fractal and fractional

Article

Topology of Locally and Non-Locally Generalized Derivatives

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Abstract: This article investigates the continuity of derivatives of real-valued functions from a topological perspective. This is achieved by the characterization of their sets of discontinuity. The same principle is applied to Gateaux derivatives and gradients in Euclidean spaces. This article also introduces a generalization of the derivatives from the perspective of the modulus of continuity and characterizes their sets of discontinuities. There is a need for such generalizations when dealing with physical phenomena, such as fractures, shock waves, turbulence, Brownian motion, etc.

Keywords: singular functions; non-differentiable functions; derivatives; continuity; fractional derivatives and integrals

MSC: 26A27; 26A15; 26A30; 33C10; 26A33

1. Introduction

Physically, a derivative can be interpreted as the rate of change in one continuous quantity compared to another, measured at space–time scale, which can be considered as "infinitesimal". On the other hand, scientific developments in the last 100 years indicate that the use of functions, non-differentiable in this classical sense, cannot be avoided when modeling natural phenomena. For instance, the idealized process of diffusing particles, the Wiener process, has non-differentiable paths. The Ornstein–Uhlenbeck process, used in the kinetic theory of gasses, assumes non-differentiable velocity fields [1]. Turbulence can also exhibit a non-differentiable acceleration field [2]. In a closely related manner, the paths in Feynman's path-integral approach to quantum mechanics are non-differentiable [3]. The deterministic approach of scale relativity theory, introduced by Nottale [4], also assumes non-differentiability of the fundamental space–time manifold. Such and other models can be viewed as an idealization indicating that the actual dynamics play out at time-scales that are incommensurable with the scale of observation of the process.

Discontinuities in spatial gradients are essential elements of the models of certain physical phenomena. Shock waves in fluid dynamics represent abrupt changes in pressure, density, and velocity. These changes lead to discontinuities in the derivatives of the flow variables [5]. The behavior of gases in phase transitions, such as the transition from gas to liquid state, can exhibit discontinuous derivatives in thermodynamic quantities like pressure and volume.

Fractals are geometrical objects featuring both self-similarity and non-differentiability. Fractal shapes are ubiquitous in nature [6]. Fractals are closely related to mathematical "monsters", such as the non-differentiable functions of Weierstrass, Takagi, Bolzano, etc. More "well-behaving" but still surprising are the singular functions of Cantor–Lebesgue [7],

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Minkowski, and the Smith–Cantor–Volterra function [8], which grow only on disconnected "Cantor dust"-types of sets. Such functions arise in a variety of problems—ranging from number theory [9,10] to probability [11]. Such concepts have been used, for example, in modeling fractures and their mechanics [12]. With the rising awareness about fractals, some extensions of calculus, e.g., of derivatives, have been put forward in order to describe such phenomena. All these objects present challenges for their description by the differential calculus apparatus—that is, by derivatives and integrals of functions.

Purely mathematically, the derivatives can be generalized in several ways. For example, a derivative can be defined as the limit of difference quotients on the accumulation sets of points ([13], ch 3, p. 105). This is a profound definition, which can be immediately applied to derivatives on Cantor sets in the scope of fractal calculus [14,15]. On the other hand, the question of the continuity on compact intervals of so-defined functions requires further clarification.

From a different perspective, assuming the usual topology of the real line, the derivatives can be generalized also by a fractionalization using the formal substitution $\Delta x \to (\Delta x)^{\beta}$ in the difference quotient. This leads to the concept of *fractional velocity* as defined by Cherbit [16]. This "quasi-differential" operator was introduced by analogy with the Hausdorff dimension as a tool to study the fractal phenomena and physical processes for which instantaneous velocity was not well defined It should be noted that a more suitable term would be a pseudo-differential operator; however, the latter has been loaded with a different meaning in the literature of Fourier analysis. Several works have demonstrated that the resulting image functions are only trivially continuous in the usual topology of the real line—that is, they vanish [17,18]. On the other hand, interesting applications to singular functions have been also demonstrated [19]. It is customary in the contemporary literature to formally substitute derivatives with non-local fractional-order operators, such as the Riemann–Liouville differ-integral. In the present contribution, I do not argue in favor of the formal fractionalization approach. Hristov convincingly argues that the appropriate fractionalization approach should start from the fractionalization of constitutive relations [20].

The definition of a *local fractional derivative* introduced by Kolwankar and Gangal [21] is based on the localization of Riemann–Liouville fractional derivatives toward a particular point of interest. This was a parallel development bridging the theory of fractional calculus. The intended use of this operator is for describing temporal evolution. Later studies demonstrate that whenever the fractional velocity exists point-wise for a given exponent, its value is equal to the local fractional derivative for the same exponent [17]. An interesting new development is the concept of fractional derivatives on fractal sets [22], which can be viewed also as fractional derivatives on Banach spaces. The triviality result discussed above limits the potential applications of the fractional velocity. A way to overcome this obstacle is to broaden the definition to Cantor-dust-type fractal sets. However, this limits the application to a particular type of fractal.

From a purely mathematical perspective, there is no reason to limit the choice of the function in the denominator of the difference quotient only to a power function. Such is the perspective of the present contribution: the form of the function is only constrained by some reasonable, from an approximation perspective, choices. Notably, it is required that a generalized Taylor–Lagrange property holds. The concept has been introduced as the *modular derivative* [23]. The present work studies the properties of this quasi-differential operator from a topological perspective.

This paper is organized as follows. Section 2 introduces general conventions. Section 3 introduces topological spaces. Section 4 characterizes the oscillation and discontinuity sets of the functions. Section 5 demonstrates applications for fractional gradients in Euclidean spaces. Section 6 studies the modular derivatives on the real line and Banach spaces, respectively. Section 7 discusses the obtained results from the perspective of the available literature.

The main contributions of this paper can be stated as follows. The discontinuity set of a given function is "small" in a precise technical sense—i.e., it is a meager set. This has implications for various applied branches of calculus—such as the vector analysis and the fractional calculus. Furthermore, the g-derivatives can induce a topology defined on their sets of change under which such g-derivatives remain continuous in the sense of Theorem 3. This is coherent with the results obtained for the calculus defined on Cantor sets.

2. Preliminaries

Definitions and Conventions

The term *variable* denotes a symbol that represents an unspecified real number. Sets are denoted by capital letters, while variables taking values in sets are denoted by lowercase. The set of real numbers is denoted by $\mathbb{R}$. The term *function* denotes a mapping from one number to another, and the action of the function is denoted as $f(x) = y$. Implicitly, the mapping acts on the real numbers: $f : \mathbb{R} \mapsto \mathbb{R}$. The co-domain of the function $f : X \mapsto Y$ is denoted as $f[X] = Y$. Everywhere, ϵ will be considered as a small positive variable or a sequence of such, depending on the context. $\mathcal{C}[A]$ denotes the space of continuous functions on the given base set A. The symbol $\mathbb{C}^n$ denotes an n-times differentiable function in an unspecified open interval.

Definition 1 (Asymptotic small $\mathcal{O}$ notation)**.** *The notation $\mathcal{O}(x^\alpha)$ is interpreted as the convention*

$$\lim_{x\to 0} \frac{\mathcal{O}(x^\alpha)}{x^\alpha} = 0$$

for $\alpha > 0$. Or, in general terms,

$$\mathcal{O}(g(x)) \Rightarrow \lim_{x\to 0} \frac{\mathcal{O}(g(x))}{g(x)} = 0$$

for a decreasing function g on a right-open interval containing 0. The notation $\mathcal{O}_x$ will be interpreted to indicate a Cauchy-null sequence possibly indexed by the variable x.

The complement of the set A is denoted as A^c.

Definition 2 (Opening and closure)**.** *A closed set A is denoted by an over bar: $\bar{A}$. The closure operation of a certain set A is denoted as* **cl** *A. The set A is closed whenever* **cl** *$A = A$.*

An opening of the set A is denoted by an open circle: A°. The interior of a set A is denoted as **int** *A. The set A is open whenever* **int** *$A = A$.*

From the above, it is also clear that $\mathbf{cl}\ A = \bar{A}$ and $\mathbf{int}\ A = A^\circ$. Furthermore, **int** and **cl** will be used in an operational sense, which allows for defining algebraic rules for computations over sets. The interior and closure are dual in the sense that one can be defined from the other:

$$\mathbf{int}\ A := X \setminus \mathbf{cl}\ A^c \tag{1}$$

The notations for closed $\bar{}$, open $\circ$, and complement ^{c} take precedence over the operator notations.

3. Topological Spaces

A topological space is most often defined by its family of open sets $\mathcal{T}$.

Definition 3. *Denote by $(X, \mathcal{T})$ a topological space over the set X generated by the open set collection $\mathcal{T}$. Furthermore,*

- *The set $E \subseteq X$ is denoted as G_δ if it is a countable intersection of open sets.*
- *The set $E \subseteq X$ is denoted as F_σ if it is a countable union of closed sets.*
- *The set $E \subseteq X$ is meager if it can be expressed as the union of countably many nowhere dense subsets of X.*
- *By duality, a co-meager set is one whose complement is meager, or, equivalently, the intersection of countably many sets with dense interiors.*

An example of a meager set are the dyadic rationals $\mathbb{Q}_2$ in $\mathbb{R}$.

