuous function $\alpha : (-\infty, 0] \to [0, \infty)$ satisfying $l = \int_{-\infty}^0 \alpha(\tau)d\tau < \infty$, the space $\mathfrak{L}_\alpha = \{y \in C((-\infty, 0], \mathbb{R}) : \int_{-\infty}^0 \alpha(\tau)\|y\|_{[\tau,0]}d\tau < \infty\}$, where $\|y\|_{[\tau,0]} = \sup_{t\in[\tau,0]} |y(t)|$. Choose $\alpha(\tau) = e^{2\tau}$ such that $\int_{-\infty}^0 e^{2\tau}d\tau = \frac{1}{2}$, and define a norm on this space by $\|y\|_{\mathfrak{L}_\alpha} = \int_{-\infty}^0 \alpha(\tau)\|y\|_{[\tau,0]}ds$. Hence, the space $(\mathfrak{L}_\alpha, \|.\|_{\mathfrak{L}_\alpha})$ is indeed a phase space that satisfies the axioms $(S_1) - (S_3)$ presented in Section 2, with $\lambda(t) = \frac{1}{2}, \delta(t) = 1, l = 2$; see [36].

Now, assume the function $\beta(t)$ to be continuous with $\beta(0) = 0$ and $\int_{-\infty}^0 e^{2\tau}\|\beta\|_{[\tau,0]}d\tau < \infty$. Thus, $\beta \in \mathfrak{L}_\alpha$. For instance, we can choose $\beta(t) = e^t - e^{3t}$, which is a continuous function with $\beta(0) = 0$, and it is clear that $\beta \in \mathfrak{L}_\alpha$, i.e., $\int_{-\infty}^0 e^{2\tau}\|\beta\|_{[\tau,0]}d\tau < \infty$.

For the first example, we illustrate Theorem 1 by choosing

$$\Psi(t, y_t) = \Big[\frac{1}{2\sqrt{t^2 + 900}} \int_{-\infty}^0 e^{2\tau}\Big(\frac{2|y_\tau|}{|y_\tau| + 1} + \sin y_\tau\Big)d\tau\ ,\ \frac{1}{120(2 + t^2)} \int_{-\infty}^0 e^{2\tau}\Big(2\tan^{-1} y_\tau + y_\tau\Big)d\tau\Big], \tag{20}$$

and

$$\mathcal{K}(t, y_t) = \frac{1}{(t + 12)^2}\Big(\int_{-\infty}^0 e^{2\tau} y_\tau d\tau + 1/3\Big), \tag{21}$$

where, in view of $(C_2)$, for $t \in [0,2]$, we find

$$\begin{aligned} \Big|\frac{1}{2\sqrt{t^2 + 900}} \int_{-\infty}^0 e^{2\tau}\Big(\frac{2|y_\tau|}{|y_\tau| + 1} + \sin y_\tau\Big)d\tau\Big| &\le \frac{1}{2\sqrt{t^2 + 900}}\Big|\int_{-\infty}^0 e^{2\tau}(2 + y_\tau)d\tau\Big| \\ &= \frac{1}{2\sqrt{t^2 + 900}}\Big|\int_{-\infty}^0 e^{2\tau}2d\tau + \int_{-\infty}^0 e^{2\tau} y_\tau d\tau\Big| \\ &\le \frac{1}{2\sqrt{t^2 + 900}}(1 + ||y_t||_{\mathfrak{L}_\alpha}) \end{aligned}$$

and

$$\begin{aligned} \Big|\frac{1}{120(2 + t^2)} \int_{-\infty}^0 e^{2\tau}\Big(2\tan^{-1} y_\tau + y_\tau\Big)d\tau\Big| &\le \frac{1}{120(2 + t^2)}\Big|\int_{-\infty}^0 e^{2\tau}(2 + y_\tau)d\tau\Big| \\ &\le \frac{1}{2\sqrt{t^2 + 900}}(1 + ||y_t||_{\mathfrak{L}_\alpha}), \end{aligned}$$

which implies that $N(t) = \dfrac{1}{2\sqrt{t^2 + 900}}$ and $\Phi(\|y\|_{\mathfrak{L}_\alpha}) = ||y||_{\mathfrak{L}_\alpha} + 1$. Similarly, $(C_3)$ is satisfied with $K_1 = \sup_{t\in[0,2]} \frac{1}{(t+12)^2} = \frac{1}{144}$, $K_2 = \sup_{t\in[0,2]} \frac{1}{3(t+12)^2} = \frac{1}{432}$.

