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New Results on a Nonlocal Sturm–Liouville Eigenvalue Problem with Fractional Integrals and Fractional Derivatives

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Abstract: In this paper, we investigate the eigenvalue properties of a nonlocal Sturm–Liouville equation involving fractional integrals and fractional derivatives under different boundary conditions. Based on these properties, we obtained the geometric multiplicity of eigenvalues for the nonlocal Sturm–Liouville problem with a non-Dirichlet boundary condition. Furthermore, we discussed the continuous dependence of the eigenvalues on the potential function for a nonlocal Sturm–Liouville equation under a Dirichlet boundary condition.

Keywords: nonlocal Sturm–Liouville problem; fractional derivative; fractional integral; continuous dependence of eigenvalues; two-parameter method

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1. Introduction

This paper discuss the nonlocal Sturm–Liouville problem

$$-y'' + q(x)y + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y = \lambda y \tag{1}$$

subject to

$$y(0) = 0 = y(1) + dy'(1), \tag{2}$$

where D^{α}_{0+} (D^{α}_{1-}) denotes the left-sided (right-sided) Riemann–Liouville fractional derivatives of order α, and $I^{\alpha}_{0+}(I^{\alpha}_{1-})$ represents the left-sided (right-sided) Riemann–Liouville fractional integrals of order α, whose definitions are given later. Here, $0 < \alpha < 1$, $q(x) \in L^2(0,1)$ is a real-valued potential function, μ and d are real constants, and λ is the spectral parameter.

From the eigenvalue properties of a class of nonlocal Sturm–Liouville problems in [1], it is known that for $0 < \alpha < 1/2$

$$\begin{cases} -y''(x) + q(x)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x) \quad \text{on } (0,1), \\ y(0) = 0 = y(1). \end{cases} \tag{3}$$

has real algebraic simple and discrete eigenvalues under certain conditions. These eigenvalues satisfy

$$-\infty < \lambda_1(\mu) < \lambda_2(\mu) < \cdots < \lambda_n(\mu) < \cdots, \lambda_n(\mu) \sim \pi^2 n^2, n \to \infty, \tag{4}$$

where $\lambda_n(\mu)$ is the n-th eigenvalue of (3). Additionally, the associated eigenfunctions form a complete orthogonal basis. Furthermore, [1] discusses the number of zeros present in the eigenfunctions, as well as the characteristics of solutions to the nonlocal Sturm–Liouville equation under specific initial conditions.

Nonlocal Sturm–Liouville problems, which incorporate both left-sided and right-sided fractional derivatives, arise from the field of nonlocal continuum mechanics (please refer to [2–5] for more details). In reference [4], the equilibrium equation governing an elastic bar of finite length, L, which includes long-range interactions among non-adjacent particles, can be expressed as

$$\frac{d^2u(x)}{dx} - \frac{\eta}{E}\mathcal{D}^\alpha u(x) = -\frac{f(x)}{E}.$$

Here, $u(x)$ denotes the axial displacement of the bar at position x, while $f(x)$ represents the longitudinal force per unit volume and η is an opportune constant of proportionality. E signifies the longitudinal modulus, and $\mathcal{D}^\alpha = D^\alpha_{0+} + D^\alpha_{1-}$, where D^α_{0+} and D^α_{1-} correspond to the left and right Riemann–Liouville fractional derivatives of order α, respectively.

Generally, a nonlocal Sturm–Liouville problem is characterized as a Sturm–Liouville-type problem that contains both integer and fractional derivatives, a topic that was the subject of extensive investigation in [6–10]. The form explored in [6–10] can be summarized as follows:

$$-y'' + q(x)y + \mu T_\alpha y = \lambda y,$$

where T_α is a self-adjoint fractional differential operator with both left-sided and right-sided fractional derivatives, such as $T_\alpha = D^\alpha_{0+} + D^\alpha_{1-}$, or $T_\alpha = D^\alpha_{0+}\,{}^cD^\alpha_{1-}$.

Additionally, the fractional Sturm–Liouville problem, which is closely related to the nonlocal Sturm–Liouville problem, is often obtained by replacing the integer derivative operators in a classical Sturm–Liouville problem, $-(p(x)y')' + q(x)y = \lambda\omega(x)y$, by the fractional derivative operators, such as

$$\mathcal{L}_\alpha y + q(x)y = \lambda\omega(x)y,$$

where $\mathcal{L}_\alpha = {}^cD^\alpha_{b-}(p(x)D^\alpha_{a+})$, or $\mathcal{L}_\alpha = D^\alpha_{a+}(p(x)^cD^\alpha_{b-})$. For more details, please refer to [11–23] and reference therein. In [23] the authors employ a change of variables to transform ${}^cD^\alpha_{b-}(p(x)D^\alpha_{a+}) + q(x)y = \lambda\omega(x)y$ into a modified version of a differential equation with a principal term structured in the classical form $-(p(x)z')' + D^{1-\alpha}_{b-}((q(x) - \lambda\omega(x))D^{1-\alpha}_{a+}z) = 0$. Thereafter, the resulting equation is similar to the one considered in this manuscript.

In this study, we present novel findings on the eigenvalue properties of (1)–(2). We first consider the eigenvalue problem of (1)–(2) with $d \neq 0$ in Section 3. We obtained results showing that the eigenvalues of (1)–(2) with $d \neq 0$ are real values, and the corresponding eigenfunctions are orthogonal. Moreover the geometric multiplicity of the eigenvalues is simple. Then we discuss the eigenvalue problem of (1)–(2) with $d = 0$ in Section 4. We introduced an auxiliary two-parameter nonlocal Sturm–Liouville problem in Section 4.1. With the aid of the eigenvalue properties of this two-parameter nonlocal Sturm–Liouville problem, we obtained the continuous dependence of eigenvalues on the potential function in Section 4.2.