Definition 4 (Topological basis)**.** *A collection τ of open sets in a topological space X is called a basis for the topology if every open set in X is a union of sets in τ.*

Definition 5 (T_1 space)**.** *A T_1 topological space (also called a Fréchet space) is a type of topological space that satisfies the following condition: For every pair of distinct points x and y in the space, there exist open sets U_x and U_y such that $x \in U_x$ and $y \notin U_x$, and $y \in U_y$ and $x \notin U_y$.*

In other words, in a T_1 space, each point is closed (i.e., the singleton set containing that point is a closed set). Some properties of T_1 spaces are the following:

- Every finite set is closed.
- The intersection of all open neighborhoods of a point is just that point.

Definition 6. *The boundary ∂ operator is defined as follows. Consider the set A; then,*

$$\partial A := (\mathbf{cl}\, A) \setminus (\mathbf{int}\, A)$$

A set is both open and closed if it has an empty boundary: $\partial A = \emptyset$.

From the definition, it is apparent that the boundary set is closed as it is a meeting of two closed sets

$$\partial A = (\mathbf{cl}\, A) \cap (\mathbf{cl}\, A^c) \tag{2}$$

The boundary operator distributes partially over unions and intersections in the sense that

$$\partial(A \cup B) \subseteq \partial A \cup \partial B \tag{3}$$

$$\partial(A \cap B) \subseteq \partial A \cap \partial B \tag{4}$$

Definition 7 (Dense set)**.** *Suppose that $D \subset (X, \mathcal{T})$. D is called dense in X if $\mathbf{cl}\, D = X$. The space $(X, \mathcal{T})$ is called separable if there exists a countable, dense set in $D \subset X$.*

The above definition is given in Encyclopedia of Mathematics [24].

Definition 8 (Nowhere dense set)**.** *A subset A of a topological space $(X, \mathcal{T})$ is called nowhere dense in X if $\mathbf{cl}\ A$ contains no nonempty open subset, that is, if the interior of its closure is empty:* $\mathbf{int}(\mathbf{cl}\, A) = \emptyset$

Definition 9 (Topological Continuity)**.** *Let X,Y be topological spaces. A function $f : X \mapsto Y$ is (topologically) continuous if and only if, for every open set $U \subseteq Y$, the pre-image set $f^{-1}(U)$ is also open; that is,*

$$U^\circ = U \Longrightarrow f^{-1}(U) = f^{-1}(U)^\circ$$

Kuratowski Closure Axioms

Topological spaces can be characterized alternatively in terms of closed sets. In order for this to be achieved it is instrumental to use the closure operator **cl**, which satisfies the Kuratowski closure axioms [25]:

Definition 10. *Consider the topological space* $(X, \mathbf{cl})$*, generated by the set* X*. For any sets* $A, B \subseteq \mathrm{Pow}(X)$*, the closure operator* $\mathbf{cl} : \mathrm{Pow}(X) \mapsto \mathrm{Pow}(X)$ *has the following properties:*

$$\mathbf{cl}\, \varnothing = \varnothing \tag{K1}$$

$$A \subseteq \mathbf{cl}\, A \tag{K2}$$

$$\mathbf{cl}\, \mathbf{cl}\, A = \mathbf{cl}\, A \tag{K3}$$

$$\mathbf{cl}(A \cup B) = \mathbf{cl}\, A \cup \mathbf{cl}\, B \tag{K4}$$

A different way of writing property (K3) is

$$\mathbf{cl}\, \bar{A} = \bar{A}$$

T_1-spaces have an additional axiom that the singleton sets are closed:

$$\mathbf{cl}\, \{a\} = \{a\} \tag{K5}$$

Remark 1. *Note that axiom K4 implies that closure operators are order preserving, since if* $A \subseteq B$*, then* $A \cup B = B$*, so* $\mathbf{cl}\, B = \mathbf{cl}\, (A \cup B) = \mathbf{cl}\, A \cup \mathbf{cl}\, B$*, which implies that* $\mathbf{cl}\, A \subseteq \mathbf{cl}\, B$*.*

4. Oscillation and Discontinuity Sets of a Function

In several prior publications, the framework of oscillation operators has been employed to characterize various extensions of derivatives, consistently assuming the topology of the real line [17,23,26]. This section recasts the same concept in the most general topological sense.

Definition 11 (Oscillation set)**.** *Consider two topological spaces* A *and* X *and the function* $f : A \mapsto X$*, and define the set of oscillations on the preimage* A *as*

$$\omega_f[A] := f[\,\mathbf{cl}\, A] \setminus \mathbf{cl}\, f[A] \tag{5}$$

Using the above definition, the following proposition holds:

Proposition 1. *If* $f \cong \mathcal{C}[A] \iff \omega_f[A] = \varnothing$*.*

Proof. The proof is immediate and follows from Hausdorff's theorem (Theorem A1, Appendix A) as the negation of its statement. □

In words, the proposition states that the oscillation set of a continuous function is empty. Based on this result, it is convenient to define the set of discontinuities of a function as follows.

Definition 12 (Discontinuity set)**.** *Define the discontinuity set as the inverse image of the oscillation set:*

$$\Delta_f[A] := f^{-1} \circ \omega_f[A]$$

We need a technical result before establishing the following theorem:

Lemma 1. *The boundary ∂A of a set A has an empty interior; hence, it is nowhere dense:*

$$\partial A^\circ = \emptyset \iff \partial A^c = X$$

Proof. By definition,

$$\partial A^\circ = X \setminus \mathbf{cl}\,(\partial A)^c$$

On the other hand,

$$(\partial A)^c = X \setminus (\bar{A} \setminus A^\circ) = (A^\circ \cap X) \cup (X \setminus \mathbf{cl}\,A) = A^\circ \cup (X \setminus \mathbf{cl}\,A)$$

Therefore, the closure is

$$\mathbf{cl}\,(\partial A)^c = \mathbf{cl}\,A^\circ \cup \mathbf{cl}\,(X \setminus \bar{A}) = \mathbf{cl}\,(\bar{A} \cup (X \setminus \bar{A})) = \mathbf{cl}\,(X \cup \bar{A}) = \mathbf{cl}\,X = X$$

Therefore, $\partial A^\circ = \emptyset$. $\square$

Lemma 2. *Let $C = \bigcup_{\alpha \in A} C_\alpha$ for an index set A. Then, the union can be written as*

$$C = \bigcup_{i=1}^{\infty} C_i \cup \bigcup_{\alpha \in A_i}^{\infty} C_\alpha$$

Proof. Let $C = \bigcup_{\alpha \in A} C_\alpha$. By the axiom of choice, one can choose a countable subset $a^1 \subset A$ and write

$$C = \bigcup_{i=1} C_{a_i^1} \cup \underbrace{\bigcup_{\alpha \in A \setminus a^1} C_\alpha}_{B_1}$$

where we defined $A_1 := A \setminus a^1$. By the same argument, one can write

$$B_1 := \bigcup_{\alpha \in A_1} C_\alpha = \bigcup_{i=1} C_{a_i^2} \cup \underbrace{\bigcup_{\alpha \in A_1 \setminus a^2} C_\alpha}_{B_2} = \dots$$

and so on. Hence, the claim follows by induction. $\square$

The main result of the section comprises the following theorem.

Theorem 1 (Discontinuity set characterization)**.** *Suppose that f is discontinuous somewhere on the set $A \subseteq X$. Then, the discontinuity set is written as*

$$\Delta_f[A] = \mathbf{cl}\,A \setminus (f^{-1} \circ \mathbf{cl}\,f[A]) \tag{6}$$

Furthermore, the following decomposition holds:

$$\Delta_f[A] = \bigcup_{\alpha \in I} \partial A_\alpha$$

for some indexing set I, where the sets ∂A_α are nowhere dense and disconnected. Therefore, $\Delta_f[A]$ is a meager set.

Proof. Observe that, by hypothesis, $\Delta_f[A]$ is not empty. The first part of the claim follows from the properties of the inverse image. Suppose that f is discontinuous on some set $A \subset X$. Then, $\mathbf{cl}\, f[A] \supset f[\mathbf{cl}\, A]$. Therefore, by the properties of the inverse image,

$$f^{-1} \circ \mathbf{cl}\, f[A] \supset \mathbf{cl}\, A$$

On the other hand, $\mathbf{cl}\, A = A \cup \partial A$. Let $B = f^{-1} \circ \mathbf{cl}\, f[A]$. Then,

$$\Delta_f[A] = \mathbf{cl}\, A \setminus B = (A \cup \partial A) \setminus B = \underbrace{(A \setminus B)}_{\varnothing} \cup (\partial A \setminus B) = \partial A \setminus B$$

since $A \subset B$. Therefore, the discontinuity set is part of the boundary of A:

$$\Delta_f[A] \subseteq \partial A$$

Further, by Lemma 1, the boundary ∂A is nowhere dense. On the other hand, $\partial A \subseteq A$. Therefore, $A = \partial A$. Since A is arbitrary, then

$$\Delta_f[A] = \bigcup_{\alpha \in I} \partial A_\alpha$$

for some indexing set I. Hence, $\Delta_f[A]$ is a union of nowhere dense sets and hence is meager.