By condition $(C_4)$, it is found that $\widehat{\theta} > 14.71361194$, such that

$$\frac{(1-\lambda_q K_1 \Lambda_3)\widehat{\theta}}{(K_2 + K_1\delta_q\|\beta\|_{\mathfrak{L}_\alpha})\Lambda_3 + \Phi(\lambda_q\widehat{\theta} + \delta_q\|\beta\|_{\mathfrak{L}_\alpha})\, N^*\Lambda_2 + \frac{q}{|\Lambda_1|}|\sigma|} > 1,$$

where $\|\beta\|_{\mathfrak{L}_\alpha} = \int_{-\infty}^{0} e^{2\tau}\|e^t - e^{3t}\|_{[\tau,0]}d\tau \approx 0.19245$, $\lambda_q = 1/2$, $\delta_q = 1$.

So, all the conditions of Theorem 1 hold, and as a result, we deduce that problem (19) has at least one solution on $(-\infty, 2]$, with $\Psi(t, y_t)$ and $\mathcal{K}(t, y_t)$ defined by (20) and (21), respectively.

In the second example, for demonstrating the applicability of Theorem 2, let us consider

$$\Psi(t, y_t) = \left[\frac{2}{36+t}\Big(\int_{-\infty}^{0} e^{2\tau}\tan^{-1} y_\tau d\tau + \cos t\Big)\,,\ \frac{(1+t)}{90}\int_{-\infty}^{0} e^{2\tau}\frac{|y_\tau|}{|y_\tau|+1}d\tau + e^{-t}\right], \quad (22)$$

and

$$\mathcal{K}(t, y_t) = \frac{e^{-t}}{(t+15)^2}\Big(\int_{-\infty}^{0} e^{2\tau}\sin y_\tau d\tau + 1\Big). \quad (23)$$

Clearly,

$$T_d(\Psi(t,y), \Psi(t,y^*)) \le \frac{2}{36+t}\|y - y^*\|_{\mathfrak{L}}.$$

So, let $\varrho(t) = \frac{2}{36+t}$, and we can easily check that $d(0, \Psi(t,0)) \le \varrho(t)$ is satisfied for almost all $t \in [0,2]$. Also, $(A_3)$ holds with $\mathcal{M} = 1/225$. Then, $\lambda_q(\|\varrho\|\Lambda_2 + \mathcal{M}\Lambda_3) \approx 0.65465994 < 1$. As a consequence of satisfying all the hypotheses of Theorem 2, problem (19) has at least one solution on $(-\infty, 2]$, with $\Psi(t, y_t)$ and $\mathcal{K}(t, y_t)$ defined by (22) and (23), respectively.

## 4. Conclusions

This paper studied a new class of FBVPs involving neutral inclusion with infinite delay, supplemented with nonlocal BCs. The existence theory of multivalued maps with infinite delay was developed by studying two cases: convex and nonconvex multivalued functions. The nonlinear Leray–Schauder alternative type for multivalued maps and Covitz and Nadler's theorems on arbitrary phase space were applied to establish the desired results. The results were demonstrated through two illustrative examples. The findings presented in this paper contributed new insights into fractional inclusions with infinite delay, leading to specific results when $\mathcal{K}(t, y_t) \equiv 0$, as noted in Remark 1. Furthermore, many efforts can be made to extend the prior results on fractional differential equations by replacing the single-valued function on the right-hand side of the equation with a multivalued one.

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