2. Preliminaries

In this section, we give some preliminary knowledge from such topics as fractional calculus and the spectral theory of nonlocal Sturm–Liouville problems, which will be used later. More detailed information can be found in [1,24].

We denote by $AC[0,1]$ the set of all the absolutely continuous, complex-valued functions on [0,1]. Let $L^2 = L^2(0,1)$ be the Hilbert space, with the usual inner product $\langle f, g\rangle$ and the norm $||f|| = \langle f, f\rangle^{1/2}$.

Definition 1. ($cf.$ [24] p. 69) *The Riemann–Liouville fractional integrals* $I^{\alpha}_{0+}f$ *and* $I^{\alpha}_{1-}f$ *of order* $\alpha \in \mathbb{C}$ *(*$R(\alpha) > 0$*) are defined by*

$$(I^{\alpha}_{0+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,\ x \in (0,1];\ (I^{\alpha}_{1-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 \frac{f(t)}{(t-x)^{1-\alpha}} dt,\ x \in [0,1),$$

where $\Gamma(\alpha)$ *is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals.*

Definition 2. ($cf.$ [24] p. 70) *Let* $0 < \alpha < 1,\ D = d/dx$*. The left-sided and right-sided Riemann–Liouville derivatives of order* α *are defined by (when they exist)*

$$(D^{\alpha}_{0+}f)(x) = D(I^{1-\alpha}_{0+}f)(x) = \frac{\frac{d}{dx}\left(\int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt\right)}{\Gamma(1-\alpha)}, x \in (0,1];$$
$$(D^{\alpha}_{1-}f)(x) = (-D)(I^{1-\alpha}_{1-}f)(x) = \frac{-\frac{d}{dx}\left(\int_x^1 \frac{f(t)}{(t-x)^{\alpha}} dt\right)}{\Gamma(1-\alpha)}, x \in [0,1).$$

Proposition 1. ($cf.$ [1] Theorem 4.1) *If* $|\mu| < \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{e^{\int_0^1 |q(t)|dt}(1+6e^{\int_0^1 |q(t)|dt})}$*, then the nonlocal initial value problem*

$$\begin{cases} -y''(x) + (q(x)-\lambda)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = 0, \quad y \in \underline{\mathcal{D}}, \\ y(0) = k_1,\ y'(0) = k_2, \end{cases} \tag{5}$$

has, at most, one solution, where μ*,* k_1*,* k_2*, and* $\lambda > 0$ *are real constants, and* $\underline{\mathcal{D}} = \{y \in L^2 : y, y' \in AC[0,1]\}$.

Proposition 2. ($cf.$ [1] Theorem 3.11) *There exists* $\mu_0 > 0$*, such that for* $|\mu| < \mu_0$*, all the eigenvalues of* $-y'' + q(x)y + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y = \lambda y$ *subject to* $y(0) = 0 = y(1)$ *are simple and satisfy*

$$-\infty < \lambda_1(\mu) < \lambda_2(\mu) < \cdots < \lambda_n(\mu) < \cdots,\ \lambda_n(\mu) \sim \pi^2 n^2,\ n \to \infty. \tag{6}$$

Definition 3. ($cf.$ [25] p.375) *Let* $\mathcal{C}(X,Y)$ *denote the set of all closed operators from* X *to* Y*. A family,* $T(\kappa) \in \mathcal{C}(X,Y)$*, defined for* κ *in a domain* D_0 *of the complex plane, is said to be holomorphic of type (A) if*

(i) $\mathcal{D}(T(\kappa)) = \mathcal{D}$ *is independent of* κ*;*
(ii) $T(\kappa)u$ *is holomorphic for* $\kappa \in \mathcal{D}_0$ *for every* $u \in \mathcal{D}$*.*

Proposition 3. ($cf.$ [25] Theorem 2.6) *Let* T *be a closable operator from* X *to* Y*, with* $\mathcal{D}(T) = \mathcal{D}$*. Let* $T^{(n)}, n = 1,2,\cdots$ *be operators from* X *to* Y *with domains containing* $\mathcal{D}$*, and let there be constants* $a, b, c \geq 0$ *such that*

$$\|T^{(n)}u\| \leq c^{n-1}(a\|u\| + b\|Tu\|), u \in \mathcal{D}, n = 1,2,\cdots \tag{7}$$

Then the series

$$T(\kappa)u = Tu + \kappa T^{(1)}u + \kappa^2 T^{(2)}u + \cdots, u \in \mathcal{D}$$

defines an operator, $T(\kappa)$*, with domain* $\mathcal{D}$ *for* $|\kappa| < 1/c$*. If* $|\kappa| < \frac{1}{b+c}$*,* $T(\kappa)$ *is closable and the closures* $\widetilde{T}(\kappa)$ *for such* κ *form a holomorphic family of type (A).*

3. Eigenvalue Problem with Non-Dirichlet Boundary Condition and $0 < \alpha < 1$

In this section, we consider the eigenvalue problem

$$\begin{cases} -y''(x) + q(x)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x) \quad \text{on } (0,1), \\ y(0) = 0 = y(1) + dy'(1), \end{cases} \tag{8}$$

where $d \neq 0$ is a real constant, and $0 < \alpha < 1$.