By construction, the boundary sets are disjoint; that is, $\partial A_\alpha \cap \partial A_\beta = \varnothing$ for distinct indices $\alpha \neq \beta$; hence, we can take the restriction

$$\Delta_f[A] \setminus \partial A_\beta = \bigcup_{\alpha \in I \setminus \beta} \partial A_\alpha$$

Therefore, for the closure, it holds that

$$\mathbf{cl}\, \Delta_f[A] \setminus \partial A_\beta = \mathbf{cl} \bigcup_{\alpha \in I \setminus \beta} \partial A_\alpha = \bigcup_{\alpha \in I \setminus \beta} \partial A_\alpha$$

since the boundary sets are closed. Let $p \in \mathbf{cl}\, \partial A_\beta = \partial A_\beta$, and we take the meeting as

$$\mathbf{cl}\, \Delta_f[A] \setminus \partial A_\beta \cap p = p \cap \bigcup_{\alpha \in I \setminus \beta} \partial A_\alpha = \bigcup_{\alpha \in I \setminus \beta} (p \cap \partial A_\alpha) = \varnothing$$

Therefore, the set is disconnected. By Lemma 2, the set can be written as a countable collection of nowhere dense sets as

$$\Delta_f[A] = \bigcup_{\alpha \in I} \partial A_\alpha \tag{7}$$

for some indexing set I, where the sets ∂A_α are nowhere dense and disconnected. Finally, $\Delta_f[A]$ is a union of meager sets and hence is meager. □

Corollary 1. *Under the same notation, suppose that $A \subset \mathbb{R}$. Then, $\Delta_f[A]$ is totally disconnected on $\mathbb{R}$, equipped with the usual topology.*

Proof. Boundary points of closed intervals in $\mathbb{R}$ are disconnected points. An arbitrary union of disconnected points is also disconnected. □

The results proven in this section have broad applications. In the next sections, we discuss two, but they are by no means exhaustive.

5. Applications to Euclidean Spaces $\mathbb{E}^n$

5.1. Gradients of Scalar Functions

Definition 13 (Gradient)**.** *Suppose that U is an open subset of the Euclidean space E^n, $x \in U$, and $f : U \to E$ is a function. Then, f is differentiable at x if there exists a linear operator denoted as $\nabla f(x)$ such that*

$$\lim_{||h|| \to 0} \frac{|f(x+h) - f(x) - h \cdot \nabla f(x)|}{||h||} = 0$$

The dot denotes the scalar product on E^n and $||.||$ denotes the norm. The operator $\nabla f(x)$ is called the gradient of f at x.

Proposition 2. *Suppose that the set of discontinuity of $\nabla f(x)$ is the set $\Delta_f \subset E^n$. Then, Δ_f is a union of meager sets of maximal topological dimension $n-1$. Furthermore, it can be written as*

$$\Delta_f = \bigcup_{k=0}^{n-1} \Delta_k, \quad \Delta_k \in E^k \tag{8}$$

Proof. The Euclidean space $\mathbb{E}^n$ equipped with the standard scalar product $(\cdot)$ has a norm $||h|| = \sqrt{h \cdot h}$. The norm generates a metric topology; hence, Theorem 1 applies. The boundary set of ∂A of $A \in \mathbb{E}^n$ is of dimension $n-1$. Therefore, the result follows by reduction. □

The above result implies that the gradient of a function ∇f can be discontinuous on different combinations of "Cantor dust" point sets, curves, and hyper-surfaces up to the $n-1$ dimensional subspace of $\mathbb{E}^n$. This observation qualitatively agrees with the results obtained for the velocity gradient and vorticity in [5,27], which describe such hyper-surfaces for the homogeneous Euler equations in fluid dynamics. Note that we have not discussed the curl of a vector-valued function since it is limited to only three dimensions. On the other hand, an application to the curl's proper generalization—i.e., the outer derivative in geometric calculus $\nabla \wedge$—is straightforward; however, including it here will dilute the scope of this paper.

Remark 2. *Fractal sets have fractal dimensions, which are real numbers different from their topological dimensions. The above result concerns only the topological dimensions and cannot be applied to fractal dimensions. For example, the topological dimension of the Cantor set is 0 (isolated points), while the Hausdorff dimension is* $\log 2 / \log 3 \approx 0.63093$.

5.2. Non-Local, Space-Fractional Derivatives

The so-developed theory is equally applicable to the continuity sets of the Riesz fractional Laplacian operator and the associated fractional gradient operator. The fractional Laplacian theory is an active area of research. The fractional Laplacian has been used to model reaction–diffusion systems, porous media, and ultrasound among others. Readers are directed to the work of Lischke et al. for a comprehensive overview [28].

Suppose that we have a scalar function $f : \mathbb{E}^d \mapsto \mathbb{R}$. The Fourier transform will be defined under the physics convention

$$\mathcal{F}[f](x) := \int_{\mathbb{R}^d} f(x) e^{ik \cdot x} dx^d = \hat{f}(k)$$

with an inverse

$$f(x) = \mathcal{F}^{-1}[\hat{f}](k) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(k) e^{-ik \cdot x} dk^d$$

where x denotes a d-dimensional vector and k denotes the wave vector.

The Riesz fractional Laplacian operator can be defined in the Fourier domain by [29]:

$$(-\Delta)^{\alpha} f(\mathbf{k}) := -||k||^{2\alpha} f(\mathbf{k}) \tag{9}$$

where the $||k|| = \sqrt{\mathbf{k} \cdot \mathbf{k}}$ is the modulus of the d-dimensional wave vector $\mathbf{k}$ and the dot denotes the usual scalar product; that is, in the Fourier space, we can identify the algebraic substitution

$$(-\Delta)^{\alpha} \mapsto -||k||^{2\alpha}$$

From this perspective, the fractional Laplacian can be considered as the gradient of another operator in the algebraic equation

$$-||k||^{2\alpha} = i\mathbf{k} \cdot i\mathbf{k^0} ||k||^{2\alpha-1}$$

where $\mathbf{k^0}$ is a unit vector $\mathbf{k^0} = \mathbf{k}/||k||$ in the Fourier space. The Riesz Laplacian can be re-expressed under a slightly different parametrization as

$$-(-\Delta)^{\alpha} \mapsto -||k||^{2\alpha} = i\mathbf{k} \cdot i\mathbf{k^0} ||k||^{2\alpha-1}, \quad 2\alpha - 1 = \beta$$

Therefore, a Riesz-type gradient can be defined algebraically by the expression

$$\nabla^{\beta} \mapsto i\mathbf{k^0} ||k||^{\beta} = i\mathbf{k}/||k||^{1-\beta}$$

for a suitable function space. This corresponds to a convolution in the spatial domain:

$$\nabla^{\beta} f(x) = \nabla I^{1-\beta} f(x)$$

and it has a physical interpretation in the scope of continuum mechanics [30].

Definition 14 (Riesz gradient)**.** *Define the Riesz gradient by the convolution*

$$\nabla^{\beta} f(x) := \nabla I^{1-\beta} f(x), \quad \beta > 0$$

where $I^{\beta} f(x) := \mathcal{F}^{-1}\left[||k||^{\beta} \hat{f}(k) \right]$ *is the fractional convolution integral.*

Therefore, we can reformulate Proposition 2 as

Proposition 3. *Suppose that the set of discontinuity of* $\nabla^{\beta} f(x)$ *is* Δ_f*. Then,* Δ_f *can be written as*

$$\Delta_f = \bigcup_{k=0}^{n-1} \Delta_k, \quad \Delta_k \in E^k$$

Using the above definition, the fractional Laplacian of a scalar function can be reformulated as

$$\nabla \cdot \nabla^{\beta} f(x) = -(-\Delta)^{(\beta+1)/2} f(x)$$

which is consistent with the formulation of the second Fick's law and hence can be interpreted on conventional physical grounds.

Remark 3. *The above definition of a fractional gradient is based on the Riesz fractional Laplacian definition, which is one of several possible choices [29]. From the discussion above, it is apparent that a fractional Laplacian can be discontinuous on different combinations of "Cantor dust" point sets, curves, and hyper-surfaces up to the* $n-1$ *dimensional subspace of* $\mathbb{E}^n$*.*

6. Applications to Modular Derivatives

Definition 15 (Topological limit)**.** *Let X, Y be topological spaces and $a \in X$, $b \in Y$. Suppose that f is a function such that $f : X \setminus \{a\} \mapsto Y$. Then, we write $\lim_{x \to a} f(x) = b$ if the auxiliary function $g : X \to Y$ such that*

$$g(x) := \begin{cases} f(x), & x \neq a, \\ b, & x = a \end{cases}$$

is topologically continuous at $x = a$ in the sense of Definition 9.

6.1. Modular Derivatives on the Real Line

As indicated in Section 1, the derivatives can be generalized in different directions. If locality is the leading requirement, then the most natural way for such generalization is to replace the assumption of local Lipschitz growth with the more general modular-bound growth.

Definition 16 (Modulus of continuity)**.** *A point-wise modulus of continuity $g_x : \mathbb{R} \mapsto \mathbb{R}$ of a function $f : \mathbb{R} \mapsto J \subseteq \mathbb{R}$ is a*

1. *Non-decreasing, non-constant continuous function;*
2. *$g_x(0) = 0$;*
3. *$|\Delta^{\pm}_{\epsilon}[f](x)| \leq K \, g_x(\epsilon)$ holds in the interval $I = [x, x \pm \epsilon] \subset J$ for some constant K.*

   A regular modulus is such that $g_x(1) = 1$.

In the subsequent sections, we will assume that all considered moduli are regular. The following definition of a modular function is adopted to avoid singular moduli.

Definition 17. *A modular function g is a regular modulus of continuity, which is also differentiable everywhere in $I = (0, L]$ for certain $L > 0$.*

Note that the dependence on the point x of the modulus of continuity is suppressed in the above definition. By duality, one can denote the set $X^{\pm}_{g}(f)$ as the set of points where g is the modulus of continuity of f. A trivial example for a modulus function is the linear modulus, which gives rise to the Lipschitz condition. This is a standard condition in the theory of ordinary and partial differential equations.

Example 1 (Hölder modulus)**.** *$g(x) = x^{\beta}$ for $0 < \beta < 1$ is a modular function, which is not differentiable at $x = 0$. This modulus determines the Hölder growth class $\mathbb{H}^{\beta}$.*

Example 2 (log-modulus)**.** *Another non-trivial example is the function $g(x) = x^{\beta} \log \frac{1}{x}$, which, for $0 \leq \beta < 1$, is a modular function.*

Example 3 (singular modulus)**.** *A third non-trivial example is the Cantor function (see below), which is its own modulus of continuity [7]. However, it is not a modular function in view of Definition 17.*

From a computational point of view, only regular, modular functions can be of practical interest. These are the elementary power functions and their products with logarithms.

Definition 18. *Define the parameterized difference operators as*

$$\Delta_\epsilon^+[f](x) := f(x+\epsilon) - f(x),$$
$$\Delta_\epsilon^-[f](x) := f(x) - f(x-\epsilon)$$

for the variable $\epsilon > 0$ and the function $f(x)$. The two operators are referred to as forward difference and backward difference operators, respectively.

Definition 19. *Define g-variation operators as*

$$v_g^{\epsilon\pm}[f](x) := \frac{\Delta_\epsilon^\pm[f](x)}{g(\epsilon)} \tag{10}$$

for a positive ϵ and a regular modular function g.

Define the modular derivative as follows:

Definition 20 (Modular derivative, g-derivative)**.** *Consider an interval $I = [x, x \pm \epsilon]$ and define the limit if it exists:*

$$\mathcal{D}_g^\pm f(x) := \lim_{\epsilon \to 0} \frac{\Delta_\epsilon^\pm[f](x)}{g(\epsilon)} = \lim_{\epsilon \to 0} v_g^{\epsilon\pm}[f](x) \tag{11}$$

for a modulus of continuity $g(\epsilon)$. The limit will be understood in a topological sense (e.g., Definition 15). The last limit will be called a modular derivative with regard to the function ***g*** *or* ***g****-derivative.*

NB! The equality of $\mathcal{D}_g^+ f(x)$ and $\mathcal{D}_g^- f(x)$ is not required.

At this point, it can be observed that the above definition is not vacuous since, for a non-singular function of the bounded variation on an interval $I = [x, x+\epsilon] \subset \mathbb{R}$,

$$\mathcal{D}_g^\pm f(x) = \lim_{\epsilon \to 0} \frac{f'(x \pm \epsilon)}{g'(\epsilon)} \tag{12}$$

by L'Hôpital's rule. In this regard, it is useful to consider the following result:

Proposition 4. *Suppose that $f \in \mathbb{C}^1$ at x and $|g'(0)| = \infty$, where g is a modular function. Then, $\mathcal{D}_g^\pm f(x)$=0.*

Proof. By L'Hôpital's rule,

$$\mathcal{D}_g^\pm f(x) = \lim_{\epsilon \to 0} \frac{f'(x \pm \epsilon)}{g'(\epsilon)} = \lim_{\epsilon \to 0} f'(x \pm \epsilon) \lim_{\epsilon \to 0} \frac{1}{g'(\epsilon)} = 0$$

since the differentiability of f implies the continuity of the derivative at x. □

From the above definition, it is clear that the modular function provides an optimal growth estimate for the function of interest. Moreover, the non-linear modulus is sensitive only to the singularities of the derivative.

Definition 21. *Consider a function f continuous on the closed interval I. The set*

$$\chi_g^\pm(f) := \{x : \; v_g^{\epsilon\pm}[f](x) \neq 0, \; \forall \epsilon > 0\} \cap I$$

is called the set of change.

By this definition, the geometrical meaning of the sets $\chi_g^{\pm}(f)$ becomes clear as the sets of points where the function f can be optimally approximated by a right and left modular function g, respectively.

Remark 4. *Together with L'Hôpital's rule, this proposition can be used in practice for computations of g-derivatives. For suitable types of functions, the process can be automated and implemented in computer algebra systems.*

6.2. Topology Induced by the Set of Change

This section characterizes the topology induced by a modular function g. To this end, we use the notion of the set of change $\chi_g(f)$.

Definition 22. *Consider the infinite bounded sequence $A = \{a_k\}_{k=1}^{\infty} \in \mathbb{R}$. Let $\bar{a} := \sup\{a_i\}$, $\underline{a} := \inf\{a_i\}$, where $\bar{a}$ and/or $\underline{a}$ are not necessarily in $\{a_k\}$. Define the Cauchy operator*

$$\mathcal{Q} : A \mapsto A \cup \{\underline{a}, \bar{a}\}, \quad \{\underline{a}, \bar{a}\} =: \delta A \tag{13}$$

The sequences, for which $\mathcal{Q}A = A$, will be called Cauchy-complete.

Definition 23 (Closure twist map)**.** *Define the closure defect map as the set-valued map*

$$q(A, B) := \{\min\left(\bar{a}, \bar{b}\right), \max\left(\underline{a}, \underline{b}\right)\}$$

Lemma 3. *The operator $\mathcal{Q}$ satisfies axioms K1–K3 for any sequence $A = \{a_k\}_{k=1}^{\infty} \in \mathbb{R}$. If $q(A, B) \subseteq \mathcal{Q}(A, B)$, then $\mathcal{Q}$ also satisfies K4 for the sequences A and B.*

Proof. Axiom K1 is satisfied vacuously since $\sup \emptyset = \inf \emptyset = \emptyset$.

Axiom K2 is satisfied since $\{a_k\} \subseteq \{a_k\} \cup \{\underline{a}, \bar{a}\} = \mathcal{Q}\{a_k\}$.

$\mathcal{Q}(\mathcal{Q}\{a_k\}) = \mathcal{Q}(\{a_k\} \cup \{\underline{a}, \bar{a}\})$. However, $\sup(\{a_k\} \cup \{\underline{a}, \bar{a}\}) = \bar{a}$ and $\inf(\{a_k\} \cup \{\underline{a}, \bar{a}\}) = \underline{a}$. Therefore, $\mathcal{Q}(\mathcal{Q}\ \{a_k\}) = \mathcal{Q}\{a_k\}$ so that axiom K3 is satisfied.

Axiom K4 is satisfied only for a certain type of sequence. Let $C = A \cup B = \{a_k\} \cup \{b_k\} = \{c_k\}$. Observe that $\bar{c} = \sup\{\bar{a}, \bar{c}\}$ and $\underline{c} = \inf\{\underline{a}, \underline{b}\}$. Then,

$$\mathcal{Q}\ C = \{c_k\} \cup \{\underline{c}, \bar{c}\} = \{a_k\} \cup \{\inf\{\underline{a}, \underline{b}\}, \sup\{\bar{a}, \bar{b}\}\}$$

On the other hand,

$$\mathcal{Q}A \cup \mathcal{Q}B = \{a_k\} \cup \{\underline{a}, \bar{a}\} \cup \{b_k\} \cup \{\underline{b}, \bar{b}\}$$

Therefore,

$$\mathcal{Q}C \cup \{\inf\left(\bar{a}, \bar{b}\right), \sup\left(\underline{a}, \underline{b}\right)\} = \mathcal{Q}(A \cup B) \cup q(A, B) = \mathcal{Q}A \cup \mathcal{Q}B$$

since, for finite sets, *min* coincides with *inf* and *max* coincides with *sup*, respectively. Therefore, if $q(A, B) \subseteq \mathcal{Q}(A, B)$, K4 holds. □

From the above, we can see that the $\mathcal{Q}$ is a closure operator for a fixed sequence.