The fractional operator $\widetilde{T}$, associated with (8), is defined by

$$\begin{aligned} &\widetilde{T}y = -y'' + qy + \mu T_{\alpha}y, T_{\alpha}y := (D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y, y \in \mathcal{D}(\widetilde{T}), \\ &\mathcal{D}(\widetilde{T}) := \{y \in L^2 : y, y' \in AC[0,1], y(0) = 0 = y(1) + dy'(1)\}. \end{aligned} \tag{9}$$

Proposition 4. *For $y_1, y_2 \in \mathcal{D}(\widetilde{T})$, it holds that*

$$\int_0^1 y_1(x) \cdot \widetilde{T}y_2(x)dx = \int_0^1 y_2(x) \cdot \widetilde{T}y_1(x)dx. \tag{10}$$

Proof. If $y_1, y_2 \in \mathcal{D}(\widetilde{T})$, by the definition of operator $\widetilde{T}$, we have $y_1(0) = 0 = dy_1'(1) + y_1(1)$

$$\begin{aligned} &\int_0^1 y_1(x) \cdot \widetilde{T}y_2(x)dx = \int_0^1 y_1(x) \cdot [-y_2''(x) + q(x)y_2(x) + \mu T_{\alpha}y_2(x)]dx \\ &= y_1(0)y_2'(0) - y_1(1)y_2'(1) + \int_0^1 y_1'(x)y_2'(x)dx + \int_0^1 q(x)y_1(x)y_2(x)dx + \mu\int_0^1 y_1(x)T_{\alpha}y_2(x)dx \\ &= dy_1'(1)y_2'(1) + \int_0^1 y_1'(x)y_2'(x)dx + \int_0^1 q(x)y_1(x)y_2(x)dx + \mu\int_0^1 y_1(x)T_{\alpha}y_2(x)dx \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 y_2(x) \cdot \widetilde{T}y_1(x)dx = \int_0^1 y_2(x) \cdot [-y_1''(x) + q(x)y_1(x) + \mu T_{\alpha}y_1(x)]dx \\ &= dy_2'(1)y_1'(1) + \int_0^1 y_2'(x)y_1'(x)dx + \int_0^1 q(x)y_2(x)y_1(x)dx + \mu\int_0^1 y_2(x)T_{\alpha}y_1(x)dx. \end{aligned}$$

It follows from $y_1, y_2 \in \mathcal{D}(\mathcal{L})$ that $y_1(0) = 0 = y_2(0)$. By integrating by parts and exchanging the order of integration, we get

$$\begin{aligned} &\int_0^1 y_1(x)T_{\alpha}y_2(x)dx = -\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 y_1(x)\frac{d}{dx}\left(\int_x^1 (t-x)^{-\alpha}\int_0^t \frac{y_2(s)}{(t-s)^{1-\alpha}}dsdt\right)dx \\ &+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 y_1(x)\left(\int_x^1 (t-x)^{\alpha-1}\frac{d}{dt}\int_0^t \frac{y_2(s)}{(t-s)^{\alpha}}dsdt\right)dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 \left(\int_0^t (t-s)^{\alpha-1}y_2(s)ds\right)\left(\int_0^t (t-x)^{-\alpha}y_1'(x)dx\right)dt \\ &+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 \left(\int_0^t (t-x)^{\alpha-1}y_1(x)dx\right)\left(\int_0^t (t-s)^{-\alpha}y_2'(s)ds\right)dt \\ &= \int_0^1 y_2(x)T_{\alpha}y_1(x)dx, \end{aligned}$$

for $y_1, y_2 \in \mathcal{D}(\widetilde{T})$, which proves (10). □

Theorem 1. *The eigenvalues of the nonlocal Sturm–Liouville eigenvalue problem (8) are real numbers.*

Proof. Let λ be an eigenvalue for (8) corresponding to eigenfunction y. Then for y and its complex conjugate $\overline{y}$, we obtain

$$\widetilde{T}y = \lambda y,\ y(0) = 0 = y(1) + dy'(1), \tag{11}$$

and

$$\widetilde{T}\overline{y} = \overline{\lambda}\overline{y},\ \overline{y}(0) = 0 = \overline{y}(1) + d\overline{y}'(1). \tag{12}$$

Multiplying two sides of (11) by $\overline{y}$ and integrating on the interval $[0, 1]$, we get

$$\int_0^1 \overline{y}(x)\widetilde{T}y(x)dx = \lambda \int_0^1 y(x)\overline{y}(x)dx. \tag{13}$$

A similar method for (12) leads to the relation

$$\int_0^1 y(x)\widetilde{T}\overline{y}(x)dx = \overline{\lambda} \int_0^1 y(x)\overline{y}(x)dx. \tag{14}$$

Using Proposition 4, the following identity is worked out using (13) and (14),

$$(\lambda - \overline{\lambda}) \int_0^1 y(x)\overline{y}(x)dx = (\lambda - \overline{\lambda}) \int_0^1 |y(x)|^2 dx = 0.$$

Since y is a nontrivial solution, $||y||^2 > 0$. Then $\lambda = \overline{\lambda}$ implies that the eigenvalue of (8) is a real number. □

Theorem 2. *The eigenfunctions of the nonlocal Sturm–Liouville eigenvalue problem* (8) *corresponding to the distinct eigenvalues are orthogonal on the interval* $[0, 1]$.

Proof. Let λ_1 and λ_2 be two distinct eigenvalues and y_1 and y_2 be the corresponding eigenfunctions. Then we obtain

$$\widetilde{T}y_1 = \lambda_1 y_1, \tag{15}$$

and

$$\widetilde{T}y_2 = \lambda_2 y_2. \tag{16}$$

Multiplying both sides of (15) by y_2 and (16) by y_1 implies the identity

$$y_2(x)\widetilde{T}y_1(x) - y_1(x)\widetilde{T}y_2(x) = (\lambda_1 - \lambda_2)y_1(x)y_2(x). \tag{17}$$

Integrating (17) on the interval $[0, 1]$, we obtain the relationship

$$\int_0^1 (y_2(x)\widetilde{T}y_1(x) - y_1(x)\widetilde{T}y_2(x))dx = (\lambda_1 - \lambda_2) \int_0^1 y_1(x)y_2(x)dx.$$

According to Proposition 4, the Formula (10) leads to the equation

$$(\lambda_1 - \lambda_2) \int_0^1 y_1(x)y_2(x)dx = 0,$$

which implies that $\int_0^1 y_1(x)y_2(x)dx = 0$ as $\lambda_1 \neq \lambda_2$. This is exactly what we want to prove. □

The following theorem obtained the geometric multiplicity of the eigenvalues for the nonlocal Sturm–Liouville problem (8).

Theorem 3. *The eigenvalues of the nonlocal Sturm–Liouville eigenvalue problem* (8) *are simple for* $|\mu| < \frac{1}{7}\Gamma(2-\alpha)(1+\alpha)$ *and* $\lambda > 0$.

Proof. Let $\psi_1(x)$ and $\psi_2(x)$ be the two eigenfunctions of the eigenvalue problem (8), with the corresponding eigenvalue being λ_0.

Denote
$$\psi(x) = \psi_1(x) - c\psi_2(x),$$
where c is an arbitrary constant.

It follows from (8) that $\psi(0) = 0$. One can check that $\psi'(x) = \psi_1'(x) - c\psi_2'(x)$.

Now we need to show that $\psi_2'(0) \neq 0$. If not, then $\psi_2(x)$ is a solution of the initial value problem (5), with $k_1 = k_2 = 0$. Hence, through Proposition 1, we conclude that $\psi_2 \equiv 0$, which is a contradiction.

Choose $c = \frac{\psi_1'(0)}{\psi_2'(0)}$. It follows that $\psi'(0) = 0$. That is, ψ satisfies the fractional initial value problem (5) with $k_1 = k_2 = 0$.

According to Proposition 1, if $|\mu| < \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{e^{\int_0^1 |q(t)|dt}(1+6e^{\int_0^1 |q(t)|dt})} < \frac{1}{7}\Gamma(2-\alpha)(1+\alpha)$, one sees that $\psi(x) \equiv 0$ on $(0,1)$, which implies that $\psi_1(x)$ and $\psi_2(x)$ are linearly dependent on $(0,1)$, which completes the proof. □

4. Eigenvalue Problem with Dirichlet Boundary Condition and $0 < \alpha < 1/2$

Due to the limited results of the initial value theory for nonlocal Sturm–Liouville problems (1), it is not possible to study the continuous dependence of eigenvalues on potential functions using the initial value theory, as in references [26–28]. We will use the two-parameter method to conduct research below.

4.1. Eigenvalue Properties of a Two-Parameter Nonlocal Sturm–Liouville Problem

In this section, we discuss the properties of the eigenvalues of the following two-parameter nonlocal Sturm–Liouville problem

$$\begin{cases} -y''(x) + q(x)y(x) + \gamma(q_1(x) - q(x))y(x) + \mu(D_{1-}^{\alpha} I_{0+}^{\alpha} + I_{1-}^{\alpha} D_{0+}^{\alpha})y(x) = \lambda y(x), \\ y(0) = 0 = y(1), \end{cases} \tag{18}$$

where $0 < \alpha < 1/2$, $q_1, q \in L^2(0,1)$, $\gamma \in [0,1]$, λ is the spectral parameter, $\mu \in (-\mu_0, \mu_0)$ is fixed, and μ_0 is defined as in Proposition 2. These properties are important to get the continuous dependence of the eigenvalues on the potential function.

Define the fractional operator, T, by

$$Ty = -y'' + qy + \mu T_\alpha y, \; T_\alpha y := (D_{1-}^{\alpha} I_{0+}^{\alpha} + I_{1-}^{\alpha} D_{0+}^{\alpha})y, \; y \in \mathcal{D}, \tag{19}$$

where

$$\mathcal{D} := \{y \in L^2(0,1) : y, y' \in AC[0,1], -y'' + qy \in L^2(0,1), y(0) = 0 = y(1)\}. \tag{20}$$

For fixed $\mu \in (-\mu_0, \mu_0)$, assume that $\lambda_n(0)$ and $n \geq 1$ are the eigenvalues of the nonlocal Sturm–Liouville problem, $Ty = -y'' + q(x)y + \mu T_\alpha y = \lambda y$, $y(0) = 0 = y(1)$, $y \in \mathcal{D}$, which satisfies (6):

$$-\infty < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0) < \cdots, \; \lambda_n(0) \sim \pi^2 n^2, \; n \to \infty.$$

Denote by $T(\gamma)$ the operator given in (18) as

$$T(\gamma)y := Ty + \gamma T_1 y = Ty + \gamma(q_1(x) - q(x))y,\ y \in \mathcal{D},\ \gamma > 0.$$