Proposition 5. *Suppose that $A \subseteq B$; then, the axiom K4 holds for $\mathcal{Q}A$ and $\mathcal{Q}B$.*

Proof. We need to demonstrate that

$$\mathcal{Q}(A \cup B) = \mathcal{Q}A \cup \mathcal{Q}B$$

Observe that

$$A \subseteq B \Longrightarrow A \cup B = B \Longrightarrow \mathcal{Q}(A \cup B) = \mathcal{Q}B$$

Also,

$$A \subseteq B \Longrightarrow \inf A \geq \inf B, \quad \sup A \leq \sup B$$

Therefore, $q(A, B) = \{\min(\bar{a}, \bar{b}), \max(\underline{a}, \underline{b})\} = \{\bar{a}, \underline{a}\}$.
On the other hand, $\{\bar{a}, \underline{a}\} \subseteq B$.
Therefore,

$$\mathcal{Q}(A \cup B) = \mathcal{Q}A \cup \mathcal{Q}B$$

□

Theorem 2 (Induced topology)**.** *Let A be a bounded sequence. Then, $(A, \mathcal{Q})$ and $(\mathcal{Q}A, \mathcal{Q})$ are T_1 topological spaces and $\mathcal{Q}$ is a closure operator for them. Every set $S \in \mathrm{Pow}(\mathcal{Q}A)$ is closed. This topology is denoted by $\mathcal{T}_{\mathcal{Q}}$.*

Proof. By Lemma 3 and Proposition 5, K1–K4 are satisfied. K5 is satisfied since $\mathcal{Q}\{a\} = \{a\}$. Therefore, $(A, \mathcal{Q})$ is a T_1 space. Furthermore, by idempotence, $(\mathcal{Q}A, \mathcal{Q})$ is a T_1 space.

For the second part, let $X = \mathcal{Q}A$; then, the boundary is $\partial X = (\mathcal{Q}X) \setminus X$. By idempotence, $\partial X = (\mathcal{Q}X) \setminus X = (\mathcal{Q}A) \setminus (\mathcal{Q}A) = \emptyset$. Therefore, the set X is closed. □

Remark 5. *The term Cauchy-complete is justified by the observation that, for a Cauchy sequence A, the set $\mathcal{Q}A \setminus A$ can have only three values—$\{\bar{a}\}$, $\{\underline{a}\}$ or $\emptyset$—depending on whether A has a minimum, a maximum, or both.*

In the next paragraphs, we give a more conventional treatment of the so-identified topology. In the conventional approach, the topology $T_{\mathcal{Q}}$, induced by $\mathcal{Q}$, can be characterized by the open sets

$$\tau_k := A \setminus \{a\}_{i=1}^{i=k} = \{t : a_i \in A, i \geq k\}, \quad k > 0$$

The points in the topological space $T_{\mathcal{Q}}$ are then the singletons $\{a_i\}$. Therefore, this topology can be recognized as the co-finite topology of the infinite (!) set A. Furthermore, one can claim the following.

Proposition 6. *The sets τ_k form a basis in the topology $\mathcal{T}_{\mathcal{Q}}$.*

Proof. To prove the statement, we need to verify two properties:

(1) Every point $x \in X$ lies in some set $t \in \tau$.

(2) For each pair of sets $t_p,\ t_q \in \tau$ and each point $x \in t_p \cap t_q$, there exists a set $t_r \in \tau$ such that $x \in t_r \subset t_p \cap t_q$. Property 1 holds as $\{a_i\} = \tau_i \cap \tau_{i+1}$. Property 2 holds for τ_p, τ_q, τ_r, such that $p < q < r$. Under this hypothesis, $t_p \cap t_q = t_q$ and $t_r \subset t_q$ for $q < r$. □

Having established the appropriate topological background, we are ready to relax the definition of $\mathcal{D}_g^{\pm} f(x)$ by requiring only that at least one of the limits

$$\lim_{\epsilon \to 0} v_g^{\epsilon\pm}[f](x), \quad x \pm \epsilon \in \mathcal{T}_{\mathcal{Q}}$$

exists in the topological sense of Definition 15. If said limit exists, we write as above $\mathcal{D}_g^{\pm} f(x) = \lim_{\epsilon \to 0} v_g^{\epsilon\pm}[f](x)$.

The main result of this section comprises the following theorem.

Theorem 3 (Topological continuity of g-derivatives)**.** *Suppose that $\chi_g(f)$ is an infinite set inducing a topology $\mathcal{T}_{\mathcal{Q}}$. Suppose that S is dense in $\chi_g(f)$. Then, the images $\mathcal{D}_g^{\pm} f(x)$ are continuous on S under $\mathcal{T}_{\mathcal{Q}}$.*

Proof. Suppose that the set S is dense in $\chi_g(f)$. Since S is dense in $\chi_g(f)$, it is Cauchy-complete, which implies that $\mathcal{Q}\ S = S$.

Let further define $B := \{y : \mathcal{D}_g^{\pm} f(x),\ x \in S\}$, where $\mathcal{D}_g^{\pm} f(x)$ exists finitely. Since $\mathcal{D}_g^{\pm} f(x)$ is finite, the action of $\mathcal{Q}$ is defined. Therefore, we can write $\mathcal{Q}\ B = B \cup \partial B$. $\mathcal{D}_g^{\pm}[\mathcal{Q}\ S] = \mathcal{D}_g^{\pm}[S] = B$. Therefore, $\mathcal{D}_g^{\pm}[S] = B \subseteq \mathcal{Q}\ B$ and, by the Hausdorff Theorem, $\mathcal{D}_g^{\pm} f(x)$ is continuous on S. □

Note that the last result does not imply the continuity of $\mathcal{D}_g^{\pm} f$ in the usual topology of $\mathbb{R}$. In contrast, strictly sub-additive modules give rise to g-derivatives, which are discontinuous in the usual topology [23]. We further specialize the argument to Hölder-continuous functions, where $g(x) = x^\beta$, $\beta < 1$. The g-derivative in this case specializes to fractional velocity denoted by $v_{\pm}^{\beta} F(x) = \mathcal{D}_g^{\pm} f(x)$. There are two composition formulas that are useful for the subsequent discussion and examples:

$$v_{\pm}^{\beta}[f \circ h](x) = v_{\pm}^{\beta} f(y)\bigg|_{y=h(x)} .[h'(x)]^{\beta} \tag{14}$$

and

$$v_{\pm}^{\beta}[h \circ f](x) = h'(y)\bigg|_{y=f(x)} .v_{\pm}^{\beta} h(x) \tag{15}$$

for a composition of the $\mathbb{H}^\beta$ function f with a differentiable function h evaluated at the argument x, depending on the order of the functions in the composition. We give some examples of singular functions that have, for sets of change, a countable subset of the Cantor set and the dyadic rationals $\mathbb{Q}_2$.

The Cantor set is the prototypical example of a totally disconnected, uncountable, perfect set. The set gives rise to the eponymous singular function.

Example 4 (Cantor singular function)**.** *On $I = [0,1]$, Cantor's singular function is the unique solution of the functional equation*

$$F(x) = \frac{1}{2}\begin{cases} F(3x), & 0 \le x \le 1/3 \\ 1 & 1/3 < x < 2/3 \\ F(3x-2)+1, & 2/3 \le x \le 1 \end{cases} \tag{16}$$

with fixed points $F(0) = 0$, $F(1) = 1$.

Cantor's function can be approximated by a possibly non-terminating iterative algorithm from the discrete floor map as follows:

$$F_N(x) = \begin{cases} \frac{1}{2}\sum\limits_{k=1}^{N-1}\frac{\lfloor 3^k x\rfloor}{2^k} + \frac{\lfloor 3^N x\rfloor}{2^N} & = \frac{1}{2}F_{N-1}(x) + \frac{\lfloor 3^N x\rfloor}{2^N}, \quad \lfloor 3^N x\rfloor \in \{0,2\} \\ \frac{1}{2}\sum\limits_{k=1}^{N-1}\frac{\lfloor 3^k x\rfloor}{2^k} + \frac{1}{2^N} & = \frac{1}{2}F_{N-1}(x) + \frac{1}{2^N}, \quad \lfloor 3^N x\rfloor = 1 \end{cases} \tag{17}$$

From the above system, it is apparent that the set of increase in the Cantor's function is a countable subset of Cantor's ternary set (and hence of measure zero); that is, $\forall x \in C$, we have

$$\chi_\alpha = \left\{ x : x = \sum_{k=1}^{N} \frac{a_k}{3^k},\ a_k \in \{0,2\} \right\} \tag{18}$$

$\forall N$ natural numbers. To calculate the fractional velocity on the co-finite topology of the Cantor set, we select a sequence such that $\forall k \ \ 0 + \epsilon_k \in C$. Such a sequence is $\epsilon = \left\{3^{-k}\right\}_{k=1}^{\infty}$. Let

$$\alpha := \log 2 / \log 3 \tag{19}$$

By the functional Equation (16), the following identity holds: $F(1/3^n) = 1/2^n$, so that $\Delta_\epsilon^+[F](0) = 1/2^n$. Therefore,

$$v_{\epsilon+}^{\alpha}[F](0) = \frac{1}{2^n} \Big/ \left(\frac{1}{3^{n\alpha}}\right) = \left(\frac{3^\alpha}{2}\right)^n = 1.$$

Therefore, $v_+^\alpha F(0) = 1$. By the functional Equation (16), $F(2/3) = F(0)/2 + 1/2$; therefore,

$$F\left(\frac{2}{3} + \frac{1}{3^n}\right) = \frac{1}{2} F\left(\frac{1}{3^{n-1}}\right) + \frac{1}{2}$$

Then,

$$\Delta_\epsilon^+[F]\left(\frac{2}{3}\right) = F\left(\frac{2}{3} + \frac{1}{3^n}\right) - F\left(\frac{2}{3}\right) = \frac{1}{2} F\left(\frac{1}{3^{n-1}}\right) = \frac{1}{2^n} \Big/ \left(\frac{1}{3^{n\alpha}}\right) = \left(\frac{3^\alpha}{2}\right)^n = 1$$

so that $v_{\epsilon+}^{\alpha}[F](2/3) = 1$

On the other hand, $v_{\epsilon+}^{\alpha}[F](1/3) = 0$.