Let $\lambda_n(\gamma)$ and $n \geq 1$ be the eigenvalues of the two-parameter nonlocal Sturm–Liouville problem

$$T(\gamma)y = Ty + \gamma T_1 y = \lambda y,\ y(0) = 0 = y(1). \tag{21}$$

Theorem 4. *Let $\mu \in (-\mu_0, \mu_0)$ be fixed. There exists $\gamma_0 > 0$, such that for $0 < \gamma < \gamma_0$, all the eigenvalues of (21) are simple and satisfy*

$$-\infty < \lambda_1(\gamma) < \lambda_2(\gamma) < \cdots < \lambda_n(\gamma) < \cdots,\ \lambda_n(\gamma) \sim \pi^2 n^2,\ n \to \infty. \tag{22}$$

Proof. By virtue of Definition 3, Proposition 3, and discussions similar to Theorem 3.8 in [1], we can prove that $\{T(\gamma), \gamma \in \mathbb{R}\}$ is a self-adjoint holomorphic family of type (A). Then for fixed $\mu \in (-\mu_0, \mu_0)$, there exists exactly one simple eigenvalue $\lambda_n(\gamma)$ of $T(\gamma)$ near each unperturbed eigenvalue $\lambda_n(0)$ for suitably small γ, since $\lambda_n(0)$ is simple. Moreover,

$$||T_1 y|| \leq \|q_1 - q\| \|y\|.$$

Therefore, the perturbation expansion near each $\lambda_n(0)$ has a positive convergence radius, ρ_n.

According to (4.74) in ([25], p. 406), the following inequality holds

$$\rho_n \geq \left(\frac{2(a + b|\lambda_n|)}{d_n} + 2b \right)^{-1}. \tag{23}$$

Then we obtain

$$\rho_n \geq \frac{d_n}{2\|q_1 - q\|}. \tag{24}$$

Here $a = \|q_1 - q\|$, $b = 0$, and d_n is the isolation distance of the eigenvalue $\lambda_n(0)$, defined as

$$d_n = \min\{|\lambda_n(0) - \lambda_{n-1}(0)|,\ |\lambda_{n+1}(0) - \lambda_n(0)|\}. \tag{25}$$

Then, if $\gamma < \rho_n$, there exists exactly one eigenvalue $\lambda_n(\gamma)$ of $T(\gamma)$, such that

$$|\lambda_n(\gamma) - \lambda_n(0)| < d_n/2.$$

Now we will prove that there exists $\gamma_0 > 0$, such that $\rho_n \geq \gamma_0$ for all $n \geq 1$. According to Proposition 2, we have $\lambda_n(0) \sim n^2\pi^2$ as $n \to \infty$. Hence,

$$d_n \sim (2n - 1)\pi^2, n \to \infty, \tag{26}$$

$$\rho_n \geq c_n := \frac{d_n}{2\|q_1 - q\|} \sim \frac{(n - \frac{1}{2})}{\|q_1 - q\|}\pi^2.$$

Let $\gamma < \delta_1$, where $\delta_1 = \frac{\pi^2}{4\|q_1 - q\|}$. Then there exists N, such that for $n > N$,

$$\rho_n \geq c_n > \delta_1 > \gamma.$$

Therefore, there exists exactly one simple eigenvalue $\lambda_n(\gamma)$ of $T(\gamma)$, such that

$$|\lambda_n(\gamma) - \lambda_n(0)| < d_n/2,\ n > N.$$

For $1 \leq n \leq N$, we choose

$$d_n = \min\{|\lambda_j(0) - \lambda_k(0)| : 1 \leq j \neq k \leq N\} := d. \tag{27}$$

By (24), we have

$$\rho_n \geq \frac{d}{2\|q_1 - q\|} := \delta_2,\ 1 \leq n \leq N.$$

Set $\gamma_0 = \min\{\delta_1, \delta_2\}$. Then

$$\rho_n \geq \gamma_0 \ \text{ for all } \ n \geq 1.$$

Denote by O_n the disc$|\lambda - \lambda_n(0)| < d_n/2,\ n \geq 1$. If $\gamma < \gamma_0$, then each O_n contains exactly one simple eigenvalue of $T(\gamma)$ for $n \geq 1$.

Let $\mathcal{A} = \cup_{n=1}^{\infty} O_n$. We need to prove that $\mathcal{A}$ contains all the eigenvalues of $T(\gamma)$.

Set $\widetilde{\mathcal{A}} = \mathbb{C} \setminus \mathcal{A}$. We will prove that for $\gamma < \gamma_0$, $\widetilde{\mathcal{A}} \subset P(T(\gamma))$, where $P(T(\gamma))$ is the resolvent of $T(\gamma)$.

Suppose $\lambda \in \widetilde{\mathcal{A}}$. If $\lambda \notin \mathbb{R}$, it follows from Theorem 3.8 in [1] that

$$\lambda \in P(T(\gamma)).$$

If $\lambda \in \mathbb{R}$, then for some $n \geq 1$, the following inequality holds

$$\lambda < \lambda_1(0) - d_1/2, \ \text{ or } \ \lambda_n(0) + d_n/2 < \lambda < \lambda_{n+1}(0) - d_{n+1}/2,$$

where d_n is defined as in (25) for $n > N$, and as in (27) for $1 \leq n \leq N$.

We now prove $\lambda \in P(T(\gamma))$.

Suppose, to the contrary, that λ is an eigenvalue of $T(\gamma)$. By Theorem 4.21 ([25], p. 408), there exist $0 < \delta < \gamma_0$ and $k \in \mathbb{N}$, such that if $\gamma < \delta$, the inequality $|\lambda - \lambda_k(0)| < d_k/2$ holds, which implies that there exists $k \in \mathbb{N}$ such that $\lambda \in O_k$.

Each O_k contains exactly one simple eigenvalue of $T(\gamma)$ for $k \geq 1$. Therefore, we obtain a contradiction. Hence, $\lambda \in P(T(\gamma))$.

For $\gamma < \gamma_0$, we obtain

$$-\infty < \lambda_n(\gamma) < \lambda_{n+1}(\gamma), n \geq 1.$$

By (25) and (27), we have

$$\lambda_n(\gamma) \sim n^2\pi^2.$$

□

4.2. The Continuous Dependence of the Eigenvalues on the Potential Function

In this section, by the aid of the two-parameter method, we investigate the continuous dependence of the eigenvalues of

$$\begin{cases} -y''(x) + q(x)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x) \quad \text{on } (0,1), \\ y(0) = 0 = y(1). \end{cases} \tag{28}$$

where $0 < \alpha < 1/2$, $q \in L^2(0,1)$, λ is the spectral parameter, $\mu \in (-\mu_0, \mu_0)$ is fixed, and μ_0 is defined as in Proposition 2.

When $\gamma = 0$, equation

$$-y''(x) + q(x)y(x) + \gamma(q_1(x) - q(x))y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x)$$

degenerates into equation

$$-y''(x) + q(x)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x),$$

and when $\gamma = 1$, equation

$$-y''(x) + q(x)y(x) + \gamma(q_1(x) - q(x))y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x)$$

can be transformed into

$$-y''(x) + q_1(x)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x).$$

Therefore, the continuous dependence of the eigenvalue of $-y''(x) + q(x)y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x)$ on the potential function $q(x)$ can be transformed into the continuous dependence of the eigenvalue of $-y''(x) + q(x)y(x) + \gamma(q_1(x) - q(x))y(x) + \mu(D^{\alpha}_{1-}I^{\alpha}_{0+} + I^{\alpha}_{1-}D^{\alpha}_{0+})y(x) = \lambda y(x)$ on the parameter γ.

Theorem 5. *Let $\mu \in (-\mu_0, \mu_0)$, $n \geq 1$, and $q_0 \in L^2(0,1)$ be fixed. For any $\varepsilon > 0$, there exists $\delta_n > 0$, such that if $||q_1 - q_0|| \leq \delta_n$ for any $q_1 \in L^2(0,1)$, then $|\lambda_{n,q_1} - \lambda_{n,q_0}| < \varepsilon$, where λ_{n,q_i} $(i = 0,1)$ are the n-th eigenvalue of $-y'' + q_i y + \mu T_\alpha y = \lambda y$ subject to $y(0) = 0 = y(1)$.*

Proof. For two-parameter nonlocal Sturm–Liouville problem

$$-y''(x) + q_0(x)y(x) + \gamma(q_1(x) - q_0(x))y(x) + \mu T_\alpha y(x) = \lambda y(x), y(0) = 0 = y(1), \quad (29)$$

$\lambda_n(\gamma)(n \geq 1)$ are corresponding eigenvalues.

It suffices to show that for any $\varepsilon > 0$, there exists $\delta_n > 0$, such that for any $q_1 \in L^2(0,1)$, if $\gamma < \delta_n$, then $|\lambda_n(\gamma) - \lambda_n(0)| < \varepsilon$.

For the sake of simplicity in writing, we dropped the variable x and the subscript n. By Theorem 4, each eigenvalue $\lambda(\gamma)$ is simple on $(0, \gamma_0)$. Choose $0 < |\Delta| \ll 1$, such that $0 < \gamma + \Delta < \gamma_0$. Assume $\lambda(\gamma)$ and $\lambda(\gamma + \Delta)$ are different eigenvalues of (29). Let eigenfunctions $\varphi(\gamma)$ and $\varphi(\gamma + \Delta)$ denote the corresponding normalized eigenfunctions of $\lambda(\gamma)$ and $\lambda(\gamma + \Delta)$, respectively. Then we obtain

$$\begin{cases} -\varphi''(\gamma) + q_0(x)\varphi(\gamma) + \gamma(q_1(x) - q_0(x))\varphi(\gamma) + \mu T_\alpha \varphi(\gamma) = \lambda(\gamma)\varphi(\gamma), \\ \varphi(0,\gamma) = 0 = \varphi(1,\gamma) \end{cases} \quad (30)$$

and

$$\begin{aligned} &-\varphi''(\gamma + \Delta) + q_0(x)\varphi(\gamma + \Delta) + (\gamma + \Delta)(q_1(x) - q_0(x))\varphi(\gamma + \Delta) + \mu T_\alpha \varphi(\gamma + \Delta) \\ &= \lambda(\gamma + \Delta)\varphi(\gamma + \Delta),\ \varphi(0, \gamma + \Delta) = 0 = \varphi(1, \gamma + \Delta). \end{aligned} \quad (31)$$