We can formally adjoin $v_{\epsilon+}^{\alpha}[F](1) = 0$ to respect the functional equation. Therefore, we obtain the functional equation system

$$v_+^\alpha F(x) = \begin{cases} v_+^\alpha F(3x), & 0 \le x \le 1/3 \\ 0 & 1/3 < x < 2/3 \\ v_+^\alpha F(3x-2), & 2/3 \le x \le 1 \end{cases} \tag{20}$$

as prescribed by the formal g-differentiation of the functional equations.

Another interesting example is De Rham's singular function, which was also rediscovered by Takacs in a different context [31]. The function depends on a real-valued parameter $a \in (0,1)$ and has, for a set of change, the dyadic rationals $\mathbb{Q}_2$, and is constant almost everywhere in $[0,1]$.

Example 5 (De Rham–Takacs singular function)**.** *In 1978, Takacs [31] introduced a new singular function defined in the unit interval, such that, for a number,*

$$x = \sum_{r=0}^{\infty} \frac{1}{2^{a_r}}$$

Where the sequence $\{a_r\}_{r=1}^{\infty}$ comprising integers is increasing, the function is defined as

$$F_\rho(x) := \sum_{r=0}^{\infty} \frac{\rho^r}{(1+\rho)^{a_r}},$$

where $F_\rho(0) = 0$, $F_\rho(1) = 1$.

If we consider the usual binary representation

$$x = \sum_{r=1}^{\infty} \frac{b_r}{2^r}, \quad b_r \in \{0,1\}$$

for $x \in [0,1]$ and restrict the discussion of the dyadic rationals $\mathbb{Q}_2$, which are dense in $\mathbb{Q}$, we can establish the following. Suppose that $0 \leq x \leq 1/2$. Then,

$$F_\rho(x) = \sum_{r=1}^{\infty} b_r \frac{\rho^r}{(1+\rho)^r} = \frac{\rho}{1+\rho} F_\rho(2x)$$

On the other hand,

$$F_\rho(1) = \sum_{r=0}^{\infty} b_r \frac{\rho^r}{(1+\rho)^{(r+1)}} = \frac{1}{\rho+1} \sum_{r=0}^{\infty} \frac{\rho^r}{(1+\rho)^r} = \frac{1}{1+\rho} \left(\frac{1}{1-\frac{\rho}{\rho+1}} \right) = 1$$

Therefore, $F_\rho(1/2) = \rho/(1+\rho)$.

For $1/2 < x < 1$, let

$$x = \frac{1}{2} + \sum_{r=1}^{\infty} \frac{b_r}{2^{r+1}}$$

as above. By a simple re-indexing of the number,

$$y = \sum_{k=1}^{\infty} \frac{1}{2^{a_k}}$$

observe that

$$F_\rho(y) = \sum_{k=1}^{\infty} \frac{\rho^{k-1}}{(1+\rho)^{a_k}}$$

Therefore,

$$F_\rho(x) = \frac{\rho}{1+\rho} + \sum_{r=1}^{\infty} b_r \frac{\rho^{r-1}}{(1+\rho)^{r+1}}$$

if we set $x = y$. On the other hand, for

$$2x - 1 = \sum_{r=1}^{\infty} \frac{b_r}{2^r}$$

the function evaluates to

$$F_\rho(2x-1) = \sum_{r=1}^{\infty} \frac{b_r \rho^{r-1}}{(1+\rho)^r}$$

Therefore,

$$F_\rho(y) = \frac{\rho}{\rho+1} + \frac{1}{\rho+1} F_\rho(2y-1)$$

To summarize,

$$F_\rho(x) = \begin{cases} \frac{\rho}{1+\rho} F_\rho(2x), & 0 \leq x \leq \frac{1}{2} \\ \frac{\rho}{1+\rho} + \frac{1}{1+\rho} F_\rho(2x-1), & \frac{1}{2} < x \leq 1 \end{cases}$$

This corresponds to the functional equation of De Rham's function $R_a(x)$ since one can identify $a = \rho/(1+\rho) < 1$ [32]. Since De Rham's function is the unique solution of its functional equation, we have established that both functions are in fact identical.

To compute the fractional velocity on the dyadic rationals $\mathbb{Q}_2$, we can formally g-differentiate the system as

$$v_{\pm}^{\beta} F_\rho(x) = \begin{cases} \frac{\rho 2^\beta}{1+\rho} v_{\pm}^{\beta} F_\rho(2x), & 0 \leq x \leq 1/2 \\ \frac{2^\beta}{1+\rho} v_{\pm}^{\beta} F_\rho(2x-1), & 1/2 < x \leq 1 \end{cases}$$

Therefore, for the fractional velocity to be finite, either

$$\frac{\rho 2^{\beta}}{1+\rho} \leq 1$$

or

$$\frac{2^{\beta}}{1+\rho} \leq 1$$

should hold. Suppose that $\rho < 1$*; then, the maximal Hölder exponent is*

$$\beta = \log_2(1+\rho)$$

At this point, the direction of differentiation should be fixed in a way that is consistent with a direct calculation. We calculate $v_{-}^{\beta} F_{\rho}(1)$*. Suppose that* $\epsilon < 1/2$

$$v_{\epsilon-}^{\beta}[F](1) = \frac{1 - F_{\rho}(1-\epsilon)}{\epsilon^{\beta}} = \frac{1}{(1+\rho)\epsilon^{\beta}}(1 - F_{\rho}(1-2\epsilon)) = \frac{1}{(1+\rho)^{k}\epsilon^{\beta}}(1 - F_{\rho}(1-2^{k}\epsilon))$$

by induction for $k > 1$*. Therefore, for* $\epsilon = 1/2^{k}$*,*

$$v_{\epsilon-}^{\beta}[F](1) = \frac{2^{k\beta}}{(1+\rho)^{k}} = 1$$

Therefore,

$$v_{-}^{\beta} F_{\rho}(x) = \begin{cases} \rho\, v_{-}^{\beta} F_{\rho}(2x), & 0 \leq x \leq \frac{1}{2} \\ v_{-}^{\beta} F_{\rho}(2x-1), & \frac{1}{2} < x \leq 1 \end{cases}$$

where also $v^{\beta} F_{\rho}(0) = 0$*.*

Conversely, if $\rho > 1$*, then the maximal Hölder exponent is*

$$\beta = \log_2\left(\frac{1+\rho}{\rho}\right)$$

We calculate $v_{+}^{\beta} F_{\rho}(0)$ *as*

$$\frac{F_{\rho}(\epsilon)}{\epsilon^{\beta}} = \left(\frac{\rho}{1+\rho}\right)^{n} 2^{n\beta} = \left(\frac{\rho}{1+\rho}\right)^{n} \left(\frac{1+\rho}{\rho}\right)^{n} = 1$$

Therefore, $v_{+}^{\beta} F_{\rho}(0) = 1$ *and*

$$v_{+}^{\beta} F_{\rho}(x) = \begin{cases} v_{+}^{\beta} F_{\rho}(2x), & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{\rho} v_{+}^{\beta} F_{\rho}(2x-1), & \frac{1}{2} < x \leq 1 \end{cases}$$

where now $v_{+}^{\beta} F_{\rho}(1) = 0$ *holds.*

A third interesting example is the Neidinger function, called also the *fair-bold gambling function* [33]. The function is based on De Rham's construction and is also constant almost everywhere in $[0,1]$.