(31)$\times\varphi(\gamma)$−(30)$\times\varphi(\gamma + \Delta)$, and integrating on [0, 1], we have

$$\begin{aligned} &(\lambda(\gamma + \Delta) - \lambda(\gamma)) \int_0^1 \varphi(\gamma)\varphi(\gamma + \Delta) \\ &= \Delta \int_0^1 (q_1(x) - q_0(x))\varphi(\gamma)\varphi(\gamma + \Delta) + \mu \int_0^1 (\varphi(\gamma)(T_\alpha \varphi(\gamma + \Delta)) - \varphi(\gamma + \Delta)(T_\alpha \varphi(\gamma))). \end{aligned} \quad (32)$$

Moreover, we obtain

$$\int_0^1 \varphi(\gamma)(T_\alpha \varphi(\gamma + \Delta)) = \int_0^1 \varphi(\gamma + \Delta)(T_\alpha \varphi(\gamma)).$$

Therefore,

$$\lambda'(\gamma) = \lim_{\Delta\to 0} \frac{\lambda(\gamma+\Delta) - \lambda(\gamma)}{\Delta} = \int_0^1 \tilde{q}\varphi^2(\gamma), \tag{33}$$

where $\tilde{q}(x) := q_1(x) - q_0(x)$.

Define

$$\tilde{Q}(x) = \int_0^x \tilde{q}(t)dt, x \in [0,1], \tilde{Q}_0 = \max_{x\in[0,1]}\{|\tilde{Q}(x)|\},$$

$$Q(x) = \int_0^x q_0(t)dt, x \in [0,1], Q_0 = \max_{x\in[0,1]}\{|Q(x)|\}.$$

Since $||\varphi(\gamma)|| = 1$, $\varphi(1,\gamma) = \varphi(0,\gamma) = 0$, then we obtain

$$|\int_0^1 \tilde{q}\varphi(\gamma)|^2 \le \frac{||\varphi'(\gamma)||^2}{4\gamma} + 4\tilde{Q}_0^2\gamma,\ |\int_0^1 q_0\varphi(\gamma)|^2 \le \frac{||\varphi'(\gamma)||^2}{4} + 4Q_0^2, \tag{34}$$

and

$$|\int_0^1 \tilde{q}\varphi(\gamma)|^2 \le \frac{||\varphi'(\gamma)||^2}{4} + 4\tilde{Q}_0^2, \tag{35}$$

Because $T_\alpha y = (D^\alpha_{1-} I^\alpha_{0+} + I^\alpha_{1-} D^\alpha_{0+})y$, by Definitions 1 and 2, we get the relationship

$$T_\alpha y = M_\alpha \left(\frac{\int_0^1 \frac{y(s)}{(1-s)^{1-\alpha}} ds}{(1-x)^\alpha} - \int_x^1 \frac{\int_0^t \frac{\frac{d}{ds}y(s)}{(t-s)^{1-\alpha}} ds}{(t-x)^\alpha} dt + \int_x^1 \frac{\int_0^t \frac{\frac{d}{ds}y(s)}{(t-s)^{\alpha}} ds}{(t-x)^{1-\alpha}} dt \right),$$

where $M_\alpha = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$. Therefore,

$$\begin{aligned} \| T_\alpha y \| &\le M_\alpha \left(\| \int_x^1 \frac{\int_0^t \frac{\frac{d}{ds}y(s)}{(t-s)^{1-\alpha}} ds}{(t-x)^\alpha} dt \| + \| \int_x^1 \frac{\int_0^t \frac{\frac{d}{ds}y(s)}{(t-s)^{\alpha}} ds}{(t-x)^{1-\alpha}} dt \| + \| \frac{\int_0^1 \frac{y(s)}{(1-s)^{1-\alpha}} ds}{(1-x)^\alpha} \| \right) \\ &= M_\alpha \left(\int_0^1 \left| \int_x^1 \frac{\int_0^t \frac{y'(s)}{(t-s)^{1-\alpha}} ds}{(t-x)^\alpha} dt \right|^2 dx \right)^{1/2} + M_\alpha \left(\int_0^1 \left| \int_x^1 \frac{\int_0^t \frac{y'(s)}{(t-s)^{\alpha}} ds}{(t-x)^{1-\alpha}} dt \right|^2 dx \right)^{1/2} \\ &\quad + M_\alpha \left(\int_0^1 \left| \frac{\int_0^1 \frac{y(s)}{(1-s)^{1-\alpha}} ds}{(1-x)^\alpha} \right|^2 dx \right)^{1/2}. \end{aligned}$$

Denoted by

$$C_1 = \left(\int_0^1 \left| \int_x^1 \frac{\int_0^t \frac{y'(s)}{(t-s)^{1-\alpha}} ds}{(t-x)^\alpha} dt \right|^2 dx \right)^{1/2},\ C_2 = \left(\int_0^1 \left| \int_x^1 \frac{\int_0^t \frac{y'(s)}{(t-s)^{\alpha}} ds}{(t-x)^{1-\alpha}} dt \right|^2 dx \right)^{1/2},$$

$$C_3 = \left(\int_0^1 \left| \frac{\int_0^1 \frac{y(s)}{(1-s)^{1-\alpha}} ds}{(1-x)^\alpha} \right|^2 dx \right)^{1/2}.$$