Example 6 (Neidinger singular function)**.** *Consider the iterated function system (IFS) for* $a \in (0,1)$*, which swaps the value of the parameter at every step of the iteration:*

$$N_n(x,a) := \begin{cases} a N_{n-1}(2x, a), & a \leftarrow 1-a,\ n \text{ even},\ 0 \leq x < \frac{1}{2} \\ (1-a)\, N_{n-1}(2x-1, a) + a, & a \leftarrow 1-a,\ n \text{ even},\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

starting from $N_0(x,a) = x$. *Define Neidinger's function* $N(x,a)$ *as the limit*

$$N(x,a) := \lim_{n\to\infty} N_n(x,a)$$

The fractional velocity of the function has been exhibited in [19]. Here, we work on the dyadic rationals $\mathbb{Q}_2$. *Formal g-differentiation of the defining IFS without regard to the parameter swapping rule results in*

$$\upsilon_+^\beta N(x,a) := \begin{cases} a2^\beta \upsilon_+^\beta N(2\,x,a), & 0 \le x < \frac{1}{2} \\ (1-a)2^\beta \upsilon_+^\beta N(2\,x-1,a), & \frac{1}{2} \le x \le 1 \end{cases}$$

The actual computation can be carried out in the following way. Starting from $A_0(x,a) = 1$, *define recursively the auxiliary IFS*

$$A_n(x,a) := \begin{cases} (1-a)2^\beta A_{n-1}(2\,x, 1-a), & 0 \le x < \frac{1}{2} \\ a2^\beta A_{n-1}(2\,x-1, 1-a), & \frac{1}{2} \le x \le 1 \end{cases}$$

Therefore, either $a = 1/2^\beta$ *or* $1 - a = 1/2^\beta$ *must hold for the IFS to converge. The maximal Hölder exponent is then*

$$\beta = \min\left(-\log_2 a, -\log_2(1-a)\right) = -\log_2 \max(a, 1-a)$$

Consider the case where $a = 1 - 1/2^\beta$. *Then,*

$$A_n(x,a) = \begin{cases} A_{n-1}(2\,x, 1-a), & 0 \le x < \frac{1}{2} \\ a/(1-a)\, A_{n-1}(2\,x-1, 1-a), & \frac{1}{2} \le x \le 1 \end{cases}$$

In a similar way, whenever $a = 1/2^\beta$, *we have*

$$A_n(x,a) = \begin{cases} (1-a)/a\, A_{n-1}(2\,x, 1-a), & 0 \le x < \frac{1}{2} \\ A_{n-1}(2\,x-1, 1-a), & \frac{1}{2} \le x \le 1 \end{cases}$$

Therefore, in the general case, we have

$$A_n(x,a) = \begin{cases} \frac{1-a}{\max(a,1-a)}\, A_{n-1}(2\,x, 1-a), & \le x < \frac{1}{2} \\ \frac{a}{\max(a,1-a)}\, A_{n-1}(2\,x-1, 1-a), & \frac{1}{2} \le x \le 1 \end{cases}$$

and

$$\upsilon_+^\beta N(x,a) = \lim_{n\to\infty} A_n(x,a)$$

From the presented calculation, we see that the sets of change in the Neidinger function are the dyadic rationals.

The plotting of the graph of the fractional velocity is challenging due to the fact that it does not vanish only on a null set in any given interval. Therefore, one could only plot the covering graph approximating the fractional velocity at a certain iteration order. This can be performed by defining recursively an IFS that, in limit, converges point-wise to the fractional velocity of the given singular function.

The construction of the Neidinger function can be generalized using the Bernoulli map. The result is another singular function, which we can call tentatively the *De Rham–Bernoulli function*. The construction can be carried out as follows.

Example 7 (De Rham–Bernoulli singular function)**.** *Suppose that* $\xi \in (0,1)$*. Starting from* $\xi[0] := \xi$*, define* $\xi[n] := 2\xi[n-1] - z[n-1]$*, where* $z[k] := \lfloor 2\xi[k] \rfloor$ *evaluates to either 0 or 1. Define the auxiliary function*

$$b(\xi) := z(1-a) + (1-z)a$$

where z is computed as above.

Finally, starting from $M_0(x, a, \xi) = x$*, define the IFS:*

$$M_n(x, a, \xi[n]) := \begin{cases} b(\xi[n]) M_{n-1}(2x, a, \xi[n-1]), & 0 \le x < \frac{1}{2} \\ (1 - b(\xi[n]))\, M_{n-1}(2x-1, a, \xi[n-1]) + b(\xi[n]), & \frac{1}{2} \le x \le 1 \end{cases}$$

Then,

$$M_\xi(x, a) := \lim_{n \to \infty} M_n(x, a, \xi[n])$$

Observe that the limit exists since $b(\xi) < 1$*.*

Indeed, suppose that $a < 1/2$*. Then,*

$$b(\xi) = z(1-a) + (1-z)a < 1 - a < 1$$

By symmetry, the same estimate holds also whenever $a > 1/2$*. Therefore, the IFS will converge point-wise to a limit.*

The fractional velocity of the above function can be computed in a similar way as above, setting $a = 1/2^\beta$*. The IFS in this case is*

$$A_n(x, a, \xi[n]) := \begin{cases} b(\xi[n]) 2^\beta A_{n-1}(2x, a, \xi[n-1]), & 0 \le x < \frac{1}{2} \\ (1 - b(\xi[n])) 2^\beta A_{n-1}(2x-1, a, \xi[n-1]), & \frac{1}{2} \le x \le 1 \end{cases}$$

starting from $A_0(x, a, \xi[0]) = 1$*. Then, in order for the IFS to converge, the maximal Hölder exponent is*

$$\beta = -\log_2 b(\xi) = -\log_2 \max(a, 1-a)$$

and the IFS transforms as

$$A_n(x, a, \xi[n]) := \begin{cases} \frac{b(\xi[n])}{\max(a, 1-a)} A_{n-1}(2x, a, \xi[n-1]), & 0 \le x < \frac{1}{2} \\ \frac{1 - b(\xi[n])}{\max(a, 1-a)} A_{n-1}(2x-1, a, \xi[n-1]), & \frac{1}{2} \le x \le 1 \end{cases}$$

Note that the form of the IFS is identical to the one in the previous example. Finally,

$$v_+^\beta M(x, a, \xi) = \lim_{n \to \infty} A_n(x, a, \xi[n])$$

A plot of the Neidinger–Bernoulli function and its fractional variation at iteration level $n = 8$ is presented in Figure 1.

A

B

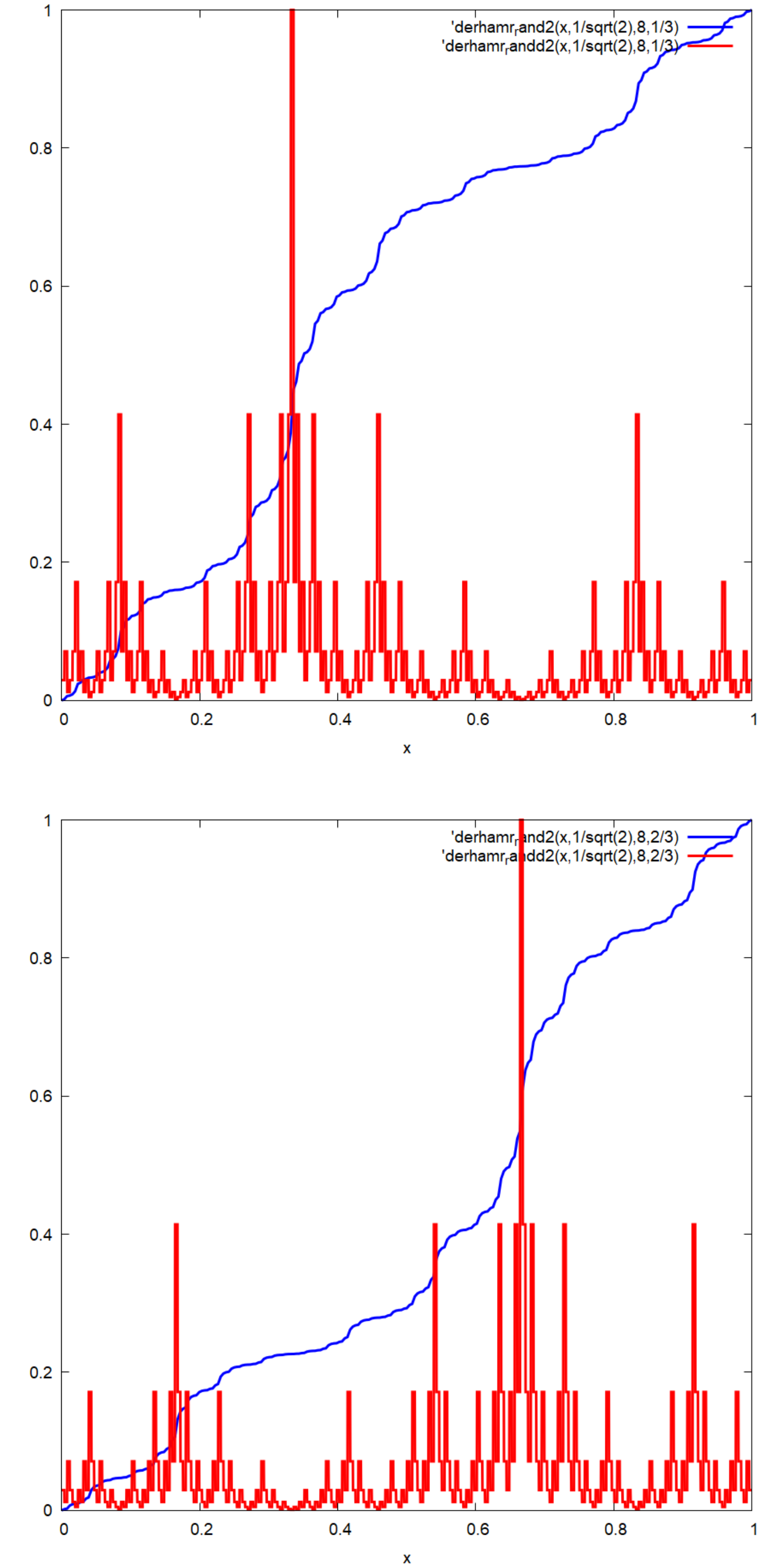

Figure 1. Neidinger–Bernouli function and its fractional variation. (**A**)—Original Neidinger construction $N(x, 1/\sqrt{2}) = M_{1/3}(x, 1/\sqrt{2})$; (**B**)—modified construction $M_{2/3}(x, 1/\sqrt{2})$.