It follows that

$$\| T_\alpha y \| \le M_\alpha (C_1 + C_2 + C_3).$$

Utilizing the Cauchy–Schwarz inequality in conjunction with the integration by parts formula many times yields the following results

$$\begin{aligned} C_1^2 &\leq \int_0^1 \left(\int_x^1 (t-x)^{-\alpha} dt \right) \left(\int_x^1 \frac{\left| \int_0^t \frac{y'(s)ds}{(t-s)^{1-\alpha}} \right|^2}{(t-x)^\alpha} dt \right) dx, \\ &\leq \frac{1}{1-\alpha} \int_0^1 \left(\int_0^t (t-x)^{-\alpha} dx \right) \left| \int_0^t \frac{y'(s)}{(t-s)^{1-\alpha}} ds \right|^2 dt \\ &\leq \frac{1}{\alpha(1-\alpha)^2} \int_0^1 \int_0^t \frac{|y'(s)|^2}{(t-s)^{1-\alpha}} dsdt \leq \frac{1}{\alpha^2(1-\alpha)^2} \parallel y' \parallel^2 \end{aligned}$$

$$\begin{aligned} C_2^2 &\leq \left(\int_0^1 \left(\int_x^1 (t-x)^{\alpha-1} dt \right) \left(\int_x^1 \frac{\left| \int_0^t \frac{y'(s)ds}{(t-s)^\alpha} \right|^2}{(t-x)^{1-\alpha}} dt \right) dx \right)^{1/2} \\ &\leq \frac{1}{\alpha^2(1-\alpha)} \int_0^1 \int_0^t \frac{|y'(s)|^2}{(t-s)^\alpha} dsdt \leq \frac{1}{\alpha^2(1-\alpha)^2} \parallel y' \parallel^2 \end{aligned}$$

$$\begin{aligned} C_3 &= \left(\int_0^1 (1-x)^{-2\alpha} dx \right)^{1/2} \left| \int_0^1 \frac{y(s)}{(1-s)^{1-\alpha}} ds \right| \\ &= \frac{1}{\sqrt{1-2\alpha}} \left| \int_0^1 \frac{y(s)}{(1-s)^{1-\alpha}} ds \right| \\ &\leq \frac{\parallel y' \parallel}{\alpha\sqrt{1-2\alpha}}. \end{aligned}$$

Hence,

$$\parallel T_\alpha y \parallel \leq M_\alpha(C_1 + C_2 + C_3) \leq C_\alpha \parallel y' \parallel,$$

where $C_\alpha = \frac{1-\alpha+2\sqrt{1-4\alpha^2}}{\Gamma(1+\alpha)\Gamma(2-\alpha)\sqrt{1-4\alpha^2}}$.

By calculation, we find that

$$| \langle T_\alpha \varphi(\gamma), \varphi(\gamma) \rangle | \leq C_\alpha \parallel \varphi'(\gamma) \parallel \parallel \varphi(\gamma) \parallel \leq \frac{1}{4|\mu|} \parallel \varphi'(\gamma) \parallel^2 + |\mu| C_\alpha^2 \parallel \varphi(\gamma) \parallel^2 . \quad (36)$$

By (30), we have

$$||\varphi'(\gamma)||^2 + \int_0^1 q|\varphi(\gamma)|^2 + \gamma \int_0^1 \tilde{q}|\varphi(\gamma)|^2 + \mu \langle T_\alpha \varphi(\gamma), \varphi(\gamma) \rangle = \lambda(\gamma). \quad (37)$$

By means of (34), (36), and (37), we obtain

$$||\varphi'(\gamma)||^2 \leq 4(\lambda(\gamma) + 4Q_0^2 + 4\tilde{Q}_0^2\gamma^2 + |\mu|^2 C_\alpha^2). \quad (38)$$

A combination of (33), (35), and (38) gives that

$$|\lambda'(\gamma)| = | \int_0^1 \tilde{q}\varphi(\gamma)|^2 \leq \frac{||\varphi'(\gamma)||^2}{4} + 4\tilde{Q}_0^2 \leq \lambda(\gamma) + c, \quad (39)$$

where $c = 4Q_0^2 + 4\tilde{Q}_0^2\gamma^2 + |\mu|^2 C_\alpha^2 + 4\tilde{Q}_0^2$. Solving the differential inequality (39), we have

$$\lambda(\gamma) + c \leq e^\gamma(\lambda(0) + c).$$

Therefore,

$$|\lambda'(\gamma)| \leq e^{\gamma}(\lambda(0)+c).$$

Hence, for any $\varepsilon > 0$, and for any $q_1 \in L^2(0,1)$, if $|\gamma| < \delta_n = \min\{\gamma_0, \frac{\varepsilon}{e^{\gamma}(\lambda(0)+c)}\}$, we have

$$|\lambda(\gamma) - \lambda(0)| = |\int_0^{\gamma} \lambda'(t)dt| \leq |\gamma|e^{\gamma}(\lambda(0)+c) < \varepsilon,$$

which completes the proof. □

5. Conclusions

In this paper, we considered a nonlocal Sturm–Liouville problem (1)–(2) with fractional integrals and fractional derivatives. We obtained that the eigenvalues of (1)–(2) with $d \neq 0$ are real values, and the corresponding eigenfunctions are orthogonal; see Theorems 1 and 2. In Theorem 3, based on these properties, we obtained results that show the geometric multiplicity of the eigenvalues is simple. Thereafter, we discussed the eigenvalue problem of (1)–(2) with $d = 0$. We led into an auxiliary two-parameter nonlocal Sturm–Liouville problem (18). In Theorem 4, we derived that the corresponding eigenvalue problem consists of a countable number of real eigenvalues, and the algebraic multiplicity of each eigenvalue is simple. With the aid of the eigenvalue properties of this nonlocal problem, we came to the conclusion that the eigenvalues are continuous with respect to the potential function; see Theorem 5.

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