6.3. Modular Derivatives on Banach Spaces

Theorem 3 formulated above can be established also in a different, more general way, and can be strengthened further. It can be used for the characterization of the modular derivatives in more general spaces. To this end, we use the topological definition of a limit.

Recall the definition of the directional derivatives:

Definition 24 (Gâteaux or directional derivative)**.** *Let X and Y be Banach spaces and let $f : X \to Y$ be a function between them; f is said to be Gâteaux differentiable if there exists an operator $T_x : X \to Y$ such that $\forall v \in X$,*

$$\lim_{h\to 0} \frac{f(x+hv) - f(x) - hT_x(v)}{h} = \frac{d}{dh} f(x+hv)\Big|_{h=0} - T_x(v) = 0$$

The operator $T_x \equiv \mathbf{D}f(x)$ is called the Gâteaux derivative of f at x.

To this end, we replace ϵ by the scalar-valued modulus function $g(\epsilon)$. The definition of a modular function at point 3 is modified as follows:

$$|f(x+\epsilon v) - f(x)| \le K\, g_x(\epsilon), \quad ||v|| = 1$$

This allows for translating the definition of g-derivatives into directional derivatives in more general spaces.

Definition 25 (modular Gâteaux derivative)**.** *Let X and Y be two Banach spaces and $f : X \to Y$ be a function between them. Denote the auxiliary variation operator by*

$$v_g^{\epsilon\pm}[f](x) := \pm \frac{f(x \pm \epsilon v) - f(x)}{g(\epsilon)}$$

Then, f is Gâteaux differentiable with respect to the modulus g if there exists an operator $T_x : X \to Y$ such that $\forall v \in X$,

$$\lim_{\epsilon\to 0} \frac{f(x+\epsilon v) - f(x) - g(\epsilon) T_x(v)}{g(\epsilon)} = \lim_{\epsilon\to 0} \Big(v_g^{\epsilon\pm}[f](x) - T_x(v) \Big) = 0$$

We write

$$\frac{d}{dg(\epsilon)} f(x+\epsilon v)\Big|_{\epsilon=0} = T_x(v)$$

The operator $T_x(v) \equiv \mathbf{D}_g f(x)$ is called the g-Gâteaux derivative of f at x in the direction v.

Theorem 4 (Topological continuity of g- Gâteaux derivatives)**.** *Suppose that S is dense in $\chi_g(f)$. Then, under the topology $\mathcal{T}_Q$, the images $\mathbf{D}_g f(x)$ are continuous on S.*

Proof. Suppose that the set A is dense in $\chi_g(f)$ and assume the co-finite topology $\mathcal{T}_Q$ on $\chi_g(f) \cap A$ as identified previously.

Observe that, for the limit operation, $\lim_{x\to a} f(x) \subseteq \mathbf{cl} f[A] \setminus f[A]^\circ = \partial f[A]$ for $x, a \in A$. Hence, if the limit exists, it is continuous by Theorem A2. Then, we specialize the argument to the right (or left for the minus sign) ϵ-neighborhood of 0 and write $\lim_\epsilon$ for the one-sided limiting operation. Finally, observe that $\mathbf{D}_g f(x) = \lim_\epsilon \circ v_g^{\epsilon\pm}[f](x)$ is a composition of two topologically continuous maps and hence is continuous. □

In this way, one could establish that Theorem 4 implies Theorem 3 if, for the underlying Banach space, is taken $\mathbb{R}$.

Moreover, the Gâteaux derivative is a generalization of the directional derivative in a Euclidean space; therefore, the result also holds for $\mathbb{E}^n$.

7. Discussion

The contributions of the present work can be discussed in several directions. In the first place, Theorem 1 generalizes the result stated in [23], which, in the previous case, was proven only for the real line.

In the second place, Theorem 3 resolves an apparent contradiction uncovered in the early literature on local fractional calculus. To illustrate the issue and its resolution, consider the Cantor function that grows on a subset $\chi_\alpha \subseteq C$, which is dense in the Cantor set C. For any $s \in \chi_\alpha$, the fractional velocity of order $\alpha = \log 2/\log 3$ is constant on χ_α and, moreover, $v^{\alpha}_{\epsilon+}[F](s) = v^{\alpha}_{+}F(s)$ =1. Therefore, we can meaningfully discuss the local fractional differential equation

$$v^{\alpha}_{+}F(s)\Big|_{s\in\chi_g} = 1$$

which, otherwise, under the usual topology of the real line, will have no solutions as proved in [17,34]. Moreover, as expected for the derivative of a constant, it vanishes:

$$v^{\alpha}_{+} \circ v^{\alpha}_{+}F(s)\Big|_{s\in\chi_g} = 0$$

In the third place, the present work exhibits a formal methodology for the computation of fractional velocities on a perfect set S, which can be summarized as follows. Identify a functional equation of the function of interest. Formally g-differentiate the equation and introduce an appropriate IFS. Establish the convergence conditions on S. This fixes the value of the maximal Hölder exponent for which the IFS converges.

As a side note, we have established that the De Rham and Takacs functions are in fact identical and can be called the De Rham–Takacs singular function. To the best of the author's knowledge, this has not been recognized in the literature.

Further directions of study could be applications of the present approach to stochastic problems, such as the Itô integral calculus, which has no explicit derivative, and comparing it to the approach identified via derivatives with regard to Brownian motion [35], rough paths [36], or Itô stochastic differentials [37].

Another area of study could be the development and validation of numerical algorithms for g-derivatives approximation. This could be achieved, for example, starting from Equation (12) and employing an appropriate convolution kernel in the spirit of [18]. Another approach could be the use of higher-order difference schemes and appropriately modifying the modular function g.

8. Conclusions

In conclusion, the presented approach enables the study of both local and non-local derivative operators within a unified topological framework. The g-derivative of a function can approximate its growth at points where the first derivative is singular. Moreover, the g-derivatives can induce a topology defined on their sets of change, under which, such g-derivatives are continuous in the sense of Theorem 3. This allows for applications in higher-order g-differential equations.

Supplementary Materials: The following supporting information can be downloaded at www.mdpi.com/xxx/s1, Maxima scripts computing Figure 1A,B and other examples.

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Appendix A. Topologically Continuous Maps

For the convenience of the reader, we recall the Hausdorff theorem [38]:

Theorem A1 (Hausdorff)**.** *Let $f : X \mapsto Y$ be a map from the topological space X to the topological space Y. Then, the following statements are equivalent:*

(a) f is continuous;

(b) Dor every subset $S \subseteq X$, $f[\mathbf{cl}\ S] \subseteq \mathbf{cl}\ f[S]$; that is, the image of the closure is a subset of the closure of the image.

The proof is reproduced for completeness of the presentation:

Proof. Suppose that f is continuous. Let S be a subset of X and $f[S] \subseteq \mathbf{cl}\ A = A$. If $x \in X$ is such that $f(x) \in Y \setminus A$, then, since f is continuous and $Y \setminus A$ is open in Y, the preimage $f^{-1}[Y \setminus A]$ is an open subset of X containing x and disjoint from S. Therefore, x is not in the closure of S.

Conversely, if f is not continuous, then there exists some open $V \subset Y$ such that the preimage $U := f^{-1}[V]$ is not open in X. Thus, there exists a point $x \in U$ such that every open set containing x fulfills $S := X \setminus U$. Thus, $x \in \mathbf{cl}\, S$ but $f(x)$ is in V and hence not in $Y \setminus V$, which is a closed set containing $f(S)$. □

Theorem A2. *The closure, interior, and boundary operators are topologically continuous maps.*

Proof. Consider $A \subseteq X$ for the topological space X. For the **cl** operator, we have

$$\mathbf{cl}\, A \subseteq \mathbf{cl}\,\mathbf{cl}\, A = \mathbf{cl}\, A$$

which is true. For the boundary operator ∂, we have

$$\mathbf{int}(\partial A) = \emptyset \Longrightarrow \partial\partial A = \mathbf{cl}\,\partial A \setminus \mathbf{int}(\partial A) = \partial A$$

Therefore,

$$\partial \bar{A} = \partial(A \cup \partial A) \subseteq \partial A \cup \partial\partial A = \partial A \cup \partial A = \partial A$$

Hence,

$$\omega_\partial[A] = \partial\, \mathbf{cl}\, A \setminus \mathbf{cl}\, \partial A = \partial \bar{A} \setminus \partial A = \emptyset$$

For **int**, we have

$$\mathbf{int}\ \bar{A} = \mathbf{int}(A \cup \partial A) = \mathbf{int}\ A \cup \mathbf{int}\ \partial A = \mathbf{int}\ A$$

Hence,

$$\omega_{\text{int}}[A] = A^\circ \setminus (A^\circ \cup \partial A^\circ) = \emptyset$$

□

Therefore, we can claim the following.

Corollary A1. *For any closure operator* **cl** *on* X*, the set of continuous maps* $\mathcal{C}$ *acting on the topological space* X *is non empty; that is,* $\mathcal{C}(X, \mathbf{cl}) \neq \emptyset$.

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